SYMMETRIC STRUCTURES ON A CLOSED CURVE

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We show the quasisymmetric topology of Ahlfors ([1], 1965) (the topology coming from uniform ratio distortion) on local homeomorphisms in one real dimension is defined when, and only when, the underlying one-manifold is provided with a "symmetric structure," one defined by using as structure pseudogroup the quasisymmetric closure of the C^1-diffeomorphisms of the real line. We show that the set of all symmetric structures on a closed curve compatible with a background quasisymmetric structure is naturally a *complete*, *complex Banach manifold*, modelled on the Banach space Λ^*/λ^*, where Λ^* and λ^* are the spaces of continuous functions F on the circle introduced by Zygmund ([17], 1945);

$$\Lambda^* : F(x + t) + F(x - t) - 2F(x) = O(t)$$

$$\lambda^* : F(x + t) + F(x - t) - 2F(x) = o(t)$$

and the complex structure is given by the Hilbert transform.

The discussion covers analytical and geometrical properties of symmetric homeomorphisms and symmetric quasicircles and suggests how the Bers' embedding technique (1965) may be used in a variety of contexts.

0. Description of results. The notion of a quasisymmetric homeomorphism of an interval into $\mathbb{R}$ is useful in the theory of Riemann surfaces and, more generally, in the theory of one real dimensional smooth dynamical systems. The set of quasisymmetric homeomorphisms is closed under composition and inverse and can be recognized locally

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by the condition that symmetric triples of points are not made too asymmetric.

The inclusion mapping of an interval I of $\mathbb{R}$ into $\mathbb{R}$ has a quasisymmetric neighborhood system. This neighborhood system is defined by declaring that a mapping f from I into $\mathbb{R}$ is near to the inclusion if it is near in the uniform topology and if the M-condition,

$$M^{-1} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M,$$

is satisfied for every symmetric triple $x - t$, x, and $x + t$ in the interval and for M equal to $1 + \epsilon$ for a small value of ϵ. In words, the neighborhood is defined by small movement of points and small distortion of symmetric triples.

This neighborhood system is not natural (or functorial) for the full pseudogroup of quasisymmetric homeomorphisms. In general, suppose f is a quasisymmetric mapping from an interval I to an interval J and that g is in a small neighborhood of the inclusion mapping for I. Then $f \circ g \circ f^{-1}$ will not necessarily be in a small neighborhood of inclusion mapping of J. It is true that the M-condition with a value of M near to 1 will be preserved for symmetric triples $x - t, x, x + t$ when t is fairly large because the mapping g is assumed to be uniformly near to the identity. However, for small values of t, the mapping f can distort symmetric triples by a definite amount and a small movement of g away from the identity can mean that, for small values of t, the mapping $f \circ g \circ f^{-1}$ no longer satisfies an M-condition with M near to 1.

There is, however, a large proper subpseudogroup of quasisymmetric mappings which preserves the neighborhood system. We call this proper subpseudogroup the pseudogroup of symmetric homeomorphisms, because it turns out to be described by a symmetry property at fine scales.

Since the adjoint action in QS is not continuous at the identity, the group QS is not a topological group. It is, however, a "partial topological group," in the sense of Section 1. In this section, we show that for any partial topological group the subgroup of elements f for which the adjoint action by f is continuous is always a closed topological subgroup. This subgroup is called the characteristic topological subgroup. The characteristic topological subgroup S of the group QS of quasisymmetric homeomorphisms of a circle turns out to be the group of symmetric homeomorphisms.

Most of the remaining sections of the paper deal with properties of S which are analogous to known properties of QS. In Section 2 we show that the symmetric subgroup S consists precisely of those mappings which satisfy a modified M-condition. In local coordinates, any element of S must satisfy an M-condition with $M = 1 + \epsilon(t)$ where $\epsilon(t)$ converges to zero with t uniformly in x. This characterization shows that, just as in the pseudogroup setting, the groups of quasisymmetric and symmetric mappings are recognized by how they distort symmetric triples of points. The difference is illustrated in Figure 1.

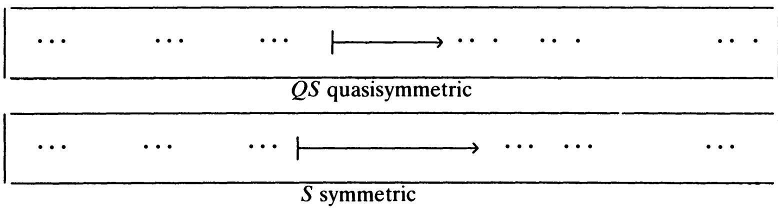

FIGURE 1

At large scales both symmetric and quasisymmetric mappings can distort symmetrically placed triples of points by a bounded amount. However, at arbitrarily small scales, quasisymmetric mappings can distort by the same bounded amount, whereas symmetric mappings must distort by lesser and lesser amounts as the scale gets smaller.

In Section 3, we look at the possible quasiconformal extensions of symmetric and quasisymmetric mappings. The Beurling-Ahlfors extension theorem tells us that every quasisymmetric homeomorphism of $\hat{\mathbb{R}}$ can be extended to a quasiconformal homeomorphism of the upper half plane. Analogously, we find that symmetric homeomorphisms are the boundary values of quasiconformal homeomorphisms of the upper half plane whose conformal distortion tends to zero at the boundary. Symmetric homeomorphisms turn out to be precisely those homeomorphisms which have boundary dilatation equal to one, in the sense of Strebel, [16]. The symmetric homeomorphisms of a circle comprise the closure, in the quasisymmetric topology, of the real analytic homeomorphisms and this closure contains the set of C^1-diffeomorphisms.

Section 4 deals with the analogue for symmetric mappings of the

Bers' embedding for quasisymmetric mappings. To describe the Bers embedding, consider a quasiconformal mapping f of the complex sphere $\hat{\mathbb{C}}$ with Beltrami coefficient equal to μ in the upper half plane and Beltrami coefficient identically equal to zero in the lower half plane. Let Ω be the image under f of the lower half plane and let Ω^* be the image under f of the upper half plane. The common boundary, C, namely, the image under f of $\hat{\mathbb{R}}$, is by definition a quasicircle. By the Riemann mapping theorem, there is a conformal mapping g of the upper half plane onto the domain Ω^*. Since $g^{-1} \circ f$ is a quasiconformal self-mapping of the upper half plane, its boundary values on $\hat{\mathbb{R}}$ are quasisymmetric. We let h be the restriction of $g^{-1} \circ f$ to $\hat{\mathbb{R}}$. From the Beurling-Ahlfors extension theorem and the measurable Riemann mapping theorem, any quasisymmetric mapping can be factored in this way and, up to an ambiguity of Möbius equivalence, this factorization is unique. It follows that every h in QS mod $PSL(2, \mathbb{R})$ uniquely determines f and the Bers' embedding is defined by

$$\mathcal{S}(h) = \text{Schwarzian derivative of } f,$$

where the Schwarzian derivative is defined by $((f''/f')' - 1/2 (f''/f')^2)dz^2$. It turns out that $\mathcal{S}$ is one-to-one and onto a bounded open set in a Banach space B. The space B consists of holomorphic quadratic differentials in the lower half plane which are pointwise uniformly bounded in the Poincaré metric, [Bers, 4]. Figure 2 gives a schematic outline of this embedding. We think of the upper and lower half planes as two complementary disks on the sphere whose common boundary is a great circle passing through the north and south poles. The mappings

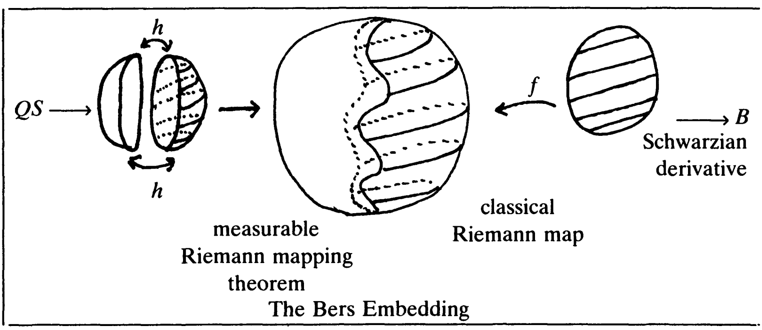

FIGURE 2

f and g are conformal mappings of standard disks onto the domains Ω and Ω^* and the common boundary, C, is a pasting locus where the rule for gluing is governed by the quasisymmetric mapping h.
We carry out an analogous procedure for $S/PSL(2, \mathbb{R})$. We find that there is also a Bers embedding in this context. Instead of using the Banach space B, it uses the closed Banach subspace B_0 of B consisting of those elements of B which vanish at the boundary of the disk.

In Section 5 we consider the coset decomposition of QS by translates of S. Viewed in QS mod $PSL(2, \mathbb{R})$, these cosets are closed complex analytic submanifolds. We show that there is a natural quotient metric on QS mod S coming from the Teichmüller metric on QS mod $PSL(2, \mathbb{R})$. The metric distance between two cosets Sf and Sg in QS mod S is given by $\log H(f \circ g^{-1})$ where H is the boundary dilatation of a mapping in the sense of Strebel, [16]. More precisely, for a quasiconformal self-mapping f of the upper half plane, we consider the equivalence class of all other quasiconformal selfmappings f_1 which agree with f on the boundary. If we can find such a mapping f_1 whose dilatation inside a neighborhood of $\hat{\mathbb{R}}$ is less than K, then $H(f) \leq K$. By definition, $H(f)$ is the largest number such that this inequality is true for every f_1 in the class of f and for sufficiently small neighborhoods of $\hat{\mathbb{R}}$.

In [1], Ahlfors characterizes quasicircles geometrically. A quasicircle C in $\hat{\mathbb{C}}$ is characterized by a "reverse triangle inequality":

$$\text{spherical dist}(a, b) + \text{spherical dist}(b, c) \leq K \text{ spherical dist}(a, c)$$

for any triple of points a, b and c on C, where b lies on the part of C which joins a to c and which has smaller spherical diameter. There is a parallel condition, shown by Becker and Pommerenke in [3], for a quasicircle to correspond to the image of the real axis under a mapping f which is asymptotically conformal on $\overline{\mathbb{R}}$. This turns out to be equivalent to saying that, under the factorization of h into $g^{-1} \circ f$, the mapping h is symmetric. By definition, we call a quasicircle of this type a symmetric quasicircle. The result of Becker and Pommerenke states that C is a symmetric quasicircle if, and only if, the reverse triangle inequality is satisfied with $K = 1 + \epsilon$ for triples of points in a sufficiently small neighborhood. At a small enough scale, the condition forces the quasicircle to move very near to a straight line. However, the condition still permits the quasicircle to be nonrectifiable and to have dense spirals.

In Section 6, we prove Becker and Pommerenke's result together with two other geometric properties which characterize the symmetry of a quasicircle. The first is the disk template property. Roughly speaking, a disk template at points p and q of a quasicircle is a disk centered at p together with a narrow canal bounded by two parallel lines equidistant from a diameter. The diameter passes through p and q. The part of the disk which is not in the canal is called land. In order to be a disk template, the quasicircle must never enter the land, the thickness of the canal must be commensurate with the distance from p to q, and the radius of the disk divided by the thickness of the canal must be large. In the proof we use the reverse triangle inequality to construct disk templates with slightly different properties, depending on three nearby points on the quasicircle. The crucial point is that, by looking at a greatly magnified scale, we are able to construct disk templates for which the radius of the disk is arbitrarily large compared to the thickness of the canal. In effect, the disk template property says that, under magnification, straight lines are the only possible limit points of the quasicircle in the Hausdorff topology on closed sets.

Our second geometric property, which is an equivalent condition for a quasicircle to be symmetric, is the extremal length property. It is a condition on the extremal lengths of conjugate curve families on the sphere punctured at four points along the quasicircle. Three of the four points are assumed to be variable and arbitrarily close together and the fourth far away.

In Section 7, we focus on the factor space, QS mod S. According to the discussion in the appendix, this factor space can be viewed as the space of symmetric structures on a circle subordinate to a given quasisymmetric structure. We show that QS mod S is a Hausdorff complex manifold modelled on the complex Banach space B/B_0. QS mod $PSL(2, \mathbb{R})$ is sometimes called universal Teichmüller space because all of the unreduced Teichmüller spaces of hyperbolic Riemann surfaces embed here. QS mod S is also universal in the sense that it contains all of the Teichmüller spaces of Fuchsian groups of the first kind.

In Section 8, we identify the tangent spaces at the identity to the manifolds S, QS, and QS/S. We find that the tangent space at the identity of QS consists of all continuous vector fields $F(x)$ on $\hat{\mathbb{R}}$ for which

$$(QS) \qquad |F(x+t) + F(x-t) - 2F(x)| \leq O(t)$$

while the tangent space at the identity of S consists of those vector fields for which

$$(S) \qquad |F(x+t)+F(x-t)-2F(x)| \leq o(t)$$

where $O(t)/t$ is bounded uniformly for x and t and $o(t)/t$ approaches 0 with t uniformly in x. This description for the vector fields for QS has already been given by Reimann in [13]. Both descriptions follow by using the known infinitesimal structure for the manifold of quasiconformal homeomorphisms of the sphere, [2], and applying the Beurling-Ahlfors extension to vector fields. It is in the space (QS) mod (S) where we look for the infinitesimal metric structure which will give the global metric determined by boundary dilatation. We would like to determine in what sense the metric is natural for the complex structure on QS mod S and hope to study this question later.

Functions in the space (QS) satisfy an $x \log x$ modulus of continuity and the spaces (S) and (QS) are studied by Zygmund in [17]. Zygmund's notations for (QS) and (S), considered as spaces of functions, are Λ^* and λ^*, respectively: he calls the space λ^* the space of "smooth" functions. It has been pointed out by Steve Kerckhoff and communicated to us by Subhashis Nag that the almost complex structure for universal Teichmüller space is just the Hilbert transform acting on vector fields. But universal Teichmüller space is QS mod $PSL(2, \mathbb{R})$. Moreover, we know that S mod $PSL(2, \mathbb{R})$ is a complex submanifold. Thus, one can conclude from Teichmüller theory that the spaces (QS) and (S) are mapped isomorphically onto themselves by the Hilbert transform. Independently of their relationship to Teichmüller theory, these facts were first proved by Zygmund in [17]. Actually, in [17] Zygmund gives three different proofs that the Hilbert transform preserves the classes (QS) and (S). For us, it is significant that the Zygmund norm is equivalent to the norm given by the infinitesimal form of Teichmüller's metric and that the Hilbert transform becomes an isometry if we use the infinitesimal Teichmüller norm on the Banach spaces (QS) and (S) and the quotient norm on $(QS)/(S)$.

In Section 9, we record some of the smoothness properties of symmetric and quasisymmetric mappings. Most of the results of this section are already contained in Carleson's paper [6].

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1. Quasisymmetric mappings viewed as a partial topological group. In this section we focus on the group of homeomorphisms of a one dimensional manifold which are quasisymmetric. We show that this group satisfies the axioms for what we call a partial topological group. Such a group always determines a characteristic closed topological subgroup. In the case of the group of quasisymmetric homeomorphisms, this subgroup turns out to be the group of symmetric homeomorphisms.

First, we introduce the notion of a quasisymmetric mapping h defined on an open interval of the real axis. We say a mapping h satisfies an M-condition on an interval if there exists a constant M such that

$$M^{-1} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \tag{1}$$

for all numbers x and t for which $x - t$, x, and $x + t$ are in the interval. We recall,

Definition 1.1. Let h be a homeomorphism mapping an open interval I of the real axis into the real axis. Then h is quasisymmetric on I if there exists a constant M such that inequality (1) is satisfied for all $x - t$, x and $x + t$ in I.

It is known that quasisymmetric homeomorphisms defined on intervals of the real line form a pseudogroup. This means that if h_1 and h_2 are quasisymmetric then so is the composition $h_2 \circ h_1$ and the inverse h_1^{-1}. Here, by the composition $h_2 \circ h_1$ we mean the mapping $h_2(h_1(x))$ defined only when x is in the domain of h_1 and $h_1(x)$ is in the domain of h_2.

The fact that quasisymmetric mappings form a pseudogroup follows from their extendability by quasiconformal mappings. Since in later parts of this paper we will look in more detail at the problem of quasiconformally extending quasisymmetric mappings, we postpone proving here the pseudogroup properties.

Denote by QS the pseudogroup of quasisymmetric mappings on the real line. We can define a QS-structure on any topological one dimensional manifold. It is an atlas of coordinate charts h_α for the manifold which is maximal subject to the following condition; the transition functions $h_\alpha \circ h_\beta^{-1}$ must be elements of the pseudogroup QS.

Let the QS-structure on the topological one-dimensional manifold be specified and let h be a homeomorphism of the manifold onto itself. We say that h is quasisymmetric (for this QS-structure) if, in terms of any charts h_1 and h_2 mapping open subsets of the manifold to intervals I_1 and I_2 of the real axis, the composition $h_2 \circ h \circ h_1^{-1}$ is a quasisymmetric mapping from I_1 to I_2. It is obvious that the set QS of homeomorphisms of the manifold which are quasisymmetric forms a group.

In order to define a system of neighborhoods of the identity, we need a finer structure subordinate to the given QS-structure. We suppose that we are given a subordinate $PSL(2, \mathbb{R})$ structure. By this we mean there is a finite open covering U_α of the manifold and charts h_α for the given QS-structure mapping U_α onto the coordinate patches $h_\alpha(U_\alpha)$, which are intervals on the real axis, such that $h_\alpha \circ h_\beta^{-1}$ is in $PSL(2, \mathbb{R})$ whenever this composition is defined. With respect to such a system of charts we define a subbasic neighborhood N_ϵ of the identity in the following way. N_ϵ is the set of h for which

a) h and h^{-1} are uniformly ϵ-near to the identity when viewed on each coordinate patch, and

b) h and h^{-1} satisfy an M-condition (1) with $M = 1 + \epsilon$, viewed on each coordinate patch.

One checks that the system of neighborhoods N defined in this way is independent of which finite system of charts for the given $PSL(2, \mathbb{R})$ structure is selected. Obviously, this system of neighborhoods is Hausdorff in the sense that the intersection of all neighborhoods of the identity is the identity itself. The system has two further properties. Namely, given a neighborhood V of the identity, there is a neighborhood U such that

i) $U \circ U \subset V$ and
ii) $U^{-1} \subset V$.

These two properties can be paraphrased as follows: given any two mappings h_1 and h_2 near to the identity, then the product $h_1 \circ h_2$ and the inverse h_1^{-1} is also near to the identity. These facts follow easily from the relationship between quasisymmetric and quasiconformal mappings described in the next section.

Although the neighborhoods of the identity can be transported around to give neighborhoods of every other element of the group, the uniform topological space so obtained is not necessarily a topological group. Since one knows that the group QS is not a topological group, it is relevant to give here a brief theory of what we call *partial topological groups*.

Definition 1.2. A partial topological group is a group with a neighborhood system at the identity which is respected by composition and inverse in the sense of i) and ii) above.

At a general point h of the group, there are two neighborhood systems, one defined by left translation by h of the neighborhood system at the identity and one defined by right translation. If N runs through the neighborhood system at the identity, then $h \circ N$ and $N \circ h$ are systems of left and right neighborhoods of h, respectively.

LEMMA 1.1. *The following conditions on a partial topological group are equivalent*:

i) *it is a topological group with the given neighborhood system of the identity*,

ii) *the left and the right neighborhood systems agree at every point*,

iii) *the adjoint map $h \mapsto f \circ h \circ f^{-1}$ is continuous at the identity for every f in the group*.

Proof. Property ii) implies that if an element f is given, then for every neighborhood U of identity, there exists a neighborhood V of the identity for which $f \circ V \subset U \circ f$. Thus $f \circ V \circ f^{-1} \subset U$. This implies property iii). Obviously this remark is reversible, so iii) implies the system of left neighborhoods of f is finer than the system of right neighborhoods. But $f \circ V \subset U \circ f$ implies that $V^{-1} \circ f^{-1} \subset f^{-1} \circ U^{-1}$ and, since one of the axioms for a partial topological group tells us that taking inverses at the identity is a continuous operation, we see that iii) implies ii).

Because multiplication and taking inverses is continuous in a topological group, obviously i) implies iii). To prove that iii) implies i), we must show, in particular, that the map $(f, g) \mapsto f \circ g$ is continuous. Assume that f and g are near to f_o and g_o in the right topology. That is, $f \circ f_o^{-1}$ and $g \circ g_o^{-1}$ are near to the identity. Note that

$$f \circ g \circ (f_o \circ g_o)^{-1} = (f \circ f_o^{-1}) \circ (f_o \circ (g \circ g_o^{-1}) \circ f_o^{-1}).$$

From iii) we see that $f_o \circ (g \circ g_o^{-1}) \circ f_o^{-1}$ is near the identity. Since $f \circ f_o^{-1}$ is also near the identity, the right hand side of this equation is near the identity since it is a product of two terms near the identity. It is also easy to show that, under the assumption that iii) is satisfied, the operation of taking inverses is continuous.

In a general partial topological group the properties of Lemma 1.1 will not be satisfied. One of the two topologies in a partial topological group will be left translation invariant and the other right translation invariant. The inverse operation interchanges these two topologies.

One can consider those elements h of a partial topological group for which the two neighborhood systems at h agree, that is, those elements h for which conjugation by h maps the neighborhood system at the identity isomorphically onto itself. These elements form a subgroup, the two topologies agree on this subgroup, and this group with this topology is a topological group. Let us call this subgroup the *characteristic topological subgroup* of a partial topological group.

If a subset of a partial topological group is invariant under the inverse operation, then it is closed for one topology if and only if it is closed for the other. In particular, one may speak without ambiguity of a closed subgroup of a partial topological group.

LEMMA 1.2. *The characteristic topological subgroup of partial topological group is a closed topological subgroup.*

Proof. Assume that the mappings $g \mapsto h_n \circ g \circ h_n^{-1}$ are continuous at the identity and that h_n converges to h_o. By the remark preceding the lemma, it does not matter whether we assume h_n converges to h_o in the left or in the right topology. Therefore, we assume that for large n, $h_n \circ h_o^{-1}$ is near to the identity. Our hypotheses tell us that three quantities are near the identity:

$$h_o \circ h_n^{-1}, \qquad h_n \circ g \circ h_n^{-1}, \qquad \text{and} \quad h_n \circ h_o^{-1}.$$

Multiplying these three together, we see that $h_o \circ g \circ h_o^{-1}$ is near the identity and, consequently, h_o is also an element of the characteristic topological subgroup. We conclude that the characteristic subgroup is closed. From the preceding lemma, the right and left topologies agree on this subgroup and it is a topological subgroup.

The group QC of all quasiconformal homeomorphisms of the complex sphere becomes a partial topological group by decreeing that neighborhoods of the identity consist of mappings f such that f and f^{-1} are uniformly near to the identity in the spherical metric and such that f has arbitrarily small dilatation. In this case the characteristic topological subgroup is $PSL(2, \mathbb{C})$.

What about the case of quasisymmetric homeomorphisms of a circle? Assume a circle on the complex sphere $\hat{\mathbb{C}}$ is given, e.g., the unit circle, S^1, or the extended real axis, $\hat{\mathbb{R}}$. The group of Möbius transformations which preserves this circle determines what we call the standard projective structure for the circle. In the case that the circle is $\hat{\mathbb{R}}$, the group $PSL(2, \mathbb{R})$ preserves the standard projective structure. If A is in $PSL(2, \mathbb{C})$ mapping $\hat{\mathbb{R}}$ onto a circle C, then the group $A \circ PSL(2, \mathbb{R}) \circ A^{-1}$ preserves the standard projective structure for the circle C. This standard projective structure is subordinate to a unique quasisymmetric structure on the circle C and we let QS be the group of homeomorphisms of the circle which are quasisymmetric for this structure. The given $PSL(2, \mathbb{R})$ structure determines a system of neighborhoods of the identity in QS which make QS into a partial topological group. We denote by S the characteristic topological subgroup of QS.

Note that if we identify $PSL(2, \mathbb{R})$ with a conjugate group, where the conjugation is induced by a Möbius transformation taking the real axis into the circle C, we have the inclusions $PSL(2, \mathbb{R}) \subset S \subset QS$.

2. A characterization of the topological subgroup S of QS. In this section we show that a quasisymmetric homeomorphism of a circle is an element of S if and only if symmetrically placed triples of points which are sufficiently close together are moved by the homeomorphism to triples of points which are nearly symmetric. In order for this statement to have invariant meaning, it is necessary to assume there is a given smooth structure, say C^1, subordinate to the quasisymmetric structure. A C^1-change of coordinates $h_\alpha \circ h_\beta^{-1}$ can then be approximated by an affine mapping $x \mapsto ax + b$. If three points x_1, x_2, x_3 are symmetric in the sense that x_2 is the midpoint of x_1 and x_3, then their images under an affine mapping will also be symmetric. Thus, three symmetric points which are very close together have images under a C^1-diffeomorphism which are nearly symmetric.

We use the standard $PSL(2, \mathbb{R})$ structure on $\hat{\mathbb{R}}$ to define symmetric triples. Select a finite system of charts $\{h_\alpha\}$ for this $PSL(2, \mathbb{R})$ structure whose domains of definition cover $\hat{\mathbb{R}}$. We say that a triple of points p_1, p_2, p_3 on $\hat{\mathbb{R}}$ are symmetric if there is a chart h_α in the finite system mapping to an interval in $\mathbb{R}$ such that all three points are in the domain of h_α and such that $h_\alpha(p_2)$ is the midpoint of $h_\alpha(p_1)$ and $h_\alpha(p_3)$. Although this notion of symmetric triple is dependent on the choice of a finite system of charts, for two different finite systems the meaning of symmetric is almost the same for triples of points which are very close together.

Now let a quasisymmetric homeomorphism h and a finite system of charts h_α be given. Assume a triple of points p_1, p_2, p_3 is symmetric for the finite system h_α. We say that this symmetric triple is allowable for h if the image of the triple under h, namely, $h(p_1)$, $h(p_2)$, $h(p_3)$, is contained in the domain of definition of at least one of the charts h_β. Let $H = h_\alpha \circ h \circ h_\beta^{-1}$ for some choice of α and β and consider the quotient

$$\frac{H(x + t) - H(x)}{H(x) - H(x - t)} = 1 + \alpha_H(x, t) \tag{2}$$

for values $h_\beta(p_1) = x - t$, $h_\beta(p_2) = x$, and $h_\beta(p_3) = x + t$ where p_1, p_2, p_3 is a symmetric triple of points allowable for h.

Definition 2.1. A mapping h has vanishing ratio distortion if there is a function $\epsilon_h(t) \geq |\alpha_H(x, t)|$ for every possible choice of $H = h_\alpha \circ h \circ h_\beta^{-1}$ where h_α and h_β are in the finite system of charts such that $\epsilon_h(t)$ converges to zero as t converges to zero.

It should be emphasized that the definition implies $\alpha_H(x, t)$ converges to zero in t uniformly in x. The notion of a mapping with vanishing ratio distortion depends only on the quotient (2) for very small values of t. Since changes of coordinates in $PSL(2, \mathbb{R})$ are almost affine on bounded intervals and at very small scales, the notion of such a mapping does not depend on which finite system of charts are selected for the given $PSL(2, \mathbb{R})$ structure. For the same reasons, the notion of vanishing ratio distortion is invariant under C^1 changes of coordinates. In the sequel, we find it useful to step between the circle $\hat{\mathbb{R}}$ or S^1 and its universal covering $\mathbb{R}$, with covering mapping $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. The exponential mapping, $\exp(2\pi i\theta)$ induces an isomorphism between $\mathbb{R}/\mathbb{Z}$ and S^1. Through this isomorphism and the projection π, the standard projective structure on S^1 lifts to a nonstandard projective structure on $\mathbb{R}$. A homeomorphism h of S^1 with vanishing ratio distortion lifts to a homeomorphism $\tilde{h}$ of the real axis such that

$$1 - \epsilon(t) \leq \frac{\tilde{h}(x + t) - \tilde{h}(x)}{\tilde{h}(x) - \tilde{h}(x - t)} \leq 1 + \epsilon(t),$$

$h(0) = 0$, $h(x) + 1 = h(x + 1)$, and $\epsilon(t)$ converges to zero with t.

THEOREM 2.1. *Let QS be the partial topological group of quasisymmetric homeomorphisms of a circle with the standard $PSL(2, \mathbb{R})$ structure. Then the characteristic topological subgroup S of QS consists precisely of those mappings which have vanishing ratio distortion.*

Definition 2.1. The group S is called the group of symmetric homeomorphisms.

Combining Lemma 1.2 with this theorem yields the following corollary.

COROLLARY 2.1. *The group of symmetric homeomorphisms is a closed topological subgroup of QS.*

There is also a second corollary to this theorem.

COROLLARY 2.2. *S is a proper subgroup of QS and, thus, QS is not a topological group. Moreover, S contains the group of C^1-diffeomorphism of the circle.*

Proof. It is easy to construct mappings in QS which are not symmetric. In fact, let $h(x) = x$ for negative x and $h(x) = 2x$ for positive x. Then h is quasisymmetric but not symmetric because at the point $x = 0$, the best possible value of M in the condition (1) is $M = 2$. This shows that QS is not equal to its characteristic topological subgroup and, hence, by Lemmas 1.1 and 1.2, QS is not a topological group. For the second statement of the corollary, assume that both h and h^{-1} have continuous derivatives. By using coordinate charts, it suffices to look at cases where h maps a finite interval to a finite interval. Then, since the left and right derivatives of a C^1 diffeomorphism defined on an interval are equal and not equal to zero, we have $|\alpha_h(x, t)| \leq \epsilon_h(t)$ where $\epsilon_h(t)$ approaches zero.

To prove Theorem 2.1 we must prove two statements. The first is that if a mapping f has vanishing ratio distortion, then conjugation by f is continuous at the identity in QS. The second is that if conjugation by f is continuous at the identity, then f has vanishing ratio distortion. We prove these statements in Lemmas 2.1 and 2.2.

LEMMA 2.1. *If f has vanishing ratio distortion and if h is quasisymmetric and near the identity, then $f \circ h \circ f^{-1}$ is near the identity.*

Proof. By the remark preceding Theorem 2.1, we can view homeomorphisms of the circle as homeomorphisms of $\mathbb{R}$ which are periodic in the sense that $h(x + 1) = h(x) + 1$. We consider symmetric intervals $[x - t, x]$ and $[x, x + t]$. Assume that h satisfies an M-condition with $M = 1 + \epsilon$. Since f and f^{-1} have vanishing ratio distortion, we can pick $\delta > 0$ so that $0 < t < \delta$ implies f and f^{-1} distort the symmetric intervals $[x - t, x]$ and $[x, x + t]$ by no more than $1 + \epsilon$. We then pick h in an N_ϵ neighborhood of the identity. The product with three factors $f \circ h \circ f^{-1}$ will distort the symmetric intervals for which $0 < t < \delta$ by an amount $1 + \epsilon'$ where ϵ' converges to 0 with ϵ.

Now assume that $t \geq \delta$. Note that δ depends on f, which is fixed, and we are still permitted to select h in a smaller neighborhood of the identity. It is known that f and f^{-1} satisfy a Hölder condition since they have quasiconformal extensions. Let c be the constant and α be the exponent for this Hölder condition, valid in some bounded interval. The assumption that h is near the identity implies that it is near to the identity in the uniform topology. Thus, for given $\delta' > 0$, if h is near enough to the identity then $|h(x) - x| \leq \delta'$ for all x in the same bounded interval.

Let $\epsilon > 0$ be given and suppose we wish to show that $f \circ h \circ f^{-1}$ satisfies an M-condition with $M = (1 + \epsilon)/(1 - \epsilon)$. Then choose $\delta' = (\epsilon\delta/2c)^{1/\alpha}$. Let $a < m < b$ be three points in the given bounded interval and m be the midpoint of a and b. Let $f^{-1}(x)$ be denoted by $\tilde{x}$ and $h(x) = x + \Delta x$ where $|\Delta x| < \delta'$. We want to consider the effect of applying $f \circ h \circ f^{-1}$ to the three numbers, a, m, and b. Since f satisfies the Hölder condition and $f(\tilde{x}) = x$, the three numbers are transformed into $a + c(\Delta\hat{a})^\alpha$, $m + c(\Delta\hat{m})^\alpha$, and $b + c(\Delta\hat{b})^\alpha$, where, in general, $\Delta\hat{x}$ is some number with absolute value less than or equal to $\Delta\tilde{x}$. Since $b - m = m - a \geq \delta$ and since $|\Delta\hat{x}| \leq |\Delta\tilde{x}| \leq \delta'$, on forming the necessary quotient to test for the M-condition, we obtain a number less than or equal to

$$\frac{1 + 2c\delta'^\alpha/\delta}{1 - 2c\delta'^\alpha/\delta}.$$

By our choice of δ' this latter expression is equal to $(1 + \epsilon)/(1 - \epsilon)$.

We now know that if h is near enough to the identity, then the M-condition with $M = (1 + \epsilon)/(1 - \epsilon)$ is satisfied for $f \circ h \circ f^{-1}$ for all symmetric triples, assuming the points of the triple lie in a given bounded interval and the three symmetric points are sufficiently close together. But by periodicity, the condition that they lie in a bounded interval is unnecessary.

LEMMA 2.2. *If conjugation by f is continuous at the identity in QS, then f has vanishing ratio distortion.*

Proof. Let f be a quasisymmetric homeomorphism which does not have vanishing ratio distortion and M be the smallest value for which f satisfies an M-condition with respect to some choice of coordinate functions whose domains of definition cover $\hat{\mathbb{R}}$. Our goal is to construct quasisymmetric homeomorphisms s arbitrarily near to the identity such that the composition $f \circ s$ does not satisfy an $(M + 1)$-condition. Then, if $f \circ s = \tilde{s} \circ f$, $\tilde{s}$ cannot be near to the identity.

Since f does not have vanishing ratio distortion, there is a sequence of points p_n in one of the coordinate patches and a decreasing sequence t_n of positive numbers converging to zero such that the ratio of f applied to the right interval $[p_n, p_n + t_n]$ to f applied to the left interval $[p_n - t_n, p_n]$ approaches a number $M_0 > 1$. If f did not distort the right hand intervals more than the left hand intervals, then a similar statement about ratios of left intervals divided by right intervals would have to be true.

For the mapping s we take the translation by t_n:

$$s(x) = s_n(x) = x + t_n.$$

Notice that in the topology on QS, the number t_n measures the distance from s_n to the identity. Since we are assuming that conjugation by f is continuous at the identity, we know that, for n large enough, we have the equation $f \circ s_n = \tilde{s} \circ f$ where $\tilde{s}$ is uniformly near to the identity and $\tilde{s}$ satisfies an M-condition with M as near to 1 as we like.

The idea is that the M-distortion of f is nearly invariant under the translation mapping s_n. Moreover, if we wish to apply the translation mapping k times, where k is a preassigned number, we can arrange for the term $\tilde{s}^{2k}$ in the equation

$$f \circ s_n^{2k} = \tilde{s}^{2k} \circ f$$

to be as near to the identity as we wish by choosing t_n near enough to 0. For the time being, let us assume that $\tilde{s}$ is exactly equal to an affine mapping. Then we obtain a chain of $2k$ adjacent intervals, each of length t_n such that, under the mapping f, the image of each interval is M_0 times as long as the image of the interval immediately to its left. Now, step away to obtain a wider perspective on the mapping f. That is, consider the effect of f on the first k of these intervals compared to its effect on the second k intervals. We find that f distorts by a factor of

$$\frac{M_0^{k+1} + M_0^{k+2} + M_0^{k+3} + \cdots + M_0^{2k}}{M_0 + M_0^2 + M_0^3 + \cdots + M_0^k} = M_0^k.$$

Clearly, by choosing M_0^k to be larger than the minimum M-condition satisfied by the quasisymmetric mapping f, we obtain a contradiction. The fact that f is not exactly invariant under the shift s_n does not affect this calculation in an essential way because adjacent intervals I and J in proportion 1 to M_0 remain, after the application of $\tilde{s}^{2k}$, in the proportion 1 to $M_0 \pm \epsilon$.

3. Quasiconformal extensions of quasisymmetric and symmetric mappings. The purpose of this section is to describe quasisymmetric and symmetric mappings defined on $\hat{\mathbb{R}}$ or on intervals in $\mathbb{R}$ in terms of their possible quasiconformal extensions. If h maps an interval onto an interval, we use the notation $\tilde{h}$ for a mapping which extends h to an open set in the complex plane containing the interval. We will usually assume the extension is invariant under complex conjugation, in the sense that $\tilde{h}(\bar{z}) = \overline{\tilde{h}(z)}$. It is known that quasisymmetric mappings and quasiconformal mappings are related in a way expressed by the following two propositions.

PROPOSITION 3A, [12]. *Let h be a homeomorphism from a closed interval I onto a closed interval J of the real axis. Then h is quasisymmetric on I if and only if there exists an extension $\tilde{h}$ of h to open sets U and V*

in the plane containing I and J such that h is quasiconformal in U. The extension $\tilde{h}$ can be taken to be invariant under complex conjugation.

This proposition follows easily from the methods contained in Lehto and Virtanen, [12, Sections 6 and 7].

PROPOSITION 3B, [1, 5]. *Assume h is a homeomorphism of $\mathbb{R}$. Then h is quasisymmetric if, and only if, there exists a quasiconformal extension $\tilde{h}$ of h to the complex plane. Now assume h is normalized to fix three points, say* 0, 1, *and* ∞. *Then if h is quasisymmetric with constant M, the quasiconformal extension $\tilde{h}$ can be selected so that its dilatation K is less than or equal to $C_1(M)$ where $C_1(M) \to 1$ as $M \to 1$. Conversely, if a normalized quasiconformal extension $\tilde{h}$ of h has dilatation K, then h is quasisymmetric with constant $M \leq C_2(K)$, where $C_2(K) \to 1$ as $K \to 1$.*

This proposition is a statement of the Beurling-Ahlfors extension theorem, [5].

We now prove two analogous propositions for symmetric mappings, which are consequences of results of Fehlmann in [7]. Before stating these propositions, we need to establish some terminology and to make some definitions.

Definition 3.1. The local dilatation $H_f(p)$ of a quasisymmetric mapping f at a point p on the real axis is the infimum of the dilatations of the possible extensions $\tilde{f}$ to neighborhoods of p.

Thus, if $H_f(p) = 1$ and if $\epsilon > 0$, then there exists an open set U containing p and a quasiconformal extension $\tilde{f}$ of f defined on U such that $\tilde{f}(x) = f(x)$ for x in $U \cap \mathbb{R}$ and such that the dilatation of $\tilde{f}$ on U is less than $1 + \epsilon$. Note that if $H_p(f) = 1$, the definition does not imply directly that there is a single extension $\tilde{f}$ whose dilatation approaches 1 at p. If the number $\epsilon > 0$ is given, the extension $\tilde{f}$ with dilatation less than $1 + \epsilon$ on U can depend on ϵ.

Definition 3.1 has an obvious extension to the case where p is replaced by a closed interval.

Definition 3.2. The local dilatation $H_f(I)$ of a quasisymmetric mapping f on a closed interval I is the infimum of the dilatations of the possible extensions $\tilde{f}$ of f to neighborhoods of the closed interval I.

Another concept is the notion of boundary dilatation for a quasisymmetric self-mapping f of $\tilde{\mathbb{R}}$.

Definition 3.3. The boundary dilatation $H(f)$ of a quasisymmetric self-mapping f of $\tilde{\mathbb{R}}$ is the infimum of the dilatations of the possible extensions $\tilde{f}$ of f to neighborhoods of $\tilde{\mathbb{R}}$ in the complex sphere.

A fourth important concept is the notion of an asymptotically conformal quasiconformal mapping.

Definition 3.4. A quasiconformal selfmapping of the complex plane which preserves the real axis is called asymptotically conformal on an interval I if its dilatation approaches 1 on I. It is called asymptotically conformal if its dilatation approaches 1 on the whole real axis.

The next two propositions give statements for symmetric mappings which are parallel to the statements given by Propositions 3A and 3B for quasisymmetric mappings. These propositions are known results in the theory of quasiconformal mapping, (see Fehlmann [7, Satz 3.1]).

PROPOSITION 3.1, [7]. *Assume h is a quasisymmetric homeomorphism from a closed interval I to a closed interval J. The following conditions on h are equivalent*:

i) *h is symmetric on I,*
ii) *h has local dilatation equal to* 1 *at every point of I,*
iii) *h has local dilatation equal to* 1 *on I,*
iv) *there is an extension $\tilde{h}$ of h which is asymptotically conformal on I.*

Remark. To clarify part iv), by saying $K(z)$ approaches 1 at I we mean that, for every $\epsilon > 0$, there is an open set V with $I \subset V \subset U$ such that $|K(z) - 1| < \epsilon$ for z in V.

PROPOSITION 3.2, [7]. *The following conditions on a quasisymmetric homeomorphism h of $\hat{\mathbb{R}}$ are equivalent*:

i) *h is symmetric on $\hat{\mathbb{R}}$,*
ii) *h has local dilatation* 1 *at every point of $\hat{\mathbb{R}}$,*
iii) *h has boundary dilatation equal to* 1,
iv) *there is an extension $\tilde{h}$ of h which is asymptotically conformal.*

COROLLARY 3.1. *Any symmetric homeomorphism of S^1 can be approximated in the quasisymmetric topology by real analytic homeomorphisms.*

Proof of Corollary. From property iv) of Proposition 3.2 we can assume that the symmetric homeomorphism is realized by a quasiconformal homeomorphism of the sphere which preserves S^1, which is invariant under reflection about S^1, and which has Beltrami coefficient μ

which is arbitrarily small in sufficiently small open neighborhoods of S^1. Define the Beltrami coefficient μ_n by

$$\mu_n(z) = \begin{cases} 0 & \text{for } (1 + n^{-1})^{-1} < |z| < 1 + n^{-1}, \\ \mu(z) & \text{elsewhere.} \end{cases}$$

A solution to the Beltrami equation with Beltrami coefficient μ_n is complex analytic near S^1 and so it restricts to a real analytic homeomorphism of S^1. Moreover, if the solution is normalized in the right way, it will approximate the given symmetric homeomorphism in the quasisymmetric topology. This is because the hypothesis on μ implies that $\|\mu - \mu_n\|_\infty$ approaches 0 as n approaches ∞.

We go on to the proof of the Propositions and begin with Proposition 3.2. Statement iv) obviously implies iii), because iv) assumes there is a single extension function $\tilde{h}$ throughout a neighborhood of $\hat{\mathbb{R}}$ with local dilatation K_z converging to 1 as z converges $\hat{\mathbb{R}}$, whereas statement iii) permits the extensions to be only local, in neighborhoods of $\hat{\mathbb{R}}$ and, moreover, it permits different extensions for different selections of the positive number ϵ. Statement iii) obviously implies ii) because iii) assumes global mappings in a neighborhood of all of $\hat{\mathbb{R}}$ and not just at each point of $\hat{\mathbb{R}}$.

To prove that statement ii) implies i), we find it convenient to view $\hat{\mathbb{R}}$ as $\mathbb{R}/\mathbb{Z}$ and change the discussion to a discussion of quasisymmetric homeomorphisms h of $\mathbb{R}$ satisfying $h(0) = 0$, $h(1) = 1$, and $h(x + 1) = h(x) + 1$. Statement ii) translates into the hypothesis that h has local dilatation equal to 1 at every point of $\mathbb{R}$. Given $\epsilon > 0$, the unit interval can be covered by a finite number of subintervals on each of which h has quasiconformal extensions with dilatation less than $1 + \epsilon$. We can find a number $\delta > 0$ so that, if $0 < t < \delta$ and x is in the unit interval, then all three of the numbers, $x - t$, x, and $x + t$, lie in one of these subintervals. To proceed with the proof that ii) implies i) we need the following lemma.

LEMMA 3.1. *Let h quasiconformal in an open set U containing a closed interval I of the real axis and assume that $h(I)$ is a closed interval J of the real axis. Assume that the dilatation of h in U is less than* $1 +$ *ϵ. Then for $x - t$, x,* and *$x + t$ in I, h satisfies an M-condition* (1) *with $M = 1 + \epsilon'$, where ϵ' converges to zero with ϵ.*

Proof. By shrinking U a small amount we can find a quasiconformal mapping $\tilde{h}$ of the whole complex plane which agrees with h on U and such that $\tilde{h}$ preserves the real axis. We take three points, $x - t$, x, and $x + t$, in I. Consider the extremal length Λ of the family of curves in the upper half plane which join the interval $[x - t, x]$ to the interval $[x + t, \infty]$. Obviously, this extremal length is equal to the extremal length of the family of curves in the upper half plane which join $[-\infty, x - t]$ to $[x, x + t]$. Since these are conjugate extremal lengths, we see that $\Lambda = 1$. Let $\tilde{\Lambda}$ be the extremal length of the family of curves in the upper half plane which join $[h(x - t), h(x)]$ to $[h(x + t), \infty]$. Grötzsch's length-area argument shows that (see [9, Chapter 1])

$$\frac{\tilde{\Lambda}}{\Lambda} \leq \int_{\mathbb{C}}\int K_z(\tilde{h})|\varphi(z)|dxdy \text{ where}$$

$$\varphi(z) = \frac{b(t)}{(z - x)(z - (x - t))(z - (x + t))}$$

and where $b(t)$ is chosen so that $\int_{\mathbb{C}}\int |\varphi(z)|dxdy = 1$. This normalization implies that $b(t) = ct$ where

$$c = 2\left(\int_{\mathbb{C}}\int \frac{dxdy}{|z(z - 1)(z + 1)|}\right)^{-1}.$$

Since $b(t)$ converges to zero as t converges to zero, for sufficiently small values of t, the complement in $\mathbb{C}$ of the open set U containing I has arbitrarily small mass with respect to the measure $|\varphi(z)|dxdy$. For z in this complement, the dilatation $K_z(\tilde{h})$ is bounded. For z in U, the dilatation $K_z(\tilde{h})$ is less than $1 + \epsilon$. Thus, we see that there is a number $\delta > 0$ so that for $0 < t < \delta$, the ratio of extremal lengths, $\tilde{\Lambda}/\Lambda$, is less than $(1 + 2\epsilon)$. This is enough to assure that

$$\frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq 1 + \epsilon'$$

where ϵ' converges to zero with ϵ. To obtain the other side of the M-condition with $M = 1 + \epsilon'$, we apply the same argument to the conjugate extremal length.

To complete the proof that ii) implies i), we apply Lemma 3.1 over and over again for smaller and smaller positive numbers ϵ. For

$$\alpha(x, t) = \frac{h(x + t) - h(x)}{h(x) - h(x - t)},$$

we obtain that $\alpha(x, t)$ converges to 1 uniformly for x between 0 and 1. Under the covering mapping from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z} \approx \hat{\mathbb{R}}$ the condition that a mapping is symmetric on the unit interval in $\mathbb{R}$ translates into the condition that the induced mapping on the quotient is symmetric.

To prove that i) implies iv), we need the Beurling-Ahlfors extension formula, [5]. Assume h is a homeomorphism of $\hat{\mathbb{R}}$ onto $\hat{\mathbb{R}}$, that h fixes 0, 1 and ∞, and that $h(x + 1) = h(x) + 1$. Consider the function $\tilde{h}(x, y) = u(x, y) + i\, v(x, y)$, where

$$(2) \qquad \begin{aligned} u(x, y) &= \frac{1}{2y} \int_{x-y}^{x+y} h(t)dt \qquad \text{and} \\ v(x, y) &= \frac{1}{y} \left(\int_{x}^{x+y} h(t)dt - \int_{x-y}^{x} h(t)dt \right). \end{aligned}$$

In order to calculate the dilatation K_z of $\tilde{h}$, we use the formula

$$K_z + K_z^{-1} = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{u_x v_y - u_y v_x}.$$

For $z = x + iy$ the calculation of $K_z(\tilde{h})$ depends only on the values of h for t between $x - y$ and $x + y$. Thus, if $|y|$ is small enough, the assumption that h satisfies an M-condition with $M = 1 + \epsilon$ for symmetric triples which are sufficiently close together implies that $K < 1 + \epsilon'$. In fact, using the formula (2) and the estimates in [1], one can show that

$$\frac{M + 1}{2M} \leq u_x \leq M, \qquad \frac{1 - M}{2} \leq u_y \leq \frac{M - 1}{2M},$$

$$1 - M \leq v_x \leq \frac{M - 1}{M}, \qquad \frac{1}{M} \leq v_y \leq M.$$

We obtain a quasiconformal extension of h to an open neighborhood of the unit interval with dilation less than $1 + \epsilon'$ in this neighborhood.

Since the extension satisfies $\tilde{h}(z+1) = \tilde{h}(z) + 1$, this extension induces an extension of the original quasisymmetric homeomorphism of $\hat{\mathbb{R}}$ whose dilatation approaches 1 at the boundary.

This completes the proof of Proposition 3.2. In the course of the proof, it is easy to see that we have made all the steps necessary to prove Proposition 3.1.

If $K(f)$ is the maximal dilatation of a quasiconformal mapping defined on some open set and if g is defined on the image of f, then $K(f \circ g) \leq K(f)K(g)$ and $K(f^{-1}) = K(f)$. These properties combined with Proposition 3.2 make it clear that QS satisfies the axioms for a partial topological group.

Proposition 3.2 gives another characterization of the subgroup S. It is the subgroup of quasisymmetric homeomorphisms which have boundary dilation equal to 1. The fact that the group of homeomorphism with boundary dilatation equal to 1 is a closed subgroup of QS can be proved directly and the proof applies to a more general situation, namely, the Teichmüller space of an open Riemann surface. If a Riemann surface R is given, the Teichmüller space $T(R)$ consists of all quasiconformal mappings f from R to a variable Riemann surface $f(R)$, factored by a certain equivalence relation. Two such mappings f and f_1 are equivalent if there is a conformal mapping A from $f_1(R)$ to $f(R)$ such that $A \circ f_1$ is isotopic to f through a curve of quasiconformal mappings each of which agrees with f on the ideal boundary of R.

The boundary dilatation $H(f)$ of an equivalence class $[f]$ in $T(R)$ is the infimum of the maximal dilatations $K(f', R - C)$ of mappings f' in the same equivalence class as f and off of compact subsets C of R.

Proposition 3.3. *Let $S(R)$ be the subset of Teichmüller space of an open Riemann surface R consisting of those Teichmüller classes $[f]$ for which $H(f) = 1$. Then $S(R)$ is a closed subset of the Teichmüller space $T(R)$.*

Proof. Suppose that $[h_n]$ is a sequence in S, that the distance from $[h_n]$ to $[h]$ measured in Teichmüller's metric approaches zero. Let g_n be a quasiconformal mapping homotopic to $h_n \circ h^{-1}$ such that $K_n = K(g_n)$ decreases to 1. Then $g_n^{-1} \circ h_n$ is homotopic to h. Since h_n is in S, we can pick a compact set C_n contained in R and a quasiconformal mapping $\tilde{h}_n$ in the same Teichmüller class as h_n such that $\tilde{h}_n$ has dilatation less than K_n off of a compact subset C_n of R. Then $g_n^{-1} \circ \tilde{h}_n$ has dilatation less than K_n^2 off of C_n. Since $K_n \to 1$, we see that the boundary dilatation of h is equal to 1.

Remarks. 1. The inclusion of S in QS is analogous to the inclusion of the Banach sequence space c_0 in the sequence space l^∞. The fact that c_0 is closed in l^∞ is nothing more than application of the fact that a uniform limit of continuous functions is continuous. Analogously, the fact that S is closed in QS reflects the definition of Teichmüller's metric in terms of maximal dilatation, which means that it is a kind of sup-norm metric.

2. In [8], Fehlmann proves an important result about local dilatation and the existence of substantial boundary points. We will return to this topic in a later section.

4. The Bers embedding for quasisymmetric and symmetric mappings. Once again, we work in the standard projective structure on a circle and the background quasisymmetric structure which it determines. The standard projective structure determines a system of neighborhoods of the identity which makes QS into a partial topological group and, recall, by definition S is the characteristic topological subgroup of QS.

If A is a Möbius transformation mapping $\hat{\mathbb{R}}$ into the circle then $A \circ PSL(2, \mathbb{R}) \circ A^{-1}$ is a subgroup of QS. We wish to study the factor space QS mod $A \circ PSL(2, \mathbb{R}) \circ A^{-1}$. We follow the conventional form, which is to use right cosets. From the point of view of considering the possible $PSL(2, \mathbb{R})$ structures on the circle subordinate to a given QS-structure, there is no essential difference between using right and using left cosets. (This point is explained further in the Appendix.) In the right coset space, a mapping f in QS is identified with $B \circ f$, where B is any element of $A \circ PSL(2, \mathbb{R}) \circ A^{-1}$.

We now review the definition of universal Teichmüller space T. T is the set of quasiconformal selfmappings of the upper half plane factored by a certain equivalence relation. Two such selfmappings f and f_1 are equivalent if there exists a Möbius transformation B in $PSL(2, \mathbb{R})$ such that $B \circ f_1$ agrees with f on $\hat{\mathbb{R}}$. There is an obvious mapping Φ from T to QS mod $PSL(2, \mathbb{R})$; Φ assigns to a quasiconformal selfmapping f of the upper half plane its the boundary values on $\hat{\mathbb{R}}$. One easily shows that Φ is well-defined and injective. The Beurling-Ahlfors extension theorem (Proposition 3B) tells us that Φ is surjective. Moreover, it tells us the equally important fact that Φ is a homeomorphism if we think of T as having the topology coming from dilatation and QS mod $PSL(2, \mathbb{R})$ as having the topology induced by M-conditions.

The Bers' embedding of $T \approx QS$ mod $PSL(2, \mathbb{R})$ is an embedding into a complex Banach space, B. For universal Teichmüller space, this Banach space consists of all functions $\varphi(z)$, holomorphic in the lower half plane, L, with bounded norm, where the norm is defined by

$$\|\varphi\| = \sup_{z \text{ in } L} |y^2\varphi(z)|. \tag{3}$$

The embedding is a map $\mathcal{S}$ from QS mod $PSL(2, \mathbb{R})$ onto a bounded open set in B. The definition of $\mathcal{S}$ involves several steps, which we list here:

i) select a representative h of an element of QS mod $PSL(2, \mathbb{R})$, which is homeomorphism of $\hat{\mathbb{R}}$,
ii) take any quasiconformal selfmapping $\tilde{h}$ of the upper half plane U such that $\tilde{h}$ restricted to $\hat{\mathbb{R}}$ is identical to h,
iii) form the Beltrami coefficient μ of $\tilde{h}$, namely, $\mu(z) = \tilde{h}_{\bar{z}}/\tilde{h}_z$,
iv) let $\tilde{\mu}(z) = \mu(z)$ for z in the upper half plane and $\tilde{\mu}(z) \equiv 0$ for z in the lower half plane and solve for f in the Beltrami equation

$$f_{\bar{z}}(z) = \tilde{\mu}(z)f_z(z)$$

to obtain a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ which is holomorphic in the lower half plane,
v) take the Schwarzian derivative φ of f in the lower half plane:

$$\varphi(z) = N' - \frac{1}{2}N^2 \text{ where } N = f''(z)/f'(z).$$

One shows that the end result, φ, depends neither on the choice of representative made in step i) nor on the quasiconformal extension $\tilde{h}$ made in step ii). By definition, we let $\mathcal{S}(h) = \varphi$. Since an element h of QS mod $PSL(2, \mathbb{R})$ is determined by the Beltrami coefficient μ of any quasiconformal extension of h to the upper half plane, we will sometimes write $\mathcal{S}(\mu)$ instead of $\mathcal{S}(h)$.

The mapping $\mathcal{S}$ from QS mod $PSL(2, \mathbb{R})$ to B is well-defined, one-to-one, complex analytic with respect to the complex structure coming from Beltrami differentials, maps onto an open set containing the ball of radius $\frac{1}{2}$ and contained in the ball of radius $\frac{3}{2}$ and has local complex analytic cross-sections. All of these properties are shown in Ahlfors [1].

We claim that all of these properties carry over if one views $\mathcal{S}$ as a mapping from the space S mod $PSL(2, \mathbb{R})$ into a complex Banach subspace B_0. By definition, B_0 is the subspace of B consisting of those φ in B such that for every $\epsilon > 0$, there is a compact subset C of L such that $|y^2\varphi(z)| < \epsilon$ for z in $L - C$. In words, B_0 consists of the elements φ of B for which the function $y^2\varphi(z)$ vanishes at the boundary. The norm for the Banach space B_0 is the same as the norm for B, namely, the norm given in formula (3).

THEOREM 4.1. *The mapping $\mathcal{S}$ described above applied to the symmetric subset S* mod $PSL(2, \mathbb{R})$ *of QS* mod $PSL(2, \mathbb{R})$ *has image in B_0. Moreover, it is well-defined, one-to-one, complex analytic with respect to the complex structure coming from Beltrami differentials, maps onto an open set containing the open ball of radius $\frac{1}{2}$ in B_0 and contained in the ball of radius $\frac{3}{2}$, and has local complex analytic cross sections.*

Proof. Most of the assertions in this proposition follow in exactly the same way they do for the mapping $\mathcal{S}$ applied to QS. The first step is to show that the Schwarzian derivative mapping $\mathcal{S}$ applied to S has image in B_0. We are indebted to the referee for providing a proof for this step which is shorter and more simple than the one which we originally gave. Let M be the open unit ball of $L_\infty(U, \mathbb{C})$ and let M_0 be the closed complex submanifold consisting of those μ in M which vanish at the boundary in the sense that for any $\epsilon > 0$ there is a compact set in U such that the supremum norm of μ outside of the compact set is less than ϵ.

We may consider $\mathcal{S}$ as mapping with domain M instead of QS mod $PSL(2, \mathbb{R})$ by defining $\mathcal{S}$ on M to be the Schwarzian derivative of a Riemann mapping of the lower half plane which is the restriction of a quasiconformal mapping of the whole plane which has Beltrami coefficient μ in the upper half plane and which is conformal elsewhere. From Proposition 3.2, it follows that the image under $\mathcal{S}$ of S mod $PSL(2, \mathbb{R})$ is the same as the image of M_0. If there is a compact set of U such that μ is identically equal to zero off this compact set, then obviously $\mathcal{S}$ applied to μ is in B_0. By the corollary to Proposition 3.2, such μ are dense in M_0. Since B_0 is closed in B and since $\mathcal{S}$ is continuous as a mapping from M to B, we conclude that $\mathcal{S}$ applied to M_0 is contained in B_0.

The next step in proving the theorem is to show why the mapping $\mathcal{S}$ has local holomorphic cross sections, when viewed as a mapping from S to B_0. We follow the same argument given by Ahlfors in [1, pp. 120–133]. Let ψ be an element of $B_0(\Omega)$. Let $\zeta \mapsto \zeta^*$ be an antiquasiconformal reflection which fixes the points of the quasicircle which is the common boundary of the domains Ω and Ω^*. We take two linearly independent solutions η_1 and η_2 of the equation $\eta''(\zeta) = -\frac{1}{2}\psi(\zeta)\eta(\zeta)$ defined throughout the domain Ω and form the function

$$g(\zeta) = \begin{cases} \eta_1(\zeta)/\eta_2(\zeta) & \text{for } \zeta \text{ in } \Omega \text{ and} \\ \dfrac{\eta_1(\zeta^*) + (\zeta - \zeta^*)\eta_1'(\zeta^*)}{\eta_2(\zeta^*) + (\zeta - \zeta^*)\eta_2'(\zeta^*)} & \text{for } \zeta \text{ in } \Omega^*. \end{cases} \tag{4}$$

Then, the formula given by Ahlfors [1, p. 132] for the Beltrami coefficient μ of g is

$$\mu(\zeta) = \frac{\frac{1}{2}(\zeta - \zeta^*)^2\psi(\zeta^*)\bar{\partial}\zeta^*}{1 + \frac{1}{2}(\zeta - \zeta^*)^2\psi(\zeta^*)\partial\zeta^*} \tag{5}$$

for ζ in Ω^* and $\mu(\zeta) \equiv 0$ for ζ in Ω. Ahlfors shows that the reflection $\zeta \mapsto \zeta^*$ has bounded first partial derivatives and that the term $|\zeta^*(\zeta) - \zeta|$ is commensurable to the Poincaré metric of the domain Ω^*. Therefore, if ψ is in $B_0(\Omega)$ and has sufficiently small norm, the μ given in this formula will be an L_∞ function which vanishes at the boundary of Ω^*. This is enough to show that $\mathcal{S}$, viewed as a mapping from S into B_0, has local holomorphic cross sections and that the image of $\mathcal{S}$ is open in B_0. All of the other parts of Proposition 4.1 can be proved by the same arguments given in [1].

5. The symmetric foliation of Teichmüller space. The Teichmüller metric on QS mod $PSL(2, \mathbb{R})$ is defined by

$$d(f, g) = \tfrac{1}{2}\log K(f \circ g^{-1}),$$

where $K(h)$ is the minimal dilatation of a quasiconformal self-mapping of Δ with the same boundary values as h. Obviously, the value $d(f, g)$ does not change if we replace f by $A \circ f$ and g by $B \circ g$, where A and B are conformal mappings and hence the metric is well-defined on cosets. From the normal families arguments for quasiconformal mappings, for given boundary values, there will always exist a quasiconformal mapping realizing the minimum of $K(h)$. It is also true that if a quasiconformal mapping has maximal dilatation equal to one then it is conformal. Combining these two facts, one sees that d is a Hausdorff metric. With respect to the metric d, any translation mapping $h \mapsto h \circ g$ is an isometry. These translation mappings are well defined on cosets in QS mod $PSL(2, \mathbb{R})$ because the equivalence between h and $A \circ h$ is translated to an equivalence between $h \circ g$ and $A \circ h \circ g$.

The continuity of the mapping $h \mapsto h \circ g$ implies that the image of the coset Sg in QS mod $PSL(2, \mathbb{R})$ is closed. It is also true that $h \mapsto h \circ g$ induces a holomorphic self-mapping of universal Teichmüller space. This is because if μ and ν are Beltrami coefficients of g and h, then the Beltrami coefficient of $h \circ g$ is

$$\frac{\mu(z) + \nu(g(z))\theta}{1 + \overline{\mu(z)}\nu(g(z))\theta},$$

where $\theta = \bar{p}/p$ and $p = g_z(z)$. If μ is held fixed, this formula varies holomorphically in ν. Thus, the cosets Sg determine closed holomorphic submanifolds which partition the manifold QS mod $PSL(2, \mathbb{R})$.

The topology coming from the metric d on QS mod $PSL(2, \mathbb{R})$ induces a topology on QS mod S. It is the finest topology which makes the projection $\pi : QS \to QS$ mod S continuous. An ϵ-neighborhood of a coset Sg is the set of cosets Sf for which

$$\bar{d}(Sf, Sg) = \inf_{s_1, s_2 \text{ in } S} d(s_1 f, s_2 g) < \epsilon. \tag{6}$$

The projection π is an open mapping and we claim that these open neighborhoods induce a Hausdorff topology. Suppose that $\inf_{s_1,s_2 \text{ in } S} d(s_1 \circ f, s_2 \circ g) = 0$. Then there are sequences s_n and $\tilde{s}_n$ of elements of S such that

$$s_n \circ f \circ g^{-1} \circ \tilde{s}_n^{-1} = \xi_n,$$

where ξ_n is a sequence of quasisymmetric mappings approaching identity. Then $f \circ g^{-1} = s_n^{-1} \circ \xi_n \circ \tilde{s}_n$ and the same method used to prove Proposition 3.3 shows that $f \circ g^{-1}$ is symmetric, which is the same as saying $Sf = Sg$.

Formula (6) determines a metric which can be expressed in terms of boundary dilatation.

LEMMA 5.1. *The expression* $\bar{d}(Sf, Sg)$ *in formula* (6) *is equal to* $\frac{1}{2}$ log H, *where* H *is the boundary dilatation* (*see Definition* 3.3) *of the mapping* $f \circ g^{-1}$ *and, therefore,* $\bar{d}$ *is a metric on the quotient space* QS mod S.

Proof. Recall that the boundary dilatation $H(f)$ of a mapping f is the infimum of maximal dilatations of quasiconformal mappings which extend f to some neighborhood of the boundary, where we permit the neighborhoods to be arbitrary. Thus, $H(f) \leq K(f)$, where $K(f)$ is the minimal dilatation of a quasiconformal extension of f to the interior of the disk. Moreover, it is obvious that if s is symmetric then $H(f \circ s) = H(s \circ f) = H(f)$. It follows that $H(f) \leq K(s_1 \circ f \circ s_2)$ for arbitrary symmetric mappings s_1 and s_2.

To prove the opposite inequality, we will show that the infimum over s in S of the quantity $K(s \circ f)$ is less than or equal to $H(f)$. Instead of working with the upper half plane and its boundary $\hat{\mathbb{R}}$, it is convenient to work with the unit disk and its boundary S^1. Suppose $\epsilon > 0$ is given and $\tilde{f}$ is an extension of f to a neighborhood $V = \{z : r < |z| \leq 1\}$ of the boundary of the unit disk Δ with dilatation less than $H(f) + \epsilon$. By taking r slightly closer to 1, we can assume the image under $\tilde{f}$ of the circle $|z| = r$ is a quasicircle. Let R be the annular domain lying between this quasicircle and the unit circle. By uniformization, there is a schlicht analytic mapping s from R onto an annulus bounded by the two circles, $|z| = 1$ and $|z| = r'$, where $0 < r' < 1$, and mapping the unit circle onto itself. Since the inner boundary of R is a quasicircle, the mapping s can be extended quasiconformally to the whole unit disk. We denote the extension by the same letter, s. Clearly, s has boundary dilatation equal to 1 and $H(s \circ f) = H(f)$. This means that we can assume that the extension $\tilde{f}$ of f, which has dilatation less than $H(f) + \epsilon$ in the neighborhood V, maps the annulus V onto the annulus $V' = \{z : r' < |z| \leq 1\}$. Obviously, by reflection, this mapping can be extended to a quasiconformal mapping with exactly the same maximal dilatation between the annuli whose outer boundaries are the unit circle and whose inner boundaries are the circles with radii r^2 and r'^2. By reflecting infinitely many times, we get a quasiconformal mapping of the whole unit disk whose dilatation is less than $H(f) + \epsilon$ and which is an extension of the boundary values of $s \circ f$.

Note that the boundary dilatation value $H(f)$ gives a different way of thinking of the metric (6) defined on the coset space QS mod S:

$$\bar{d}(Sf, Sg) = \tfrac{1}{2} \log H(f \circ g^{-1}). \tag{7}$$

We have proved the following theorem.

THEOREM 5.1. *The cosets Sg of the symmetric subgroup S in QS* mod $PSL(2, \mathbb{R})$ *partition QS* mod $PSL(2, \mathbb{R})$ *by leaves which are closed with respect to the metric d. These cosets are complex submanifolds of QS. The coset space QS* mod *S is a metric space with the finest topology for which the projection mapping $\pi : QS \to QS$* mod *S is continuous.*

Remark. Since $\frac{1}{2} \log H(f)$ is the metric distance between the identity and f in the quotient space QS mod S, the boundary dilatation H is a continuous function on this quotient space. We will later see that QS mod S has a natural complex structure. We believe that the boundary dilatation metric $\bar{d}$ is, in some sense, natural for this complex structure. A key tool in Teichmüller theory is the Reich-Strebel inequality for $K(f)$, the minimal dilatation of a quasiconformal extension of the quasisymmetric mapping f. The inequality for $K(f)$ is

$$\frac{1}{K(f)} \le \int_\Delta \int \frac{\left|1 - \mu(z) \dfrac{|\varphi(z)|}{\varphi(z)}\right|^2}{1 - |\mu(z)|^2} |\varphi(z)| dxdy, \tag{8}$$

where $\varphi(z)$ is any holomorphic function in the unit disk for which $\|\varphi\| = \int_\Delta \int |\varphi(z)| dxdy = 1$ and μ is the Beltrami coefficient of any quasiconformal extension of the quasisymmetric mapping f. There is an analogous inequality for the boundary dilatation $H(f)$. We define a sequence φ_n with $\|\varphi_n\| = 1$ to be degenerating if φ_n converges uniformly to zero on compact subsets of Δ. Then the same argument used to prove the above inequality shows that

$$\frac{1}{H(f)} \le \liminf_n \int_\Delta \int \frac{\left|1 - \mu(z) \dfrac{|\varphi_n(z)|}{\varphi_n(z)}\right|^2}{1 - |\mu(z)|^2} |\varphi_n(z)| dxdy,$$

where φ_n is any degenerating sequence with $\|\varphi_n\| = 1$ and where μ is any Beltrami coefficient of any quasiconformal extension of any quasisymmetric mapping of the form $s_1 \circ f \circ s_2$ where s_1 and s_2 are symmetric. This inequality has consequences for the infinitesimal theory of the metric determined by $\frac{1}{2}\log H$ the quotient space QS mod S. Important elements in the study of the quotient metric $\bar{d}$ on QS mod S will be the "frame mapping condition" of Strebel [16] and Fehlmann's theorem on the existence of substantial points, [8]. The frame mapping condition tells us that when the quotient mapping from QS mod $PSL(2, \mathbb{R})$ to QS mod S contracts the distance between the origin and a given point, then that point has a representative of Teichmüller form.

6. Quasicircles and symmetric quasicircles. Let Ω and Ω^* be two Jordan domains, complementary in $\hat{\mathbb{C}}$, whose common boundary is a Jordan curve C. Let f and g be the Riemann mappings which map the lower half plane L onto Ω and the upper half plane U onto Ω^*, respectively. Since these mappings extend continuously to the common boundaries, the mapping $g^{-1} \circ f$ restricted to $\hat{\mathbb{R}}$ is a homeomorphism. We define the Jordan curve C to be a quasicircle or a symmetric quasicircle if the composition $g^{-1} \circ f$ is quasisymmetric or symmetric, respectively.

Because of the Beurling-Ahlfors extension theorem, a Jordan curve is a quasicircle precisely if it is the image of the real axis under a quasiconformal homeomorphism of the sphere which is conformal in the lower half plane. In a parallel manner, it follows that a Jordan curve is a symmetric quasicircle precisely if it is the image of the real axis under a quasiconformal homeomorphism of the sphere which is conformal in the lower half plane and which has dilatation K_z approaching 1 as z approaches $\hat{\mathbb{R}}$.

In [1], Ahlfors characterizes geometrically which Jordan curves are quasicircles. Let a and c be points on the curve, which divide the curve into two arcs. Let b be a third point lying on whichever of these two arcs has smaller spherical diameter. Ahlfors' geometric condition is that there exists a constant K such that

$$\text{(9)} \quad \text{spherical dist}(a, b) + \text{spherical dist}(b, c) \leq K \text{ spherical dist}(a, c),$$

for any three points a, b, and c selected in this way. We call this inequality the *reverse triangle inequality*.

There is a parallel condition for symmetric quasicircles due to Becker and Pommerenke, [3]. We say that a Jordan curve C has the *strong reverse triangle property* if for every point p in C and every $K > 1$, there exists a neighborhood N of p such that inequality (9) is satisfied for any three points a, b and c lying in N with b between a and c. The nearer K is to 1 in inequality (9) the nearer the part of the curve between a and c is to a geodesic segment.

For our purposes, it is more convenient to work with the Euclidean metric. The notions of quasicircle and symmetric quasicircle are invariant under change of coordinate by a Möbius transformation. If the Jordan curve passes through ∞, we can select a Möbius transformation which moves ∞ to a finite point. For bounded Jordan curves, it is just as well to work with the parallel inequality with the spherical metric replaced by the Euclidean metric:

$$|a - b| + |b - c| \leq \lambda |a - c|. \tag{10}$$

Definition 6.1. A bounded Jordan curve C has the *strong reverse triangle property* if, for every p in C *and for every* $\lambda > 1$, *there exists a neighborhood* N of p, such that inequality (10) holds for all points a, b and c lying in N and on C with b between a and c.

If the quasicircle passes through ∞, then for the point $p = \infty$ we require the same condition to be satisfied after a change of coordinates by a Möbius transformation which moves ∞ to a finite point. Notice that inequality (10) defines an ellipse with foci at a and c. We will call the number λ^{-1} the eccentricity of this ellipse.

We will prove the following theorem of Becker and Pommerenke.

THEOREM 6.1, [4]. *A quasicircle is a symmetric quasicircle if, and only if, it has the strong reverse triangle property.*

In the course of proving this result, we give two other equivalent geometric conditions on quasicircles, the *extremal length property* and *the disk template property*. To describe the extremal length property consider Figure 3.

Assume a, b, c and d are consecutive points on the quasicircle C. Let α_{ab} be the arc of C joining a to b. Define the arcs α_{bc}, α_{cd}, and α_{ad} analogously. Let γ_1 be a Jordan curve separating α_{ab} from α_{cd}. Let γ_2 be a Jordan curve separating α_{bc} from α_{ad}. Let $\tilde{\Lambda}_1$ be the extremal length on $\hat{\mathbb{C}} - \alpha_{ab} \cup \alpha_{cd}$ of the family of curves homotopic to γ_1 and let $\tilde{\Lambda}_2$ be

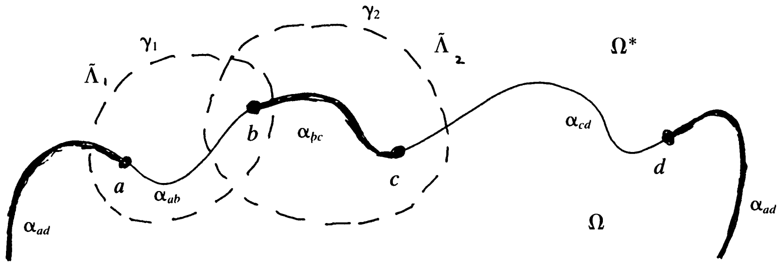


FIGURE 3

the extremal length on $\hat{\mathbb{C}} - \alpha_{ad} \cup \alpha_{bc}$ of the family of curves homotopic to γ_2.

In the course of the proof of the theorem we will see that no matter what four points a, b, c, and d are selected, the product of the extremal lengths, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, is always greater than or equal to 4.

Definition 6.2. A bounded quasicircle C has the *extremal length property* if, for every point d on C, every p in $C - \{d\}$ and every $\epsilon > 0$, there is a $\delta > 0$, such that whenever a, b, c are consecutive points in $C - \{d\}$ and a, b, and c are in a δ-neighborhood of p, then $\tilde{\Lambda}_1\tilde{\Lambda}_2 < 4 + \epsilon$.

We first prove that symmetry of a quasicircle is equivalent to the extremal length property.

THEOREM 6.2. *Let Ω and Ω^* be complementary Jordan domains with common boundary equal to the quasicircle C. Let f be a Riemann mapping from the lower half plane to Ω and g a Riemann mapping from the upper half plane to Ω^*. Then a necessary and sufficient condition for the mapping $g^{-1} \circ f$ restricted to $\hat{\mathbb{R}}$ to be symmetric is that the quasicircle C has the extremal length property.*

Proof. We begin by proving extremal length property is sufficient. Assume that x_1, x_2 and x_3 are symmetrically placed points on the real axis with $f(x_1) = a$, $f(x_2) = b$, $f(x_3) = c$, $f(\infty) = d$ where f is a Riemann mapping from the lower half plane to the Jordan domain Ω. By precomposing g with an element of $PSL(2, \mathbb{R})$, we may take $g(\infty) = d$. Let $h(x_1) = x_1'$, $h(x_2) = x_2'$, $h(x_3) = x_3'$, $h(\infty) = \infty$ where $h = g^{-1} \circ f$ and $g(\infty) = d$. Given $\delta > 0$, if x_1, x_2, x_3 are sufficiently close together and in the interval from -2 to 2 and near enough to a given point x, then the three points a, b, and c will be close together and near to $f(x)$. We will show that this forces

$$1 - \epsilon' \le \frac{x_3' - x_2'}{x_2' - x_1'} \le 1 + \epsilon',$$

where ϵ' converges to zero with ϵ in Definition 6.2. This will show that h is symmetric in the interval from -2 to 2. If $A(x) = x^{-1}$, the same proof will show that $A \circ h \circ A^{-1}$ is symmetric on the interval from -2 to 2 and, hence, that h is symmetric.

Let Λ_1 be the extremal length of the family of arcs in the lower half plane which join the interval $[x_1, x_2]$ to the interval $[x_3, \infty)$. Since f maps the lower half plane conformally onto Ω, Λ_1 is also equal to the extremal length of the family of arcs in Ω which join α_{ab} to α_{cd}. Similarly, we let Λ_2 be the extremal length of the family of arcs in the lower half plane which join the interval $(-\infty, x_1]$ to the interval $[x_2, x_3]$. Thus, Λ_2 is also equal to the extremal length of the family of arcs in Ω which join the part of the quasicircle between α_{ad} to α_{bc}. Λ_1' and Λ_2' are defined in a similar way with respect to the upper half plane and the points x_1', x_2' and x_3'. The comparison principle for extremal lengths gives the following two inequalities:

$$\tilde{\Lambda}_1 \ge \Lambda_1 + \Lambda_1' \quad \text{and} \quad \tilde{\Lambda}_2 \ge \Lambda_2 + \Lambda_2'.$$

We start with three points x_1, x_2, and x_3 on the real axis which are symmetric in the sense that x_2 is the midpoint of x_1 and x_3. Then, it is obvious that Λ_1 and Λ_2 are both equal to 1 and, on multiplying these two inequalities, we get

$$\tilde{\Lambda}_1 \tilde{\Lambda}_2 \ge (1 + \Lambda_1')(1 + \Lambda_2') = 2 + \Lambda_1' + \Lambda_1'^{-1}. \tag{11}$$

The equality on the right of (11) follows because $\Lambda_1' \Lambda_2' = 1$. From the hypothesis that the quasicircle has the extremal length property we know that $\tilde{\Lambda}_1 \tilde{\Lambda}_2$ is near to 4. Thus, both Λ_1' and Λ_2' are near to 1. It follows that the points x_1', x_2', and x_3' are nearly symmetric and the mapping h is symmetric.

We now prove that the condition is necessary. Let the point d in C, p in $C - \{d\}$ and $\epsilon > 0$ be given and assume the homeomorphism h equal to $g^{-1} \circ f$ restricted to $\hat{\mathbb{R}}$ is symmetric. By precomposing f and g with elements of $PSL(2, \mathbb{R})$, we may assume $f(\infty) = g(\infty) = d$ and $h(\infty) = \infty$. Since f^{-1} is continuous, we are allowed to assume that the points x_1, x_2, x_3, whose images under f are a, b, and c, are arbitrarily close together. Since $g^{-1} \circ f$ is symmetric, we know f has a quasiconformal extension to the whole plane with a Beltrami coefficient μ which is zero in the lower half plane and which vanishes as z approaches $\hat{\mathbb{R}}$. Let α_1 be a Jordan curve in $\mathbb{C}$ which separates $[x_1, x_2]$ from $[x_3, \infty]$. The extremal length L_1 of α_1 on $\mathbb{C} - [x_1, x_2] \cup [x_3, \infty]$ is realized by an extremal metric $\rho(z)|dz|$ of the form $\rho(z)|dz| = |\varphi(z)|^{1/2}|dz|$ where $\varphi(z)dz^2$ is

$$\varphi(z)dz^2 = \frac{Cdz^2}{(z - x_1)(z - x_2)(z - x_3)}.$$

We select the constant C so that $\|\varphi\| = 1$. Grötzsch's inequality tells us that

$$\frac{\tilde{\Lambda}_1}{L_1} \leq \int_{\mathbb{C}}\int K_z|\varphi(z)|dxdy.$$

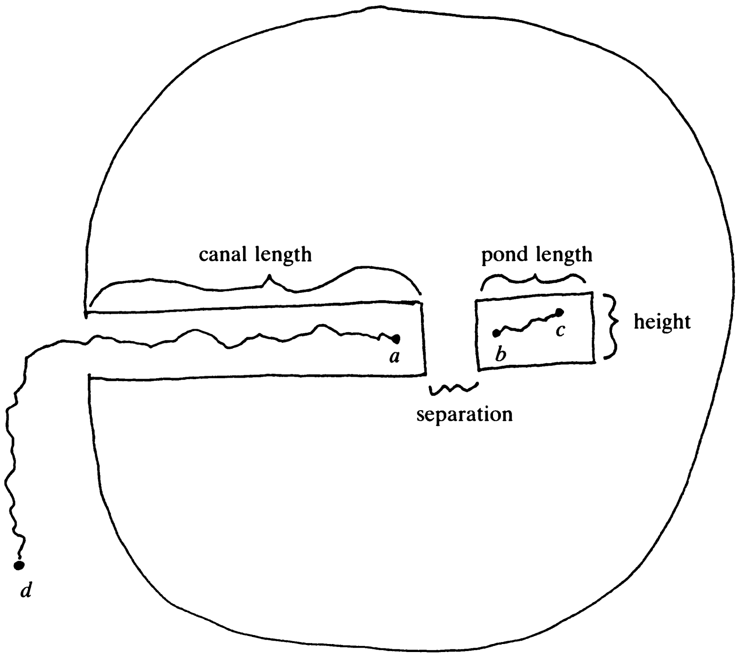
An obvious calculation shows that nearly all of the mass of the measure $|\varphi(z)|dxdy$ is contained in the δ'-neighborhood of x_1, x_2, and x_3. On the other hand, we have a fixed bound on K_z outside of this δ'-neighborhood and inside this neighborhood K_z is bounded by $1 + \epsilon$. Hence, the ratio in $\tilde{\Lambda}_1/L_1$ is bounded by $1 + 2\epsilon$. Obviously, we have the parallel result for the ratio $\tilde{\Lambda}_2/L_2$. Since the product $L_1L_2 = 4$, we conclude that $\tilde{\Lambda}_1\tilde{\Lambda}_2 < 4 + \tilde{\epsilon}$, where $\tilde{\epsilon}$ converges to zero with ϵ. This completes the proof of Theorem 6.2.

In order to prove Theorem 6.1, we need the notion of a disk template. We refer to the drawing in Figure 4.

We think of a disk template D as a disk-shaped island out of which we have excavated a canal and a pond. It is a disk template for the quadruple a, b, c, d on the quasicircle C if

i) a is at the center of the disk,
ii) the arc α_{bc} is contained in the pond, and if
iii) the arc α_{ad} is contained either in the canal or the exterior of the disk.

Thus the straight line segment from a to b lies on a diameter of the disk and we assume that the top and bottom of the canal and the pond are



A disk template for a, b, c, d on the quasicircle C
FIGURE 4

line segments equidistant from and parallel to this diameter. The height, canal length, separation and pond length are defined from Figure 4.

Assume a Riemann mapping f from Ω onto L and a neighborhood U of $d = f(\infty)$ are given. We define the triple a, b, c to be allowable for f and U if a, b, and c are images of symmetrically placed points $x - t$, x, and $x + t$ and if a, b and c are not in U. Since C is a quasicircle, given f and U, we know there exists a constant B which bounds all possible ratios of the three numbers $|a - b|$, $|b - c|$, and $|a - c|$ whenever a, b, c is an allowable triple for f and U.

Definition 3. The quasicircle C has the *disk template property* if, for every Riemann mapping f from L onto Ω and neighborhood U of $f(\infty) = d$, every $\epsilon > 0$, and every positive integer N, there exists a $\delta > 0$ such that if a, b, c is an allowable triple for f and U with $|a - b| = \delta' \leq \delta$, then there exists a disk template for a, b, c and d with

canal length $\geq \delta'(\log N)/2B$,
canal height $=$ pond height $\leq \epsilon\delta'$,

pond length between $B^{-1}\delta'$ and $B\delta'$,
separation $\geq (1 - \epsilon)\delta'$.

In order to establish Theorem 6.1, we show that the strong reverse triangle property, the extremal length property, and the disk template property are all equivalent for quasicircles. We use the same notation as before: f is a Riemann map from the lower half plane to Ω, g is a Riemann map from the upper half plane to Ω^*, and the quasicircle C is the common boundary of the two complementary Jordan domains, Ω and Ω^*.

THEOREM 6.3. *The following conditions on a quasicircle C are equivalent*:

i) *the mapping $g^{-1} \circ f$ restricted to $\hat{\mathbb{R}}$ is symmetric*,
ii) *C has the strong reverse triangle property*,
iii) *C has the disk template property*,
iv) *C has the extremal length property*.

Proof. We have already shown that iv) is equivalent to i). Now we show that i) implies ii). In the definition of the strong reverse triangle property assume the point p on C is selected and the neighborhood N is small enough so that $C - N$ is nonempty. Let d be in $C - N$. By applying a Möbius transformation we can assume $d = \infty$. We must show that $|a - b| + |b - c| < \lambda|a - c|$ for consecutive triples a, b, c on $C - \{d\}$ and sufficiently near to p.

Observe that a neighborhood system of the closed straight line segment from a to c is determined by inequality (10) with parameter $\lambda > 1$. It is useful to have a second system of closed neighborhoods of this line segment. Let d_{ac} be the Poincaré metric on the triply punctured sphere $\hat{\mathbb{C}} - \{a, c, \infty\}$. Let N_ϵ be the set of points b whose distance from this line segment measured in the metric d_{ac} is less than or equal to ϵ. Then $N_\epsilon \cup \{a, c\}$ is also a system of neighborhoods of the closed line segment joining a to c.

We assume that $h = g^{-1} \circ f$ restricted to $\hat{\mathbb{R}}$ is symmetric and that f has a quasiconformal extension to the whole plane with a Beltrami coefficient μ which is zero in the lower half plane and which vanishes as z approaches $\hat{\mathbb{R}}$. Let x_1, x_2, and x_3 be three consecutive points on the real axis, whose images under f are the three points a, b, and c. The curve $f^{t\mu}(x_2)$, as a function of t, takes the value x_2 when $t = 0$ and b when $t = 1$. By renormalizing, we can assume that $f^{t\mu}$ fixes x_1, x_3 and

∞ for $0 \leq t \leq 1$. Then, measured in the Poincaré metric on $\hat{\mathbb{C}} - \{x_1, x_3, \infty\}$, the distance from x_2 to b is no more than $\frac{1}{2} \log K_0$, where K_0 is the minimal dilatation of a quasiconformal mapping in the same homotopy class as f^μ on $\hat{\mathbb{C}} - \{x_1, x_2, x_3, \infty\}$. To get an upper bound on K_0, we use the fact that

$$K_0 = \sup_\alpha \frac{\Lambda(f^\mu(\alpha))}{\Lambda(\alpha)}, \tag{12}$$

where $\Lambda(\alpha)$ is the extremal length of the family of curves which are freely homotopic to α on $\hat{\mathbb{C}} - \{x_1, x_2, x_3, \infty\}$ and the supremum is taken over all simple closed curves α which do not pass through x_1, x_2, or x_3 and which separate two of the points $\{x_1, x_2, x_3, \infty\}$ from the other two. This supremum would be a difficult number to estimate were it not for the fact that we know exactly the form of the extremal metrics for these extremal length problems. The extremal metrics $\rho(z)|dz|$ are of the form $\rho(z)|dz| = |\varphi(z)|^{1/2}|dz|$ where $\varphi(z)dz^2$ is a holomorphic quadratic differential on $\hat{\mathbb{C}} - \{x_1, x_2, x_3, \infty\}$. This means that

$$\varphi_\theta(z)dz^2 = \frac{Ce^{i\theta}dz^2}{(z - x_1)(z - x_2)(z - x_3)}. \tag{13}$$

We select the constant C so that $\|\varphi_\theta\| = 1$. Grötzsch's inequality tells us that

$$\frac{\Lambda(f^\mu(\alpha))}{\Lambda(\alpha)} \leq \iint K_z|\varphi_\theta(z)|dxdy \tag{14}$$

where θ is chosen so that the horizontal closed trajectories of φ_θ are in the same homotopy class as α. Let p be a point on the real axis and $\epsilon > 0$ be given. Then, by hypothesis, there is a neighborhood N of p such that $K_z \leq 1 + \epsilon$ for all z in N. Now, let x_1, x_2, and x_3 be points on the real axis and very near to p. Nearly all of the mass of the measure $|\varphi(z)|dxdy$ is contained in the open set N as x_1, x_2, and x_3 approach p. Also, we have a fixed bound on K_z outside of the neighborhood N. Hence, the ratio in (13) is bounded by $1 + 2\epsilon$ and this estimate is uniform in α. Thus, the number K_0 in (12) is bounded by $1 + 2\epsilon$. We combine this result with our observation that the system of elliptical neighborhoods of the segment from a to c, defined by (10), are generated by the Poincaré neighborhoods, N_ϵ. It follows that if a mapping of the real axis is symmetric, then the induced quasicircle C has the strong reverse triangle property.

Our next objective is to show that ii) implies iii), that is, that if C has the strong reverse triangle property, then it has the disk template property. Let $x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}$ be $2n + 1$ equally spaced points on the real axis and let $a_j = f(x_j)$, $0 \leq j \leq n$, and $a_{n-2} = a$, $a_{n-1} = b$, $a_n = c$, where f is the Riemann mapping of the lower half plane onto Ω. Assume $f(x_k)$ is not in U, where U is the given neighborhood of $d = f(\infty)$. Apply the reverse triangle inequality (10) with

$$\lambda = 1 + \frac{1}{2}\left(\frac{\epsilon}{nB^{1+\log_2 n}}\right)^2. \tag{15}$$

The hypothesis implies the existence of a positive number δ such that, for $0 < \delta' \leq \delta$ and for $|a_{j-1} - a_j| < \delta$, then the arc of C joining a_{j-1} to a_j is contained entirely in the ellipse with foci at a_{j-1} and a_j and eccentricity equal to λ^{-1}. Since the Riemann mapping f extends continuously to the real axis, we can assume that the equally spaced points x_j are near enough together so that $|a_j - a_{j-1}| < \delta$, for each j.

Let l_j be the straight line segment joining a_{j-1} to a_j. Let θ_j be the angle of turning from l_{j-1} to l_j. Obviously, the total turning θ between the line segment l_1 and the line segment l_n satisfies $|\theta| \leq |\theta_1| + \cdots + |\theta_n|$ and the total length of the polygonal path made up of l_1 through l_n is less than or equal to $|l_1| + \cdots + |l_n|$. We wish to estimate the vertical displacement of a_0 from the straight line through $a_{n-1} = b$ and $a_n = c$. This displacement is bounded by $|\sin\theta|$ multiplied by $|l_1| + \cdots + |l_n|$.

Let $\delta' = |a - b| = |l_{n-1}| = |a_{n-1} - a_{n-2}|$. The bound B on the ratios of distances of the form $|a_k - a_{k-j}|$, $|a_{k-j} - a_{k-2j}|$ and $|a_k - a_{k-2j}|$ leads to the bound $B^{\log_2 n}|l_{n-1}| = B^{\log_2 n}\delta'$ for $|l_1| + \cdots + |l_n|$. For the method of proof of this estimate, see the proof of Lemma 6.1. The other factor in determining this displacement is $\sin\theta$. Since $|\sin\theta| \leq |\theta|$, it suffices to estimate each of the angles θ_j. We apply the law of cosines to the triangle with vertices at a_{j-2}, a_{j-1}, and a_j. On letting $|l_j| + |l_{j-1}| = (1 + \tilde{\epsilon})|a_{j-2} - a_j|$, for sufficiently small θ_j, we obtain

$$\theta_{j-1}^2 \leq 2\,\frac{|a_{j-2} - a_j|^2}{|l_j||l_{j-1}|}\,\tilde{\epsilon}.$$

The bound B on the ratios $|a_{j-2} - a_j|/|l_{j-1}|$ and $|a_{j-2} - a_j|/|l_j|$ yields

$$\theta_{j-1}^2 \leq 2\tilde{\epsilon}B^2.$$

Therefore, the total turning between l_1 and l_n is less than or equal to $2^{1/2}nB\tilde{\epsilon}^{1/2}$. Applying this result and the assumption that $\tilde{\epsilon}$ is less than or equal to $\frac{1}{2}(\epsilon/nB^{1+\log_2 n})^2$, we obtain the total vertical displacement is less than or equal to $\epsilon\delta'$. By taking a slightly smaller value of λ, we obtain a disk template with canal height and pond height less than or equal to $\epsilon\delta'$. Moreover, the arc α_{bc} is contained in the pond and the arc of the quasicircle C between a_0 and $a_{n-2} = a$ is contained in the canal. Since $\delta' = |a - b|$, it is clear that the separation between the canal and the pond is of the order $(1 - \epsilon)\delta'$.

We must still arrange for the arc of the quasicircle between ∞ and a_0 not to reenter the disk template. In order to accomplish this, we must take a finer array of equally distributed points $x_0, \ldots, x_{2m}$, where m is much larger than n. For sufficiently large m, if the arc between ∞ and a_0 reenters the disk template centered at a_{m-2} and running back along n points, then the quasicircle C would not satisfy the bound B on the ratios of distances between images of symmetric points.

Finally, we must show that the canal length divided by δ' approaches ∞. Since the polygonal path with vertices at a_j, $0 \leq j \leq n - 2$, are trapped in a canal which is narrow compared to the length of any one of its sides, this fact is a consequence of the following lemma.

LEMMA 6.1. *Let $x_0, \ldots, x_n$ be points on the real axis which partition the closed interval $[x_0, x_n]$ into n adjacent subintervals, $J_i = [x_{i-1}, x_i]$. Assume every pair of adjacent intervals satisfies*

$$B^{-1}|J_i| \leq |J_{i-1}| \leq B|J_i|.$$

Assume further that every pair of pairs of adjacent intervals satisfies

$$B^{-1}(|J_{i-2}| + |J_{i-1}|) \leq (|J_i| + |J_{i+1}|) \leq B(|J_{i-2}| + |J_{i-1}|).$$

Assume further that every pair of k-tuples of adjacent intervals satisfies an analogous inequality for $k \leq n/2$. Then $|x_n - x_0| \geq B^{-1}(\log_2 n)|J_1|$.[1]

---

[1]From quasiconformal mapping, the term $\log_2 n$ can be replaced by n^α for some $\alpha > 0$.

Proof. Let m be the largest integer for which $2^m \leq n$. The hypotheses imply that $|J_2| \geq B|J_1|$,

$$|J_3| + |J_4| \geq B^{-1}(|J_1| + |J_2|),\ |J_5| + \cdots + |J_8| \geq B^{-1}(|J_1| + \cdots + |J_4|), \ldots, \text{etc.}$$

It follows that

$$|J_1| + \cdots + |J_{2^m}| \geq (1 + B^{-1})^m |J_1|. \tag{16}$$

The lemma follows from the facts that $|x_n - x_0|$ is bigger than or equal to the left hand side of (16), that $B \geq 1$, and that $m \geq \log_2 n - 1$.

Notice that we have used only a few of the assumed inequalities. Using the other inequalities, a stronger conclusion can be drawn. However, the result given here suffices for our purposes. Since the straight line segments connecting a_{i-1} to a_i are trapped between the sides of a canal which is narrow compared to $|a_{i-1} - a_i|$, we see that the canal length of the disk template is bigger than or equal to $(\log_2 n)\delta'/2B$. It is obvious that the pond length is between $B^{-1}\delta'$ and $B\delta'$.

The final step is to show that iii) implies iv), that is, that the disk template property implies the extremal length property. To do this we compare three different extremal lengths of the family of curves homotopic to γ_1 on different Riemann surfaces. Λ_1 is the extremal length of the family of curves homotopic to γ_1 in the disk template. $\tilde{\Lambda}_1$ is the extremal length of the family of curves homotopic to γ_1 on the Riemann sphere minus two subarcs α_{ad} and α_{bc} of the quasicircle C. L_1 is the extremal length of the family of curves homotopic to γ_1 on the Riemann sphere minus four points, a, b, c, and d. Obviously, $\Lambda_1 \geq \tilde{\Lambda}_1 \geq L_1$. Without loss of generality, we can assume that $d = \infty$.

Of course, the corresponding extremal lengths defined for the curve γ_2 satisfy $\Lambda_2 \geq \tilde{\Lambda}_2 \geq L_2$. To complete the proof of the theorem, we need two additional lemmas:

LEMMA 6.2. *There is a function $c(\epsilon)$ approaching 1 as ϵ approaches 0 such that $c(\epsilon)L_1 \geq \Lambda_1$ and $c(\epsilon)L_2 \geq \Lambda_2$.*

LEMMA 6.3. *The product of the extremal lengths L_1 and L_2 approaches 4 as a, b, and c move through admissible triples into smaller* *and smaller neighborhoods of a given point p on C if p is not equal to d.*

Both of these lemmas are proved by examination of the covering of $\hat{\mathbb{C}} - \{a, b, c, \infty\}$ by the plane $\mathbb{C}$ factored by the lattice subgroup, $n + m\tau$. This covering is induced by the elliptic $\mathcal{P}$-function. Let $\mathcal{P}(\frac{1}{2}) = e_1$, $\mathcal{P}(\frac{1}{2} + \frac{1}{2}\tau) = e_2$ and $\mathcal{P}(\frac{1}{2}\tau) = e_3$. We can assume that τ is selected so that, after postcomposing $\mathcal{P}$ by an affine transformation, $e_1 = a$, $e_2 = b$, and $e_3 = c$. Of course, $\mathcal{P}(0) = \infty$. The homotopy class of γ_1 lifts to the homotopy class of a line passing through the left side and the right side of the period parallelogram. The homotopy class of γ_2 lifts to the homotopy class of the line passing through the bottom and top sides of the period parallelogram. The extremal lengths L_1 and L_2 are given by

$$(17) \quad L_1 = \frac{2}{\tau_2} \quad \text{and} \quad L_2 = \frac{2(\tau_1^2 + \tau_2^2)}{\tau_2}, \qquad \text{where } \tau = \tau_1 + i\tau_2.$$

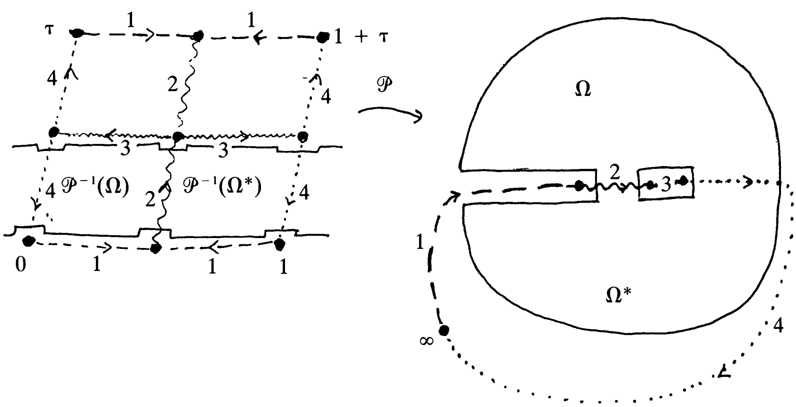


FIGURE 5

From inspection of Figure 5, it is clear that the disk template for a, b and c with b and c lying in the pond lifts to a strip in the plane minus the lattice of half periods whose extremal length on the quotient torus satisfies $\Lambda_1/L_1 \leq c(\epsilon)$. Moreover, the parallel statement is true for a disk template with a and b lying in a pond. This completes the proof of Lemma 6.2. Lemma 6.3 follows from (17) and the fact that as a, b and c move through admissible triples to smaller and smaller scales, the relative distance from b to a or c stays bounded in terms of the distance from a to c and the reverse triangle inequality with λ arbitrarily close to 1 forces b to be arbitrarily near to the straight line segment joining a to c. This completes the proof of Theorem 6.3.[2]

Theorems 6.1, 6.2 and 6.3 give geometric ways to construct symmetric quasicircles. There is also an analytic way to construct a wide class of symmetric quasicircles. Let $\varphi(z)$ be a rational function, holomorphic in the unit disk Δ, with following properties:

i) $\varphi(z)$ has at most simple poles and these occur only on S^1,
ii) $\sup_{z \text{ in } \Delta}|(1 - |z|^2)^2\varphi(z)| < 2$.

Let $f(z) = \eta_1(z)/\eta_2(z)$ where η_1 and η_2 are two linearly independent solutions of the equation $\eta''(z) = -\frac{1}{2}\varphi(z)\eta(z)$ for z in the unit disk Δ. Let $\Omega = f(\Delta)$. We know that property ii) by itself is enough to force Ω to be a quasidisk. The additional assumption that $\varphi(z)$ is rational with at most simple poles on S^1 implies that the cusp form $(1 - |z|^2)^2\overline{\varphi(z)}$ defined on the unit disk Δ vanishes as $z \to S^1$. Let Ω^* be the domain complementary in $\hat{\mathbb{C}}$ to Ω and let g be the Riemann mapping from Δ onto Ω^*.

COROLLARY TO PROPOSITION 4.1. *Let f and g be constructed as above and assume φ satisfies* i) *and* ii). *Then the induced mapping $g^{-1} \circ f$ restricted to S^1 is a symmetric homeomorphism of S^1 and $f(S^1)$ is a symmetric quasicircle.*

7. The complex manifold structure on the space of symmetric structures. At the beginning of Section 4, we already defined the Banach spaces B and B_0. As our model Banach space for the complex structure on the space of symmetric structures, we use the quotient space B/B_0 with the quotient norm. Recall that the complex structure for Teichmüller space is determined by declaring that the mapping $\pi : \mu \mapsto f_\mu$ is holomorphic, where the holomorphic dependence on μ is determined by the complex structure on the open unit ball of the complex vector space of L_∞-Beltrami differentials. Thus a mapping h defined from QS mod $PSL(2, \mathbb{R})$ is holomorphic if, and only if, the composition $h \circ \pi$ is holomorphic.

[2]There is a unique line in the Teichmüller space of the sphere punctured at a, b, c, and ∞ along which the product L_1L_2 is minimum. The straight line segment from a to c is the image in Riemann space of that line. This is a special case of a the minimum geodesic principle for transversely realizable measured foliations proved by Gardiner and Masur in [10].

To show that QS mod S is a complex manifold, we construct a holomorphic chart for points $S\tilde{g}$ in a neighborhood of a fixed right coset Sg. We define a mapping $\hat{\mathcal{S}}$ from QS mod S into B/B_0. $\hat{\mathcal{S}}$ is defined in exactly the same way as $\mathcal{S}$ in steps i) through v) at the beginning of Section 4, with the exception that $\hat{\mathcal{S}}$ is viewed as defined on the right cosets of S in QS with image in B/B_0. To show that $\hat{\mathcal{S}}$ is well defined, we must show that $\mathcal{S}(s \circ g)$ differs from $\mathcal{S}(g)$ by an element of the Banach space B_0. The Cayley identity for the Schwarzian derivative implies

$$\mathcal{S}(s \circ g) = g^*(\mathcal{S}(g_* s)) + \mathcal{S}(g). \tag{18}$$

We claim that the first term on the right hand side of (18) is an element of B_0 and that, therefore $\hat{\mathcal{S}}$ is well-defined.

To prove the claim, let μ and ν be the Beltrami coefficients of quasiconformal extensions of g and s to the interior of S^1. Let g_μ and s_ν be these extensions. Let g^μ be a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ with Beltrami coefficient equal to μ in the unit disk Δ and identically equal to 0 in the exterior of the disk. Let $\tilde{s}$ be the quasiconformal homeomorphism of $\hat{\mathbb{C}}$ with the property that $\tilde{s} \circ g^\mu$ has the same Beltrami coefficient as $s_\nu \circ g_\mu$ in Δ and $\tilde{s} \circ g^\mu$ has Beltrami coefficient identically equal to zero in the exterior of Δ. Let h be a conformal mapping of $g^\mu(\Delta)$ onto Δ. Then, $h \circ g^\mu = g_\mu$ up to postcomposition by a Möbius transformation. Moreover, if r is the Riemann mapping of $\tilde{s} \circ g^\mu(\Delta)$ onto Δ, then

$$s_\nu \circ g_\mu = r \circ \tilde{s} \circ g^\mu = r \circ \tilde{s} \circ h^{-1} \circ h \circ g^\mu = r \circ \tilde{s} \circ h^{-1} \circ g_\mu.$$

Cancelling g_μ from the right and left side of this equation, we find that $s_\nu \circ h = r \circ \tilde{s}$. Since postcomposition by a conformal mapping does not change a Beltrami coefficient, the Beltrami coefficient of $r \circ \tilde{s}$ equals the Beltrami coefficient of $\tilde{s}$. On the other hand, precomposition by a conformal mapping changes the Beltrami coefficient in an obvious way. The Beltrami coefficient $\tilde{\nu}$ of $\tilde{s}$ is equal to the pullback by the Riemann mapping h to the domain $g^\mu(\Delta)$ of the Beltrami coefficient ν in Δ. That is,

$$\tilde{\nu} = \nu(h(z)) \frac{\overline{h'(z)}}{h'(z)}.$$

Therefore, $\tilde{\nu}$ is a Beltrami coefficient with boundary dilatation 1 in the domain $g^{\mu}(\Delta)$.

The first term on the right hand side of (18) is

$$g^*(\mathcal{S}(g_*s)) = ((\text{Schwarzian deriv. of } \tilde{s}) \circ g^{\mu})g^{\mu'}(z)^2, \tag{19}$$

where z is in the exterior of the unit disk. Now, Proposition 4.1 obviously extends to the case where QS and S are defined for a quasicircle instead of S^1. In particular, the Schwarzian derivative of the mapping $\tilde{s}$, which has boundary dilatation equal to 1 on the domain $g^{\mu}(\Delta)$, is an element of the space of bounded cusp forms which vanish at the boundary of the complementary domain. Since composing with g^{μ} and multiplying by $g^{\mu'}(z)^2$ is an isometry on spaces of bounded cusp forms and since this operation preserves cusp forms which vanish at the boundary, it follows that the right hand side of (19) is an element of B_0 and the claim is proved. We have the commutative diagram:

$$\begin{array}{ccc} QS & \xrightarrow{\mathcal{S}} & B \\ \pi \downarrow & & \downarrow p \\ QS \bmod S & \xrightarrow{\hat{\mathcal{S}}} & B/B_0 \end{array}$$

We do not know whether $\hat{\mathcal{S}}$ is globally one-to-one. Nonetheless, it follows from the existence of the cross sections constructed in Section 4 that $\hat{\mathcal{S}}$ is locally one-to-one and a homeomorphism from a neighborhood of any point in QS mod S onto a neighborhood of its image in B/B_0.

To see that $\hat{\mathcal{S}}$ is locally one-to-one, pick a small neighborhood of the identity in QS which by π is mapped onto a neighborhood of the identity in QS mod S. Suppose that $\mathcal{S}(h_1) = \mathcal{S}(h_2) + \varphi_0$ for two functions h_1 and h_2 in this neighborhood and that φ_0 is in B_0. Let $h_i = f_i \circ g_i$, $i =$ 1 or 2, be the Riemann factorizations of these two mappings. Then

$$\text{Schwarzian}(g_1 \circ g_2^{-1}) = \varphi_0(g_2^{-1}(z))(g_2^{-1}(z))'^2.$$

The right hand side of this equation is a cusp form in g_2 (lower half plane) which vanishes at infinity and which has small norm. Therefore, there is a mapping f, asymptotically conformal on g_2 (upper half plane) and conformal on g_2 (lower half plane), whose Schwarzian derivative is equal to the right hand side. We conclude that $g_1 = f \circ g_2$. Thus, $f_1 \circ g_1 = f_1 \circ f \circ f_2^{-1} \circ f_2 \circ g_2$ and so $h_1 = f_1 \circ f \circ f_2^{-1} \circ h_2$. But $f_1 \circ f \circ f_2^{-1}$ is asymptotically conformal in the upper half plane, which means that h_1 is in the class Sh_2.

The mapping $\hat{\mathcal{S}}$ provides a manifold structure on QS mod S; the transition functions between overlapping neighborhoods are holomorphic because one can be moved into the other by composition on the right and right composition is a holomorphic mapping. We have the following theorem.

THEOREM 7.1. *The coset space* QS mod S *is a complex analytic manifold modelled on the Banach space* B/B_0. *Moreover, the projection mapping* $\pi : QS \to QS$ mod S *is holomorphic.*

8. Vector fields for symmetric and quasisymmetric mappings. From Teichmüller theory and from Sections 6 and 7, QS mod $PSL(2, \mathbb{R})$, S mod $PSL(2, \mathbb{R})$ and QS mod S form complex manifolds. It is also true that QS and S are differentiable manifolds. In this section we identify function spaces which are the tangent spaces at the identity to QS and S. The tangent at the coset S of the complex manifold QS mod S is then the quotient of these two function spaces. The tangent space for QS has already been described by M. Reimann in [13].

It is convenient to work with $\hat{\mathbb{R}}$ as a specific realization of a circle. Then these tangent spaces can then be viewed as spaces of real-valued functions $F(x)$ defined on the real axis. According to [13], the tangent space for QS consists of functions $F(x)$ satisfying a local condition

$$(QS) \qquad F(x + t) + F(x - t) - 2F(x) = O(t) \quad \text{and}$$

$$|F(x)| \leq O(x^2) \quad \text{as} \quad x \to \infty,$$

where the constant in $O(t)$ is uniform in x and t. The growth rate for F has the effect of making the function $F \circ L/L'$, where $L(x) = 1/x$, satisfy the same $O(t)$-condition on bounded intervals that F satisfies on bounded intervals. We let (QS) be the space of functions satisfying both of these properties.

For S, the tangent space consists of functions $F(x)$ which are in

(QS) and such that F and $F \circ L/L'$ both satisfy the local condition

$$(S) \qquad F(x + t) + F(x - t) - 2F(x) = o(t),$$

where $o(t)/t$ approaches zero uniformly for x in any bounded interval. Functions satisfying (S) are the "smooth" functions in the sense of Zygmund, [17]. The facts that the tangent spaces to QS and S are described in this way can be concluded from general principles for vector fields of quasiconformal motions developed by Sullivan and Thurston in [14]. Our purpose now is to show that these descriptions also follow from the infinitesimal theory of the Beltrami equation developed by Ahlfors and Bers and the Beurling-Ahlfors extension theorem.

From Ahlfors and Bers [2], we know that for any L_∞ complex valued function $\mu(z)$ defined for z in $\mathbb{C}$ there is a curve of quasiconformal homeomorphisms $f^{t\mu}$ of $\hat{\mathbb{C}}$ defined for $|t| < \|\mu\|_\infty^{-1}$ such that

i) $f^{t\mu}(z)$ is the identity mapping when $t = 0$,
ii) $f^{t\mu}$ is quasiconformal with Beltrami coefficient equal to $t\mu$,
iii) $f^{t\mu}$ is uniquely determined by $t\mu$ if it is normalized to fix 0, 1, and ∞,
iv) $f^{t\mu}$ is holomorphic as a function of t and its derivative in t is given by the following formula:

$$f^{t\mu}(z) = z + tF(z) + O(t^2),$$

where the constant in $O(t^2)$ is uniform for z in compact sets. If the solution $f^{t\mu}$ is normalized to fix three points, say a_1, a_2, and a_3, then the function $F(z)$ is given by

$$(20) \quad F(z) =$$

$$-\frac{(z - a_1)(z - a_2)(z - a_3)}{\pi} \iint \frac{\mu(\zeta)\,d\xi\,d\eta}{(\zeta - a_1)(\zeta - a_2)(\zeta - a_3)(\zeta - z)}.$$

The function $F(z)$ is uniquely determined by the conditions:

a) $F(a_1) = F(a_2) = F(a_3) = 0$ and $F(z)$ has a growth rate of the form $|z^2|$ as $z \to \infty$, and
b) the partial derivative $F_{\bar{z}}(z)$ exists in the generalized sense and is equal to $\mu(z)$.

We see that any complex valued function $F(z)$ satisfying properties a) and b) and having a bounded $\bar{\partial}$-derivative is a tangent vector to a curve of quasiconformal homeomorphisms $f^{t\mu}$, holomorphic in t and normalized to fix a_1, a_2 and a_3. Conversely, any holomorphic curve of homeomorphisms fixing a_1, a_2 and a_3 has a tangent vector $F(z)$ of this form.

In the special case that the normalization is at 0, 1, and ∞ formula (20) reduces to

$$(20') \qquad F(z) = -\frac{z(z-1)}{\pi} \iint \frac{\mu(\zeta)\,d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)} \quad \text{and}$$

and the growth of $F(z)$ near ∞ is bounded by a constant times $||z|\log|z||$. It is a consequence of formula (20) that $F(z)$ has a $||z|\log|z||$ modulus of continuity at every point in the plane.

Now suppose we are given a curve of quasisymmetric mappings $h^t(x)$, normalized to fix 0, 1 and ∞, which is the identity for $t = 0$ and which is differentiable with respect to t for the manifold structure on QS. Since the manifold QS has sections in the space of Beltrami coefficients, this implies there is a differentiable curve of Beltrami coefficients ν_t which are symmetric about the real axis such that h^t is the restriction to the real axis of a normalized quasiconformal mapping f^t whose Beltrami coefficient is ν_t. The differentiable curve of Beltrami coefficients ν_t satisfies $\nu_t = t\mu + o(t)$ for some L_∞ complex valued function μ and the function $\dot{F}$ given by formula (20) is a tangent vector to the curve of quasisymmetric mappings h^t. A simple calculation using formula (20′) shows that

$$(21) \quad \frac{F(z+t) + F(z-t) - 2F(z)}{t}$$

$$= -\frac{t}{\pi} \iint \frac{\mu(\zeta)\,d\xi d\eta}{(\zeta-(z+t))(\zeta-(z-t))(\zeta-z)}$$

$$= -\frac{1}{\pi} \iint \frac{\mu(t\zeta+z)\,d\xi d\eta}{(\zeta-1)(\zeta+1)(\zeta)}.$$

Since μ is bounded and the other part of the integrand is integrable, the whole expression is bounded independently of t and z and, therefore, $F(x)$ is in the function space (QS). Moreover, if $\mu(z)$ vanishes as z

approaches the real axis, then for sufficiently small values of t, the function $\mu_1(\zeta) = \mu(t\zeta + z)$ will have sup norm less than ϵ in any given compact set and will be bounded by the bound for μ outside of the compact set. Since the mass of the measure $|d\xi d\eta/(\zeta - 1)(\zeta + 1)(\zeta)|$ vanishes at ∞, we see that the assumption that μ vanishes at the real axis implies that $F(x)$ is in the function space (S).

Now, suppose we are given a real-valued function $F(x)$ in the function space (QS) or (S). We wish to create a differentiable curve of quasisymmetric or symmetric mappings which has $F(x)$ as its tangent vector. From the Ahlfors-Bers theorem on the existence of solutions to the Beltrami equation, it will be sufficient to extend $F(x)$ to a complex valued function with bounded $\bar{\partial}$-derivative. To this end, we apply the Beurling-Ahlfors extension formula to the function $F(x)$. Denoting the extension of F by the same letter, we let $F(x, y) = U(x, y) + i\,V(x, y)$, where

$$U(x, y) = \frac{1}{2y}\int_{x-y}^{x+y} F(t)dt \quad \text{and} \quad V(x, y)$$

$$= \frac{1}{y}\left[\int_{x}^{x+y} F(t)dt - \int_{x-y}^{x} F(t)dt\right].$$

To show that the $\bar{\partial}$-derivative of $U + iV$ is bounded, we must show that $U_x - V_y$ and $U_y + V_x$ are bounded. But

$$U_x = \frac{1}{2y}[F(x + y) - F(x - y)],$$

$$U_y = -\frac{1}{2y^2}\int_{x-y}^{x+y} F(t)dt + \frac{1}{2y}[F(x + y) + F(x - y)],$$

$$V_x = \frac{1}{y}[F(x + y) - 2F(x) + F(x - y)], \qquad \text{and}$$

$$V_y = -\frac{1}{y^2}\left[\int_{x}^{x+y} F(t)dt - \int_{x-y}^{x} F(t)dt\right] + \frac{1}{y}[F(x + y) - F(x - y)].$$

To estimate $U_x - V_y$ and $U_y + V_x$, we must estimate expressions of the form

$$\frac{1}{b-a}\left|\frac{1}{b-a}\int_a^b F(t)dt - \frac{1}{2}F(a) - \frac{1}{2}F(b)\right| \tag{22}$$

in terms of the bound B in

$$\left|\frac{F(x+t)+F(x-t)-2F(x)}{t}\right| \le B. \tag{23}$$

LEMMA 8.1. *If* (23) *is satisfied for all* x *and all* t, *then the quantity* (22) *is bounded by* $B/2$.

Proof. Note that inequality (23) can be rewritten as

$$\left|\frac{F(x+t)-F(x)}{t} - \frac{F(x)-F(x-t)}{t}\right| \le B$$

and it therefore says that for any symmetric triple $x - t$, x, and $x + t$, the difference between the right chordal slope and the left chordal slope is less than or equal to B. To estimate (22), let us first assume that $a = 0$, $b = 1$ and $F(0) = F(1) = 0$. In this case we claim that the maximum value of $|F(x)|$ for x in $[0, 1]$ cannot be more than $B/2$. Assume that the maximum value is taken on at $x = \xi$ and that $F(\xi) > B/2$. By symmetry around the vertical line $x = 1/2$, we can assume $0 < \xi \le 1/2$. Consider the symmetric triple 0, ξ, 2ξ. The slope of the chord through $(0, F(0))$ and $(\xi, F(\xi))$ is bigger than B. Since $F(\xi)$ is a maximum, the slope of the chord through ξ and 2ξ is negative. The difference between these two slopes is more than B and we have a contradiction of (23). We conclude that $|F(\xi)| \le B/2$.

For the second step in our proof, we again assume that $F(0) = 0 = F(1)$ and that F satisfies (23) for all x and t. Then it is obvious that

$$\left|\int_0^1 F(t)dt\right| \le |F(\xi)| \le B/2.$$

The third step is to observe that if $A(x) = cx + d$ is an affine mapping, then F satisfies (23) if, and only if, $F + A$ does. Similarly, the quantity in (22) is bounded by a given number for F if, and only if, it is bounded by the same number with F replaced by $F + A$. In particular, in order to prove (22) is bounded by $B/2$, we may assume that $F(a) = F(b) = 0$. Obviously, we may apply a translation so that a is translated to 0 and the statement that (22) is less than or equal to $B/2$ becomes

$$\left|\frac{1}{b}\int_0^b F(t)dt\right| \leq \frac{B}{2}\,b.$$

On making the substitution $t = bs$, we see that this last inequality is equivalent to

$$\left|\int_0^1 b^{-1}F(bs)ds\right| \leq B/2.$$

But it is easy to check that if $F(x)$ satisfies (23) then $G(x) = b^{-1}F(bx)$ also satisfies (23). Hence the inequality of the lemma is a consequence of our observations about the special case when $a = 0$, $b = 1$, $F(0) = F(1) = 0$.

Remark. One can easily show the converse to this lemma, namely, if the quantity in (22) is bounded independently of a and b, then F is in (QS).

To obtain a bound on the $\bar\partial$-derivative of $U + iV$, using the lemma we see that U_y is bounded and V_x is bounded from the fact that F satisfies (23). The same kind of argument shows that $U_x - V_y$ is bounded. One also can see that if the number B is replaced by a quantity $B(t)$ which approaches 0 as t approaches zero, then

$$\mu(z) = \frac{\partial}{\partial \bar z}(U + iV)$$

approaches zero as y approaches 0. Putting all of these results together, we obtain the following result.

THEOREM 8.1. *The tangent space at the identity to the manifold QS is the space of functions satisfying the boundedness condition (QS). The tangent space at the identity to S is the space of functions satisfying the boundedness condition (S). Finally, the tangent space at the identity to the complex manifold QS* mod *S is the quotient space $(QS)/(S)$. Here (QS) is the space of functions F which satisfy $F(x + t) + F(x - t) - 2F(x) = O(t)$, locally and uniformly in x, and (S) is the space of functions F which satisfy $F(x + t) + F(x - t) - 2F(x) = o(t)$, locally and uniformly in x.*

9. Smoothness properties of symmetric mappings. By the Grötzsch argument one knows that K-quasiconformal homeomorphisms and, thus, their quasisymmetric boundary values, satisfy a Hölder condition $|h(x) - h(y)| \leq C|x - y|^\alpha$ where $\alpha = 1/K$ and C depends on the region. It follows that symmetric homeomorphisms satisfy every α-Hölder condition for $\alpha < 1$. More is true, however.

If h satisfies an M-condition with M equal to $1 + \epsilon(t)$, then h has a modulus of continuity

$$C \times \exp\left(C_o \int_x^1 \frac{\epsilon(t)}{t}\, dt\right). \tag{24}$$

In order to prove (24), we may assume we are proving continuity at $x_0 = 0$ and that $h(0) = 0$ and $h(1) = 1$. This is because if h satisfies an M-condition then $L_1 \circ h \circ L_2$ satisfies the same M-condition for any affine transformations L_1 and L_2. Assume x is near x_0, $x_0 = h(x_0) = 0$ and $h(1) = 1$. Now use the inequality $h(x) \leq [M/(M + 1)]h(2x)$, which follows from the M-condition. Putting $x = 2^{-n}$ and $M = 1 + \epsilon(t)$, we obtain

$$h\left(\frac{1}{2}\cdot\frac{1}{2^{n-1}}\right) \leq \frac{1}{2}\cdot\left(1 + \epsilon\left(\frac{1}{2^n}\right)\right)\cdot h\left(\frac{1}{2^{n-1}}\right).$$

By induction, this yields

$$h(2^{-n}) \leq 2^{-n}(1 + \epsilon(2^{-1})) \cdots (1 + \epsilon(2^{-n})).$$

Taking logarithms and assuming, without loss of generality, that $\epsilon(t)$ is a monotone function of t, we obtain (24) by comparing the integral $\int_x^1 \frac{\epsilon(t)}{t}\, dt$ to a Riemann sum.

In [6], Carleson shows that $\int_0^1 \frac{\epsilon(t)}{t}\, dt < \infty$ implies h is C^1 and $h'(x)$ has a modulus of continuity

$$\int_0^x \frac{\epsilon(t)}{t}\, dt.$$

Thus h is $C^{1+\alpha}$ if and only if $\epsilon(t) = t^\alpha$. He also shows that under the weaker condition that $\int_0^1 \frac{\epsilon(t)^2}{t}\, dt < \infty$, h is nonsingular with respect to Lebesgue measure and h' belongs locally to L^2. Conversely, he shows that each of these conditions is sharp. In other words, if $\int_0^1 \frac{\epsilon(t)}{t}\, dt = \infty$, he constructs a quasisymmetric mapping h satisfying an M-condition with $M = 1 + \epsilon(t)$ which is not C^1. And, if $\int_0^1 \frac{\epsilon(t)^2}{t}\, dt = \infty$, he constructs a quasisymmetric mapping h with $M = 1 + \epsilon(t)$ which is not absolutely continuous.

Appendix on structures and coset spaces. If $\mathcal{G}$ is a pseudogroup of homeomorphisms of the real line, namely, a collection of partially defined homeomorphisms closed under composition, we can define $\mathcal{G}$-structures on topological one-manifolds. These are maximal atlases of coordinate charts with overlap transformations in $\mathcal{G}$.

If $\mathcal{H} \subset \mathcal{G}$ is a subpseudogroup, we can define $\mathcal{H}$ structures β subordinate to a given $\mathcal{G}$ structure α and we write $\beta \subset \alpha$.

Given a $\mathcal{G}$ structure α on the circle, say, we can define $G_\alpha(S^1)$, the group of global homeomorphisms of S^1 which are locally in $\mathcal{G}$ for the structure α. The group $G_\alpha(S^1)$ acts on the $\mathcal{H}$ structures $\beta \subset \alpha$ by "transport of structure": the β-chart φ is sent to the β'-chart ψ where $\psi = \varphi \circ f$, where f is in $G_\alpha(S^1)$. The isotropy group of this action at the structure β is just $H_\beta(S^1)$, that is, the group of global homeomorphisms of S^1 which are locally in $\mathcal{H}$ for the structure β.

If all β-structures subordinate to α are related by the transport of structure, then the set of β-structures subordinate to α is isomorphic to either of the left or right coset spaces $G_\alpha(S^1)$ factored by $H_\beta(S^1)$.

In [11], Kuiper calculates the possible projective structures on the circle, up to action by $\mathrm{Diff}(S^1)$ by transport of structure. This calculation has been reproduced since [11] at various times.

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