Proceedings of the International Congress of Mathematicians
Berkeley, California, USA, 1986

Quasiconformal Homeomorphisms in Dynamics, Topology, and Geometry

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Dedicated to R. H. Bing

This paper has four parts. Each part involves quasiconformal homeomorphisms. These can be defined between arbitrary metric spaces: $\varphi\colon X \to Y$ is K-quasiconformal (or K qc) if

$$H(x) = \limsup_{r\to 0} \frac{\sup|\varphi(x) - \varphi(y) \quad \text{where } |x-y| = r \text{ and } x \text{ is fixed}}{\inf|\varphi(x) - \varphi(y)| \quad \text{where } |x-y| = r \text{ and } x \text{ is fixed}}$$

is at most K where $|\ |$ means distance. Between open sets in Euclidean space, $H(x) \leq K$ implies φ has many interesting analytic properties. (See Gehring's lecture at this congress.)

In the *first part* we discuss Feigenbaum's numerical discoveries in one-dimensional iteration problems. Quasiconformal conjugacies can be used to define a useful coordinate independent *distance between real analytic dynamical systems* which is decreased by Feigenbaum's renormalization operator.

In the *second part* we discuss de Rham, Atiyah-Singer, and *Yang-Mills theory* in its foundational aspect *on quasiconformal manifolds.* The discussion (which is joint work with Simon Donaldson) connects with Donaldson's work and Freedman's work to complete a nice picture in the structure of manifolds—each topological manifold has an essentially unique quasiconformal structure in all dimensions—except four (Sullivan [**21**]). In dimension 4 both parts of the statement are false (Donaldson and Sullivan [**3**]).

In the *third part* we discuss the **C**-analytic classification of *expanding* analytic transformations near *fractal invariant sets.* The infinite dimensional Teichmüller space of such systems is embedded in the Hausdorff measure theories possible for the transformation on the fractal. These possible Hausdorff measure theories of fractals are nicely encoded in the theory of Gibbsian measure classes or Gibbs states.

In the *fourth part* we give a characterization of constant curvature among variable *negative curvature* in terms of a measure theoretical dynamical property equivalent to uniform quasiconformality for the geodesic flow. A dynamical equivalent of $-\frac{1}{4} < k \leq -1$ pinching is utilized.

I. Feigenbaum's renormalization operator and the quasiconformal Teichmüller metric. Mitchell Feigenbaum [7] made some remarkable numerical discoveries concerning the iteration of families f_a of real quadratic-like functions, namely those which fold the line smoothly with a nondegenerate quadratic critical point, for example $f_a(x) = -x^2 + a$. These discoveries may be summarized as follows:

(i) If a parameter variation f_a creates basic $2, 4, 8, 16, \ldots$ period doubling, the periods actually double at parameter values a_n which converge at a definite geometric rate to a limit a_∞,

$$|a_\infty - a_n| \sim \text{ constant}(4.6692\ldots)^{-n}.$$

(ii) The mapping for the limiting parameter value a_∞ has a Cantor set X to which almost all bounded orbits tend, and X has universal geometric properties like (a) Hausdorff dimension $X = .53\ldots$, and (b) X can be defined by an intersection of families of intervals $I_n = \{I_1^n, I_2^n, \ldots, I_{2^n}^n\}$ where ratios of sizes $|I_i^n|/|I_j^{n+1}|$ converge exponentially fast to universal ratios I_α, labeled by α which is any one-sided string of 0's and 1's.

Khanin, Sinai, and Vul [14] formulated the statement of Feigenbaum's convergence in this way—index the intervals of the nth level containing the critical point and critical value by 0 and 1, respectively. Index the remaining $2^n - 2$ intervals by time evolution ($f(I_k^n) = I_{k+1}^n$, $k + 1 < 2^n$), and think of the index in its 2-adic expansion, $k = \varepsilon_0 + \varepsilon_1 2 + \cdots + \varepsilon_{n-1}2^{n-1}$.

Then the ratios $|I_{i_n}^n|/|I_{j_n}^{n+1}|$ converge where $j_n = i_n$ or $j_n = i_n + 2^n$ to universal ratios I_α where α is any *2-adic integer* if the **final** coefficients of the expansions of $j_n = \varepsilon_0 + \varepsilon_1 2 + \cdots + \varepsilon_n 2^n$ agree on larger and larger **final** segments, and α is defined by $\alpha = \lim_{n\to\infty}(\varepsilon_n + \varepsilon_{n-1}2 + \cdots + \varepsilon_0 2^n)$.

After making these first two numerical discoveries (and one more described below) Feigenbaum formulated a renormalization picture to describe these phenomena. The renormalization operator is obtained by iterating the transformation twice, restricting this iterate to an interval about the original critical point, and then renormalizing to obtain a real quadratic-like mapping on a fixed size interval. Studying this operator R numerically Feigenbaum found a third phenomenon:

(iii) $R^n f_\infty$ converges to a universal function g, where f_∞ denotes the mapping of the given family corresponding to the parameter value a_∞ mentioned above. The function g is a fixed point of the operator R,

$$g(x) = \lambda g \cdot g(x/\lambda), \qquad \lambda = -2.50290\ldots$$

(the Cvitanovic-Feigenbaum equation).

Since Feigenbaum's work, there has been more numerical work revealing similar phenomena in other dynamical situations, e.g., Cvitanovic–period tripling, etc, Shenker—circle mappings with critical point, Widom—boundaries of Siegel disks, Milnor—infinitely many points of the Mandelbrot set, and others. There has also been work trying to prove theoretical theorems modelling Feigenbaum's discovery (Campanino, Collett, Eckmann, Epstein, Khanin, Lanford, Ruelle, Sinai, Tresser, Vul, Wittwer, and many others). For example, see Epstein [**5**] for a function theory proof of the existence of the universal function g satisfying the Cvitanovic-Feigenbaum equation. Khanin, Sinai, and Vul [**14**] proved that for the universal function g the ratio of interval lengths converge exponentially fast to ratios I_α as indicated above. Lanford [**15**] proved (using rigorous numerical analysis) that the spectrum of the operator R linearized at g has one point outside the open unit circle $\{-4.6692\dots\}$. Lanford's work yields Feigenbaum's picture of the renormalizaton operator R on *some neighborhood* $\mathcal{L}$ of the fixed function g. Also, it proves the point (i), $|a_\infty - a_n| \sim \text{constant}(4.6692\dots)^{-n}$, for the $x \to -x^2 + a$, family

Several questions remain.

PROBLEM 1. Prove the second Feigenbaum discovery (ii) that the Feigenbaum attractor X has universal geometric structure for a general class of mappings. (This universal structure can be described by Gibbsian measure class as in Part III using $E\colon X \to X$ where E is an expanding map which is the union of $x \to \lambda x$ and $x \to \lambda g(x)$ on left and right pieces respectively of X.)

PROBLEM 2. Justify Feigenbaum's third numerical discovery (iii) by extending the local stable manifold (due to Lanford's work) of the Feigenbaum renormalization operator to a global stable manifold.

PROBLEM 3. Find a conceptual, more geometrical treatment of Feigenbaum's three points (i), (ii), (iii) yielding a new proof of Lanford's theorem on the spectrum and, hopefully, proofs for various generalized Feigenbaum phenomena heretofore only treated numerically.

We will study these problems using quasiconformal homeomorphisms to define a coordinate free distance between *complexifications* of real quadratic maps, a definition due to Douady and Hubbard [**4**]. A *complex quadratic-like mapping* is a pair (π, i) where $\pi\colon \overline{V} \to V$ is a two-sheeted covering with one branch point onto a simply-connected Reimannian surface V, and $i\colon \overline{V} \to V$ is a conformal embedding of $\overline{V}$ into V with compact closure. Given (π, i) we consider the quadratic-like mapping $f\colon V_1 \to V$ given by the composition $\pi \cdot i^{-1}$ where $V_1 = i\overline{V}$.

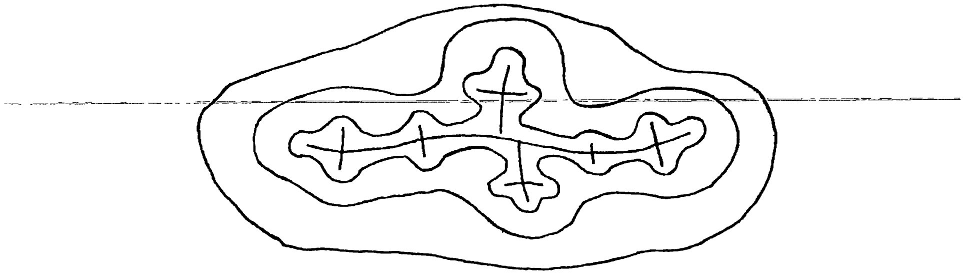


FIGURE 1

Note that $f^{-1}V = V_1$ and if $V_n = f^{-n}V$, then $K(f) = \bigcap_n V_n$ is a maximal and compact invariant set for the iteration of f on V_1.

We say two complex quadratic-like mappings f_1, f_2 are (i) *analytically equivalent* if they are complex analytically conjugate on neighborhoods of the compact invariant sets, and (ii) at *distance at most* $\log K$ if there is a K quasiconformal homeomorphism conjugating f_1 and f_2 between neighborhoods of the invariant sets. Define distance(f_1, f_2) as the infimum of such $\log K$.

THEOREM. *If* distance$(f_1, f_2) = 0$, *then* f_1 *and* f_2 *are analytically equivalent on neighborhoods of their invariant sets.*

SKETCH OF PROOF. If the invariant set of f is connected one can associate (Douady and Hubbard [**4**]) a real analytic expanding map h of the circle (of exterior prime ends of the connected invariant set). The real analytic conjugacy class of h essentially determines the equivalence class of f and is in turn determined by the sizes of eigenvalues at its periodic points (see Part III of this paper or Shub and Sullivan [**19**]). However, the K *qc* conjugacy between f_1 and f_2 yields a Hölder continuous conjugacy φ between h_1 and h_2, $|\varphi(x) - \varphi(y)| \leq C|x-y|^\alpha$, where C depends on the size of the neighborhood and α only depends on K. If distance$(f_1, f_2) = 0$ this relationship implies the eigenvalues of h_1 equal to those of h_2. So h_1 and h_2 are real analytically conjugate and this (plus one more consideration which is automatic) implies f_1 and f_2 are $\mathbf{C}$ analytically conjugate on some neighborhood.

If the invariant set is not connected it is expanded by f and we may use Part III directly. Q.E.D.

Let $\mathcal{F}$ be the metric space of equivalence classes of complex quadratic-like mappings at a finite distance from the Feigenbaum universal function (defined by $\lambda g \circ g(x/\lambda) = g(x)$.) (One knows the complexification of g is quadratic-like; see Epstein [**6**], for example.) The space $\mathcal{F}$ has rectifiable paths between any pair of points whose length is the distance (this uses the measurable Reimannian mapping as in the *qc* deformations section of Sullivan [**20**]).

Feigenbaum's renormalization operator may be defined on $\mathcal{F}$. Since g is quadratic-like, the equation $\lambda g \circ g(x/\lambda) = g$ shows $g \circ g$ is quadratic-like near the original critical point. Thus if f is *qc* conjugate to g, $f \circ f$ is *qc* conjugate to $g \circ g$ and so it is quadratic-like also on some disk around the critical point. It is easy to see that regarding $f \circ f$ as a quadratic-like mapping only depends (up to equivalence) on the critical point chosen and not on the disk. More generally, there are canonically defined renormalization operators defined on full path components of the space of quadratic-like mappings with the Teichmüller metric whenever one mapping of the component can be renormalized. Here is our result on Problem 2.

THEOREM. *In the qc path component* $\mathcal{F}$ *of the universal Feigenbaum map, there is a canonical renormalization operator* $R\colon \mathcal{F} \to \mathcal{F}$ *defined on representatives by* $f \to f \circ f$ *(restricted to a neighborhood around the original critical point). The operator* R *is strictly distance decreasing for the above Teichmüller* *metric, $R[g] = [g]$, and for any $[f] \in \mathcal{F}$ the orbit of $[f]$ under iterates of R tends to $[g]$.*

NOTE. A real analytic mapping whose complexification is quadratic-like, and whose critical orbit has the kneading sequence of the Feigenbaum map, lies in the space $\mathcal{F}$.

COROLLARY. *The unique complex quadratic-like solution of the Cvitanovic-Feigenbaum functional equation is $[g]$.*

PROOF OF THEOREM. The nonincreasing distance property of R follows from the definitions. It follows from Lanford's results [**15**] that R is contracting on some neighborhood U of $[g]$. From the existence of distance paths connecting $[g]$ to any other point $[f]$ it follows $\mathcal{F} = \bigcup_n R^{-n}U$. Then the strictly decreasing property follows from Royden's interpretation of the Teichmüller metric as the Kobayaski metric.

DISCUSSION OF PROBLEM 1. Curt McMullen has observed (private communication) that the convergence of the theorem can be lifted to the level of representatives. To see this one looks at the size of $[f]$, the supremum of moduli of the annuli $V - V_1$ over representatives.

Feigenbaum and others have calculated the spectrum of R numerically. The part inside the unit circle satisfies $|\lambda| \leq (2.50290\ldots)^{-1}$. This inequality applies to R acting on representative functions up to linear rescaling.

The author and Feigenbaum [**8**] have proved that if on the level of representatives $R^n f \to g$ at an exponential rate $\leq (2.50290\ldots)^{-n}$ in the C^3 topology, then the Cantor set for f is $C^{1+\beta}$ diffeomorphic to that of g by a map which is a conjugacy on the Cantor sets.

SPECULATION. This summarizes our results about the third problem. Regarding the first two problems and others in iteration theory we conjecture that there is an infinite-dimensional *Teichmüller mapping* theorem in these dynamical contexts which may be used to show directly that all the renormalization operators strictly decrease distance and have fixed points (compare Milnor's conjectures [**16**]).

II. Analysis on quasiconformal manifolds and Yang-Mills fields. A quasiconformal manifold is a topological manifold provided with a maximal atlas of charts U_α where the overlap transformations $\varphi_{\alpha\beta}$ are quasiconformal homeomorphisms between open sets of Euclidean n-space. One knows that if $n \neq 4$ all topological n-manifolds have such qc charts. Also, if $n \neq 4$ the qc structure is unique up to homeomorphism arbitrarily close to the identity. (See Sullivan [**21**] for these and the same theorems for bi-Lipschitz homeomorphisms.)

In joint work with Simon Donaldson [**3**] we have tried to show enough global analysis exists on qc manifolds to replace the word smooth by the word quasiconformal in many of the latter's theorems. Then adding in Freedman's work one finds [**3**] many topological 4-manifolds do not have qc structures and many pairs of homeomorphic qc (even smooth or C-algebraic) 4-manifolds are not qc homeomorphic. We discuss that global analysis now.

If M is a smooth compact Riemannian n-manifold one has a $*$-operator from k-forms to $(n-k)$-forms, pointwise norms defined by

$$|w| = (w \wedge *w/\text{Riemannian volume})^{1/2}$$

and local and global L^p norms defined by $(\int |w|^p\, dm)^{1/p}$. The topological vector spaces so defined only depend on the underlying differentiable structure (even the Lipschitz structure). A set of these norms, one for each k, $\|\omega\| = (\int |\omega|^{n/k})^{k/n}$ where ω is a k-form, is unchanged if the metric is changed conformally. Also, $\|\omega \wedge \eta\| \le \|\omega\| \cdot \|\eta\|$ by Hölder $(k/n + l/n = (k+l)/n)$ so we have a natural graded Banach algebra $\Omega(\|\cdot\|)$ of forms associated to the underlying conformal structure of the Riemannian manifold. It follows that the graded Banachable algebra Ω of forms is locally well defined under qc changes of coordinate using the fact that a K-quasiconformal homeomorphism φ (oriented) satisfies

(i) φ is differentiable a.e.;

(ii) Jacobian $\varphi > 0$ a.e.; and

(iii) $|d\varphi|^n \le K$ Jacobian φ a.e.

The deeper fact (Gehring [**9**]) that Jacobian φ is locally p-summable for some $p = p(K) > 1$ implies the dense subalgebra $\Omega' \subset \Omega$ consisting of all k-forms with coefficients locally p-summable for some $p > n/k$ is also qc invariant.

Now we turn to the exterior differential.

PROPOSITION. *The unbounded operator defined in the distributional sense $\Omega \xrightarrow{d} \Omega$ in a local chart commutes with the action of qc homeomorphisms φ,*

$$d(\varphi^* \omega) = \varphi^*\, d\omega.$$

SKETCH OF PROOF. Using Chapter 3 of Morrey [**17**], Ziemer [**28**] shows the class of continuous functions f with $\|df\| < \infty$ (i.e., df is n-summable) is qc invariant, and $d(\varphi^* f) = \varphi^*\, df$ where df means the distributional total differential. Form the subalgebra of forms generated by such f, df. It has a d and may be used as a testing algebra to define a qc invariant distributional d. A smoothing argument (Vaisala [**27**, p. 80]) shows this qc differential d is the same as the smooth distributional d. Q.E.D.

Now define the *p-regular forms* on a qc manifold to be the set of forms in $\Omega' \subset \Omega$ whose exterior d is also in Ω', namely

$$p\text{-}regular\ forms\ on\ qc\ manifold = \Omega' \cap d^{-1}\Omega'.$$

Note that a p-regular function f is one such that df has coefficients in L^p for some $p > n$. Thus f is Hölder of exponent $(p/n - 1)$ (Morrey [**17**]). A similar result holds for h-regular functions defined below.

In a coordinate system consider a smoothing operator on forms $\omega \to \int t^* \omega\, d\mu = R\omega$ where μ is a smooth measure on the translation group. Using [Lie derivative] = [d(contraction) + (contraction)d] one finds by integration a chain homotopy between R and the identity: $R - I = dS + Sd$, where S is (i) a derivation of degree -1, and (ii) a singular integral operator sending k-forms with p-summable coefficients into $(k-1)$-forms with coefficients having first partials p-summable, $p > 1$. Thus by the Sobolev embedding $L_1^p \subset L^q$ where $1/q = 1/p - 1/n$ we see that S carries Ω' into Ω', and Ω into Ω (except for Ω_n). This yields a Poincaré lemma and the following

THEOREM. *The de Rham cohomology of* $(\Omega' \xrightarrow{d} \Omega')$ *for a qc manifold agrees with the usual cohomology.*

Now consider a trivialized R^k-bundle over a chart U, gauge transformations $g\colon U \to O(k)$ which are regular (in the above sense and the sense below), $k \times k$ skew symmetric connection matrices of regular 1-forms θ, and the corresponding curvature forms $\Omega = d\theta + \theta \wedge \theta$. Changing the trivialization by a gauge transformation g induces the familiar changes $\theta \to g^{-1}dg + g^{-1}\theta g = \theta^g$, $\Omega \to g^{-1}\Omega g = \Omega^g$. If $g = \exp \xi t$, the infinitesimal change in θ is $d\xi - [\xi, \theta]$ where $[\ ,\ ]$ means commutator.

NONABELIAN POINCARÉ LEMMA. Take $\xi = S\theta$; then the infinitesimal change in θ is $dS\theta - [S\theta, \theta] = dS\theta + S\,d\theta - S\,d\theta - S(\theta \cdot \theta)$ (since S is a derivation)$=$ $-\theta + R\theta - S\Omega$.

Using this we can show a regular connection form can be regauged to reduce $\|\theta\|$ so that it is dominated by a constant $\cdot\|\Omega\|$.

The second notion of regularity is h-regularity of ω which means the amount of $\|\ \|$ norm (for ω and $d\omega$) in a ball of radius r is at most Cr^α for some constants C, α. This notion of h-regularity is qc-invariant and the S operator preserves this h-regularity because Calderon Zygmund kernels do (Peetre [**17.5**]). Now consider locally h-regular connections on 4-manifolds whose curvatures satisfy, in addition, the *quasi-self duality condition*:

$$\textit{K-quasi Yang-Mills condition} \quad |\Omega|^2 \le K \operatorname{tr} \Omega \wedge \Omega / \text{ volume element}$$

relative to a measurable locally qc Euclidean metric.

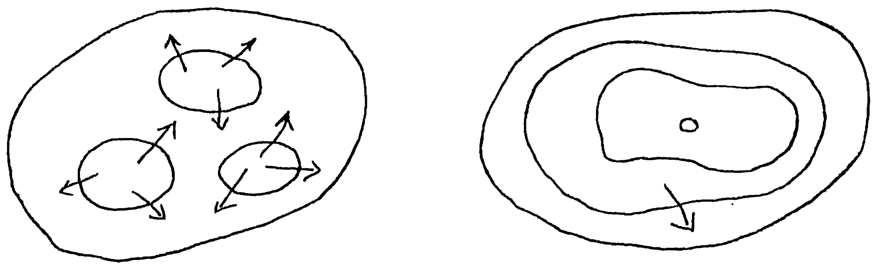
If θ and $d\theta$ (equivalently θ, Ω) are h-regular in B^4-point, then in concentric annuli $\{2^{-n} \le r \le 2^{-n+1}\}$ one regauges the connection so that $\|\Omega\|$ controls $\|\theta\|$. If CS is the Chern Simons form $\operatorname{tr}(\Omega \wedge \theta - \frac{1}{3}\theta \cdot \theta \cdot \theta)$, one has $dCS = \operatorname{tr} \Omega \wedge \Omega$, and a Stokes theorem argument in the concentric annuli shows Ω, and thus the new regauged θ is h-regular over the point with C, α controlled by the norm $\|\ \|$ (assumed sufficiently small) of Ω on B^4-point.

These remarks allow one to have Karen Uhlenbeck's (compactness/noncompactness) picture [**26**] for any sequence of h-regular K-quasi Yang-Mills connections on a compact qc four manifold.

One may also develop an Atiyah-Singer Index theory for the signature operator with coefficients in a bundle over a qc M^{4l}. One uses the de Rham complex up to the middle

$$(\cdots \to \Omega^h \to \Omega^{h+1} \to \cdots \to \Omega^{2l-1} \to \Omega^{2l} \xrightarrow{*})$$

where the last arrow uses the measurable locally quasiconformally Euclidean metric to project $(d\Omega^{2l-1})$ onto $\frac{1}{2}$ the space Ω^{2l} (where $*\omega = \omega$). One may



each little disk maps onto the larger disc

f is a degree d covering

FIGURE 2

tensor with the bundle E and use S to restore the fact that one loses the $d^2 = 0$ property in the tensoring process.

In this way one obtains a Fredholm index provably independent of $*$ on M^{4k} and the connection in E (compare Teleman [**24**]).

With Uhlenbech's picture and the basic Atiyah-Singer theory in place one may develop at least one Donaldson type result (say a là Fintushel-Stern [**8.5**]) to show the *qc* theory of 4-manifolds is different from the theory of topological 4-manifolds.

ACKNOWLEDGEMENT. I am indebted to the participants in the quasiconformal gauge theory conference at the University of Texas, in particular Bill Beckner, John Gilbert, Bob Gompf, Gary Hamrick, and Bruce Palka.

REMARK. A relation between this section and dynamics is suggested by Alain Connes's theory of noncommunitative differential geometry where operators of Shatten class n/k are utilized.

III. A rigidity theorem for *qc* deformations of expanding systems and Gibbs states. A **C**-analytic expanding system is determined by a **C**-analytic map $f: U_1 \to U$ where U_1 is a domain properly embedded in the Riemann surface U and f is a $(d > 1)$-sheeted onto covering. Such systems are classified by analytic conjugacy near the compact invariant set $K_f = \bigcap_n f^{-n}U$. For example, see Figure 2.

To motivate the connections with quasiconformality we state the following

THEOREM. (1) *Any sufficiently small* **C**-*analytic perturbation of f near K_f defines another expanding system which is quasiconformally conjugate to f near the invariant set.*

(2) *Any topological conjugacy between the compact invariant sets of expanding systems agrees with a qc conjugacy between neighborhoods.*

(3) *All expanding systems qc conjugate near the invariant sets to a given one can be constructed by deforming the complex structures on the Riemann surface $U - U_1$ for some presentation (U, U_1) using the measurable Riemann mapping theorem.* (*See Sullivan* [**20**] *and* [**22**].)

The analytic classification of expanding systems of a given topological dynamics type on the invariant set is a kind of Teichmüller theory. The dimension is infinite because there are infinitely many complex moduli given by the complex eigenvalues of f-periodic points of K_f (which are dense in K_f and thus infinite; to avoid trivial cases and exceptions we suppose f/K_f is *leo* (locally eventually onto): for each neighborhood V in K_f, $f^n V \supset K_f$ for some n).

PROPOSITION. (i) *The δ-Hausdorff measure μ_δ of K_f is finite and positive;* (ii) *there is a unique real analytic conformal metric defined on K_f for which μ_δ is invariant by f, $f_*\mu_\delta = \mu_\delta$.*

SKETCH OF PROOF. One first shows the δ-Hausdorff measure is finite and positive where $\delta =$ dimension K_f, measured, say, in some real analytic metric (see Sullivan [**22**] or Ruelle [**18**]). If ν denotes this measure, one studies the density function ρ_n of $\nu_n = f_*^n \nu$ relative to ν. By calculation one sees that the ρ_n form a compact family of continuous functions and the limits are real analytic. (The value of ρ_n at x is a sum of d^n terms $\sum_{y\in f^{-n}x} \omega(y, fy, \dots)^\delta$ where ω is the product of reciprocal linear element derivatives of f along the orbit of y up to x. The Hausdorff dimension δ is the power that makes these sums finite and because f is expanding, the functions ω are absolute values of C-analytic functions with fixed domain and exponentially decreasing range.) Q.E.D.

We say an expanding system (f, K_f) is *linear* if

(i) the curvature of the natural metric (defined in the proposition) is identically zero near K_f;

(ii) the absolute value of the derivative of f in the metric is locally constant on K_f; otherwise we say the system is *nonlinear.*

EXAMPLE. $z \to z^n$ near $|z| = 1$ is linear.

REMARK. For linear systems there are flat C-analytic charts defined near K_f so that the complex derivative $f'(z)$ is locally constant near K_f.

Our rigidity theorems concern the *nonlinear* expanding systems.

THEOREM. *Let (f, K_f) and (g, K_g) be two* C*-analytic expanding systems not both linear. Then there is a* C*-analytic conjugacy between (f, K_f) and (g, K_g) which restricts to a given Borel map $h\colon K_f \to K_f$ satisfying $fh = hg$ if*

(i) *h is a homeomorphism and moduli of eigenvalues at a periodic points associated by h are equal; or if*

(ii) *h is a nonsingular transformation between Hausdorff measure classes on K_f and K_g, respectively* ($\dim K_f = \dim K_g$ *is a consequence here not an assumption*).

REMARK. Both of these statements are false if both systems are linear.

COROLLARY. *The infinite-dimensional Teichmüller theory of complex analytic expanding systems is embedded in the Hausdorff measure theory of the fractal invariant sets.*

SKETCH OF PROOF. In the canonical metric consider the Jacobian of f, Jf relative to the invariant measure in the Hausdorff measure class. Consider the *Jacobian invariant*: domain of dynamics $\xrightarrow{J(f)}$ Hilbert cube defined by $x \to (Jf(x), Jf(fx), \dots)$. We show that J is locally injective somewhere unless we are in the linear case. We successively deduce (1) h is measure preserving by ergodicity so $Jf = Jg \circ h$; (2) h is somewhere locally Lipschitz; (3) h is continuous everywhere (and measure preserving); (4) h is real analytic; (5) h is complex analytic.

The idea of all these is to use the expanding dynamics (as in Part IV) to see the improved quality of h.

We reduce part (i) to part (ii) by showing h must preserve the Hausdorff measure class. Q.E.D.

The Hausdorff measure theory of such dynamical fractals can be understood in terms of Gibbsian measure classes and Gibbs states. For simplicity consider one topological model, the shift on the Cantor set X of one-sided strings on two symbols, $\{.\varepsilon_0\varepsilon_1\dots\} \xrightarrow{T} \{.\varepsilon_1\varepsilon_2\dots\}$. A (Hölder) Gibbsian measure class for $T\colon X \to X$ is a measure class determined by a probability measure ν on X so that (i) $\nu(A) > 0$ if and only if $\nu(TA) > 0$; and (ii) φ the log Jacobian of T rel ν (definable by (i)) satisfies $\sup|\varphi(x) - \varphi(y)| \leq c\alpha^{-n}$ whenever x and y agree for the first n-symbols. Let $C^{k,\alpha}$ denote $\{f\colon D^{k-1}\log Df$ is α-Hölder$\}$ where $k = 1, 2, 3, \dots$ $\alpha \in (0, 1]$.

THEOREM. *Let ν determine a (α-Hölder) Gibbsian measure class for $T\colon X \to X$. Then for each $\delta \in (0,1)$ there is a Hölder continuous embedding of X in the real line $X \subset R$ and a $C^{1,\alpha}$ expanding map $f\colon R \to R$ defined on a neighborhood of X so that*

(i) *X is the maximal invariant set of f in the neighborhood and f/X is the shift T;*

(ii) *ν is the δ-Hausdorff measure of X, and therefore δ is the Hausdorff dimension of X;*

(iii) *any such $C^{1,\alpha}$ geometric realization of $T, f\colon X \to X$ is determined up to bi-Lipschitz conjugacy by δ and the Gibbsian measure class determined by ν;*

(iv) *if $f\colon X \to X$ is $C^{k,\alpha}$ ($k \geq 2$, $k = \infty$, or $k = w$) and $f'' \neq 0$ (in the metric where Hausdorff measure is invariant) at some point of X, then $f\colon X \to X$ is determined up to $C^{k,\alpha}$ conjugacy near X by the Gibbsian measure class ν. (δ is determined also.)*

REMARKS. (1) One knows (Bowen [1]) that a (Hölder) Gibbsian measure class is determined by the Jacobians at the periodic points. Also there is a canonical representative using the unique invariant measure. An important consequence of the first is that the set of Gibbsian measure classes is isomorphic to a locally closed subset in a Banach space. (2) All topological models based on one-sided subshifts of finite type can be similarly treated.

NOTE ADDED IN PROOF. The author has recently found that such $C^{1,\alpha}$ expanded Cantor sets C have a Hölder continuous *scale function* $\sigma\colon C^* \to R$ where $C^* = \{\dots\varepsilon_3\varepsilon_2\varepsilon_1.\}$ if $C = \{\varepsilon_1\varepsilon_2\varepsilon_3\dots\}$, $\varepsilon_i = 0$ or 1. The scale function is independent of the smooth C^1 coordinate system being defined as asymptotic limits of ratios of lengths of intervals at stage n to lengths of containing intervals at stage $n-1$. Two expanding systems which are $C^{k,\alpha}$ ($k = 1, 2, 3, \dots$ α

in $(0,1]$) are $C^{k,\alpha}$ conjugate iff they have the same scale functions. Also any Hölder continuous function occurs as the scale function in some $C^{1,\alpha}$ expanding model. This caveat to the expanding theory is very useful for understanding the Feigenbaum discovery. (See Part I.)

IV. Quasiconformality in the geodesic flow of negatively curved manifolds. If M is a compact negatively curved $n+1$ manifold, then there is defined a topological n-sphere at infinity in the universal cover on which $\Gamma = \pi_1 M$ acts with every orbit dense. One knows that the sphere carries a *qc* manifold structure in which Γ *acts uniformly quasiconformally* if and only if there is a *constant negatively curved compact manifold of the same homotopy type* as M. Compare Gromov [**11**], Sullivan [**23**], and Tukia [**25**]. (More precisely, one shows the uniformly *qc* action is *qc* conjugate to a conformal action.) One hopes and conjectures that one may always find this quasiconformality when $n+1 = 3$, and one knows for $n+1 = 4, 5, 6, \ldots$ that it is generally impossible even for manifolds with sectional curvatures almost equal to -1 (Gromov and Thurston [**12**]).

Here we describe a necessary and sufficient condition for this quasiconformality in a more precise sense when the curvature is pinched $-\frac{1}{4} < k \leq -1$. In this case one knows the sphere at infinity has a natural C^1-structure; see Green [**10**] and Hirsch-Pugh [**13**] and the following text. (Gromov has asked if the sphere at infinity has a C^2 structure only in the locally symmetric case.)

The pinching condition implies that the horospheres in the universal cover have extrinsic curvatures satisfying $\frac{1}{2} < k \leq 1$. This implies that the geodesic flow has eigenvalues in its expanding manifolds satisfying $\frac{1}{2} < \log \lambda \leq 1$. An elementary calculation shows that composing expansions with these eigenvalue inequalities yields a composition F whose derivative only varies in Lipschitz manner (in $Gl(n)$) along an arc so that the length of its F image is ≤ 1. The point is that the Lipschitz constant is independent of the length of the composition.

We call this property of composed expansions the *quasilinearity principle.*

PROPOSITION. *In a $-\frac{1}{4} < k \leq -1$ pinched Riemannian manifold the geodesic flow on its expanding horospherical foliation satisfies the quasilinearity principle.*

This proposition leads to the C^1-structure on the sphere at infinity. It also allows one to characterize uniform quasiconformality of the action of Γ on the C^1-sphere at infinity.

THEOREM. *The following are equivalent in the $-\frac{1}{4} < k \leq -1$ pinched compact Riemannian manifold.*

(i) *The $\pi_1 M = \Gamma$ action on the C^1-sphere at infinity is uniformly quasiconformal.*

(ii) *The geodesic flow is uniformly quasiconformal on its expanding horospheres.*

(iii) *The* Γ *action on the tangent spaces of the sphere at* ∞ *is measurably irreducible.*

(iv) *The geodesic flow acting on the tangent spaces of the expanding horospheres is measurably irreducible.*

Measure irreducibility means there is no measurable field of proper subspaces of the tangent spaces which is a.e. invariant by the relevant action.

SKETCH OF PROOF. (1) The orbits of Γ on the sphere at infinity are in one-to-one correspondence with the leaves A_+ of forward asymptotic geodesics in the unit tangent bundle. Each leaf of the foliation A_+ is a family of horospheres swept out by the geodesic flow. Then the A_+ foliation is an R-extension of a foliation with polynomial growth leaves, the horospheres. For such foliations (yielding *amenable* equivalence relations) Zimmer [**29**] has shown any associated $Gl(n)$ cocyle has a *measurable reduction* or it is *measurably equivalent* to an *associated cocycle of similarities.*

(2) In the latter case one can show the measurable invariant similarity structure is continuous by expanding a small neighborhood of an almost continuity point (on the sphere at infinity). One uses (a) the quasilinear principle and (b) the existence of a natural metric on the similarity structures on one tangent space to enlarge a neighborhood with high percentage very small oscillation. (Because (b) is lacking for subspaces one cannot use this argument to show that the measurable reduction of (1) is continuous. In fact, this conclusion is certainly false for all the odd dimensional examples of Gromov-Thurston [**12**] because an even sphere has no continuous tangent subbundle.) If Γ preserves a continuous similarity structure the action is uniformly quasiconforml relative to the C^1-structure. This shows (iii)$\Rightarrow$(i).

(3) The rest of the implications do not use the $-\frac{1}{4} < k \leq -1$ pinching: (i)$\Rightarrow$(ii) and (iv)$\Rightarrow$(iii) are formal, (ii)$\Rightarrow$(i) is a picture, and for hyperbolic manifolds (iv) is known, so (i)$\Rightarrow$(iv).

PROBLEM. Do the conclusions of the theorem imply the curvature is actually constant? (Part III suggests something of this sort.)

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