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Brownian Motion and Harmonic Functions on the Class Surface of the Thrice Punctured Sphere

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1. INTRODUCTION

Let M be the sphere punctured at 0, 1, ∞ and equipped with the metric of constant curvature -1 it receives from its *universal covering* by the Poincaré disc M_2, so that it appears as the 3-horned sphere of Fig. 1. M_2 is obtained from M by fixing a base point x_0 and covering the general point $x_1 \in M$ by the deformation classes of paths starting at x_0 and ending at x_1. The covering group G_2 is free of non-abelian rank 2: it is generated by the 3 cycles indicated in Fig. 1, subject to the single relation $g_\infty g_1 g_0 = 1$. The coarser classification of paths according to their winding numbers about the 3 cusps of M gives rise to an intermediate covering: the so-called *class-surface* M_1. The covering group G_1 of M_1 over M is simply G_2 *made commutative* by quotienting out the commutator subgroup K of G_2; the latter is the covering group of M_2 over M_1. G_1 is naturally identified with the 2-dimensional lattice $\mathbb{Z}^2$. Lyons and McKean [1] proved that *the Brownian motion on* M_1 *is transient* by direct estimation of the winding numbers of the Brownian motion on M, correcting and amplifying McKean [2]. The purpose of the present paper is to give a less quantitative but more geometrical and simpler proof of this fact, together with the proof of a new fact: that M_1 *does not carry any non-constant positive harmonic functions.*

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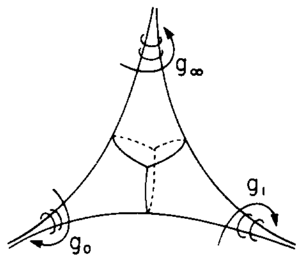

FIGURE 1

This means, so to say, that *the Brownian motion on M_1 has only one mode of running off to ∞*. The proof requires only the clear geometrical picture of M_1 expounded in Section 2 together with elementary probabilistic reasoning. Section 3 deals with the harmonic functions. The transience is confirmed in Section 4.

AMPLIFICATION 1. *M has finite volume and M_1 is a $\mathbb{Z}^2$ cover, so it is natural to conjecture that, in general, $\mathbb{Z}^2$ covers of finite-volume Riemann surfaces have only constant positive harmonic functions.*

This is false: indeed, the transience of the Brownian motion on M_1 means that the latter has a finite Green's function h. Removal of the fiber of $M_1 \to M$ associated with a fixed pole of h and of its projection on M leaves a 4-times punctured sphere below and a $\mathbb{Z}^2$ cover above on which h is a *bona fide* non-constant harmonic function.

AMPLIFICATION 2. *M_1 may be viewed as the curve $\{e^x = e^y + 1\}$*[2].

Demailly [3][2] has proved that any holomorphic function on M_1 of polynomial growth in x and y extends holomorphically to C^2 with the *same* growth. In this connection, note that the Green's function h on M_1 has a many-valued harmonic conjugate k on M_1 punctured at the pole so that $f = \exp[-2\pi(h + \sqrt{-1}\,k)]$ is a many-valued bounded holomorphic function on M_1, h being positive and of the form $-(1/2\pi)\log r$ near the pole. The ambiguity of f is due solely to the homology of M_1. The latter is described by $\mathbb{Z}^\infty$, as can be seen in Fig. 4, the moral being that *a big abelian cover can produce bounded holomorphic functions where none existed before.*

One of the results in Lyons and Sullivan [4], however, asserts that an abelian cover of a recurrent surface has no bounded harmonic functions.

[1] $x = \log z$, $y = \log(z-1)$ for $z \in M$.

[2] Reference by the kindness of P. Malliavin.

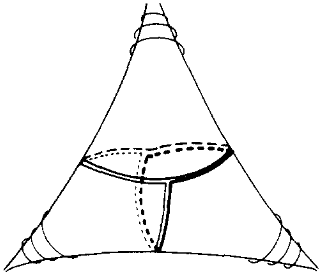

FIGURE 2

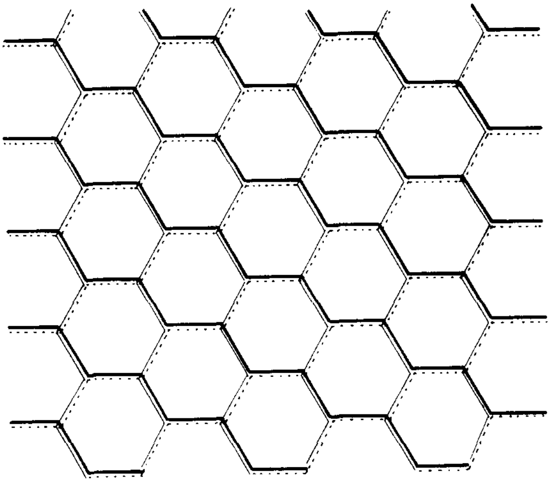

FIGURE 3

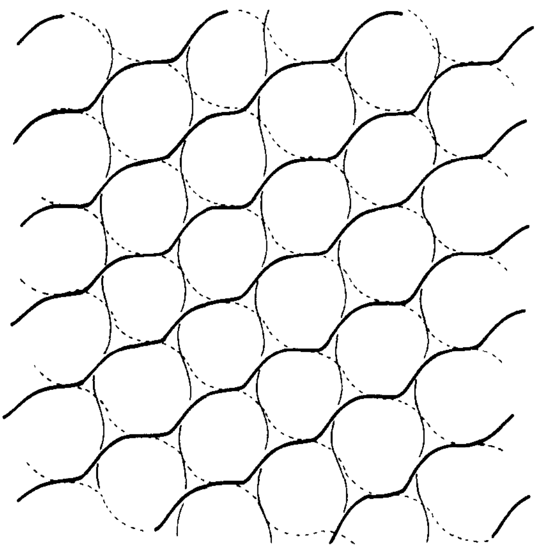

FIGURE 4

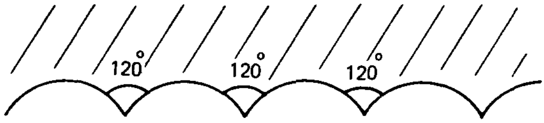

FIGURE 5

2. GEOMETRY OF M_1

A clear picture of M_1 may be obtained as follows. M is dissected into 4 pieces, as in Fig. 2, by means of 3 broken geodesics, each with 120° corners front and back: 3 of the pieces are identical non-compact *cusps*; the residual compact *ribbon* is bordered by the 3 broken geodesics. Think of the ribbon as very narrow and unfold it on the class surface M_1: The 3 bordering curves unfold into 3 families of non-intersecting lines with 120° corners. They form a hexagonal pattern, as in Fig. 3, spanned, as in fig. 4, by a twisted *covering ribbon* dividing in two in the vicinity of each corner. The class surface is now completed by gluing along each broken line a broken-bordered half-plane, as in Fig. 5, representing the unfolding of the adjacent cusp; such a half-plane is called a *fin*. The covering group $G_1 = \mathbb{Z}^2$ acts by rigid motions on the whole preserving the hexagonal tesselation.

3. HARMONIC FUNCTIONS

Let $h(x)$ be a positive harmonic function on M_1: *it is to be proved that it is constant.* Let g be any element of the covering group $\mathbb{Z}^2$ and x any point of the covering ribbon. Then gx is also a point of the covering ribbon at distance[3] $d(gx, x) \leqslant c_1$ from x, so that $h(gx) \leqslant c_2 h(x)$ with a universal constant c_2 provided by Harnack's inequality, independently of h and x. Now g can move points in the fins a long way: for example, g_∞ represents rotation about the cusp of ∞, and if you begin far out in another cusp you have to travel for miles. Nevertheless, *the estimate $h(gx) \leqslant c_2 h(x)$ holds with the same constant in the fins as well.*

Grant this for the moment and let h be a minimal harmonic function. This means that any harmonic function dominated by a multiple of h *is* a multiple of h, so that $h(gx) = c_3(h) \cdot h(x)$. Let g signify 1 rotation about a cusp, e.g., $g = g_\infty$, and let x lie in one of the associated fins. The latter is a half-plane, bordered as in Fig. 5, and is preserved by g. The latter acts by horizontal

[3] The hyperbolic distance is pulled up from M. c_1, c_2, etc., stand for constants depending only upon the geometry of M; constants depending upon h are written $c(h)$.

translation, and it follows from the Poisson representation of positive harmonic functions in a half-plane that

$$\sum_{n\neq 0} n^{-2}h(g^n x) = h(x) \sum_{n\neq 0} n^{-2}c_3^n(h) < \infty.$$

But this forces $c_3(h) = 1$, so $h(gx) = h(x)$, and as the same is true for any other cusp, h is seen to be not a *bone fide* function on the class surface but merely a function on the base space M with possible singularities at the 3 punctures 0, 1, and ∞. The proof is finished by the remark that the only such *harmonic* functions are constant, as is well-known and easily proved by means of Green's formula and the a priori estimate $h(x) \leqslant c_4(h)|\log r|$ in the vicinity of a singularity.

It remains to propagate the estimate $h(gx) \leqslant c_2 h(x)$ from the covering ribbon to the fins. Now in any fin, $h(x)$ can be expressed by an integral along the border with respect to harmonic measure *plus* a pole at ∞,

$$h(x) = \int p(x, dy)\, h(y) + c_5(h)\, x_2,$$

in which $c_5(h) \geqslant 0$ and x_2 is the harmonic function vanishing on the border which behaves as $[1 + O(1)] \times$ (height) at $\sqrt{-1}\,\infty$, as you will see by straightening out the border with a Riemann map. The desired propagation of $h(gx) \leqslant c_2 h(x)$ from border to fin is now self-evident *provided* $c_5(h) = 0$, the mean-value property $h = \int ph$ applying equally to $h(x)$ and to $h(gx)$.

The final step is now to prove $c_5(h) = 0$. The universal cover M_2 is identified with the Poincaré disc. Think of h as a function on M_2, invariant under the action of the covering group K of M_2 over M_1, and express it as a Poisson integral *up there*, supposing $c_5(h) > 0$. Then $h \geqslant c_5(h)\, x_2$ implies that the representing mass distribution on the circle $S^1 = \partial M_2$ has atoms on the orbit of K representing the fiber of S^1 over the point $\sqrt{-1}\,\infty$ in the fin, and as these atoms transform as they must for the invariance of h under K, so the Poisson integral produces from *them alone* a K-invariant harmonic function on M_2, alias a minimal harmonic function h_1 on M_1, having the same growth $c_5(h)\, x_2$ at $\sqrt{-1}\,\infty$ in the fin. *This is not possible*: The generator g of the cusp group $\mathbb{Z}^1 \subset G_1 = \mathbb{Z}^2$ acts by horizontal translation in the fin and preserves the fiber of $S^1 = \partial M^2$ covering $\sqrt{-1}\,\infty$, $h_1(gx)$ being minimal and having *the same compartment as* $h(x)$ *at* $\sqrt{-1}\,\infty$. The mass of $h_1(gx)$ is now located on that same fiber, and as this mass must transform in the previous manner to ensure the invariance of $h_1(gx)$ under K, so $h_1(gx)$ can only be a multiple $c_6(h_1)$ of h_1. But $c_6(h_1) = 1$ in view of $h_1(gx) \sim h_1(x)$ at $\sqrt{-1}\,\infty$, and now the end is near: $h_1(gx) = h_1(x)$, so that h_1 drops down from M_1 to the Z-covering surface of the *thrice-punctured* sphere, the plane with an arithmetical array of singularities, and as the plane Brownian motion $x(t): t \geqslant 0$ does not perceive single points, the existence of the limit of the positive martingale $h_1 \circ x(t)$ together with the recurrence of the plane Brownian motion forces the constancy of h_1. This contradicts $c_5(h) > 0$, completing the proof that M_1 admits no positive harmonic functions except the constants.

4. Transience of Brownian Motion on M_1

The covering ribbon of Fig. 4 is bisected by a hexagonal skeleton. The Brownian motion of M_1 is now started on the skeleton and one notes *the next hitting place on the skeleton after reaching the border of the ribbon.* The outward step (from skeleton to border) is like the passage of a plane Brownian motion $x_1 + \sqrt{-1}\, x_2$ from $x_2 = 0$ to $x_2 = \pm 1$ and is small, while the inward step (from the border back) is like the passage from $x_2 = +1$ to $x_2 = 0$ and is large: in the first case, the distribution of the horizontal displacement satisfies $E[e^{\delta x_1}] < \infty$ if $|\delta| < \pi/2$; in the second, it is distributed by the Cauchy law $[\pi(1 + x_1^2)]^{-1}\, dx_1$. The geometry of M_1, as depicted in Fig. 4, now suggests that *the chain of hitting places on the skeleton, so produced, is transient*: from most points of the skeleton, the short step out lands you on the border of one of the 2 adjacent fins and the long step back lands you far away; only near the corners are 3 fins close enough to be reached by a short step, so this more complicated situation will be less frequently met and will not change things much. The situation may be caricatured by a walk on $\mathbb{Z}^2$ with independent Cauchy-distributed steps taken horizontally or vertically according to the outcomes of a standard coin tossing game. The probability of landing in the box $(-1 \leqslant x_1 < 1) \times (-1 \leqslant x_2 < 1)$ after n steps is

$$2^{-n} \sum_{k=0} \binom{n}{k} \int_{-1}^{1} \frac{k}{\pi} (k^2 + x_1^2)^{-1}\, dx_1 \int_{-1}^{1} \frac{n-k}{\pi} [(n-k)^2 + x_2^2]^{-1}\, dx_2$$

$$\leqslant \frac{2^{-n+1}}{n\pi} + \frac{2^{-n}}{\pi^2} \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{k(n-k)} \leqslant c_7 n^{-2};$$

so the caricature is transient, and one may hope that the actual hitting chain is, too.

The proof is postponed in favor of the remark that *the transience of the full Brownian motion on M_1 follows from that of the chain of hits*: in fact, if the former were recurrent, then it would return infinitely often to a small disc D_1 of M_1 via the boundary of a slightly larger disk D_2. A *loop* is a segment of the Brownian path starting at ∂D_1 and ending at the next passage to ∂D_1

via ∂D_2. Put D_2 on a fin for clarity. Then, on any loop, there is a positive probability of executing a complete (outward and inward) step of the hitting chain and landing on a prescribed piece of the skeleton, and as the chain of loops is metrically transitive,[4] *this must happen with a positive frequency, violating the transience of the chain of hits.*

The final step is now to prove the transience of the chain, but first a modification with a view of technical simplicity: instead of the skeleton, use the unfoldings of circles about the 3 cusps of M providing a fattish smoothly bordered ribbon A; also take a wider ribbon B of the same kind, invariant under the action of $G_1 = \mathbb{Z}^2$, and consider the new chain of hits on ∂A via ∂B. Plainly the previous reasoning applies; so it suffices to prove the transience of this modified chain. $\mathbb{Z}^2$ acts on A with compact fundamental region F: a smooth hexagonal ribbon with identifications. The chain is now viewed as a chain of hits $y_n : n \geqslant 0$ of ∂F together with labels $g_n : n \geqslant 0$ from $\mathbb{Z}^2$ indicating the parts of the tesselation $G_1 \partial F$ to which the hits are to be ascribed.

Let $p_{ab}(g)$ be the probability of the step from one point $(x_0, g_0) = (a, 0)$ of $\partial F \times G_1$ to another $(x_1, g_1) = (b, g)$, *conditional upon* $x_1 = b$. Then

$$P\{g_n = g \mid g_0 = 0, x_0, x_1, \dots, x_n\} = \sum \prod_{i=1}^{n} p_{x_{i-1}x_i}(g_i),$$

the sum being extended over $g_1 \cdots g_n = g$, and so

$$P(g_n = 0) = E \int \prod_{i=1}^{n} \hat{p}_{x_{i-1}x_i}(k)\, d^2k,$$

in which

$$\hat{p}_{ab}(k) = \sum_{\mathbb{Z}^2} p_{ab}(g) \exp(2\pi\sqrt{-1}\, g \cdot k)$$

and the integral extends over the 2-dimensional torus $(-\frac{1}{2} \leqslant k_1 < \frac{1}{2}) \times (-\frac{1}{2} \leqslant k_2 < \frac{1}{2})$.[5] The problem is to prove that $\sum_{n=0}^{\infty} P(g_n = 0) < \infty$, as would be seen from the estimate

$$|\hat{p}_{ab}(k)| \leqslant 1 - c_8 |k|$$

with $0 < c_8 < 1$ and $0 < c_9 < 2$; indeed, the estimate implies

$$\sum P(g_n = 0) \leqslant \sum \int [1 - c_8 |k|]^n\, d^2k = c_9 \int \frac{d^2k}{|k|} < \infty.$$

[4] See, for instance, Ito and McKean [5].

[5] The trick is adopted from Guivarche [6]; it goes back to Polya [7] in connection with the 2-dimensional symmetric random walk.

Let us confirm the estimate. Let $G(x, y)$ be the Green's function of $M_1 - A$ grounded along ∂A. The harmonic density of $y \in \partial A$ as viewed from $x \in \partial B$ is the flux $(\pm 1)\, \partial G/\partial n$ at y, so that

$$P[x_1 \in db, g_1 = g \mid x_0 = a, g_0 = 0] = \int_{\partial B} H(a, dx)(\pm 1)(\partial G/\partial n)(x, y) \cdot db,$$

in which H is the harmonic measure of ∂B viewed from outside and db is the element of length on ∂A. Plainly, $a \to P[x_1 \in db, g_1 = g \mid x_0 = a, g_0 = 0]$, as a function of a, is the restriction to ∂F of a function harmonic in the neighborhood, so that it is independent of a up to a factor c_{10} depending solely upon the geometry of M. Likewise the dependence upon b: $G(x, y)$ is harmonic in y near the (smooth) border ∂A on which it vanishes, and as B and A are preserved by the action of $G_1 = \mathbb{Z}^2$, so $\partial G/\partial n$ is independent of $b \in \partial F$ up to a similar factor c_{11}. *This is the key to the proof.* Now it is plain from the Cauchy-like nature of the hitting chain that $p_{ab}(g)$ may be underestimated, independently of a and b, by a small multiple $[c_{12}]$ of the Cauchy-like distribution

$$\begin{aligned} p_-[g = (n_1, n_2)] &= \tfrac{1}{2} c_{13}^{-1}(1 + n_1^2)^{-1} \qquad (n_1 \in \mathbb{Z}, n_2 = 0) \\ &= \tfrac{1}{2} c_{13}^{-1}(1 + n_2^2)^{-1} \qquad (n_1 = 0, n_2 \in \mathbb{Z}) \end{aligned}$$

with normalizer $c_{13} = \sum (1 + n^2)^{-1}$, so that

$$|\hat{p}_{ab}(\theta)| \leqslant 1 - c_{13} + c_{13} \cdot \frac{1}{2} c_{13}^{-1} \left| \sum \frac{e^{2\pi\sqrt{-1}\, n_1 k_1}}{1 + n_1^2} + \sum \frac{e^{2\pi\sqrt{-1}\, n_2 k_2}}{1 + n_2^2} \right|.$$

The required estimate follows from the Poisson summation formula: if $0 \leqslant k \leqslant \frac{1}{2}$, then

$$c_{13}^{-1} \sum \frac{e^{2\pi\sqrt{-1}\, nk}}{1 + n^2} = (1 + e^{-2\pi})^{-1}(e^{-2\pi k} + e^{-2\pi} e^{2\pi k}) \leqslant 1 - c_{14} k$$

with, *e.g.*, $c_{14} = 2\pi(1 - e^{-2\pi})(1 + 2^{-2\pi})^{-1}$.

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