L'Enseignement Mathématique, t. 29 (1983), p. 15-25

MANIFOLDS WITH CANONICAL COORDINATE CHARTS: SOME EXAMPLES [1])

by Dennis SULLIVAN and William THURSTON

We will consider examples of low dimensional manifolds with inversive, projective, and affine structures (see below). The geometry of the associated developing maps is a problem like the qualitative study of dynamical systems involving as it does the "infinite composition" of finitely many operations. Our goal will be to answer in the negative certain questions about affine manifolds— by producing examples where the developing map is either not a covering of its image or has a rather complicated image.

These structures, classically called locally homogeneous spaces [1, 3, 8] or spaces with a flat Cartan connection [4], are determined by "canonical coordinate charts" on manifolds and may be defined in general as follows:

One starts with a model manifold A (see table below) and a transitive group $\mathscr{A}$ (usually a Lie group) of real analytic homeomorphisms of A. Then one constructs all possible manifolds by choosing open sets of A and pasting these together using restrictions of homeomorphisms from the given group $\mathscr{A}$ of analytic homeomorphisms. Such a manifold M is called an $\mathscr{A}$-manifold. More precisely, an $\mathscr{A}$-manifold is a manifold M together with an atlas $\kappa_i : U_i \to A$ such that the changes of charts are restrictions of elements of $\mathscr{A}$; a "canonical chart" is any chart on M which is $\mathscr{A}$-compatible with these.

We will consider the following cases:

---

[1]) Our first draft of this paper was done in January 1977. The present version contains notes and clarifications by N. Kuiper bringing the paper to a more precise form. The authors acknowledge their thanks.

structure	dimension	model manifold A	group $\mathscr{A}$
affine	2, 3, 4	affine space	affine motions
inversive	2	two sphere	group generated by inversions in circles
projective	2	real projective plane	projective transformations

We are basically interested in the compact case.

The dynamical nature of such a structure on M arises from the ability to "roll" or "develop" the manifold M along paths of M (by pasting the open sets of A) into the model manifold A. We can do this in particular for closed paths representing generators of the fundamental group $\pi_1 M$ of M. The words in $\pi_1 M$ then determine a dynamical system on A.

Starting from one of the open sets $U \subset A$, the development produces a covering space $M' \to M$, a representation $\pi_1 M \xrightarrow{\rho} G$ called the holonomy and a structure preserving immersion, the developing map $M' \xrightarrow{d} A$, which is equivariant via ρ with respect to the action of $\pi_1 M$ on M' and $\mathscr{A}$ on A.

The ambiguity in the development $M' \xrightarrow{d} A$ is the choice of one canonical chart $\kappa : U \to A$ about a base point $x \in U$ on M, uniquely determined up to multiplication on the right by an element of $\mathscr{A}$. In other words, the development d may be found by choosing an arbitrary structure-preserving map in one patch, then extending this choice by analytic continuation. This process works globally because $\mathscr{A}$ is a group of globally analytic diffeomorphisms.

In Note 1 at the end of this paper Ehresmann's neat definition of the development is given.

In Note 2 one sees how an $\mathscr{A}$-structure can be viewed as a fibrebundle $A \to E \to M$ with fibre A and structure group $\mathscr{A}$, fixed cross section $s(M)$, and a foliation $\mathscr{F}$ *transverse to the fibres and to* $s(M)$; the foliation $\mathscr{F}$ defines a "parallel transport" of the "tangent" fibres such that, the holonomy is in $\mathscr{A}$.

In Note 3 one finds the development of curves in a manifold with Cartan connection as described by Ehresmann in [4], and how this specialises for flat connections to the above developing map into one fibre.

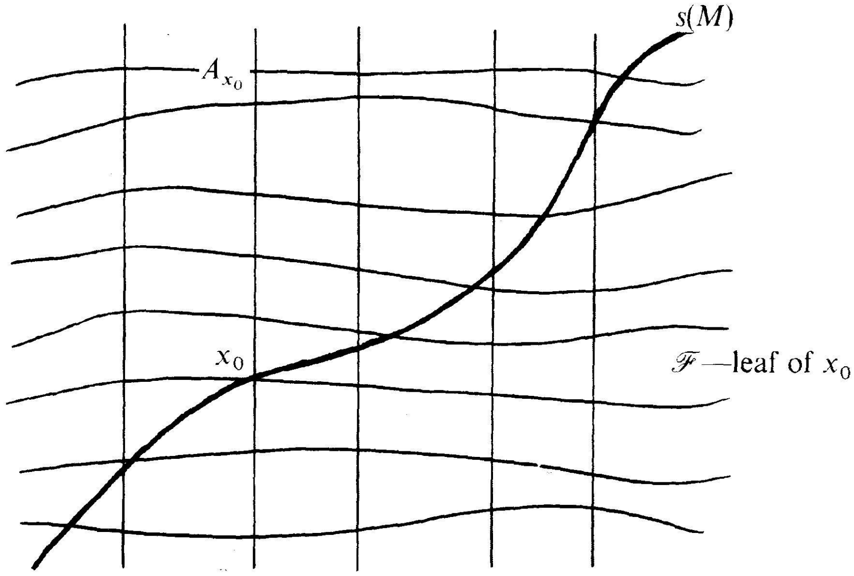


FIGURE 1

Before going to our examples we remark that considerations of the developing map (an immersion of manifolds of equal dimension),

$$\begin{array}{ccc} M' & \xrightarrow{d} & A \\ {\scriptstyle p}\downarrow & & \\ M & & \end{array}$$

immediately shows such things as

i) there are no compact manifolds with finite fundamental group which have affine structures.

ii) the only compact n-manifolds with finite fundamental group with projective or inversive structures are actually covered by the n-sphere S^n.

Actually i) is true whenever the model manifold A is non-compact and ii) is true whenever the model manifold A has a compact universal covering like S^n.

In this context we remark that it is not totally unreasonable to hope that *all 3-manifolds have canonical charts relative to some subgroup of analytic homeomorphisms of* S^3. This statement by the above remark implies a strong form of the Poincaré conjecture; yet the statement itself only involves dimension 3 and not the fundamental group explicitly.

Now we turn to our 2-dimensional examples.

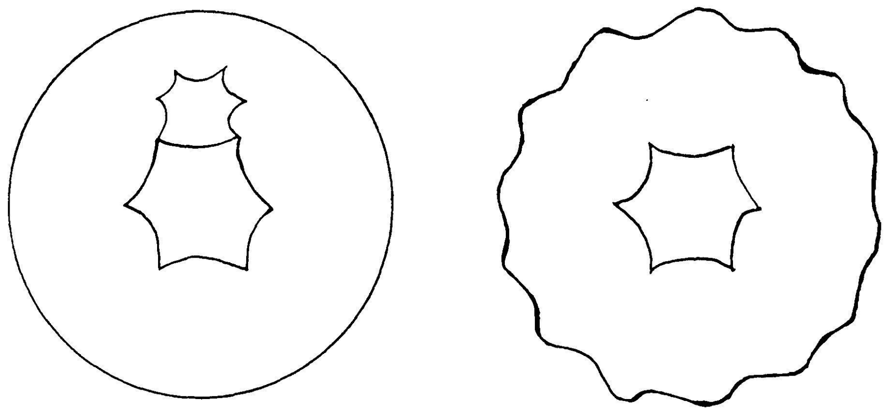
INVERSIVE 2-MANIFOLDS

Inversive structures on orientable two manifolds of genus >1 form a rich theory properly containing for example the classical subject of Fuchsian and Kleinian surface groups.

If $Sl(2, \mathbf{C}) / \pm 1 = Gl(2, \mathbf{C})/Gl(1, \mathbf{C})$ is the group of fractional linear transformations of $\mathbf{CP}^1$, that is the group of orientable inversive (conformal) transformations of S^2, and Γ is a discrete subgroup acting freely and discontinuously on a connected open set $\Omega \subset S^2$, then Ω/Γ is a 2-manifold M with inversive structure. M' is just Ω and the developing map is an embedding.

Example 1. If Γ is a Fuchsian group, that is, Ω is an open (round) disk in $\mathbf{C} \subset S^2$, then the inversive structure is actually a hyperbolic structure—corresponding to a metric of constant negative curvature. The structure is inversive and projective at the same time.

Example 2. If Γ as in Example 1 is deformed slightly (a so-called quasi-Fuchsian group; see [9]) then Ω remains an open disk whose boundary can be a rather remarkable non rectifiable Jordan curve. This curve has no tangent at a dense set.



Example 1 Example 2

FIGURE 2

Example 3. Let Γ be generated by two general hyperbolic elements of sufficient strength so that the union of the fundamental domains of each covers the entire sphere. Then Ω is S^2 minus a Cantor set and Ω/F is a compact conformal 2 manifold whose developing image is Ω. (Shottky group)

In Figure 3, r_1, r_2 and r_3 are inversions (reflections) in three circles and Γ consists of all products of an even number of these inversions. Γ is generated by $r_1 r_2$ and $r_1 r_3$. A fundamental domain is $D \cup r_1 D$, $D = D_1 \cup D_2$. The Cantor set appears clearly on the line of symmetry m.

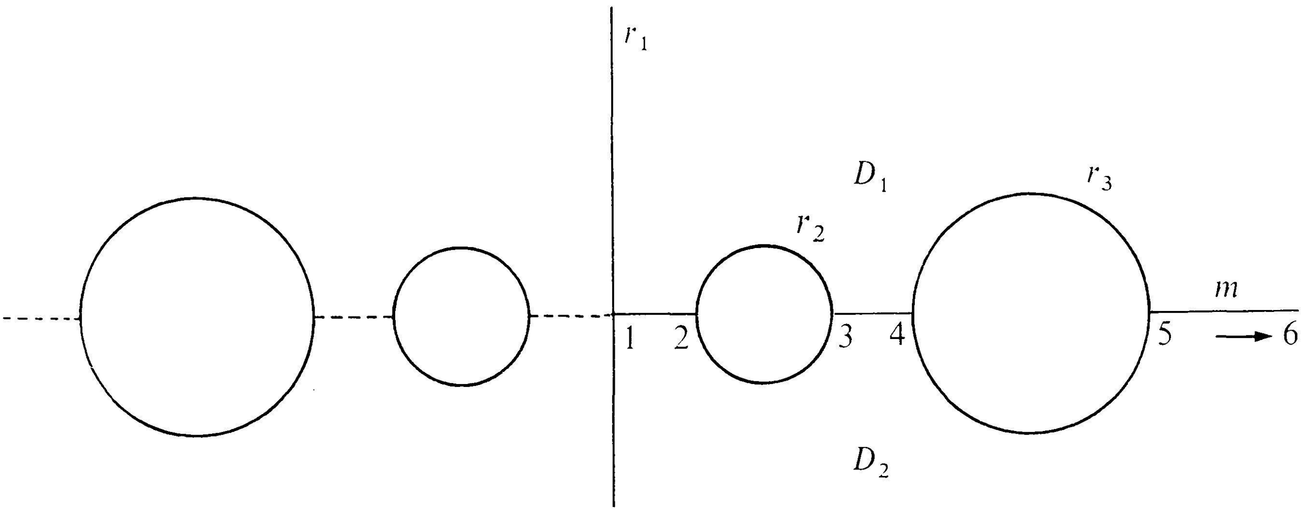


FIGURE 3

Example 4. A class of examples not always arising from Kleinian groups as above can be achieved as follows. Let γ be the boundary of an immersed disk in S^2. Approximate γ by a closed immersed curve again bounding an immersed disk constituted of $2g + 2$ (for some integer $g > 0$) successive arcs of circles meeting at right acute angles (Fig. 4). The new disk with scalloped edges has a conformal structure from the immersion and four of these may be assembled to obtain an inversive 2-manifold of genus g. This *topological* assemblage is suggested in Figure 5.

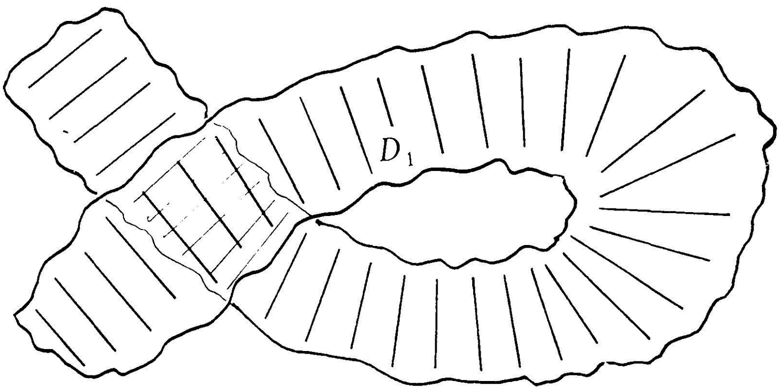


FIGURE 4

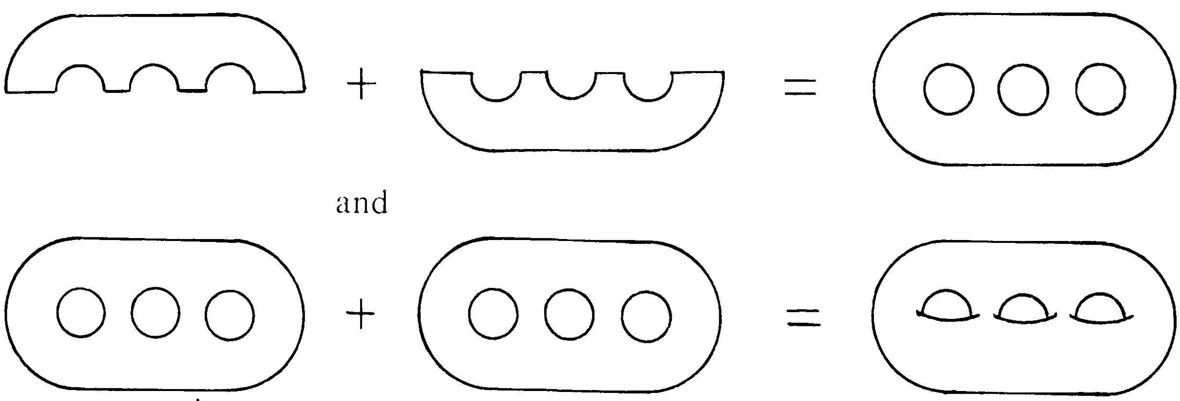


FIGURE 5

Note this construction uses inversion in circles, and four angles at a vertex add up to achieve the non singular conformal structure. Also note the original immersed disk may be chosen (for g big enough) to cover S^2 completely (in a very complicated way) and then the developing map $M' \to S^2$ cannot be a covering. In Figure 6 an example with immersed disk D with 6 vertices ($g = 2$) is suggested, where the developing map covers clearly S^2 completely.

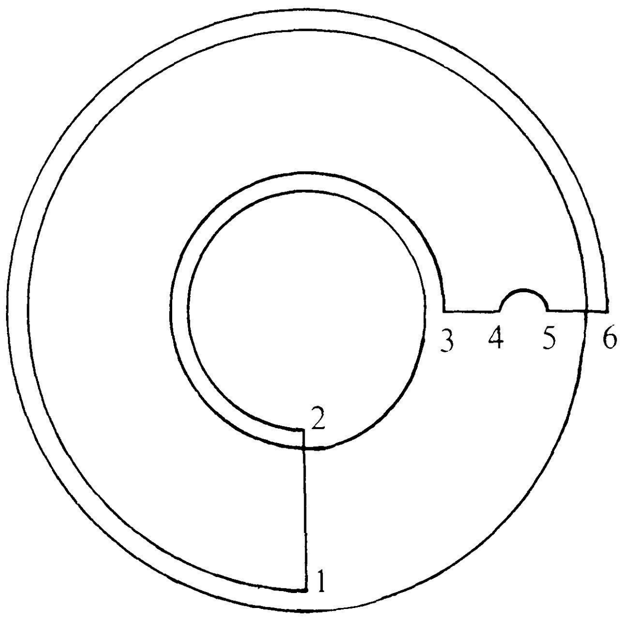


FIGURE 6

We note conversely that if the developing map $M' \to S^2$ is not onto (see Fig. 3, where D_1 is the initial disk, for an example) then the developing map is rather remarkably a covering of its image (Gunning [6]). The idea of the proof is the following—if the image omits at least three points, (exactly one or two points is easy) M' has a Poincaré metric of constant negative curvature preserved by the holonomy group of Moebius transformation acting on the image. Then the developing map becomes an isometric immersion of a complete manifold and thus a covering map.

Example 5. There are interesting projective structures on the torus constructed as follows. Start with a *generic* linear flow on the projective plane (with a source, a sink, and a saddle in point B in Fig. 7*a*) and choose an immersed curve transverse to the flow lines (Fig. 7*b*). Note that such curves may be based on a word in 2 symbols for example *ccaaaa* in Figure 7, and *ccaaacacaa* in Figure 8, where the closed curve on $\mathbf{RP}^2$ is drawn on the open band that universally covers the Moebius band, projective plane minus point B.

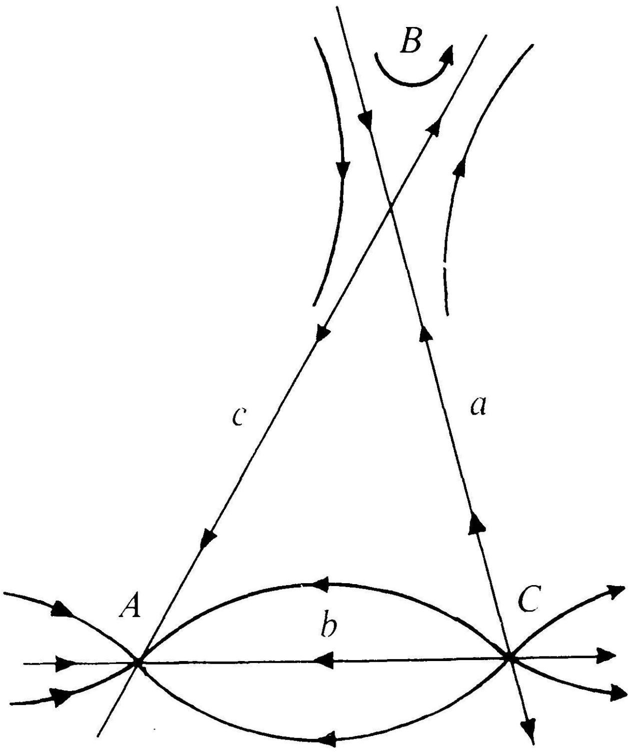


FIGURE 7a

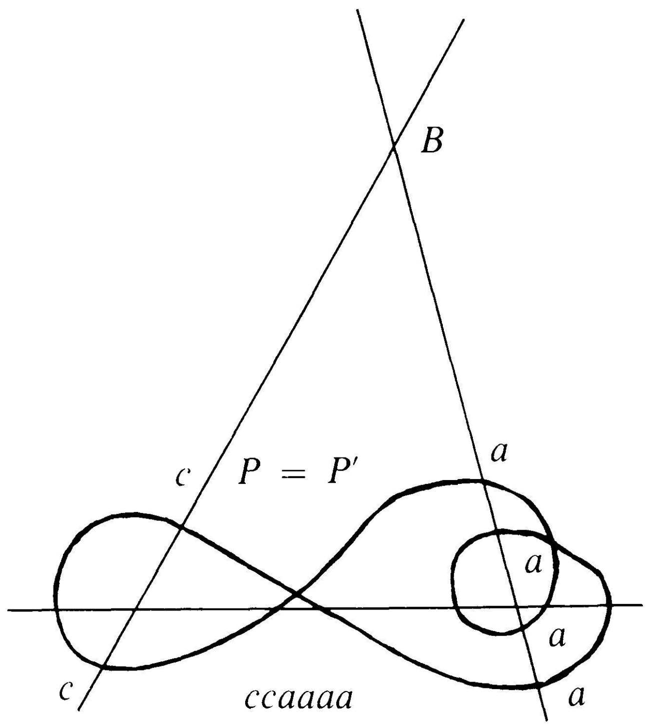


FIGURE 7b

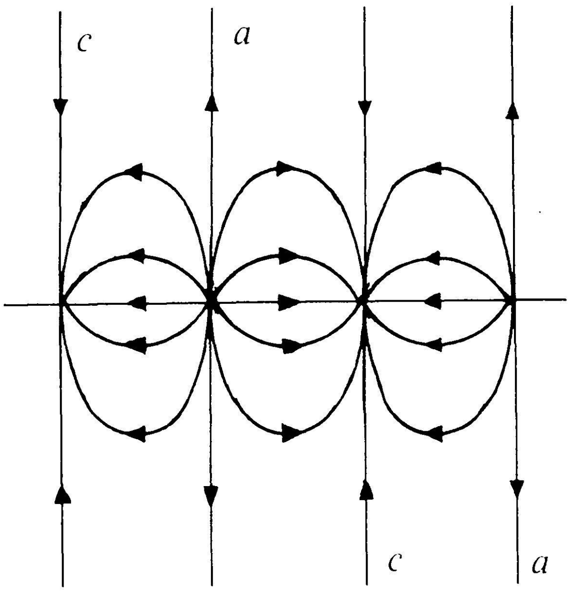


FIGURE 8a

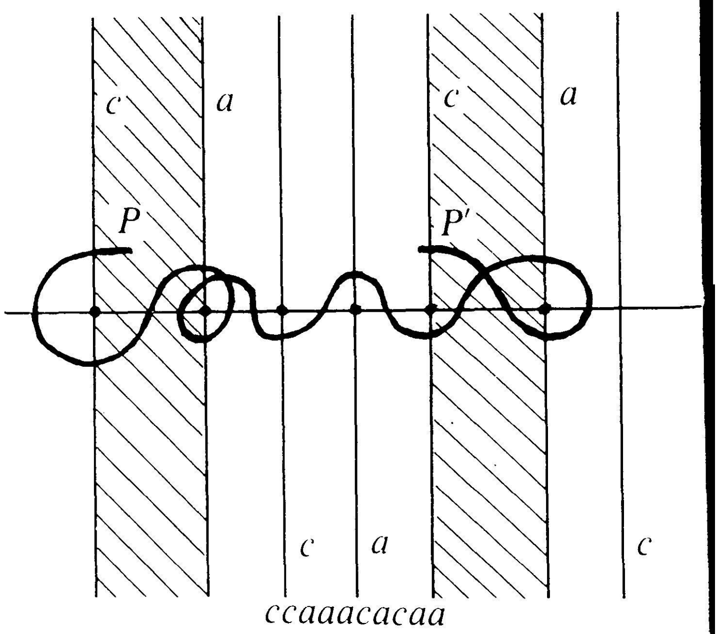


FIGURE 8b

Flowing the curve along for time t sweeps out a thickening of the immersed curve, an immersed annulus. We may identify the two boundary components of the annulus by the time t map, a locally projective isomorphism.

The identification space is a projective structure on the torus M whose developing map is the map: $M' = S^1 \times R \to \mathbf{RP}^2$, obtained by spreading the immersed curve around by the flow for all time $t \in \mathbf{R}$.

The developing map is not a covering and the image is the projective plane minus three points for any word different from aa or cc. Note that the covering space M' is obtained by gluing, each time along one of the two segments of a or c, as many copies of open sectors bounded by the lines a and c, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})\ \alpha < \beta < \gamma,\quad t = 1$.

Remark. Following the curve from its initial point P to its endpoint P', one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \to (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1:

$$g : (x, y, z) \to (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$.

Affine structures in 2, 3, and 4 dimensions

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops over the part outside the coordinate axes of $\{X > 0\} \cup \{Z > 0\} \subset \mathbf{R}^3 = \{(x, y, z)\}$, but not as a covering. In these examples the 3-manifold M is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line $\mathbf{CP}^1 = \mathbf{C}^2 - \{0\}/\mathbf{C}^*$, that is to say an orientable conformal or inversive transformation of $S^2 = \mathbf{CP}^1$, lifts to a complex affine transformation of $V = \mathbf{C}^2 - \{0\}$, unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of $\mathbf{C}$-dimension 2, and this construction is the analogue of the above over $\mathbf{C}$, thinking of S^2 as $\mathbf{CP}^1$ and the conformal transformations as the $\mathbf{C}$-projective transformation.

Again compactness is achieved if we divide by $\alpha = 2$. Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in R^4 omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of $\mathbf{C}^2 - 0$). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). *Ehresmann* defined the *development map* as follows. Let $\mathscr{P} \to M$ be the principle $\mathscr{A}$-bundle over M, whose points are germs $[x, \kappa]$ of canonical charts $\{x \in U \subset M, \kappa\colon U \to A\}$. Define a new topology $\mathscr{F}(\mathscr{P})$ in the set $\mathscr{P}$ by taking as open set the germs at all points $x \in U$ of any given chart $\kappa\colon U \to A$. The natural map $d\colon \mathscr{F}(\mathscr{P}) \to A$ is an immersion. Choose one component of $\mathscr{F}(\mathscr{P})$ and call it M'. The restriction $d\colon M' \to A$ is a development map. The restriction of the natural fibre bundle projection $p\colon \mathscr{F}(\mathscr{P}) \to M$ is a covering $M' \to M$.

NOTE 2 (see page 16). *The fibre bundle picture.* For the simple *local* discussion of *one canonical chart* $U \subset A$, we can describe a trivial fibre bundle $E_U = U \times A \to U$ by assigning to any $x \in U$ the "heavily osculating" model space $A_x = A$. The manifold U is embedded as the diagonal cross section. $s(U) = \{(y, y)\} = \mathrm{diag}(U \times U) \subset U \times U \subset U \times A$. Its points are the points of tangency of fibre and base manifolds. Finally a foliation $\mathscr{F}$ is defined as the one with horizontal leaves $U \times \{v\} \subset E_U = U \times A$, for $v \in A$.

For the *global* discussion of an $\mathscr{A}$-structure on a manifold M, we assume $\mathscr{A}$-compatible canonical charts that are topological embeddings $\kappa\colon U \hookrightarrow A$ for small open sets $U \subset M$. A point of the fibre bundle space E over M is by definition a triple

$$\{x, \kappa, v\},$$

where $x \in U \subset M, \kappa : U \to A$ is a canonical chart and $v \in A$, *modulo equivalence* by the action of $\mathscr{A}$ given by $g : \{x, \kappa, v\} \rightrightarrows \{x, \kappa', v'\}$ where $\kappa' = g \circ \kappa$, $v' = gv$, $g \in \mathscr{A}$. In E, M is embedded as the "diagonal cross section" $s(M)$, whose points are represented by triples $\{x, \kappa, \kappa(x)\}$. The foliation $\mathscr{F}$ has the local "horizontal" leaves represented by triples $\{U, \kappa, v\}$. For contractible closed curves starting and ending at $x_0 \in M$ in the base space M, the holonomy of the foliation is of course the identity map of the fibre A_{x_0}. As a consequence for closed curves in general, starting and ending at x_0 the holonomy gives the representation of $\pi_1 M$ into the group $\mathscr{A}$ acting on A_{x_0}. "Parallel displacement" of the points of $s(M)$ along the lifting in $\mathscr{F}$-leaves of curves in the base space ending at x_0, determines the development map $M' \to A_{x_0}$.

NOTE 3 (see page 16). *Flat Cartan connections.* Manifolds with canonical $(\mathscr{A}, A)$-charts are the flat cases (without torsion and without curvature) of *manifolds M with a general* $(\mathscr{A}, A)$*-connection.* They are defined in [4] as follows

(1) A fibre bundle $A \to E \to M$ with fibre A over M

(2) A fixed cross section $s(M)$

(3) An n-plane field ξ in E transversal to the fibres and transversal to the fixed cross sections, such that

(4) The holonomy obtained by lifting a closed curve starting and ending at $x_0 \in M$, into all curves tangent to ξ, belongs to $\mathscr{A}$ acting on A_{x_0}. It is in general different for homotopic curves. It is flat if contractible closed curves have trivial holonomy (= identity).

The development of a curve ending in x_0 in M, is obtained by dragging along ξ the corresponding points of $s(M)$ until they arrive in the fibre A_{x_0}. In the flat case homotopic curves with common initial and end points give the same image of the initial point in the end fibre and the development map is achieved.

REFERENCES

[1] BENZECRI, J.-P. a) Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. de France 88* (1960), 229-332.
b) Variétés localement affines. *Sém. de top. et de géom. differ. dirigé par C. Ehresmann 2* (1961), 35 pages.

[2] CARTAN, E. *Leçons sur la géométrie des espaces de Riemann.* Gauthier-Villars, 1928. Chapitre III, §2.

[3] EHRESMANN, Ch. Sur les espaces localement homogènes. *Enseign. Math. 35* (1936), 317-333.

[4] —— Les connexions infinitésimales dans un espace fibré différentiable. *Colloque de topologie du Centre Belge de Recherches Mathématiques* (1950), 29-55.

[5] GOLDMAN, W. M. *Affine manifolds and projective geometry on surfaces.* Thesis, Princeton, 1977.

[6] GUNNING, R. Special Coordinate coverings of Riemann Surfaces. *Math. Ann. 170* (1967), 67-86.

[7] NAGANO, T. and K. YAGI. The affine structures on the real two-torus. *Osaka Journal of Mathematics 11* (1974), 181-210.

[8] WHITEHEAD, J. M. C. Locally homogeneous spaces in differential geometry. *Ann. of Math. 33* (1932), *Complete works*, volume 1, p. 267.

[9] FRICKE, R. und F. KLEIN. *Vorlesungen über Automorphe Funktionen.* Teubner, 1897, volume 1, p. 440.

(Reçu le 7 avril 1982)

Dennis Sullivan

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette
France

William Thurston

Princeton University
Princeton, 08540, New Jersey