Topology Vol. 20, pp. 209–218
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GEOMETRY OF LEAVES

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(*Received for publication* 26 *June* 1980)

1. INTRODUCTION

A LEAF OF a foliation of a compact manifold† has a complete Riemannian metric determined up to quasi-isometry. In fact, since a quasi-isometry is by definition a diffeomorphism with global bounds on how much it can stretch or shrink a tangent vector, any two metrics on the manifold are quasi-isometric and the same must hold for the metrics they induce on the leaf.

From this point of view it is interesting to ask: what do leaves of foliations of compact manifolds look like? We will prove here, for example, that the quasi-isometry types of the "Jacob's ladder", the "infinite jail cell window"[10] and the "infinite jungle gym" (see Fig. 1) cannot occur in foliations of S^3, or in fact in an orientable foliation of any manifold with second Betti number zero. These surfaces are diffeomorphic to the "infinite Loch Ness monster" which does occur in S^3[2]. We use the following criterion, proved in §5.

THEOREM. *A 2-dimensional leaf L of subexponential growth type in an orientable foliation of a compact manifold M satisfying* $H_2(M; R) = 0$ *must have average Euler characteristic zero.*

These properties of leaves are defined in §2 and §3 where it is proved that they only depend on the quasi-isometry type.

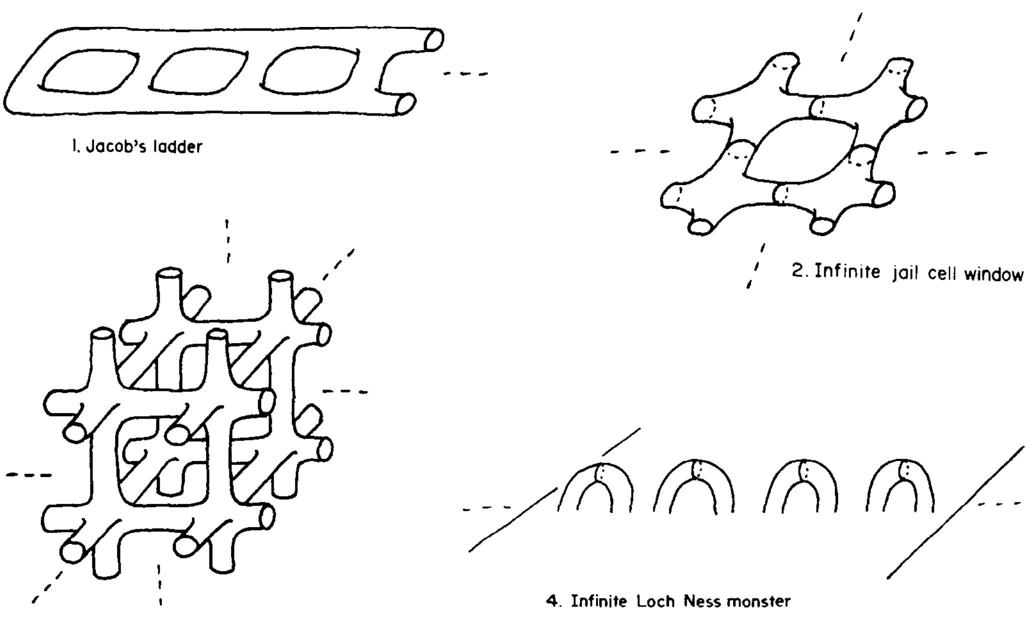

Fig. 1.

†Differentiability hypotheses: the manifold should be smooth and the foliation tangent to a continuous distribution; in particular each leaf is a manifold of class C^1.

This theorem may be viewed as an analogue on the level of cycles of the well known homological fact that a compact leaf K in a 2-dimensional foliation $\mathfrak{F}$ of S^3, or for that matter of any M satisfying $H_2(M; R) = 0$, must have Euler characteristic zero. (The Euler characteristic of K can be calculated by evaluating the Euler class of the tangent bundle of $\mathfrak{F}$ on the homology class of K.)

The proof of our theorem is related to Plante's idea that leaves of subexponential growth are like cycles. For a transversally smooth foliation a proof can be given using the curvature form of the tangent bundle of the foliation (a form representing the Euler class) and applying the Gauss–Bonnet theorem to a suitable exhausting family of submanifolds of L, for instance one constructed from the plaques of the flow-box decomposition of §4. (See the remarks at the end of §5.) Our argument is based on vector fields rather than curvature in order to avoid the transversal smoothness hypothesis.

This type of question from other viewpoints has been considered by Cantwell and Conlon[2], Goodman[3] and Sondow[9].

2. QUASI-ISOMETRIES, GROWTH TYPE

To repeat the definition more explicitly, a C^1-diffeomorphism $f: L \to L'$ between two Riemannian manifolds is a *quasi-isometry* if there exist positive constants k, K such that $k\|v\| \le \|f_* v\| \le K\|v\|$ for any vector v tangent to L. (This terminology is compatible with "quasiconformal" as defined, e.g. in [1].)

Now the *growth type* of a positive function g defined for all positive t is the equivalence class of g under the relation of mutual dominance, where g_1 is dominated by g_2 if there exist positive constants a, b, c, d such that $g_1(t) \le ag_2(bt + c) + d$; and the growth type of a Riemannian manifold L is defined to be the growth type of the function $g(r)$ = volume $B_r(x)$, where $B_r(x)$ is the set of points at distance $\le r$ from x, a fixed point in L. The growth type does not depend on x. This feature is well known[6] and has been used extensively in studying foliations by Plante[7, 8] (see also [4]).

It follows from the inequalities at the beginning of this section that $f(B_r(x)) \subset B_{Kr}(f(x))$ and $B_{kr}(f(x)) \subset f(B_r(x))$ and therefore[7, 11] that *the growth type is invariant under quasi-isometry.*

We will say that a function has *growth type of degree n* if it is equivalent as above to a polynomial of degree n (these types, for different n, are clearly distinct); and *subexponential* growth type if it does not dominate the exponential function. With this terminology the four surfaces of Fig. 1 have growth type of degree 1, 2, 3, 2 respectively. They are all subexponential.

3. AVERAGE EULER CHARACTERISTIC OF A SURFACE

Continuing with the notation above, suppose L is 2-dimensional. We will say that L has *average Euler characteristic zero* if there exists a sequence of connected submanifolds-with-boundary $L_0 \subset L_1 \subset \dots$ with the following properties.

(1) The L_i are *comparable to* the $B_r(x)$ for some (and consequently any) $x \in L$, in the sense that there is a constant Q and a sequence of radii $r_0, r_1, \dots \to \infty$ such that $B_{r_i}(x) \subset L_i \subset B_{Qr_i}(x)$.

(2) $\lim\limits_{i\to\infty} \dfrac{\chi(L_i)}{\text{area}\,(L_i)} = 0$. ($\chi$ is the Euler characteristic.)

Suppose now $f: L \to L'$ is a quasi-isometry as above, and consider the sequence $f(L_0) \subset f(L_1) \subset \dots$ of submanifolds of L'. The chain of inclusions

$$B_{kr_i}(f(x)) \subset f(B_{r_i}(x)) \subset f(L_i) \subset f(B_{Qr_i}(x)) \subset B_{KQr_i}(f(x))$$

shows them to be comparable to the $B_r(f(x))$. The equation

$$\left|\frac{\chi(f(L_i))}{\text{area } f(L_i)}\right| = \left|\frac{\chi(L_i)}{\text{area } f(L_i)}\right| \leq \frac{1}{k^2}\left|\frac{\chi(L_i)}{\text{area } L_i}\right|$$

implies that

$$\lim_{i\to\infty} \frac{\chi(f(L_i))}{\text{area } f(L_i)} = 0.$$

(We have used the fact that f can shrink areas by a factor of at most k^2.) Thus: *average Euler characteristic zero is a quasi-isometry invariant.*

Example. Consider surface 1 above. We may suppose the representative of the quasi-isometry class to be such that the $B_n(x)$ for $n \in Z$ are as shown in Fig. 2. Then $\chi(B_n) = 1 - 2n$. Any connected subspace of L containing B_n must have Euler characteristic $\leq \chi(B_n)$, so for any sequence of L_i comparable to the $B_r(x)$, each

$$|\chi(L_i)| \geq |\chi(B_{[r_i]})| = 2[r_i] - 1$$

where $[r]$ is as usual the greatest integer $\leq r$. On the other hand there exist positive constants a and b such that area $L_i \leq$ area $B_{Q([r_i]+1)} \leq a[r_i] + b$, since this surface has growth type of degree 1. It follows that

$$\lim_{i\to\infty} \frac{|\chi(L_i)|}{\text{area } L_i} \geq \frac{2}{a} > 0$$

and that this surface does not have average Euler characteristic zero.

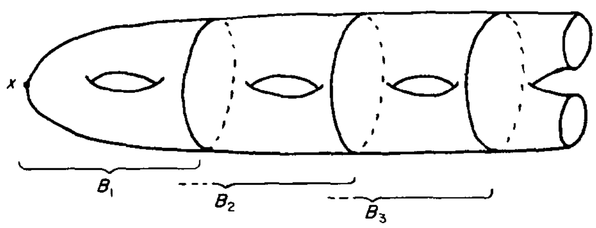

Fig. 2.

Similar calculations show that surfaces 2 and 3 also do not have average Euler characteristic zero, but that surface 4 does. Together with the growth type data, this shows that the four surfaces represent four distinct quasi-isometry types.

4. LEAVES OF FOLIATIONS OF COMPACT MANIFOLDS

We will construct in this section, for a leaf L of a 2-dimensional foliation of a compact M^{k+2}, a covering by closed discs with globally bounded geometry and relative geometry. The combinatorial properties of this covering can be thought of as giving a concrete form to the relation between the topology of M and the quasi-isometry type of L.

First we remark that under the differentiability hypotheses noted in §1, integrability means that there exist local $C^{1,0}$-*homeomorphisms* $f: R^2 \times R^k \to M$ (homeomorphisms such that $\partial f/\partial x_1$ and $\partial f/\partial x_2$ exist and are continuous with respect to $x_1, \ldots, x_{k+2}$) such that $\partial f/\partial x_1$ and $\partial f/\partial x_2$ locally span the distribution, i.e. are tangent to the foliation. It follows that each point of M has a closed neighborhood $C^{1,0}$-homeomorphic to the product of discs $D^2 \times D^k$, with each *plaque* $D^2 \times \{u\}$, $u \in D^k$, lying on a leaf. Such a neighborhood will be called a *closed flow-box* of the foliation.

PROPOSITION 4.1. *A compact manifold M with a 2-dimensional foliation admits a finite covering $B_1, \ldots, B_N$ by closed flow-boxes such that plaques of different flow-boxes intersect generically, namely*

(1) *Boundaries of plaques intersect transversely or not at all.*

(2) *There are no triple intersections of boundaries of plaques.*

COROLLARY 4.2. *Let $\{P_\lambda\}$ be the collection of all the plaques of $B_1, \ldots, B_N$. Then because each flow-box is compact, and because there are only finitely many, there exist*

(1) *A lower bound $\epsilon_0 > 0$ on the distance, measured along the boundary of a plaque, between intersection points with boundaries of other plaques.*

(2) *A lower bound $\delta_0 > 0$ on the area of a non-empty sector of a plaque, where a* sector *of a plaque P is a subset of the form $P \cap P_{\lambda_1} \cap \cdots \cap P_{\lambda_j} \cap P'_{\lambda_{j+1}} \cap \cdots \cap P'_{\lambda_{j+k}}$, prime denoting complement, for $j, k \geq 0$ (see Fig.* 3).

(3) *A positive Lebesgue number μ for the covering of the foliation by plaques.*

Furthermore, independently of conditions 1 and 2, there exist upper bounds C and D on the circumference and diameter of any plaque.

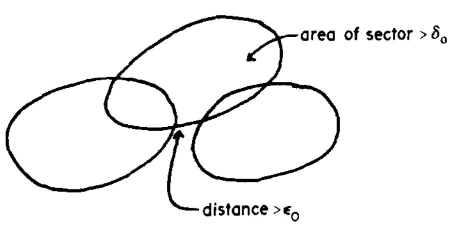

Fig. 3.

Proof of Proposition 4.1. In codimension one a finite covering by closed flow-boxes $B_1, \ldots, B_N$ may be constructed by choosing discs in leaves and sliding them back and forth along short arcs perpendicular to the foliation. Then if a plaque of B_i and a plaque of B_j intersect generically, the same will hold for every intersection between a plaque of B_i and one of B_j, so all non-generic intersections of plaques may be removed by suitable small displacements of $B_1, \ldots, B_N$.

In higher codimension we must proceed differently, since there rarely is a transversal foliation, and what is generic for plaques may not be generic for flow-boxes. This is schematically shown in Fig. 4, which illustrates what might happen in codimension one without a normal field: no small change in the flow-boxes A_1 and A_2 can eliminate tangency of plaque boundaries.

Begin with a finite covering $A_1, \ldots, A_L$ of M by closed flow-boxes with the property that each one is contained in the interior of a larger flow-box: for every $i = 1, \ldots, L$ there exists A'_i and $\varphi_i: D^2 \times D^k \to A'_i$ such that $A_i = \varphi_i(D_i^2 \times D_i^k)$ for discs $D_i^2 \subset \operatorname{Int} D^2$ and $D_i^k \subset \operatorname{Int} D^k$.

Set $B_1 = A_1$. Suppose flow-boxes $B_1, \ldots, B_K$ have been defined satisfying 1 and 2 and covering $A_1 \cup \cdots \cup A_{p-1}$. For each $v \in D_p{}^k$, construct a flow-box as follows. If the boundary of the plaque $\varphi_p(D_p{}^2 \times \{v\})$ has no tangencies or triple intersections with

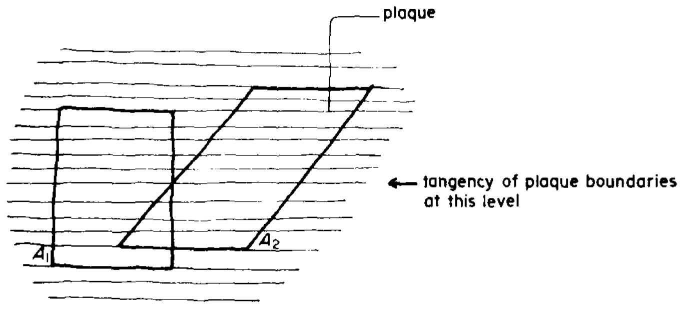

Fig. 4.

boundaries of plaques of $B_1, \ldots, B_K$, set $D_v^2 = D_p^2$. If it has a non-generic intersection, choose $D_v^2 \subset D^2$ to be a disc *concentric to* and containing D_p^2, and which has none. Now since the boundary of $\varphi_p(D_v^2 \times \{v\})$ intersects boundaries of plaques of $B_1, \ldots, B_K$ either not at all or transversely, and has no triple intersections, the same will hold for $\varphi_p(D_p^2 \times \{t\})$ for every $t \in D_v^k \subset D^k$, a sufficiently small k-ball of positive radius about v. Set $A_v = \varphi_p(D_v^2 \times D_v^k)$. A finite number $D_{v_1}{}^k, \ldots, D_{v_n}{}^k$ of these k-balls cover D_p^k with their interiors. Set $B_{K+1} = A_{v_1}, \ldots, B_{K+n} = A_{v_n}$. Now by construction the B_{K+i}-plaques intersect the B_j-plaques generically for $j \leq k$, and since the $D_{v_i}{}^2$ are all concentric, there are no intersections between boundaries of B_{K+i}-plaques (unless it should happen that $D_{v_i}{}^k \cap D_{v_j}{}^k \neq \emptyset$ and $D_{v_i}{}^2 = D_{v_j}{}^2$; this can be eliminated by expanding say $D_{v_j}{}^2$ slightly). Furthermore $A_p \subset B_{K+1} \cup \cdots \cup B_{K+n}$. Proceed in this manner until $A_{p+1}, \ldots, A_L$ are covered. (See Fig. 5.)

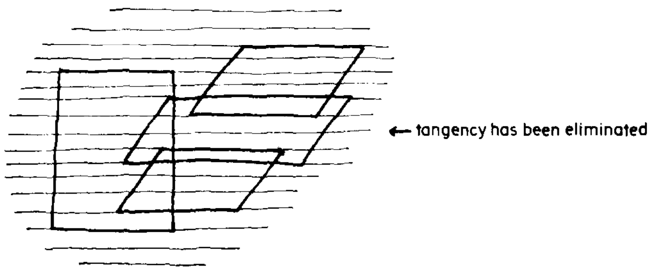

Fig. 5.

An easy modification of this argument proves the following proposition, which will be useful in establishing our main result. Suppose that M carries a continuous vectorfield V everywhere tangent to the foliation. The zeroes of V can be assumed to lie on a k-dimensional submanifold Z of M, and it may be further assumed that each leaf of the foliation intersects Z in a set of isolated points. (In fact, we may approximate the distribution T tangent to the foliation by a smooth distribution S in such a way that (*) orthogonal projection $p: S \to T$ is non-singular. Then the usual transversality argument yields a smooth vectorfield $V' \subset S$ with zeroes forming a k-dimensional $Z \subset M$. Any non-generic tangency of Z with the foliation may be eliminated by an isotopy of the foliation small enough to preserve (*). Then $V = p(V')$ has the properties claimed.)

PROPOSITION 4.3. *Given a compact manifold M with a 2-dimensional foliation and a vectorfield V as above, then M admits a finite covering $B_1, \ldots, B_N$ by closed flow-boxes such that plaques of different flow-boxes intersect generically (as previously defined) and such that no zeroes of V occur on plaque boundaries.*

Definition. Given a continuous, non-zero vectorfield W defined in a neighborhood of the unit circle S^1 in R^2, and two points θ_0 and θ_1 on the circle, we will say that W has a ***flip*** between θ_0 and θ_1 if W is a positive multiple of $\partial/\partial\theta$ at θ_0 and a negative multiple of $\partial/\partial\theta$ at θ_1, or vice versa.

Let F represent the total number of consecutive flips of W counted around the circle. If $\mathrm{Rot}(W, S^1)$ is the total angular rotation of W with respect to the tangent to the circle (i.e. 2π times the winding number of $W(\theta)\,e^{-i\theta}$ when $W|_{S^1}$ is represented by a complex-valued function on $[0, 2\pi]$), then $|\mathrm{Rot}(W, S^1)| \leq \pi F$. In fact for any segment $A \subset S^1$ and for any parametrization, we have

$$|\mathrm{Rot}(W, A)| \leq \pi(F+2).$$

Recall that the vectorfield V is nonzero in a neighborhood of the boundary of any plaque P of the flow-box covering given by Proposition 4.3.

COROLLARY 4.4. *For the flow-box covering of Proposition* 4.3 *there exists, besides the bounds ϵ_0, δ_0, μ, C, D established above, a bound ρ on the rotation of V along any segment of a plaque boundary.*

Proof. For any plaque P the number $F(P)$ of flips of V around ∂P is finite, and $\pi(F(P)+2)$ bounds $|\mathrm{Rot}(V, A)|$ for any segment $A \subset \partial P$. The Corollary follows from continuity of V, compactness of the flow-boxes and their finite number.

5. AVERAGE EULER CHARACTERISTIC OF SUBEXPONENTIAL LEAVES

THEOREM. *A 2-dimensional leaf L of subexponential growth type in an orientable foliation of a compact manifold M satisfying $H_2(M; R) = 0$ must have average Euler characteristic zero.*

Remark. The "infinite Loch Ness monster" has average Euler characteristic zero, as already noted: it has homology growing like r and area growing like r^2. The appendix exhibits examples, for each $n \geq 2$, of a foliation of S^3 containing leaves with homology growing like r^n, and area like r^{n+1}.

Proof of Theorem. We may suppose that the foliation carries a tangent vectorfield V satisfying the hypotheses of Proposition 4.3. Let $\{P_\lambda\}$ be the set of plaques of the flow-box covering given by that proposition.

Let us pick a point $x \in L$, set $H_0 = P_{\lambda_0}$, a plaque containing x, and for $i \geq 1$ set $H_i = \cup P_\lambda$, the union taken over all λ such that $P_\lambda \cap H_{i-1} \neq \emptyset$.

Assertion 1. The H_i are comparable to the $B_r(x)$.

Assertion 2. $\liminf_i \dfrac{\text{length } \partial H_i}{\text{area } H_i} = 0.$

These assertions will be proved later. Using Assertion 2 and the method of Plante[7] we may extract from the H_i a subsequence $H_{i_0} = L_0$, $H_{i_1} = L_1, \ldots$ satisfying

$$(*) \quad \lim_{j\to\infty} \frac{\text{length } \partial L_j}{\text{area } L_j} = 0$$

and which defines a real 2-cycle $\lim_{j\to\infty} \dfrac{L_j}{\text{area } L_j}$ representing an asymptotic homology class of L.

The Euler characteristic $\chi(L_j)$ is given by applying to L_j and V the rule

$$2\pi\chi(N) = -\operatorname{Rot}(W, \partial N) + 2\pi \sum_{p\in N} \operatorname{Index}(W, p)$$

which holds for any surface N carrying a vectorfield W nonzero along ∂N and with isolated zeroes at points P in the interior. (The derivation of this rule is an easy exercise.)

Assertion 3. $\lim_{j\to\infty} \dfrac{\operatorname{Rot}(V, \partial L_j)}{\text{area } L_j} = 0.$

Assertion 4. $\lim_{j\to\infty} \dfrac{1}{\text{area } L_j} \sum_{p\in L_j} \operatorname{Index}(V, p) = 0.$

These two assertions imply that $\lim_{j\to\infty} \dfrac{\chi(L_j)}{\text{area } L_j} = 0$. Since Assertion 1 implies that the L_j are comparable to the $B_r(x)$, the theorem is proved if the four assertions are true.

Proof of Assertion 1. We use the constants D, μ from Corollary 4.4, and observe that $B_{\mu i}(x) \subset H_i \subset B_{Di}(x)$.

Proof of Assertion 2. We use the constants C, δ_0 from Corollary 4.4. First remark that

$$\text{length } \partial H_i \le \frac{C}{\delta_0} \text{ area } (H_i - H_{i-1}).$$

In fact, in the set of new plaques added to H_{i-1} to form H_i, the only ones contributing to the boundary are those "independent" plaques which are not covered by any other plaques intersecting H_{i-1}, and each of these contributes to the area of $H_i - H_{i-1}$ at least one sector which is contributed by no other plaque. Thus length $\partial H_i \le C$(number of independent plaques) and area$(H_i - H_{i-1}) \ge \delta_0$(number of independent plaques), yielding the desired inequality.

Next we observe that $\liminf_i \dfrac{\text{area } (H_i - H_{i-1})}{\text{area } H_i} = 0$. If it were otherwise, there would be an i_0 and an $\alpha > 0$ such that $\dfrac{\text{area } (H_i - H_{i-1})}{\text{area } H_i} > \alpha$, for $i > i_0$. Consequently

$$\text{area } B_{Di} > \text{area } H_i > (1+\alpha)^{i-i_0} \text{ area } H_{i_0},$$

and L could not have subexponential growth.

Proof of Assertion 3. We use the constants ϵ_0, ρ from Corollary 4.4. The boundary of L_j is made up of segments of plaque boundaries separated by corners. Since the corners are at least ϵ_0 apart the total number of corners (and of segments) appearing in ∂L_j is bounded by $1/\epsilon_0$ length ∂L_j. The rotation along any segment is bounded by ρ. The rotation at any corner is bounded by π. The assertion now follows from (*).

Proof of Assertion 4. Here is where the hypothesis on the Betti number is used. The index of V at p is defined as usual by $\text{Index}(V, p) = 1 + (1/2\pi)\,\text{Rot}(V, \partial D)$ where D is a disc about p containing no zeroes in $D - \{p\}$. The intersection number of L and Z at p is defined by approximating L in D with a surface intersecting Z transversely, and counting the intersection numbers so obtained. It is not difficult to see that *these two numbers are equal.* Consequently

$$\lim_{j\to\infty} \frac{1}{\text{area } L_j} \sum_{p \in L_j} \text{Index}(V, p)$$

is the intersection number of Z with the asymptotic cycle $\lim_{j\to\infty} \frac{L_j}{\text{area } L_j}$. Since the manifold has no real 2-dimensional homology or cohomology, this number must be 0.

Remarks. (1) The hypothesis $H_2(M; R) = 0$ could be replaced by the weaker requirement that the Euler class X of the tangent bundle of the foliation (which is Poincaré dual to Z) be zero in real homology. (2) Under stronger differentiability conditions one can be more explicit. Let us fix a Riemannian metric on M. Suppose for example that the foliation is C^∞, so that it makes sense to talk about the geodesic curvature of the boundary of a plaque and so that the class X may be represented along the leaves by the Gauss–Bonnet integrand ω. Then there will exist, besides the bounds ϵ_0, δ_0, μ, C, D given by Corollary 4.2, and the upper bound A on the area of any plaque, global upper bounds κ_0 and K_0 on the absolute value of the geodesic curvature of plaque boundaries and of the Gaussian curvature of the leaves of the foliation.

Choose $x \in L$ and let $B_0 = B_{r_0}(x) \subset B_1 = B_{r_1}(x) \subset \cdots \subset L$ be an exhausting sequence defining an asymptotic cycle in the sense of [7]. This means in particular that

$$\lim_{i\to\infty} \frac{\text{length } \partial B_i}{\text{area } B_i} = 0.$$

Define L_i to be the union of all the plaques intersecting B_i. Then the sequence $L_0 \subset L_1 \subset \cdots$ satisfies

$$\text{(a)} \quad B_{r_i}(x) \subset L_i \subset B_{r_i+D}(x)$$

(this implies that the L_i are comparable to the B_{r_i})

$$\text{(b)} \quad \text{area}(L_i - B_i) \leq \frac{A^2}{\mu\delta_0} \text{length } \partial B_i$$

$$\text{(c)} \quad \text{length } \partial L_i \leq \frac{CA}{\mu\delta_0} \text{length } \partial B_i.$$

Suppose now that the value of X on the asymptotic cycle $\lim_{i\to\infty} \frac{B_i}{\text{area } B_i}$ is zero, i.e. that $\lim_{i\to\infty} \frac{1}{\text{area } B_i} \int_{B_i} \omega = 0$. By the Gauss–Bonnet Theorem, the Euler characteristic $\chi(L_i)$ may be evaluated by

$$2\pi\chi(L_i) = \int_{L_i} \omega + \int_{\partial L_1} \kappa \, ds + \sum_{p \in \partial L_i} \theta_p$$

where κ is the geodesic curvature and θ_p is the exterior angle at a corner p of ∂L_i. Note that $\left|\int_{\partial L_i} \kappa \, ds\right| \leq \kappa_0$ length ∂L_i and that $\left|\sum_{p \in \partial L_i} \theta_p\right| \leq \frac{\pi}{\epsilon_0}$ length ∂L_i since the minimum distance between consecutive corners is ϵ_0.

Finally,

$$\lim_{i\to\infty} \frac{|\chi(L_i)|}{\text{area } L_i} \leq \lim_{i\to\infty} \frac{1}{2\pi} \frac{1}{\text{area } L_i} \left(\left|\int_{B_i} \omega\right| + \left|\int_{L_i - B_i} \omega\right| + \left(\kappa_0 + \frac{\pi}{\epsilon_0}\right) \text{length } \partial L_i \right),$$

and

$$\lim_{i\to\infty} \frac{1}{\text{area } L_i} \left|\int_{B_i} \omega\right| \leq \lim_{i\to\infty} \frac{1}{\text{area } B_i} \left|\int_{B_i} \omega\right| = 0$$

$$\lim_{i\to\infty} \frac{1}{\text{area } L_i} \left|\int_{L_i - B_i} \omega\right| \leq \lim_{i\to\infty} \frac{1}{\text{area } B_i} K_0 \frac{A^2}{\mu\delta_0} \text{length } \partial B_i = 0$$

$$\lim_{i\to\infty} \frac{1}{\text{area } L_i} \text{length } \partial L_i \leq \lim_{i\to\infty} \frac{1}{\text{area } B_i} \frac{CA}{\mu\delta_0} \text{length } \partial B_i = 0$$

so L has average Euler characteristic zero.

Acknowledgements—It is a pleasure to acknowledge helpful conversations with James Heitsch, David Stone and Walter Neumann, and to thank Larry Conlon for sending us a copy of the manuscript of [2].

REFERENCES

1. L. AHLFORS: *Lectures on Quasiconformal Mappings.* van Nostrand, Princeton, New Jersey (1966).
2. J. CANTWELL and L. CONLON: Leaves with isolated ends in foliated 3-manifolds. *Topology* **16** (1977), 311–322.
3. S. GOODMAN: Closed leaves in foliated 3-manifolds. *Proc. Nat. Acad. Sci.* **71** (1974), 4414–4415.
4. G. HECTOR: Leaves whose growth is neither exponential nor polynomial. *Topology* **16** (1977), 451–459.
5. H. B. LAWSON, Jr.: Foliations. *Bull. Am. Math. Soc.* **80** (1974), 369–418.
6. J. MILNOR: A note on curvature and fundamental group. *J. Diff. Geom.* **2** (1968), 1–7.
7. J. PLANTE: A generalization of the Poincaré–Bendixson theorem for foliations of codimension one. *Topology* **12** (1973), 177–181.
8. J. PLANTE: Foliations with measure preserving holonomy. *Annals of Math.* **102** (1975), 327–361.
9. J. SONDOW: When is a manifold a leaf of some foliation? *Bull. Am. Math. Soc.* **81** (1975), 622–624.
10. M. SPIVAK: *A Comprehensive Introduction to Differential Geometry I.* Brandeis University (1970).
11. D. SULLIVAN: Inside and outside manifolds. *Proc. Int. Cong. of Mathematicians.* Vancouver (1974).

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APPENDIX: EXAMPLES OF LEAVES

For each $n \geq 2$ we exhibit here a class-C^∞ foliation of S^3 containing leaves L_n with homology growing like r^n, and area like r^{n+1}.

Let D_n represent a closed 2-disc less the interiors of n distinct discs in its interior.

Draw a set $\gamma_1, \ldots, \gamma_n$ of non-intersecting arcs, so that each hole is joined to the outside boundary. Parametrize S^1 by $0 \leq t < 1$.

The Cartesian product $S^1 \times D_n$ is foliated by the $\{t\} \times D_n$. Modify this foliation by slicing along $S^1 \times \gamma_i$ and joining the t-level on one side of the cut to the $(t + e^i)$-level on the other; and do this for each $i = 1, \ldots, n$. The new leaves in $S^1 \times D_n$ are all identical. Their area and homology both grow like polynomials of degree n: each leaf can be placed along an n-dimensional lattice in the way shown in Fig. 6 for $n = 2$.

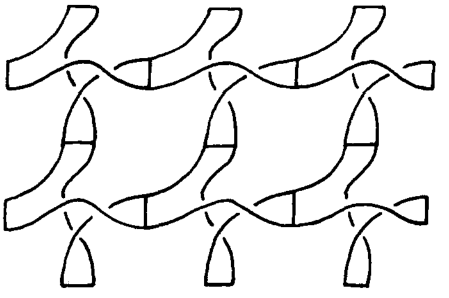

Fig. 6.

The next step is to modify this foliation by drawing the leaves up so as to make them tangent to the boundary. Then add n Reeb components to fill the interior holes; this gives a foliated solid torus. Joining one more Reeb component will then yield a foliation of S^3; and this can all be done smoothly [5].

The quasi-isometry class L_n of the modified leaves can be represented for $n = 2$ by adding a half-plane along each of the edges in the figure, and for higher n similarly. The area in L_n will now grow like a polynomial of degree $n + 1$, since a ball of radius r will encounter edges of total length proportional to r^n, and will cut into each of the added half-planes to a height proportional, on the average, to r.