Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorf dimension two,

in celebration of Nico Kuiper's sixtieth birthday.

Do not REMOVE

by Dennis Sullivan

If Γ is a finitely generated discrete subgroup of linear fractional transformations $\{z \to \frac{az+b}{cz+d}\}$, one defines since Poincaré the limit set $\Lambda = \Lambda(\Gamma)$ to be the set of cluster points of the orbit under Γ of any fixed point $z_0 \in \mathbb{C}$. One knows Λ is a closed subset of the Riemann sphere $\mathbb{C} \cup \infty$ which is either everything or a closed nowhere dense subset of $\mathbb{C} \cup \infty$.

A famous problem in the theory of such groups is Ahlfors question (1965): If the limit set $\Lambda(\Gamma)$ is not the entire Riemann sphere, is the two-dimensional Lebesgue measure** of $\Lambda(\Gamma)$ equal to zero?

This question is still unresolved, although Ahlfors himself (1967) settled the question affirmatively for groups which have a fundamental domain of finitely many sides for their action on the 3-dimensional hyperbolic space. More recently, Thurston (1978), has resolved the question affirmatively for groups which have infinite sided fundamental domains but which are limits in a strong sense of groups with finite sided fundamental domains. These limiting groups play a central role in the study of finitely generated groups so Thurston's theorem is a decisive advance.

In this paper we will show on the contrary that for some of these limit groups* the Hausdorf measure of the limit set for the gauge function $r^2 \log 1/r$ is positive. In particular the Hausdorf dimension of the limit set is equal to two. (§6).

*Those with no short closed geodesics.

**It will be useful to remember that planar measure is Hausdorf measure using the gauge function r^2.

The proof makes use of a canonical measure μ on the limit set (§5) which satisfies

$$* \qquad \gamma^{*}\mu = |\gamma'|^{2}\mu \qquad \gamma \in \Gamma$$

where $|\gamma'|$ is the linear distortion of the spherical metric on S^2, the boundary of the unit ball model of hyperbolic 3-space. We conjecture that in these examples μ is the Hausdorf measure for the gauge function

$$r^{2}(\log 1/r)^{1/2}(\log \log \log 1/r)^{1/2} .$$

In the first three sections we develop general estimates concerning positive harmonic functions on some end of a Riemannian manifold. For example, any such positive harmonic function which is proper either grows at most linearly or the end of the manifold is infinitely often arbitrarily skinny. (§1).

§1 Growth of Positive Harmonic Functions

We consider a harmonic function h on a complete oriented Riemannian manifold M. We assume that $M_{+} = h^{-1}[0,\infty)$ has a compact boundary and that h/M_{+} is proper; that is $h^{-1}[a,b]$ is compact for $0 \le a < b < \infty$. We further assume that M_{+} has bounded geometry locally; that is each point of M^{+} (away from the boundary) has a ball neighborhood of radius 1 which is a geometrically bounded distortion of a unit ball in Euclidean space.

Theorem 1: Under the above assumptions on h and M_{+} the gradient of h is uniformly bounded on M_{+}.

Corollary: The growth of h in M_+ is at most linear in the distance from a fixed point of M_+.

Proof of Corollary: Choose a geodesic connecting the fixed point and the arbitrary point. The change in h along the geodesic is by the theorem no more than a constant times the length of the geodesic.

Proof of theorem 1: We assemble some general facts about harmonic functions. By the "values at distance one" we mean the values in a neighborhood of radius one about some point.

i) The size of the first and second derivatives of a harmonic function at a point are controlled by the values at distance one. (This follows from the average value property and the bounded geometry.)

ii) For a positive harmonic function the value at a point p controls the values at distance one. (This is a form of Harnack's inequality for bounded geometry.)

iii) The gradient of a harmonic function defines a volume preserving flow. (Laplacian $h = 0$ means divergence (gradient h) = 0).

Now i) implies:

iv) The size of the second derivative at a point p is controlled by the values of the first derivative at distance one. (Subtract off a constant to make $h(p) = 0$. This doesn't change any derivatives. Now the values at distance one are controlled by the values of the first derivative at distance one which therefore by i) control the value of the second derivative at p . This proves iv).).

And ii) implies:

v) The growth of a positive harmonic function or its gradient is at most that of some fixed exponential. (By ii) the values control the values at distance one which by i) control the values of the derivatives. Now integrate.)

Finally iii) implies:

vi) If A and B are compact hypersurfaces which bound a region R, then the flux of grad h across A equals the flux of grad h across B. (We mean the integral of the normal components of the gradients are equal. This follows from Stokes theorem, using iii).).

Now we are ready to prove the theorem:

1) Consider the level sets of h. These are compact since h is proper and for almost all values of h they are hypersurfaces. Also the levels for two values a and b bound the region $h^{-1}[a,b]$. The gradient is normal to the levels so by vi) we have $\int_{h^{-1}(a)} |\text{grad } h|$ is some constant independent of which value a we consider (as long as $h^{-1}a$ is a hypersurface).

2) Suppose at some point p the gradient is very large (compared to constants involved in the bounded local geometry of M_+ and the properties i) and ii).) We claim the gradient is either much larger or much smaller in a unit neighborhood B of p. (If not, consider the "fibring" of B by the levels of h. Since the gradient is uniform on B the coarea formula shows that some good level has a definite area. But then this contradicts 1) because the integral of $|\text{grad } h|$ over this piece would be too large.)

3) We want the first conclusion of 2) that the gradient is much larger at a point of B. So assume the second possibility that it is much smaller at a nearby point. Now if the value of grad h at p is

N , the argument above could have used a neighborhood of any fixed size after taking N large enough to get the contradiction. Thus we see the gradient changes from size N to less than $1/2\ N$ say in a very small distance. In order for this to happen, the second derivative must be much larger than N in the neighborhood. Thus by iv) the gradient itself must be much larger at a nearby point.

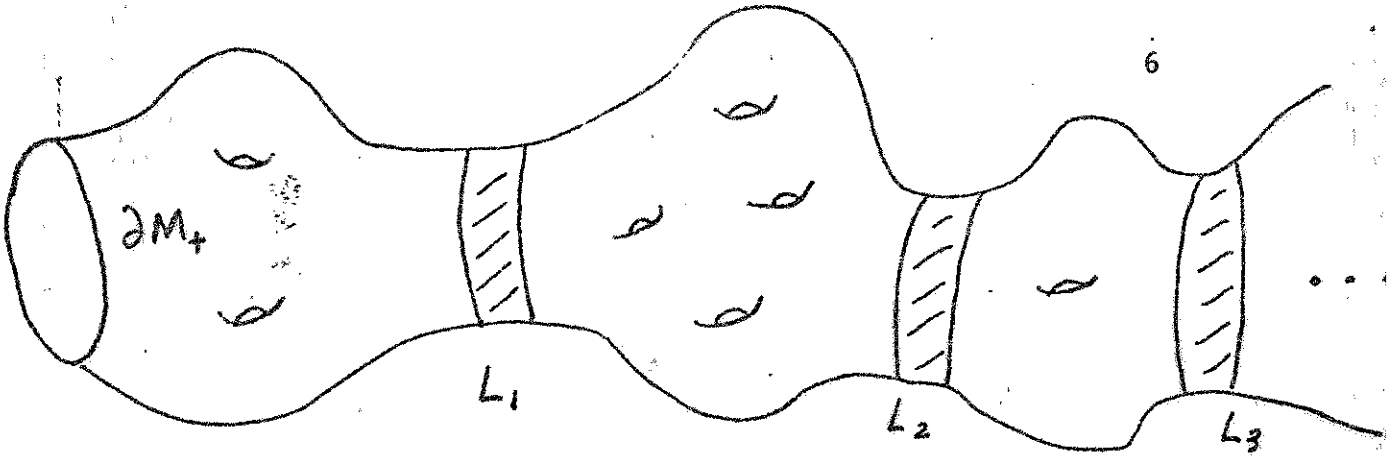
4) Using 2) and 3) repeatedly we construct a sequence of points $p = p_0, p_1, p_2, \ldots$ so that the gradient increases along these at least by a fixed factor (arbitrarily large) and distance $(p_{i+1}, p_i) \leq 1$. This contradicts v) and the theorem is proved.

§2 The Infimum of Positive Superharmonic Functions.

To apply §1 we need to show that certain harmonic functions are proper. In this section we abstract an argument from Thurston's notes [T, 8.12.3] for this purpose. We suppose we have a complete Riemannian manifold M_+ with a compact boundary and regions $L_1, L_2, \ldots$ (called bands) so that

i) the bands $L_1, L_2, \ldots$ contain unit width neighborhoods of compact hypersurfaces $S_1, S_2, \ldots$ all homologous to the boundary of M_+ and which tend to ∞ in M_+ .

ii) the volume of the respective bands is bounded $(\text{vol } L_i \leq c)$.



Now let h be any positive superharmonic function on M that is, h is positive and the gradient of h is volume non-increasing.

<u>Theorem</u> 2: <u>The infimum of</u> h <u>is assumed on the boundary of</u> M_+ .

<u>Proof</u>: (Thurston [T, 8.12.3]) 1) Consider the flow $\varphi_t(x)$ corresponding to the vector field -grad h . If $A \subset \{\varphi_t(x) ; t \geq 0\}$, let T_A be the time the trajectory spends in A which has Riemannian length ℓ_A .

Applying Schwartz,

$$(\ell_A)^2 = \left(\int_A \frac{1}{\sqrt{\text{grad h}}} \sqrt{\text{grad h}}\, d\ell \right)^2$$

$$\leq \left(\int_A \frac{1}{\text{grad h}} \right)\left(\int_A \text{grad h} \right)$$

$$= T_A \quad (\text{variation of h on A}).$$

Since h is positive and decreasing for positive time one obtains

(*) $$\boxed{T_A \geq (\ell_A)^2/h(x) .}$$

2) The inequality (*) shows the flow φ_t is defined for all time unless the trajectory exits at the boundary.

Now take any point x and let B be a small ball about x. We want to show almost all trajectories starting in B exit at the boundary. If not there is a set of such trajectories of positive measure which stay in M_+ for all time. Since the volume is <u>nondecreasing</u> under the flow φ_t, and since the regions between the bands and the boundary are compact, these trajectories must cross infinitely many bands.

The inequality implies each flow line spends time in crossing k-bands on the order of k^2 (since each has width ≥ 1). Since the volume of k-bands is of the order k and the flow is volume non-decreasing we have a contradiction.

3) Thus almost all trajectories starting from B exit at the boundary. So choose a sequence of such starting points $x_i \to x$. Then $h(x_i) \to h(x)$, and $h(x_i) \geq h(y_i)$ for some $y_i \in \partial M_+$. Thus $h(x) \geq \inf h(y)$ for $y \in \partial M_+$. This proves theorem 2.

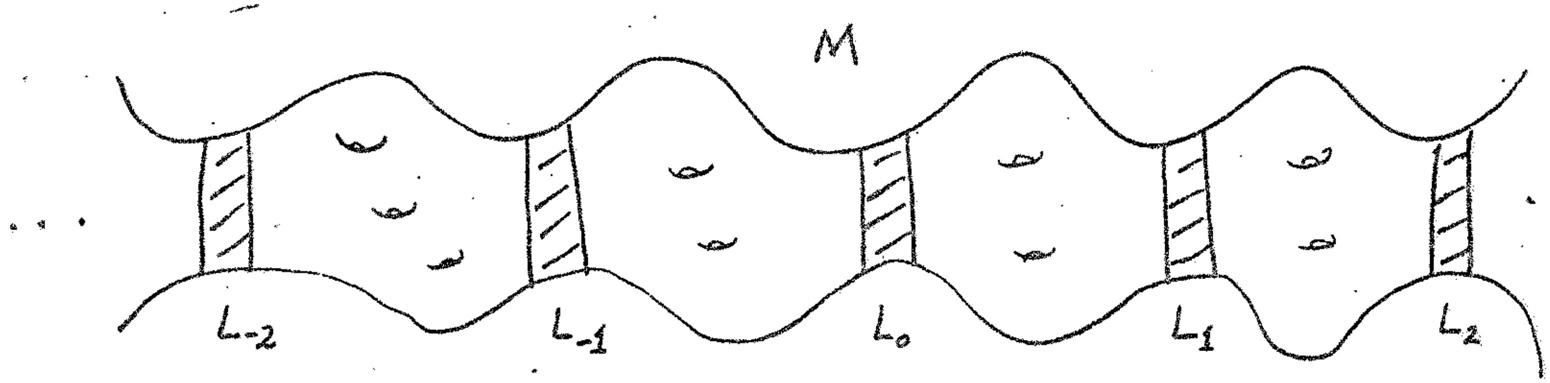
Now we assume in addition that the bands L_i are connected, have bounded diameters and the local geometry of M_+ is bounded for points in the bands. Then we have the

<u>Corollary 2.1: If h is a positive harmonic function on M_+ then either h is bounded or $h(x) \to \infty$ if $x \to \infty$ in M_+; that is, h is proper.</u>

<u>Proof</u>: Consider the minimum m_i of h in each band L_i. Using the additional assumptions on L_i and properties i) and ii) in the proof of theorem 1 we can get a constant c so that the maximum M_i of h in the band L_i satisfies $M_i \leq cm_i$. (Namely $|\text{grad } h| \leq ch$.)

By the maximum principle if the M_i are bounded h is bounded on M_+. Otherwise $M_i \to \infty$ which implies $m_i \to \infty$. Applying Theorem 2 to the part $M_+(i)$ of M_+ outside the band L_i shows m_i is the minimum of h restricted to $M_+(i)$. This proves Corollary 2.1.

Now suppose we have a manifold with at least 2 ends and bounded volume homologous bands going to ∞ in both directions.



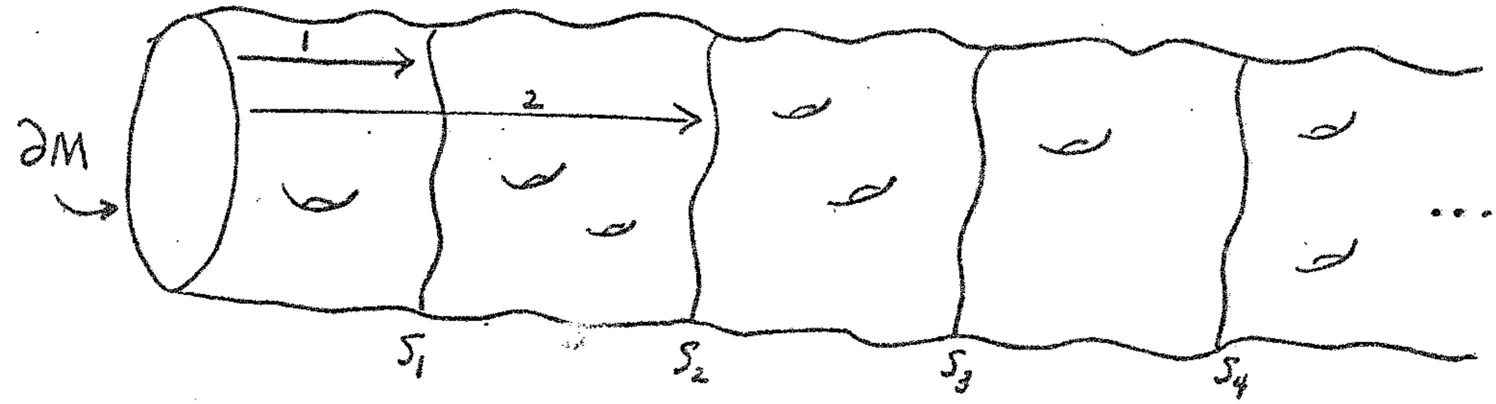
Corollary 2.2: Under these assumptions M admits no positive non constant superharmonic function.

Proof: The infimum of h is approached by the values of h in the part of M on one side or another of a given band. Applying Theorem 2 to this side shows the minimum is achieved in this band. This violates the minimum principle which is valid for positive superharmonic functions. So corollary 2.2 is proven.

Historical Note. Variants of the corollary were proven by Ahlfors (C.R.A.S. Paris, 1936) in the context of Riemann surfaces.

§3 Positive Harmonic Functions on Quasi-cylinders

Now we make more geometric assumptions about our manifold with boundary M . Not only is the geometry of M locally bounded but for each n the hypersurface at some distance d from ∂M in the interval $[n, n+1]$ has bounded diameter. We call such manifolds "quasi-cylinders".



Any manifold obtained from the product $V \times [0,\infty)$, where V is a compact closed Riemannian manifold, by a bounded distortion of the metric is a quasi-cylinder.

Theorem 3: A "quasi-cylinder" M admits a non-constant positive harmonic function. Any such function h satisfies for some constant $c < \infty$,

$$\frac{1}{c} \text{ distance } (x,\partial M) \leq h(x) \leq c \text{ distance } (x,\partial M) .$$

Proof: 1) The construction of such an h is quite general and uses none of the geometric hypotheses. For each n construct a harmonic function which is 0 on the boundary and n on the nth surface S_n . Such a function is ≥ 0 by the minimum principle.

Multiplying by constants we obtain $h(p) = 1$ for some fixed $p \in M$. By Harnack's inequality applied to fixed compact neighborhood of p we have a family of functions with bounded gradient. There is then a convergent subsequence on this neighborhood. Taking larger and larger neighborhoods and subsequences yields a global nonnegative harmonic function which is zero on ∂M and 1 at p. Now add a positive constant.

2) By corollary 2.1 such a function is either proper or bounded. By theorem 1 or Harnack the gradient of h is bounded. This proves the upper bound.

To get the lower bound we use the additional geometric hypotheses.

We suppose the function h is non-constant. Then almost all trajectories exit at the boundary by theorem 2. In particular the flux of -grad h at the boundary, $\int_{\partial}$ (grad h) $\cdot$ (normal), is non-zero. Then the flux at each of the homologous surfaces S_n is this same non-zero constant. Since the S_n's have bounded area the function $|\text{grad } h|$ must have a definite value at some point $p_n \in S_n$ (otherwise $\int_{S_n} |\text{grad } h|$ would be too small). So if R_n denotes the region between ∂M and S_n we have $\int_{R_n} (\text{grad } h)^2 \geq cn$ because we get a definite contribution around each p_n. Applying Green's formula

$$\int_{R_n} (\text{grad } h)^2 = \int_{\partial} h \cdot \text{grad } h + \int_{S_n} h \cdot \text{grad } h$$

yields h on S_n must be at least cn because the first term is bounded and grad h is bounded. Since diameter S_n is bounded and grad h is bounded h only varies by a bounded amount on S_n. This proves theorem 3.

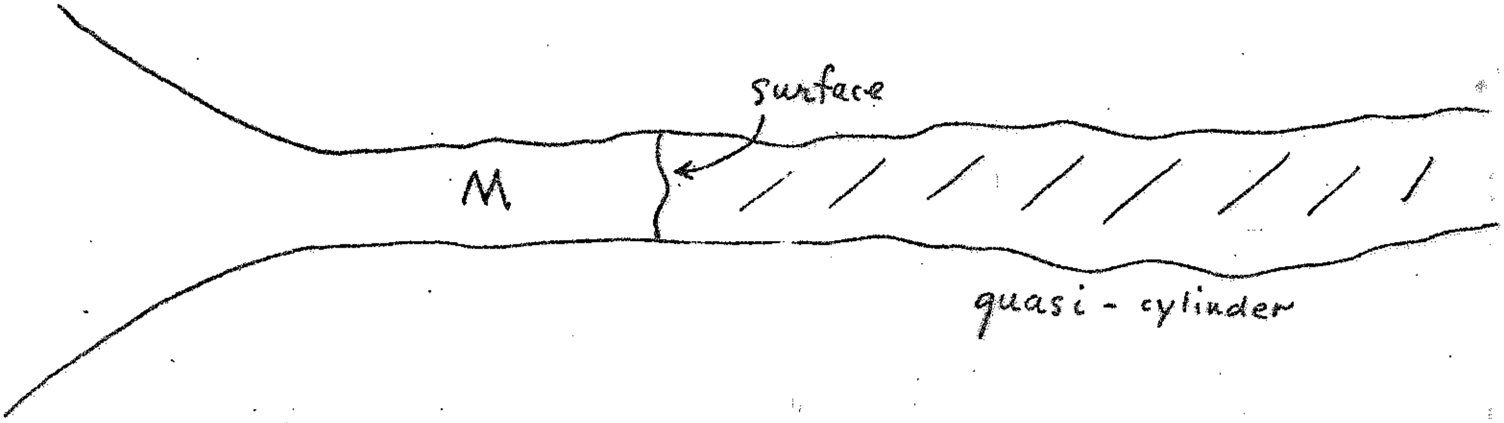
§4 The Critical Exponent of Certain Hyperbolic Manifolds

We consider 3-dimensional hyperbolic manifolds M which satisfy

i) $\pi_1(M)$ is isomorphic to the fundamental group of a compact surface of genus g .

ii) the limit set of $\Gamma = \pi_1 M$ in $\mathbb{C} \cup \infty$ is not all of the Riemann sphere.

iii) there is a separating surface in M so that the part of M on one side is a quasi-cylinder . (§3) .



We will explain in §7 how such manifolds (which we refer to as hyperbolic half cylinders) arise in Thurston's discussion of limits of geometrically finite quasi-fuchsian groups. There are uncountably many such manifolds so that no two have metrics related by a bounded distortion.

We recall the critical exponent $\delta(\Gamma)$ of $\Gamma = \pi_1 M \subset \{z \to \frac{az+b}{cz+d}\}$. By [S, §2, Corollary 4] this may be defined as the infimum (which is achieved) of real numbers α so that there is a finite measure μ on the limit set $\Lambda(\Gamma)$ satisfying

(*) $$\gamma^* \mu = |\gamma'|^\alpha \mu , \quad \gamma \in \Gamma .$$

Theorem 4 (Sullivan and Thurston): The critical exponent of a hyperbolic half cylinder is equal to two, $\delta(\Gamma) = 2$.

Proof: The proof makes use of one of the ideas from the existence proof of Thurston's hyperbolic structures on 3-manifolds which fibre over the circle.

Namely, we consider base points $x_1, x_2, \ldots$ which travel out the cylindrical end of M. Geometrically, a region of fixed radius about x_i converges to a region of that radius in a hyperbolic full cylinder (at least for a subsequence) (see [T, 9.1]).

We can apply Corollary 2.2 to see that the full cylinder supports no non-constant positive superharmonic function.

On the other hand a measure satisfying (*) for $0 < \alpha < 2$ determines a positive eigenfunction of the Laplacian with eigenvalue $\alpha(\alpha - 2)$; see [S, §7]. Such a function may be normalized to be one at x_i, and we can take a limit to obtain a positive eigenfunction on the hyperbolic full cylinder. Since $\alpha(\alpha - 2) \neq 0$ this limit function is not constant and superharmonic (since $\alpha(\alpha - 2) < 0$), a contradiction.

Remark: The critical exponent $\delta(\Gamma)$ is originally defined (see [S] for connections to previous work) as the critical exponent of the Poincaré series $g_s(x,y) = \sum_{\gamma\in\Gamma} \exp(-s \text{ distance } (x,\gamma g)$ where x and y lie in hyperbolic 3-space and $g_s(x,y)$ converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$.

§5 The Canonical Measure Associated to a Hyperbolic Half Cylinder

We assume Γ is the discrete group of linear fractional transformations determined by a hyperbolic half cylinder as defined in §4.

Theorem 5: There is on the limit set of Γ one and only one probability measure satisfying

$$\gamma^*\mu = |\gamma'|^2\mu , \quad \gamma \in \Gamma .$$

Proof: Since the critical exponent $\delta(\Gamma)$ equals 2 by theorem 4 at least one such measure exists (see §4, and [S, §1, §2]).

For uniqueness we show that any such μ is ergodic. For then if ν is another, so is $m = \frac{1}{2}(\mu + \nu)$, which is also ergodic. Then the Radon ratio of μ and m is constant. Thus $\mu = m$ since both are probability measures. Similarly $\nu = m$ and so $\mu = \nu$.

Now such a μ determines a Γ - invariant positive harmonic function h on hyperbolic 3 - space. If A is a Γ - invariant subset of the limit set which has positive μ , then μ/A also defines a positive Γ - invariant harmonic function h_A . If $\mu(A) < 1$, then along a μ positive set of rays the ratio h_A/h tends to zero. (More generally $h_A/h \to$ characteristic function of A for μ almost all rays.)

But this contradicts theorem 3 which implies any two non-constant positive harmonic functions are in a bounded ratio at all points of M_+ (which for definiteness we take to be the convex hull of the limit set of Γ modulo the action of Γ .) This proves theorem 5.

Corollary (Thurston): The area of the limit set of a hyperbolic half cylinder is equal to zero.

Proof: If the area were positive then the unique measure of theorem 5 would be Lebesgue measure. But then the associated positive harmonic function would be bounded, contradicting theorem 3.

§6 The Hausdorf Dimension of the Limit Set

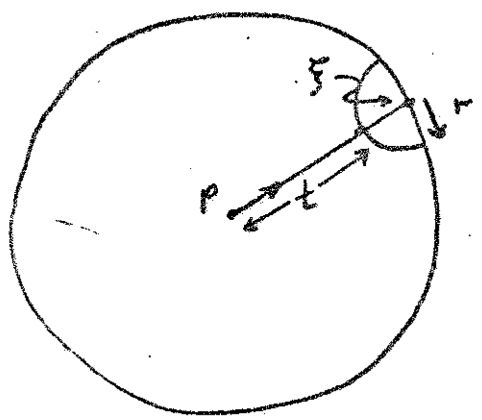
We continue studying the limit set of a hyperbolic half cylinder M using the canonical measure μ of §5 and the estimate on positive harmonic functions of §3. Let $\mu(\xi,r)$ denote the μ mass of a disk of radius r on the sphere centered at ξ in the limit set $\Lambda(\Gamma)$.

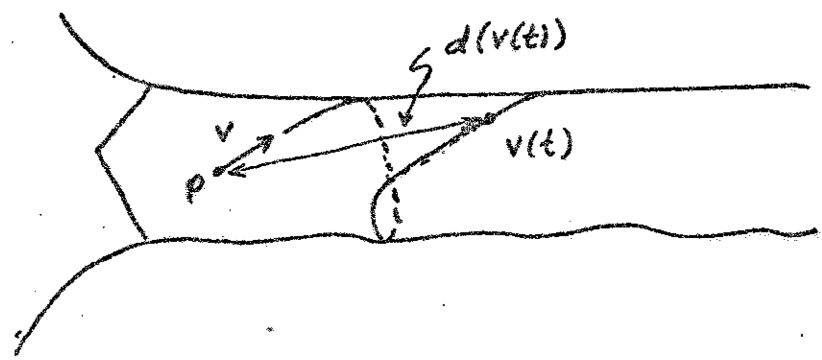
Theorem 6: We have the inequality

$$\mu(\xi,r) \leq \text{constant } r^2 \log 1/r$$

for all ξ, r.

Proof: Let the center of the ball model of hyperbolic 3-space corresponds to $p \in M$. We consider geodesics emanating from p. Let $d(v(t))$ denote the hyperbolic distance from p to $v(t)$, the point achieved after traveling time t starting in the direction v. Let $\xi = \xi(v)$ be the point on the sphere in the direction of v and $r = r(t)$ be e^{-t}.





Using the definition of h yields $\mu(\xi,r) \leq cr^2 h(v(t))$. (Since $h(v(t)) = \int |\gamma'|^2 d\mu$ where γ^{-1} is the hyperbolic isometry moving p along the geodesic towards ξ a distance t .)

Since $d(v(t)) \leq t$, $h(v(t)) \leq ct$ by theorem 3 which implies $h(v(t)) \leq c \log 1/r$. Thus $\mu(\xi,r) \leq$ constant $r^2 \log 1/r$ for all ξ and r . This proves theorem 6.

Corollary: The Hausdorf dimension of the limit set $\Lambda(\Gamma)$ is equal to two. In fact the Hausdorf measure of $\Lambda(\Gamma)$ relative to the gauge function $r^2 \log 1/r$ is positive.

Proof: If $\psi(r) = r^2 \log 1/r$ and $r_1, r_2, \ldots$ are the radii of any covering of $\Lambda(\Gamma)$ by balls of radii $r_1, r_2, \ldots$ and centers $\xi_1, \xi_2, \ldots$, then

$$1 = \mu(\Lambda(\Gamma)) \leq \sum_i \mu(\xi_i, r_i) \leq \sum_i \psi(r_i) .$$

By definition the Hausdorf measure H_ψ , which is constructed from infinium of such expressions, is ≥ 1 .

Clearly if $\varepsilon > 0$ $\psi(r) \leq r^{2-\varepsilon}$ eventually so the Hausdorf measure with gauge function $r^{2-\varepsilon}$ is also ≥ 1 . Thus the Hausdorf dimension $\geq 2 - \varepsilon$ for all $\varepsilon > 0$. This proves the corollary.

Remark: We suppose the Hausdorf measure for the gauge function $r^2 \log 1/r$ is actually infinity. We conjecture in fact that a finite positive Hausdorf measure can only result using the gauge function $r^2 (\log 1/r)^{1/2} (\log \log \log 1/r)^{1/2}$.

§7 Existence of Hyperbolic half Cylinders

A discrete subgroup of hyperbolic isometries isomorphic to π_1 (compact surface) is called quasi-fuchsian if the limit set is a topological circle. Bowen (1978) proved this circle is actually round or the Hausdorf dimension is > 1.

Now such groups are determined up to isomorphism, Bers (1965), by two points in Teichmuller space of the surface (corresponding to the two domains of discontinuity modulo the group). Moreover if one of the points in Teichmuller space approaches ∞ limit groups were constructed by Bers.

For example in Jorgensen's description [J] of the punctured torus case, Teichmuller space is the Poincaré disk and the geometry of the hyperbolic 3-manifold corresponding to the limit group is controlled by the tail of the continued fraction expansion of a limiting point on the boundary of the disk.

A hyperbolic half cylinder results (we ignore the cusp) iff the partial convergents are bounded. Thus there are uncountably many distinct examples but they form a set of measure zero in the space of all possible limits.

For the higher genus compact surface case there is an exactly analogous picture thanks to the geometric work of Thurston. The limit groups are labeled by an ending lamination on the $6g-7$ dimensional sphere Thurston boundary of Teichmuller space. There is a generalized continued fraction expansion for this ending lamination (see Kerchoff's proof of M. Keane's conjecture). A hyperbolic half cylinder conjecturally results when the convergents are bounded, and this is proven in infinitely many cases (e.g. periodic cases corresponding to fibred hyperbolic 3-manifolds).

Finally, how does the Hausdorf dimension of the quasi-fuchsian limit sets behave as the group limits on one of these hyperbolic half cylinders.

Theorem 7: *If Γ_t is a family of quasi-fuchsian surface groups varying continuously and converging algebraically when $t \to \infty$ to a hyperbolic half cylinder, and if D_t is the Hausdorf dimension of the quasi-circle limit set corresponding to Γ_t, then D_t varies continuously and $D_t \to 2$ as $t \to \infty$.*

Proof: If Γ_{t_1} and Γ_{t_2} are quasi conformally conjugate by a qc homeomorphism φ with small dilation, then D_{t_1} is close to D_{t_2} because there is a Hölder estimate for φ with exponent near 1.

This is what we mean by continuous variation of D_t.

Now we prove $D_t \to 2$. It is enough by [S, Theorem 7] to prove the critical exponents $\delta(\Gamma_t) \to 2$. By [T, Theorem 9.2] the limit sets of Γ_t converge in the sense of the Hausdorf metric to that of the limit group. If $\sup \delta(\Gamma_t) < 2$, we could construct a measure μ on the limit set of the limit group satisfying $\gamma^*\mu = |\gamma'|^\alpha \mu$ for $\alpha < 2$. This contradicts theorem 4 of this paper. Thus in fact for any sequence of $t_i \to \infty$ $\sup \delta(\Gamma_{t_i}) = 2$. This proves theorem 7.

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