HYPERBOLIC GEOMETRY AND HOMEOMORPHISMS

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The idea that one can radially identify Euclidean space R^n with the interior of the finite ball B^n has been very useful. In de Rham's book of 1955 a smoothing procedure for currents is based on the fact that the translation group of R^n damps out to the identity in a C^∞ fashion at the boundary of B^n. Thus familiar convolution in R^n can be coordinate wise placed on a general smooth manifold. In 1963 Connell's work on approximating a homeomorphism by a piecewise affine homeomorphism was based on the similar idea that a homeomorphism a bounded distance from the identity on R^n becomes the identity at the boundary at B^n. In 1968 the spectacular results of Kirby, Edwards-Kirby and Kirby-Siebenmann incorporated the Connell idea with the ingenious use of the torus T^n to arrive at periodic and thus bounded homeomorphisms of R^n. (See quotation in "radial engulfing" below.)

We now propose the idea that one may further profit from the idea that the ball B^n with its Euclidean geometry is conformally equivalent to hyperbolic space, the carrier of non-Euclidean geometry.

Analogous to de Rham, we note that heat diffusion in hyperbolic space damps out to the identity at the boundary of B^n and we can construct coordinate wise on general manifolds a smoothing procedure with conformal symmetry properties.

Analogous to Connell, we have the simple proposition that a quasi-isometry[1] of hyperbolic space which is a bounded hyperbolic distance from the identity determines a quasi-isometry of B^n (in its Euclidean metric) which is the identity at the boundary. This follows easily since the conformal factor is essentially the Euclidean distance to the boundary, namely,

(hyperbolic element)$\sim$*(Euclidean element)*.*(Euclidean distance to to boundary)*

Similarly, a bounded quasi-conformal[2] homeomorphism of hyperbolic space determines a quasi-conformal homeomorphism of the closed ball which is the identity at the boundary.

Analogous to the Kirby immersion torus devise we have the following

i) there are discrete groups of isometries of hyperbolic space so that the quotient is a compact manifold. (This is classical but non-trivial for $n > 5$, see "hyperbolic space forms" below.)

ii) for each of these there is a finite cover which is parallelizable in the complement of a point. (At the moment the proof of this fact makes use of deep properties of etale homotopy theory in characteristic p, see below.)

[1]*A homeomorphism which distorts the metric by only a bounded amount. Or φ and φ^{-1} satisfy a uniform Lipschitz condition.*

[2]*A homeomorphism which distorts the conformal geometry by only a bounded amount.*

iii) thus we have desired "hyperbolic manifolds" in each dimension $2, 3, 4, \ldots, n, \ldots$ to apply Kirby's immersion device.

Namely, we have M^n so that $M^n - pt$ immerses in R^n, M^n is compact, and the universial cover of M^n is hyperbolic space. We note these manifolds are not related in simple inductive manner as are the torii T^n. For $n = 2$, any surface of higher genus works here.

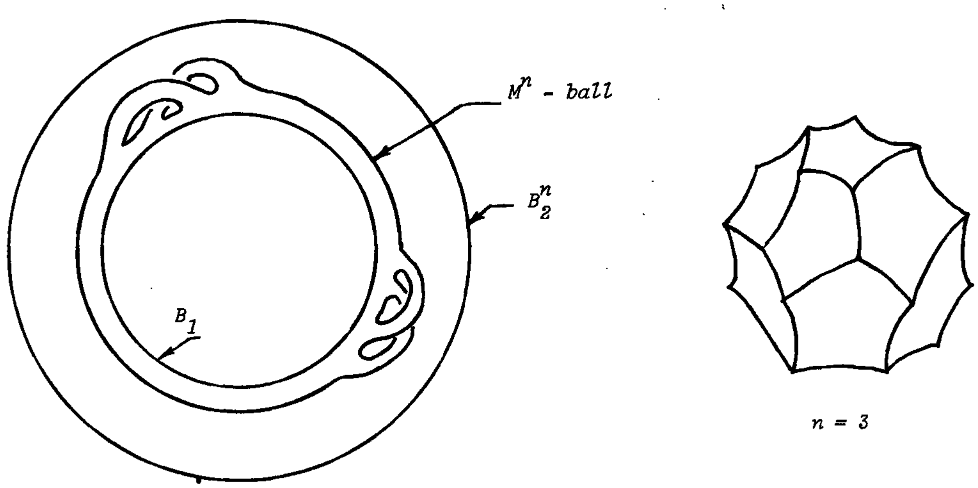

Fig. 1.

For $n = 3$ one can imagine a discrete group defined by reflecting a symmetrical dodecahedion with dihedral angles equal to $\pi/2$.

iv) Now following Kirby (see figure 1) we can immerse M^n - ball in B_2^n and extend a homeomorphism defined near image $(M^n$ - ball$)$ and sufficiently close to the identity to approximately M^n - ball. Thinking of the deleted ball in polar coordination we can furl (see "furling" below) to obtain commutation with a radial homothety and extend over the ball by infinite repetition. We have extended the quasi-isometry defined near B_1 and close to the identity to a global quasi-isometry of M close to the identity.

We lift to hyperbolic space apply the remark above about bounded quasi-isometries of hyperbolic space to obtain a quasi isometry of B^n which is the identity at the bdry and containing the original on B_1. This is the 0-handle case required for the construction of isotopies as in the Edwards - Kirby, paper (Annals of Math 93 (1971) pp. 63-88.)

v) For the k-handle case we choose a group as in iv) for hyperbolic space of dimension $(n - k)$. We extend this group by geodesic perpendicularity to a group in hyperbolic n-space. Then the fundamental domain has the form $D^k \times D^{n-k}$ ($k = 1$, $n = 3$ is shown in figure 2)

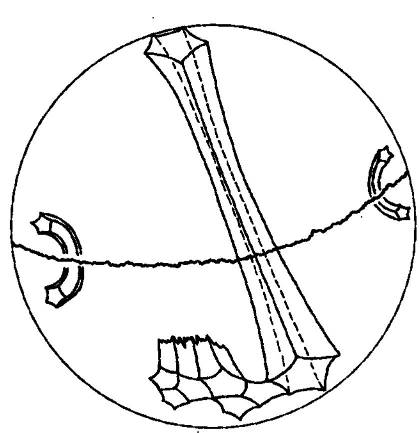

Fig. 2.

Then we treat a homeomorphism defined near the core of a k-handle which is close to the identity and equal to the identity near the bdry. Firstly, we use the Kirby immersion devise as in iv) to obtain an extension to the (fundamental domain - $(n-k)$ ball) x D^k compatible with the group. Then since we have the identity near the bdry we again have an n-dimensional hole which can be filled in by furling and infinite repetition as in iv). We extend by the group and apply the bounded quasi-isometry remark to obtain a quasi-isometry of B^n which is the identity at the bdry and agrees with the original near the core of the k-handle.

Using a fine handle decomposition and iv) and v) inductively we find that in the context of quasi-isometries or quasi-conformal homeomorphisms of any smooth manifold

THEOREM 1. a) A homeomorphism near a compact domain and pt wise close to the identity can be extended to a global homeomorphism which is the identity outside a slightly larger neighborhood.

b) Sufficiently close homeomorphisms are connected by a path of homeomorphism, that is they are isotopies.

c) The construction of isotopics is local, relative, compatible with parameters, etc., that is we have the Cernavskii, Kirby-Edwards isotopy theory)

To have the proof for quasi-conformal also we note that everything works the same. We note also there is the stronger extension fact (Gehring) that any quasi-conformal mapping of interior B^n extends to the quasi-conformal on the closed B^n and the boundary maps is also quasi-conformal in dim $n - 1$

Remark. The schoenflies theorem (a collared $(n - 1)$ sphere in S^n bounds a ball), the Annulus theorem (the region between two collared $(n - 1)$ spheres is S^{n-1} x I) and the component problem (an orientation preserving homeomorphism of a ball into R^n is connected to the identity by a path of homeomorphisms) come out nicely in these quasi conformal and quasi isometric contexts.

The isotopy extension of Theorem 1 shows a) connected sum is well defined which allows one to easily give Mazur's proof of Schoenflies with the infinite bad point being homothetic and all right for the quasi contexts, b) the component problem follows since we have derivatives at almost all points. Thus we can use Milnor's isotopy $\{e^t \cdot \varphi e^{-t}\}$ near a point to change the map to nearly its derivative and extend this change by theorem 1. Firstly, as usual Schoenflies and the component problem implies the Annulus theorem given isotopy extension (Theorem 1). So we have

COROLLARY 1. In the quasi-conformal and quasi-isometry contexts we have the Schoenflies theorem, the Annulus theorem, and the component problem in all dimensions.

Of course in the topological context the Annulus theorem is unknown in dimension 4 (while Schoenflies is known) and in the *pl* or smooth context Schoenflies is unknown in dimension 4 (while the component problem is known). Only part of the corollary is new (see Väisälä).

Another corollary is the following - by the above quasi-conformal homeomorphisms of R^n are stable and thus (using Connell) approximable by piecewise affine homeomorphism in dimensions ≥ 5, (which demonstration we have reached without using Kirby's Annulus theorem).

Now we will change terminology and refer to quasi-isometries as Lipschitz homeomorphisms. We want to discuss approximating arbitrary homeomorphism by Lipschitz homeomorphism. For dimensions less than 4 there is a good classical theory (Moise).

Dimension 4 is unknown and remains so. For dimensions greater than 4 we get full positive results using Connell's radial engulfing and Kirby's Annulus theorem.

As usual it will suffice to study a homeomorphism of a handle $D^k \times D^{n-k}$ into R^n which is Lipschitz near $(\partial D^k \times pt)$. Using Connell's radial engulfing (see below) we spread the Lipschitz property across the k-handle at the cost of an ϵ discrepancy on nghd $(D^{k-1} \subset \partial D^k)$. This discrepancy is Lipschitz and can by Theorem 1 or v) be extended to a Lipschitz homeomorphism which is the identity outside a small nghd of the discrepancy. The composition of the latter with the former gives the desired relative Lipschitz approximation on the handle.

Applying this result inductively to a fine handle decomposition we have the

THEOREM 2. A homeomorphism defined near a compact connected region L of R^n, $n \neq 4$, into R^n which is Lipschitz near a non void sub compact K can be approximated pt wise by a Lipschitz homeomorphism which is unchanged near K.

Remark. We need K to be non void to do the zero handles. If there is no K we use Kirby's annulus theorem for the zero handles to obtain

COROLLARY 2. Theorem 2 is also true when K is void

COROLLARY 3. Topological manifolds of dimension $\neq 4$ have Lipschitz coordinate systems. Such locally Euclidean Lipschitz structures are unique up to homeomorphism close to the identity.

Corollary 3 follows by standard arguments using Theorem 2 and corollary 2 and fine handle decompositions in coordinate neighborhoods.

CONCLUDING REMARKS

The unique Lipschitz structure on topological manifolds allows one certain geometrical and analytical methods. The discussions in Whitney's book "Geometric Integration" and Federer's book "Geometric measure Theory" are invariant under Lipschitz changes of coordinates.

Thus we have the theory of Hausdorf measure and dimension on k-dimensional subsets $0 \leq k \leq n$. This implies general position, (intersection empty) and complementary dimension transversality (zero dimensional intersection) and almost everywhere transversality (general dimension) on the level of cycles or currents, (federer).

We have also a class of differential forms closed under wedge product, exterior differentiation, and transforming by Lipschitz mappings. These Whitney forms (called flat forms in his book) provided the original motivation[1] for wanting a Lipschitz structure. Locally Whitney forms are defined by the property that they and their exterior derivative (in the sense of distributions) are forms with bounded measurable coefficients. Dualizing Whitney forms give us currents.

On a Lipschitz manifold the L^2-spaces of forms are well defined (as pre-Hilbert spaces) with d as a closed unbounded

[1] *Also very important was Siebenmann's observation (1970) that his counterexample to the pl Hautvermutung was not a counterexample to the analogous Lipschitz statement.*

operator. One can contemplate an index theory which would lead to a new analytic proof of Norvikov's theory about Pontryagin classes. There is a local Gauss Bonnett formula $\chi_M = \int_M U/\text{diagonal}$ where U is a locally defined n form on $M \times M$ defined for example by smoothing the diagonal current (p. 80 de Rham). Furthermore, this equation can be proved directly and analytically taking as χ_M the Euler characteristic for Whitney form cohomology.

There are many other mathematical contexts with a topological component where hyperbolic space and its bdry plays an interesting and important role. One might mention

i) Nielson theory - providing a natural homeomorphism

$$\text{Aut } (\pi_1 \text{ (surfaces)}) \to (\text{quasi-conformal homeo } S^1)$$

ii) Teichmuller space and its boundary

iii) Thurston's canonical surface transformations

iv) Kleinian groups and non-compact hyperbolic 3-manifolds

v) Thurston's canonical hyperbolic structure on many compact 3-manifolds

vi) Anosov flows (geodesic flow on negative curved manifolds) their structural stability, ergodicity, etc.

vii) Mostow's theorem on the rigidity of hyperbolic space forms of finite volume $(n \geq 3)$ and Gehring's quasi-conformal mappings in space

viii) Fefferman's theorem on complex analytic homeomorphisms being smooth up to the boundary.

ix) Margulis and Gromov theorems in discrete groups and Riemannian geometry

x) Furstenburg's work on random walks and Poisson boundary (Corresponding to the beautiful fact that a random continuous path hits a definite point at ∞ with probability 1.)

At the time of the research it was mainly i), vi), and vii) which motivated the ideas here. It is a pleasure to acknowledge the inspiring discussions with Gromov and Thurston on hyperbolic geometry and Edwards and Siebenmann on homeomorphisms.

Further topological references may be found in the body and bibliography of the book by Kirby and Siebenmann "Foundational Essays on Topological manifolds, smoothings, and triangulations." Annals of Math Studies Princeton Univ. Press 1977. For quasi-conformal mappings the body and bibliography of Mostow "Quasi-conformal mappings in n-space and the regidity of hyperbolic space forms" Publications Mathematiques I.H.E.S. No. 34 (1968): (is a good source).

FURLING: M. Brown 1964 unpublished. Also L. Siebenmann, "Pseudo Annuli and invertible cobordisms", Archiv der Math. 19 (1968) 28-35.

THEOREM. If X and Y are compact Hausdorf spaces and h is a almost vertical homeomorphism of $X \times I$ onto a nghd of $Y \times 0$ in $Y \times R$, then there is a homeomorphism g so that the following commutes

$$\begin{array}{ccc} X \times I' & \xrightarrow{h} & Y \times R \\ \downarrow i & & \downarrow j \\ X \times S^1 & \xrightarrow{g} & Y \times S^1 \end{array}$$

Remark. The proof is ingenious but only two lines. If X and Y are metric and h is a quasi-isometry so is g. Similar remarks apply to the quasi-conformal context. Here I is an interval, is a smaller interval, I' is determined by an inclusion $I' \subset S^1$ and j by the projection $R \to S^1$.

HYPERBOLIC SPACE FORMS (Compact and almost parallelizable)

i) Let Γ denote matrices preserving the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}\, x_{n+1}^2$ with entries of the form $n + m\sqrt{2}$, n,m integers. Then Γ is a group of isometries of hyperbolic space, which is discrete and has compact quotient. All this follows easily using the field automorphism of $Q\sqrt{2}$, and the compactness criterion for lattices in R^n - minimal separation bounded from below and volume bounded from above. (Algebraic group theory is not required as it is for the more general result of Borel's paper in the first volume of Topology.)

A subgroup of finite index (e.g. $(n + m\sqrt{2})_{ij} \equiv \delta_{ij} \bmod 3$) in Γ will have no torsion and define a manifold. This type of construction is the only known way to get compact hyperbolic space forms in n-dimesnions.

ii) It is not clear when such manifolds are almost parallelizable. They are not parallelizable in even dimensions and their analogues in complex hyperbolic space have non-trivial rational Pontryagin classes. However, there is a fortunitous general result which saves us. Namely, "Over a finite polyhedron a vector bundle with *discrete* structure group in a *real orthogonal* group becomes continuously trivial in some finite cover." We can apply this result because a real line bundle over our manifold is the quotient of the positive light cone by the group. Thus its tangent bundle is trivial in some finite cover by the general result. It follows that *some finite cover of a hyperbolic space form is stably parallelizable* (which is even more than almost parallelizable by an elementory obstruction argument).

Now the general result follows because the complexification of a real orthogonal group $O(p,q)$ has the homotopy type of $O(n)$, $n = p + q$. Etale homotopy theory in finite characteristics can be applied (as in Deligne, Sullivan "Fibres complex ..." Compte Rendu t. 281) p. 1081 (1975), to study principal $O(n,\mathbb{C})$ bundles with discrete structure groups (and see they become trivial in finite covers).

The role of the orthogonal group here is crucial not only for the proof but also for the truth of the assertion about bundles. For Millson has recently constructed flat bundles with group $Sl(n,r)$ so that the second Stiefel Whitney class is non-trivial in every finite cover.

CONNELL'S RADIAL ENGULFING: This is lemma 3 of E. H. Connell "Approximating stable homeomorphisms by piecewise linear ones." Annals of Maths 78 (1963) pp. 326-338. It is stated there for $n \geq 7$ where engulfing in codimension 4 follows from straightforward general position and induction. The result is just as true for $n \geq 5$ by the more complicated double induction argument needed for engulfing in codimension 3. This was explained to me by Siebenmann but I don't know a reference.

THEOREM. On Euclidean space of dimension ≥ 5 there is a homeomorphism which is the identity on one ball and stretches a larger ball over a third larger ball, which almost preserves radii, and which is piecewise linear relative to any given structure.

It is perhaps historically interesting to add the last para- of this prescient and relevant paper.

"Suppose T_1 and T_2 are two arbitrary pl structures on E^n. It is known (except in dimension 4) that $\exists$ a homeomorphism $h : E \to E$ which is pl from T_1 to T_2 (see Stallings and Moise). If h could be chosen as a bounded homeomorphism, then by Lemma 5 h would be stable and it would follow that all orientation preserving homeomorphisms are stable. Thus *the annulus conjecture in dimension* , *would be true.* Conversely, if the annulus conjecture were true in all dimensions it would follow from the procedures of this paper that (for $n \geq 7$) h could be chosen to be bounded. Thus *the annulus conjecture is roughly equivalent to this strong form of the Hauptvermutung* for Euclidean space where h is to be chosen as bounded from the identity."

In light of the fact that five years later Kirby (1968) reduced the Annulus conjecture to a "periodic Hauptvermutung" for Euclidean space (and the latter was sufficiently true by surgery (1966) to finish) these remarks of Connell seem almost prophetic.