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THE HOMOLOGY THEORY OF THE CLOSED GEODESIC PROBLEM

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The problem—*does every closed Riemannian manifold of dimension greater than one have infinitely many geometrically distinct periodic geodesics*—has received much attention. An affirmative answer is easy to obtain if the fundamental group of the manifold has infinitely many conjugacy classes, (up to powers). One merely needs to look at the closed curve representations with minimal length.

In this paper we consider the opposite case of a *finite* fundamental group and show by using algebraic calculations and the work of Gromoll-Meyer [2] that *the answer is again affirmative if the real cohomology ring of the manifold or any of its covers requires at least two generators.*

The calculations are based on a method of the second author sketched in [6], and he wishes to acknowledge the motivation provided by conversations with Gromoll in 1967 who pointed out the surprising fact that the available techniques of algebraic topology loop spaces, spectral sequences, and so forth seemed inadequate to handle the "rational problem" of calculating the Betti numbers of the space of all closed curves on a manifold. He also acknowledges the help given by John Morgan in understanding what goes on here.

The Gromoll-Meyer theorem which uses nongeneric Morse theory asserts the following. Let $\Lambda(M)$ denote the space of all maps of the circle S^1 into M (not based). Then there are *infinitely many* geometrically distinct periodic geodesics in *any* metric on M *if* the Betti numbers of $\Lambda(M)$ are unbounded. (The round 2-sphere shows the condition is not actually necessary.)

In [6] a description of the *minimal model of* $\Lambda(M)$ is given in terms of the minimal model of M, and is valid if, for instance, M is simply connected. This gives an explicit algorithm for calculating the Betti numbers of $\Lambda(M)$. This algorithm yields the fact that the Betti numbers of $\Lambda(M)$ are always nonzero in an infinite number of dimensions (which can be taken to lie in an arithmetic sequence.)

We explicate and extend this study of $\Lambda(M)$ here by

(i) giving the description and proof of the algorithm for the homology when $\pi_1 M = \{e\}$, and by

(ii) showing that the Betti numbers of $\Lambda(M)$ are unbounded if and only if

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the real cohomology ring of M requires at least two generators.

Thus the question of infinitely many geodesics is still open when either

(i) $\pi_1(M)$ is infinite but has only finitely many conjugacy classes (up to powers), or

(ii) $\pi_1 M$ is finite but the real cohomology of the universal cover is of the form

$$\{1, x, x^2, x^3, \cdots, x^n\} \quad \text{with} \quad x^{n+1} = 0\,;$$

for example, spheres and projective spaces have these properties.

In the latter case, Klingenberg [4] claim the conjecture for the *generic* metric, and we include a specific calculation of the rational cohomology ring of $\Lambda(M)$ for possible use in a further Morse study for the special metric.

Remark on covers. If $\tilde{M} \to M$ is a finite regular cover, then the cohomology ring of M is isomorphic to the subring of $H^*\tilde{M}$ which is fixed by the finite group of deck transformations. If $H^*(\tilde{M})$ has one generator an easy argument shows H^*M has one generator. Thus, if $\pi_1 M$ is finite the cohomology ring of M or one of its covers requires two generators if and only if this is so for the universal cover.

Also an easy geometric argument shows the property of having infinitely many distinct periodic geodesics is shared by a manifold and its finite covers. Thus in the study of the finite π_1 case in the calculations below we may actually assume $\pi_1 M = \{e\}$ and the cohomology ring requires at least two generators. We will see that the Betti numbers of $\Lambda(M)$ are unbounded and the asserted theorem will follow.

The description of the space of all closed curves on M. In [6] and [7] an algebraic description of homotopy problems via differential algebras and differential forms was given. The nature of this description is such that if a proferred formula for $\Lambda(M)$ has the correct algebraic properties it must be correct. This method works here and in other contexts as well, for example in the study of Gelfand-Fuks cohomology [7].

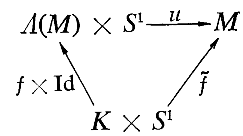
To each simply connected space M (or even a nonsimply connected space of nilpotent homotopy type) the theory associates (see [6], [7], [1]) a special differential algebra over $\boldsymbol{Q}$ (or $\boldsymbol{R}$) which describes its rational (or real) homotopy type, [8]. The cohomology of this special differential algebra, called the minimal model of M, is the cohomology of M, and the generators of the algebra (which is free of relations besides graded commutativity) give a dual basis of the rational homotopy groups of M. If X is any space, the homotopy classes of maps of X into the rational homotopy type M_Q of M, [8], is in one to one correspondence with the homotopy classes of maps of the minimal model of M into the rational de Rham complex of X. By a homotopy between two maps of differential graded algebras $\mathscr{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{B}$ we mean a dga map $\mathscr{A} \xrightarrow{H} \mathscr{B}(t, dt)$

where $\mathscr{B}(t, dt)$ means $\mathscr{B}$ with a variable t adjoined in degree zero and dt is its differential in degree one, and f and g are obtained by composing H with the two evaluations $\mathscr{B}(t, dt) \rightrightarrows \mathscr{B}$, obtained by setting $t = 0$, $dt = 0$ and $t = 1$, $dt = 0$; see [1]. In the statement about maps the forms on X may be replaced by any other dga which maps to the forms by a map inducing an isomorphism on cohomology. If X is nilpotent the minimal model of X itself is one such dga.

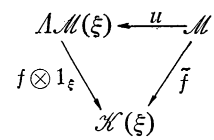
Now consider $\Lambda(M)$ the space of all maps of the circle into M. A map f of an arbitrary space K into $\Lambda(M)$ is the same as a map $\tilde{f}$ of $K \times S^1$ into M. In fact we have a universal map

$$\Lambda(M) \times S^1 \xrightarrow{u} M$$

and a commutative diagram determining this correspondence between f and $\tilde{f}$:



A correct formula for the minimal model of $\Lambda(M)$ will be given by the dga which bears the analogous relation to $\mathscr{M}$, the minimal model of M. In fact, we look for a dga $\Lambda\mathscr{M}$ and a universal dga map $\mathscr{M} \xrightarrow{u} \Lambda\mathscr{M}(\xi)$ (where $\mathscr{A}(\xi)$ means adjoining a closed one-dimensional generator to the dga $\mathscr{A}$) so that the universal property expressed by



$\mathscr{K}$ arbitrary

is satisfied. That is, dga maps $\mathscr{M} \xrightarrow{\tilde{f}} \mathscr{K}(\xi)$ correspond to dga maps $\Lambda(\mathscr{M}) \xrightarrow{f} \mathscr{K}$ in a one to one fashion, the connection being provided by the commutative diagram.

We proceed as follows. If $\mathscr{M}$ is the free commutative algebra $\Lambda(x_1, x_2, \cdots; d)$ on $x_1, x_2, \cdots$ with differential d, let $\Lambda\mathscr{M}$ have generators $x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots$ where $\dim \bar{x}_i = \dim x_i - 1$. Here it is convenient that $\mathscr{M}$ is simply connected and has no generators in degree one, in which case more discussion is required [9]. Note that the symbol "Λ" is used in three distinct ways.

The universal map $\mathscr{M} \xrightarrow{u} \Lambda(\mathscr{M})(\xi)$ is defined by $x_i \rightarrow x_i + \xi\bar{x}_i$. We must define $\bar{d}$ in $\Lambda(\mathscr{M})$ so that u is a dga map. Notice that if $x \rightarrow x + \xi\bar{x}$ and $y \rightarrow y + \xi\bar{y}$, then

$$xy \to (x + \xi\bar{x})\cdot(y + \xi\bar{y}) = xy + \xi(\bar{x}y + (-1)^{|x|}x\bar{y}) ,$$

where $|x| = \dim x$. So $x \to$ coefficient of ξ in $u(x)$ defines a derivation $\mathcal{M} \xrightarrow{i} \Lambda\mathcal{M}$ of degree minus one over the natural inclusion $\mathcal{M} \subset \Lambda\mathcal{M}$.

If we write u in the form

$$x \to x + \xi\cdot i(x) ,$$

then $dx \to dx + \xi\cdot i(dx)$ and for u to commute with d we must have

$$dx + \xi\cdot i(dx) = \bar{d}(x + \xi\cdot i(x)) .$$

This is equivalent to the two equations:

$$\bar{d}x = dx ,$$
$$i\cdot(dx) + \bar{d}(ix) = 0 , \qquad \text{for all } x \in \mathcal{M} .$$

Thus we define $\bar{d}$ in $\Lambda(\mathcal{M})$ so that $\mathcal{M}$ is a differential subcomplex and the relation $\bar{d}i + id = 0$ holds. Hence we obtain the following result:

Given the dga $\mathcal{M} = \Lambda(x_1, x_2, \cdots, d)$, *define the* dga $\Lambda\mathcal{M} = \Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots, \bar{d})$, *with* $\dim \bar{x}_i = \dim x_i - 1$, *by* $\bar{d}x_i = dx_i$ *and* $\bar{d}\bar{x}_i = -idx_i$ *where* $i\colon \mathcal{M} \to \Lambda\mathcal{M}$ *is the unique derivation of degree* -1 *extending* $x_i \to \bar{x}_i$. *Then the map* $\mathcal{M} \xrightarrow{u} \Lambda(\mathcal{M})(\xi)$ *defined by* $x_i \to x_i + \xi\bar{x}_i$ *is a* dga *map which is universal and sets up a one to one correspondence between* dga *maps* $\mathcal{M} \xrightarrow{\bar{f}} \mathcal{K}(\xi)$ *and* $\Lambda\mathcal{M} \xrightarrow{f} \mathcal{K}$ *via the diagram*:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{u} & \Lambda(\mathcal{M})(\xi) \\ & {\scriptstyle \bar{f}}\searrow \quad \swarrow{\scriptstyle f\otimes 1_\xi} & \\ & \mathcal{K}(\xi) & \end{array} \qquad \mathcal{K} \text{ arbitrary}$$

At this point we could pass to homotopy classes of dga maps, and assert that $\Lambda\mathcal{M}$ defines the correct homotopy functor and so must be the minimal model of $\Lambda(M)$. We will detail a more explicit argument.

The universal geometric map $\Lambda(M) \times S^1 \to M$ corresponds to a map of minimal models $\mathcal{M} \xrightarrow{g} \mathcal{M}(\Lambda(M))(\xi)$ implying by the above algebraic universality the existence of a dga map

$$\Lambda(\mathcal{M}) \xrightarrow{\pi} \mathcal{M}(\Lambda(M)) .$$

Now since M is simply connected, $\Lambda(M)$ is at least nilpotent [8], so the homotopy groups of $\Lambda(M)$ correspond to the free generators of $\mathcal{M}(\Lambda(M))$. Now the fibration $\Omega M \to \Lambda(M) \to M$, where ΩM is the based loop space, has a natural cross section, the constant maps of S^1 into M, so that $\pi_i\Lambda(M) \approx \pi_i\Omega M \oplus \pi_i M$. Of course, $\pi_i\Omega M \approx \pi_{i+1}M$ so that $\mathscr{M}(\Lambda(M))$ is the free algebra on the generators x_i of $\mathscr{M}$ (which is a d-subalgebra of $\mathscr{M}(\Lambda(M))$) and another set y_i obtained by shifting the x_i down one:

$$\mathscr{M}(\Lambda(M)) = \Lambda(x_1, y_1, x_2, y_2, \cdots) ,$$

with $\dim y_i = \dim x_i - 1$. If one examines the minimal model form of the universal geometric map $\mathscr{M} \xrightarrow{g} \mathscr{M}(\Lambda(M))(\xi)$, one finds that in terms of all these identifications $x_i \to x_i + \xi y_i$ modulo decomposables of $\mathscr{M}$. It follows that the map $\Lambda(\mathscr{M}) \xrightarrow{\pi} \mathscr{M}(\Lambda(M))$ described above must be an isomorphism.

In summary we state the

Theorem. *If M is a simply connected finite complex, and M has minimal model $\Lambda(x_1, x_2, \cdots, d)$ then the space of closed curves on M has model $\Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots, d)$ where $\dim \bar{x}_i = \dim x_i - 1$, d is defined by $di + id = 0$, and i is the derivation of $\Lambda(x_1, x_2, \cdots)$ into $\Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots)$ defined by $ix_j = \bar{x}_j$.*

Remark. Note that $\Lambda(x_1, x_2, \cdots)$ is a d-subcomplex of $\Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots)$, and the image of d in $\Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots)$ is contained in the ideal of $(x_1, x_2, \cdots)$. Thus the induced d in $\Lambda(\bar{x}_i)$ is zero. This algebraic picture corresponds to the natural fibration

$$\Omega M \to \Lambda(M) \to M$$

since the model of $\Omega(M)$ is $\Lambda(\bar{x}_1, \bar{x}_2, \cdots; d = 0)$. (The minimal model of any H-space is just the free algebra in the dual homotopy groups with the differential identically zero.)

In the following, we use the notation (Λ, d) for the minimal model of a simply connected finite complex M, and (Λ', d') for the minimal model of the space of all maps of S^1 into M. The generators in even (resp. odd) degrees will be denoted by $x_1, x_2, \cdots$ (resp. $y_1, y_2, \cdots$). We shall prove the

Theorem. *The following properties are equivalent:*

(i) *the cohomology algebra of M requires at least two generators,*

(ii) *the Betti numbers of the space of all maps of S^1 into M are unbounded.*

Before proceeding to the details, we give the structure of the argument. Suppose the generators of the model of M (arranged by degree) begin with

$$x_1, \cdots, x_n; y_1; x_{n+1}, \cdots, x_r; y_2, \cdots$$

where y_1 and y_2 are the first two exterior generators, and the x_i are the first polynomial generators.

In Λ', the classes $\left\{\left(\prod_{i=1}^{n} \bar{x}_i\right)\bar{y}_1^{\alpha}\right\}$, $\alpha = 1, 2, \cdots$, show that for any nontrivial finite complex, $H^*(\Lambda')$ is nonzero in an infinity of dimensions, [6]. This uses

$dx_i = 0$, $i = 1, \cdots, n$ and $dy_1 = Q(x_1, \cdots, x_n)$. If we knew $dx_j = 0$ for $j = n+1, \cdots, r$, then the classes $\left\{\left(\prod_{i=1}^{r} \bar{x}_i\right)\bar{y}_1^\alpha \bar{y}_2^\beta\right\}$ with integers α and β would give us the unbounded Betti numbers. (y_1 must be present if $H^*(M)$ is finite dimensional and distinct from the ground field, and y_1 and y_2 must be present except when H^*M is generated by one element (Propositions 1, 2, 3).

If $dy_1 \neq 0$, then $d^2 = 0$, $dx_1 = 0, \cdots, dx_n = 0$ and $dx_{n+1} = Q_{n+1}(x_1, \cdots, x_n)y_1, \cdots, dx_r = Q_r(x_1, \cdots, x_{r-1})y_1$ imply $Q_j = 0$, $j = n+1, \cdots, r$, and the above argument works. If $dy_1 = 0$, this argument does not work. However, if dy_2 were also zero, we could use the classes $\{\bar{y}_1^\alpha \bar{y}_2^\beta\}$ with integers α, β to obtain enough cohomology. We do know $dy_2 = Q(x_1, \cdots, x_r)$ so that $d\bar{y}_2 = \sum_i Q_i(x_j, \bar{x}_j)x_i$ in Λ' by the formula. If we set $x_i = 0$, $i = 1, \cdots, r$ in Λ', $d\bar{y}_2$ becomes zero, the above classes work, and an inductive argument shows this quotient differential graded algebra has unbounded Betti numbers only if Λ' does (Proposition 4).

1. Some results about differential graded algebras

We will consider differential graded algebras $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ over a ground field $k = \boldsymbol{Q}$ or $\boldsymbol{R}$, endowed with a differential d of degree one. These algebras are the tensor product of a polynomial algebra graded in even degrees and an exterior algebra generated in odd degrees. We assume that $\Lambda_0 = k$, and that, for every $z \in \Lambda$, dz belongs to $\Lambda^+ \cdot \Lambda^+$ where $\Lambda^+ = \bigoplus_{n>0} \Lambda_n$. Actually, if there are generators in degree one, we assume in addition that all the generators can be ordered $z_1, z_2, \cdots$ so that each dz_i is a polynomial $Q_i(z_1, z_2, \cdots, z_{i-1})$ in earlier generators. If a finite complex M is simply connected, the minimal models of M and $\Lambda(M)$ satisfy these conditions.

Proposition 1. *Let (Λ, d) be such a differential graded algebra. We have the following equivalences:*

(1) *the cohomology algebra $H^*(\Lambda, d)$ is generated by one element,*

(2) *(Λ, d) has one of the following types:*

(a) *Λ is generated by one element,*

(b) *Λ is generated by two elements x and y, where x is a polynomial generator with $dx = 0$, and y is an exterior generator with $dy = \lambda x^h$, $\lambda \in k^*$, and an integer $h \geq 2$.*

Proof. (2) $\Rightarrow$ (1) is easy. If $\Lambda = k(z)$, then $H^*(\Lambda) = \Lambda = k(z)$. If $\Lambda = k[x] \otimes k(y)$, then $H^*(\Lambda) = \dfrac{k[x]}{x^h k[x]}$. To show (1) $\Rightarrow$ (2), we assume first that the generators of lowest degree of Λ are exterior. By (1) there is only one such generator and we have $dy = 0$. Another generator of Λ of lowest degree is also closed, again a contradiction to (1). So we have $\Lambda = k(y)$.

Assume now that the generators of lowest degree of Λ are polynomial. Again there is only one such generator, say x_1, and $dx_1 = 0$. If Λ has other generators, we consider the generators of lowest degree among the generators distinct from x_1. If these generators are polynomial $x_2, \cdots$, we have $dx_2 = 0$ and this is not possible by (1). If these generators are exterior, let y_1 be one of them. Then we have $dy_1 = \lambda_1 x_1^h$, where $\lambda_1 \in k^*$, and h is an integer ≥ 2. In the set of generators distinct from x_1 and y_1, we look at the generators of lowest degree. If such a generator x_2 is of even degree, we have $dx_2 = \mu_2 x_1^\alpha y_1$, with $\mu_2 \in k$, $\alpha \geq 1$. Since $d^2 x_2 = 0$, we have $0 = \mu_2 \lambda_1 x_1^{\alpha+h}$, which implies that $dx_2 = 0$ and contradicts (1). Therefore, if the set of generators of Λ distinct from x_1 and y_1 is not empty, the generators of lowest degree in this set are exterior. Let y_2 be one of them. Then we have $dy_2 = \lambda_2 x_1^m$, where m is an integer ≥ 2 and $\lambda_2 \in k$. It follows that $d(y_2 - (\lambda_2/\lambda_1)x_1^{m-h}y_1) = 0$. Then $H^*(\Lambda)$ should contain the class of x_1 and the class of $y_2 - (\lambda_2/\lambda_1)x_1^{m-h}y_1$, contradicting (1). So we have

$$\Lambda = k[x_1] \otimes k(y_1) \qquad \text{with } dx_1 = 0 \ , \ dy_1 = \lambda_1 x^h \neq 0 \ .$$

Proposition 2. *Let (Λ, d) be a differential graded algebra. Let $\mathcal{O}$ be the ideal of Λ, generated by the exterior generators, and let $A = \Lambda/\mathcal{O}$. If y is an exterior generator of Λ such that the image of dy in A is nonzero, we have $H^*(\bar{\Lambda}, \bar{d}) = H^*(\Lambda, d)$, where $\bar{\Lambda} = \Lambda/(y, dy)\Lambda$, and $\bar{d}$ is the induced differential on $\bar{\Lambda}$.*

Proof. See also [3]. Let ξ be an element of Λ such that $d\xi \in (dy, y)\Lambda$. Then $d\xi = y\alpha + d(y\beta)$. We shall prove that $\operatorname{Ker} d \cap y\Lambda = 0$. Let $y\gamma$ be such that $d(y\gamma) = 0$. Then we have $d(y)\gamma + yd\gamma = 0$. Since $dy = a + \alpha$ with $\alpha \in \mathcal{O}$ and $a \neq 0$, $a \in A$, we show easily that $\gamma \in y\Lambda$, and $y\gamma = 0$. Thus we have $d\xi = d(y\beta)$ and

$$\operatorname{Ker} \bar{d} = (\operatorname{Ker} d \oplus y\Lambda)/(dy, y)\Lambda \ .$$

Since $\operatorname{Im} \bar{d} = (\operatorname{Im} d + y\Lambda)/(dy, y)\Lambda$ and $\operatorname{Im} d \subset \operatorname{Ker} d$, we have

$$H^*(\bar{\Lambda}, \bar{d}) = \frac{\operatorname{Ker} \bar{d}}{\operatorname{Im} \bar{d}} = \frac{\operatorname{Ker} d \oplus y\Lambda}{\operatorname{Im} d \oplus y\Lambda} = \frac{\operatorname{Ker} d}{\operatorname{Im} d} = H^*(\Lambda, d) \ .$$

In the remainder of this section, we assume that Λ has a finite number of generators in each degree. So, for all $n \in N$, Λ_n and $H^n(\Lambda)$ are finite dimensional vector spaces.

We can consider the Poincaré series

$$S_\Lambda(T) = \sum_{n\geq 0} (\dim_k \Lambda_n)T^n \ , \qquad S_{H^*(\Lambda)}(T) = \sum_{n\geq 0} (\dim_k H^n(\Lambda))T^n \ .$$

Definition. Let $S(T) = \sum_{n\in N} a_n T^n$ and $S'(T) = \sum_{n \in N} b_n T^n$ be two formal series with real coefficients, we say that $S(T) \leq S'(T)$ if and only if we have $a_n \leq b_n$, for every $n \in N$.

Proposition 3. *Let (Λ, d) be a differential graded algebra, and y an exterior generator of degree $2r+1 \geq 3$ such that $dy = 0$. Then we have*

$$S_{H^*(\Lambda/y\Lambda,\bar{d})}(T) \leq S_{H^*(\Lambda,d)}(T)/(1-T^{2r}) ,$$

where $\bar{d}$ is the induced differential on the quotient.

Proof. Let π be the canonical morphism $\Lambda \to \Lambda/y\Lambda$. If $s \in N$, we define $\Lambda[-s] = \bigoplus_{n \in N} \Lambda[-s]_n$ by

$$\Lambda[-s]_n = 0 \quad \text{for } n < s , \qquad \Lambda[-s]_n = \Lambda_{n-s} \quad \text{for } n \geq s .$$

Let φ be the map $\Lambda/y\Lambda[-2r-1] \to \Lambda$ defined by

$$\varphi(\xi) = \xi y \qquad \text{for all } \xi \in \Lambda/y\Lambda[-2r-1] .$$

We check easily that the sequence

$$0 \longrightarrow \Lambda/y\Lambda[-2r-1] \xrightarrow{\varphi} \Lambda \xrightarrow{\pi} \Lambda/y\Lambda \longrightarrow 0$$

is an exact sequence of differential graded algebras. So we have the long exact sequence of cohomology:

$$\begin{array}{c}
\cdots \xrightarrow{d_n} H^n(\Lambda/y\Lambda[-2r-1]) \xrightarrow{H^n(\varphi)} H^n(\Lambda) \\
\| \\
H^{n-2r-1}(\Lambda/y\Lambda) \\
\longrightarrow H^n(\Lambda/y\Lambda) \xrightarrow{d_{n+1}} H^{n+1}(\Lambda/y\Lambda[-2r-1]) \longrightarrow \cdots \\
\| \\
H^{n-2r}(\Lambda/y\Lambda)
\end{array}$$

Let $K_n = \operatorname{Im} H^n(\varphi)$ and $K = \bigoplus_{n \in N} K_n$. The long exact sequence splits and gives

$$0 \to K_n \to H^n(\Lambda) \to H^n(\Lambda/y\Lambda) \to H^{n-2r}(\Lambda/y\Lambda) \to K_{n+1} \to 0 ,$$

then we have, for every $n \in N$,

$$\begin{aligned}
\dim_k K_n - \dim_k H^n(\Lambda) + \dim_k H^n(\Lambda/y\Lambda) \\
- \dim_k H^{n-2r}(\Lambda/y\Lambda) + \dim_k K_{n+1} = 0 .
\end{aligned}$$

Hence

$$(1+T) \times (1/T) \times S_K(T) - S_{H^*(\Lambda)}(T) + (1-T^{2r})S_{H^*(\Lambda/y\Lambda)}(T) = 0 ,$$

or

$$S_{H^*(\Lambda/y\Lambda)(T)} = [1/(1 - T^{2r})]S_{H^*(\Lambda)}(T) - [(1 + T)/(1 - T^{2r})] \times (1/T) \times S_K(T) .$$

Since $S_K(T)$ has positive coefficients, the product $[(1 + T)/(1 - T^{2r})] \times (1/T) \times S_K(T)$ also has positive coefficients, and we have the inequality.

Proposition 4. *Let (Λ, d) be a differential graded algebra, and x a polynomial generator of degree $2s$ such that $dx = 0$. Then we have*

$$S_{H^*(\Lambda/x\Lambda,\bar{d})}(T) \leq (1 + T^{2s-1})S_{H^*(\Lambda,d)}(T) ,$$

where $\bar{d}$ is the differential deduced from d by quotient.

Proof. Let $\mu\colon \Lambda[-2s] \to \Lambda$ be the multiplication by x. The sequence $0 \to \Lambda[-2s] \xrightarrow{\mu} \Lambda \xrightarrow{\pi} \Lambda/x\Lambda \to 0$, where π is the canonical morphism, is an exact sequence of differential graded algebras. Then we can write the long exact sequence

$$\cdots \longrightarrow H^{n-2s}(\Lambda) \xrightarrow{H^n(\mu)} H^n(\Lambda) \longrightarrow H^n(\Lambda/x\Lambda) \longrightarrow H^{n-2s+1}(\Lambda) \longrightarrow \cdots .$$

Let $I_n = \operatorname{Im}(H^n(\mu))$, and $I = \oplus I_n$. For every $n \in N$, we have

$$0 \to I_n \to H^n(\Lambda) \to H^n(\Lambda/x\Lambda) \to H^{n-2s+1}(\Lambda) \to I_{n+1} \to 0 ,$$

which gives us

$$(1 + T) \times (1/T) \times S_I(T) - S_{H^*(\Lambda)}(T) + S_{H^*(\Lambda/x\Lambda)}(T) - T^{2s-1}S_{H^*(\Lambda)}(T) = 0 ,$$

or

$$S_{H^*(\Lambda/x\Lambda,\bar{d})}(T) = (1 + T^{2s-1})S_{H^*(\Lambda,d)}(T) - (1 + T) \times (1/T) \times S_I(T)$$

with $S_I(T) \geq 0$.

2. Proof of the theorem

Since M is a simply connected finite complex, its minimal model Λ has a finite number of generators in each degree and $\Lambda_1 = 0$. Also, $H^*(\Lambda)$ is a finite dimensional vector space, and the algebra $H^*(\Lambda)$ has at least one generator except when $\Lambda = k$.

(ii) $\Rightarrow$ (i). If $H^*(\Lambda, d)$ is generated by one element, we use Proposition 1 and make a direct calculation to show that the Betti numbers of the space of all maps of S^1 into M are bounded. See Addendum.

(i) $\Rightarrow$ (ii). (a) We claim that when M is a finite complex, the algebra $H^*(M)$ requires at least two generators if and only if the minimal model Λ of M has at least two exterior generators.

We first remark that Λ has exterior generators since $H^*(\Lambda)$ is finite. By

Proposition 1, if Λ has at least two exterior generators, then $H^*(\Lambda)$ has at least two generators.

Conversely, assume that Λ has one exterior generator. Let $\Lambda = k[x_i]_{i\in I} \otimes k(y)$ where the x_i are polynomial generators, and $dy = P(x_1, \cdots, x_n)$. If $dy \neq 0$, we have, by Proposition 2,

$$H^*(\Lambda) = H^*(\Lambda/y, dy)\Lambda) = k[x_i]_{i\in I}/Pk[x_i] .$$

Since $k[x_i]_{i\in I}/Pk[x_i]$ is a finite dimensional vector space, it is easy to prove that card $I \leq 1$. (The results of [5] about the dimension of local rings can be extended to graded algebras.) Then we should have $\Lambda = k[x] \otimes k(y)$ with $dy = \lambda x^h$, $\lambda \in k^*$, $h \geq 2$, and therefore $H^*(\Lambda, d)$ is generated by one element. If $dy = 0$, we have, by Proposition 3,

$$S_{k[x_i]_{i\in I}}(T) \leq [1/(1 - T^{2r})]S_{H^*(\Lambda)(T)} = P(T)/(1 - T^{2r}) ,$$

where $P(T) \in Z[T]$, and $2r + 1 = \deg y$. Let $A = k[x_i]_{i\in I} = \bigoplus_{n\geq 0} A_n$. The above inequality shows that the dimension of A_n is bounded, independent of n. This proves card $I \leq 1$. If $\Lambda = k[x] \otimes k(y)$ with $dy = 0$, we have necessarily $dx = 0$, contradicting the fact that $H^*(\Lambda)$ is finite dimensional. So we have $\Lambda = k(y)$ with $\deg y$ odd, and $H^*(\Lambda) = \Lambda$ is generated by one element. This proves our claim.

(b) Let y_1 and y_2 be the first two exterior generators of Λ. We denote the generators of Λ by increasing degrees: $x_1, \cdots, x_n; y_1; x_{n+1}, \cdots, x_r; y_2; \cdots$, where the $(x_i)_{1\leq i\leq r}$ are the first polynomial generators (perhaps $r = 0$, or $n = 0$). We have $dx_1 = \cdots = dx_n = 0$.

1st case: $dy_1 \neq 0$. Then $dy_1 = P_1(x_1, \cdots, x_n)$, $n \geq 1$, and $dy_2 = P_2(x_1, \cdots, x_n, x_{n+1}, \cdots, x_r)$ where $P_1 \in k[x_1, \cdots, x_n]$ and $P_2 \in k[x_1, \cdots, x_r]$. Let $m \in [n, r - 1]$, and assume that $dx_1 = \cdots = dx_m = 0$. We shall prove that $dx_{m+1} = 0$.

We have $dx_{m+1} = Q(x_1, \cdots, x_m)y_1$ where $Q \in k[x_1, \cdots, x_m]$. Since we have $d^2x_{m+1} = 0$, we deduce that $Qdy_1 = 0$, so that $Q = 0$. This proves that $dx_1 = \cdots = dx_r = 0$.

In Λ', the elements $\left(\prod_{i=1}^r \bar{x}_i\right)\bar{y}_1^\alpha \bar{y}_2^\beta$ where $(\alpha, \beta) \in N^2$ are cycles, because $d'\bar{x}_i = 0$, $d'\bar{y}_1 \in (\bar{x}_1, \cdots, \bar{x}_n)\Lambda'$ and $d'\bar{y}_2 \in (\bar{x}_1, \cdots, \bar{x}_r)\Lambda'$. It is easy to see that $\operatorname{Im} d' \subset (x_i, y_i)\Lambda'$. Thus the elements $\left(\prod_{i=1}^r \bar{x}_i\right)\bar{y}_1^\alpha \bar{y}_2^\beta$ are homologically independent.

If $\deg y_1 = 2r_1 + 1$, and $\deg y_2 = 2r_2 + 1$, then let $m = \text{l.c.m.}\ (r_1, r_2)$. For every $N \in N$, in $H^{2Nm+\Sigma_{i=1}^r \deg \bar{x}_i}(\Lambda')$ there are $N + 1$ elements $\left(\prod_{i=1}^r \bar{x}_i\right)\bar{y}_1^\alpha \bar{y}_2^\beta$ which are homologically independent. Hence the dimensions of $H^r(\Lambda', d')$ are unbounded.

2nd case: $dy_1 = 0$. If $dy_2 = 0$, let $\deg y_1 = 2r_1 + 1$, $\deg y_2 = 2r_2 + 1$,

$m = \text{l.c.m.}\ (r_1, r_2)$. Then for every $N \in \boldsymbol{N}$ in $H^{2Nm}(\Lambda')$ we have $N+1$ elements $\bar{y}_1^{\alpha}\bar{y}_2^{\beta}$, where $(\alpha, \beta) \in \boldsymbol{N}^2$, which are linearly independent. So the dimensions of $H^n(\Lambda', d')$ are not bounded. Assume $dy_2 = P(x_1, \cdots, x_r) \neq 0$. Then we have

$$dx_1 = 0\ , \quad \cdots\ , \quad dx_n = 0\ , \quad dx_{n+1} = Q_{n+1}(x_1, \cdots, x_n)y_1\ , \quad \cdots\ ,$$
$$dx_i = Q_i(x_1, \cdots, x_{i-1})y_1, \quad \cdots\ , \quad dx_r = Q_r(x_1, \cdots, x_{r-1})y_1\ , \qquad (r \geq 1)\ ,$$

where Q_i is a polynomial of degree ≥ 1. Let $2s_i$ be the degree of x_i for $i = 1, \cdots, r$. Let $\Lambda'_{(1)} = \Lambda'/x_1\Lambda'$ with the differential $d'_{(1)}$ induced by d'. Then we have $d'_{(1)}x_2 = 0$ and, by Proposition 4,

$$S_{H^*(\Lambda'_{(1)}, d'_{(1)})}(T) \leq (1 + T^{2s_1 - 1})S_{H^*(\Lambda', d')}(T)\ .$$

So consider the successive quotients

$$\Lambda'_{(2)} = \Lambda'/(x_1, x_2)\Lambda'\ , \quad \cdots\ , \quad \Lambda'_{(r)} = \Lambda'/(x_1, \cdots, x_r)\Lambda'\ .$$

Let $d'_{(r)}$ be the differential on $\Lambda'_{(r)}$ induced by d'. By an inductive argument, we see that

$$S_{H^*(\Lambda'_{(r)}, d'_{(r)})}(T) \leq \prod_{i=1}^{r} (1 + T^{2s_i - 1})S_{H^*(\Lambda', d')}(T)\ .$$

So, if the dimensions of $H^n(\Lambda'_{(r)})$ are not bounded, then the dimensions of $H^n(\Lambda')$ will be unbounded also. Since $dy_2 = P_2(x_1, \cdots, x_r)$ where P_2 is a polynomial of degree ≥ 2, by the formula we have $d'\bar{y}_2 = \sum_{i=1}^{r} \frac{\partial P_2}{\partial x_i}\bar{x}_i$; thus we have $d'_{(r)}\bar{y}_2 = 0$. The elements $\bar{y}_1^{\alpha}\bar{y}_2^{\beta}$ with $(\alpha, \beta) \in \boldsymbol{N}^2$ are closed in $\Lambda'_{(r)}$ and are homologically independent. So the dimensions of $H^n(\Lambda'_{(r)}, d'_{(r)})$ are unbounded. This completes the proof of the theorem.

Addendum. Now we calculate the cohomology ring of $\Lambda(M)$ when H^*M has one generator.

1. If H^*M is the exterior algebra on one generator y in degree $2n+1$, then clearly $H^*\Lambda(M)$ is $\Lambda(y, \bar{y})$ the exterior algebra on y tensor the polynomial algebra on $\bar{y}$ in degree $2n$.

2. If $H^*M = \{1, x, x^2, \cdots, x^n\}$ with $x^{n+1} = 0$ and degree $x = 2k$, then the model of M is $\Lambda(x, y; dy = x^{n+1})$, and we have to calculate $H^*(\Lambda', d')$ where $\Lambda' = \Lambda(x, y, \bar{x}, \bar{y})$ with $d'x = d'\bar{x} = 0$, $d'y = x^{n+1}$ and $d'\bar{y} = (n+1)x^n\bar{x}$.

By Proposition 2 we can set $y = 0$ and $d'y = x^{n+1}$ equal to zero, and calculate the cohomology of $\bar{\Lambda} = \Lambda(x, \bar{x}, \bar{y})$ where $x^{n+1} = 0$, $dx = d\bar{x} = 0$ and $d\bar{y} = x^n\bar{x}$ (the constant can be ignored by replacing $\bar{y}$ by $\bar{y}/(n+1)$). So in $\bar{\Lambda}$ the cycles in positive degrees make up the ideal of x and $\bar{x}$, while the boundaries make up the ideal of $x^n\bar{x}$. The reduced cohomology ring of $\Lambda(M)$ can then be described as the quotient of the ideal of x and $\bar{x}$ in $\bar{\Lambda}$ modulo the ideal of

$x^n\bar{x}$. This quotient is isomorphic to the finite dimensional ring $\Lambda^+(x, \bar{x})/(x^{n+1}, x^n\bar{x})$ (without unit) tensor the polynomial algebra (with unit) on one generator $\bar{y}$ in degree $2(k(n+1)-1)$ where degree $x = 2k$, degree $\bar{x} = 2k-1$. Clearly, the Betti numbers of $\Lambda(M)$ are bounded.

Notice that the reduced cohomology ring of $\Lambda(M)$ is totally nilpotent (every $(n+1)$ foldp roduct is zero). For example, for $M = S^2$ we obtain the zero ring on additive generators in dimensions $1, 2, 3, 4, \cdots$ for $H^*\Lambda(M)$. This degeneracy in the ring structure belies the structure of the homotopy groups of $\Lambda(M)$ which has total rank 4 (over Q).

This difference is made up by a rich structure of M assey products or higher order cup-products. All this information is carried by the minimal model which is simpler to describe for these spaces than the cohomology ring itself.

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