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A counterexample to the Periodic Orbit Conjecture in codimension 3

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1. Introduction

Let M be a manifold with a flow, such that every orbit is a circle. We address ourselves to the following question: Is the time-of-first-return function locally bounded? An equivalent question is: Is the length of the circular orbits a locally bounded function? (See Epstein [3] for a discussion of the consequences of an affirmative answer, and for other equivalent statements.) The *Periodic Orbit Conjecture* was the conjecture that the answer is affirmative for all such flows on compact manifolds. This conjecture is known to be false (Sullivan [5] and [6]).

An analogous question was first tackled by Reeb in his thesis [4], where it arose as a natural part of the investigation of stability (the Reeb Stability Theorems). Here the flow is replaced by a foliation. The condition, that each orbit should be a circle, becomes the condition that each leaf is compact. The question now is: Is the volume of the leaf a locally bounded function?

In codimension one, Reeb shows how the concept of holonomy leads to an affirmative answer, since, if everything is oriented, the volume of the leaf is a continuous function. Reeb gave an example of a foliated non-compact manifold, in which every leaf is compact, and the volume of the leaf is a locally unbounded function. Epstein [2] produced a real analytic flow on a non-compact 3-manifold with locally unbounded time-of-first-return function. However Epstein [2] shows that on a *compact* 3-manifold, the time-of-first-return function must be bounded. By modifying the proof in [2], this result has been extended to foliations of codimension two, such that each leaf is compact (see Edwards, Millett, and Sullivan [1] or Vogt [7]). The methods used to obtain this affirmative answer in codimension two are intricate and for some time after the publication of [2], it seemed

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reasonable to hope that better methods would be found, which would yield results in other codimensions.

However, in [5] and [6] Sullivan produced an example of a flow on a compact 5-manifold, with every orbit a circle, but with unbounded time-of-first-return. Sullivan also gives in [5] and [6] an exposition of a subsequent example by Thurston which is real analytic. Thurston's example is also explained in an appendix written by D. B. A. Epstein for a forthcoming book by A. Besse (pseudonym) on Riemannian manifolds, all of whose geodesics are closed curves.

Stronger justification for the intricacies involved in Epstein's proof, [2], is provided in a forthcoming paper by E. Vogt [8]. In this paper Vogt produces examples in which the Epstein hierarchy ends at any pre-assigned countable ordinal. It follows that the complications foreshadowed in [2], which Epstein has to rule out in dimension 3, can in fact occur in higher codimensions.

The present paper produces an example like that of Sullivan or Thurston, but in codimension 3 instead of 4. (Given an example we can always increase either the dimension or the codimension, by multiplying by a circle.) That is to say, we produce a flow on a compact 4-manifold, in which each orbit is a circle, and the circles have unbounded length. Moreover we improve on the analyticity of Thurston's example; ours is polynomial. Our compact 4-manifold M is the inverse image of 0 under a polynomial map $F\colon \mathbf{R}^7 \to \mathbf{R}^3$, and 0 is a regular value for F. Our vector field, generating the flow, is given by polynomials defined on the whole of $\mathbf{R}^7$. (Note that the flow itself can not possibly be given by polynomials, because a polynomial cannot be periodic with respect to any variable.)

In [6], Sullivan gives some plausibility arguments due (jointly) to Epstein and Hirsch, to show that a flow on a compact 4-manifold, with every orbit a circle, cannot have a bad set which is a manifold. (The *bad set* is the set on which the time-of-first-return function is locally unbounded.) A rigorous proof of this is lacking. In the present paper, the bad set is the union of four 3-spheres and four 2-dimensional tori. Each 3-sphere has the Hopf flow on it, and the flow on each torus is a circular flow around one of the two generating circles. These are arranged like a necklace with eight beads, 3-sphere alternating with torus. An adjacent 3-sphere and torus have exactly one circle in common. Thus the bad set in this case is a nice polyhedron. This raises a second question: In the case of an analytic foliation with all leaves compact, is the length of the hierarchy bounded by the codimension?

Two methods used in this paper for constructing manifolds may possibly be of wider interest. Suppose N is a manifold with boundary B and suppose $\phi\colon N \to [0, \infty)$ is a function such that $B = \phi^{-1}(0)$. Then we *double* N by taking the set of points (x, t) in $N \times R$ satisfying $\phi(x) = t^2$. We *spin* N about B by taking the set of points $(x, u_1, u_2) \in N \times \mathbf{R}^2$, satisfying $\phi(x) = u_1^2 + u_2^2$. If we happen to be in a situation where a well-behaved ϕ is part of the data, these constructions will obviously be more natural than previously defined methods of doubling or spinning.

2. Finding the example—a qualitative description

In order to make the various features of our example comprehensible, we will guide the reader along a path where the need for each successive complication is clear. (Needless to say, the historical development did not follow this natural path. In particular the discussion we give here understates the role played by other mathematicians and their insights. For example, the theorem of Edwards, Millett, and Sullivan [1] giving homological information about the bad set provided a powerful motive for putting into the bad set at least one S^3 with the Hopf flow on it.)

Our objective is to construct a compact manifold with a flow such that each orbit is a circle, and such that the time-of-first-return function τ is a (locally) unbounded function.

First attempt. (See Figure 1.)

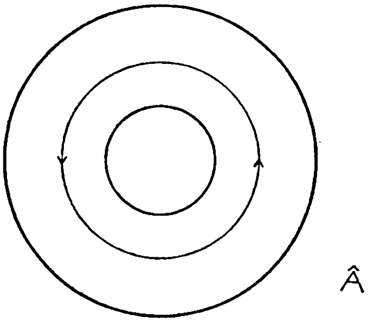

FIGURE 1

The most naive thing to do is to take a 2-dimensional disk $\hat{D}$ in the plane, with a circular flow of the form $(r, \theta, t) \to (r, \theta + t\phi(r))$. Then the orbits are circles C_r of radius r. We throw away a small disk around the origin, because the origin is a fixed point, and we obtain an annulus $\hat{A}$. In order to have $\tau(r, \theta) = 2\pi/\phi(r)$ unbounded, we let $\phi(r)$ tend to zero as r tends to 1.

First attempt fails. An orbit at the outer boundary C_1 is a single point, and not a circle, as we require.

Second attempt. We multiply the "horizontal" annulus $\hat{A}$ by a "vertical" circle S, and arrange for the flow to have, in addition to the flow already described, a constant vertical component. Then the orbit through a point lying over C_1 will be just a copy of the circle S.

Second attempt fails. Over the circle C_r, we have a torus $C_r \times S$ which is invariant under the flow. The flow on this torus is linear, and the slope is vertical for $r = 1$. For $r < 1$ the slope is not vertical. Therefore the slope must be irrational for some value of r, and the orbit is not a circle.

The third attempt. We need a better idea to force orbits to be closed: That idea is *symmetry*. We construct a field of vectors $v(x)$ on $\hat{A}(x \in \hat{A})$ tangent to S. In Figure 2, these vectors are drawn, for convenience, as if they were tangent to $\hat{A}$, but in fact they are tangent to S, and orthogonal to $\hat{A}$.

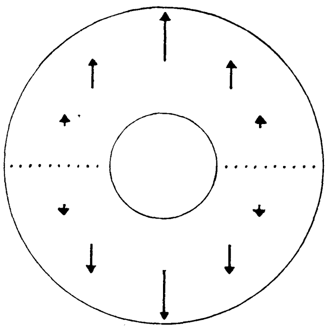

FIGURE 2

Since $S = \mathbf{R}/Z$, the vectors all lie in a 1-dimensional space. We require any movement in the vertical direction to be cancelled out by an equal and opposite movement at a later time. Explicitly, we assume that the velocity along C_r in the horizontal direction depends only on r, and not on θ. We also assume that $v(x) = -v(-x)$. If α is any arc of C_r, then as the point covers α, the vertical movement is proportional to $\int_\alpha v(x)dx$. If $\alpha = C_r$, then the symmetry of $v(x)$ ensures that the integral is zero. Thus, if $r < 1$, each orbit is a circle covering C_r exactly once.

Third attempt fails. Since $v(x) = -v(-x)$ there must be at least two zeros of v on each circle C_r. These are shown by the row of dots in Figure 2. Therefore the flow is stationary over the two points $(1, 0)$ and $(-1, 0)$ in $\hat{A}$.

However, we may take heart from the fact that we now have an example on a *non-compact* manifold, namely $(\hat{A}\backslash\{(1, 0), (-1, 0)\}) \times S$. This example is essentially the same as that of Reeb [4] or Epstein [2]. The time-of-first-return function is not only unbounded (which is trivial to arrange on a non-compact manifold), but also *locally* unbounded on C_1.

Fourth attempt. We retain the symmetry $v(x) = -v(-x)$, but we insist that v should never be zero. In order to achieve this, we replace Figure 2 by Figure 3, which has yet to be made meaningful. This is done by

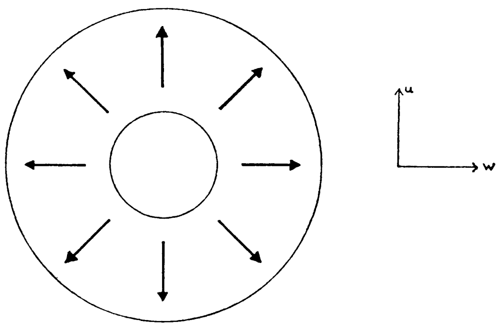

FIGURE 3

multiplying $\hat{A}$ not by a circle $S = \mathbf{R}/Z$, but by a torus $T = \mathbf{R} \times \mathbf{R}/Z \times Z$. The arrows in Figure 3 are once again orthogonal to $\hat{A}$, although they are drawn tangent to $\hat{A}$. The arrows represent vectors in $\mathbf{R}^2$, the tangent space of T. As in the third attempt, the condition $v(x) = -v(-x)$ ensures that if $r < 1$, then each orbit is a circle over C_r. Such orbits lie in $C_r \times T$.

Fourth attempt fails. Over a given point of C_1, the flow on the torus T is the projection of the flow on $\mathbf{R}^2$ along lines of a fixed slope. As the slope varies, the flow on T will vary through an irrational flow, and then the orbits are not circles.

Fifth attempt (success). We give a rough description at this point: In later sections of the paper we will be very precise.

We have a parametrized family of tori, one for each point of $\hat{A}$. We use the symbols u and w to denote coordinates in $\mathbf{R}^2$, so that (u, w) is a typical point. The idea is to let the tori, which we write as $S_x^u \times S_x^w$ $(x \in \hat{A})$, vary in size. Here S_x^u and S_x^w are circles. Now if we insist that for each $x \times \hat{A}$ we obtain a true torus, then the slope of $v(x)$ on C_1 will be frozen at a constant rational value, making $v(x) = -v(-x)$ impossible. However, if we allow the radius of one of the circles, say S^u, to shrink down so as to be zero along an arc Q of C_1, and if the vector $v(x)$ always has a non-zero component in the w-direction, for x in Q, then we get a well-defined flow on the space thus described. Moreover the slope of $v(x)$ will be unfrozen within this limitation.

In order to convince oneself that the union of such parametrized tori gives a manifold, rather than a manifold with a singularity, we contemplate the construction called *spinning*, due to E. Artin. If we take a rectangle and spin it around one edge, we get a solid cylinder. The edge becomes the axis of the cylinder, and disappears into the interior of the solid cylinder. The other three boundary edges spin to produce new boundary pieces, each of dimension two. As a point on the rectangle moves nearer to the axis, the circle it generates becomes smaller, until the circle actually shrinks to a point, when the axis is reached.

Thus, the situation described above, with a family of tori over $\hat{A}$, can be achieved by first spinning around the arc Q of C_1, and then multiplying by a circle. This shows that our family of parametrized tori gives rise to a 4-dimensional manifold. Over the arc Q, the vector $v(x)$ may be varied in direction, as long as the w-component is always non-zero. Over each point of Q, we have a copy of the circle S^w, and the flow simply goes around this circle.

The idea now is to use the spinning operation along several disjoint arcs of C_1. In between these arcs, we will keep the slope of $v(x)$ constant and rational, thus getting a rational flow on the torus $S^u_x \times S^w_x$. On each of the arcs, either S^u_x or S^w_x will be shrunk to a point (but not both), and $v(x)$ will either have the w-component never zero or the u-component never zero. Thus Figure 3 is replaced by Figure 4.

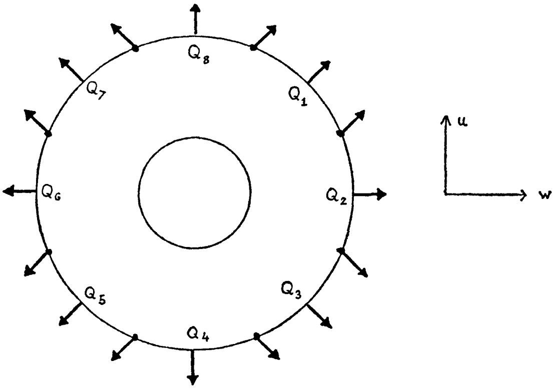

FIGURE 4

On each of the arcs Q_1, Q_3, Q_5, Q_7, the slope of $v(x)$ is constant and is equal to ± 1. On the arcs Q_2 and Q_6, $v(x)$ has a non-zero w-component. On the arcs Q_4 and Q_8, $v(x)$ has a non-zero u-component. We spin the annulus simultaneously around the arcs Q_2 and Q_6, thus generating the circles S^u which shrink down to zero radius precisely on Q_2 and Q_6. We also spin simultaneously around Q_4 and Q_8, thus generating the circles S^w, with zero radius precisely on Q_4 and Q_8. This construction gives a 4-manifold with five boundary components, corresponding to Q_1, Q_3, Q_5, Q_7 and the interior circle of the annulus. We double the manifold to remove the boundary.

This completes the qualitative description of the flow. Off C_1, orbits are circles because of the symmetry $v(x) = -v(-x)$. Over Q_1, Q_3, Q_5, and Q_7, where the inverse image of a point is a torus, the flow on the torus has slope ± 1, so every orbit is a circle. Over Q_2, Q_4, Q_6, and Q_8, where the inverse image of a point is a circle (or two circles, because of doubling), the orbit is equal to that circle.

The part of the manifold over Q_i, for $i = 1, 3, 5$ or 7, is a 3-sphere. The part over Q_i, for $i = 2, 4, 6$ or 8, is a 2-dimensional torus, obtained by doubling an annulus.

The time of first return function is locally unbounded at points of C_1, because it is equal to $2\pi/\phi(r)$ or C_r if $r < 1$, as described above in the first attempt.

3. Finding the equations for the manifold

We now show how to translate the informal description of Section 2 into polynomial formulas. We note that analytic continuation prevents different types of phenomena from occurring along the eight arcs of a round circle,

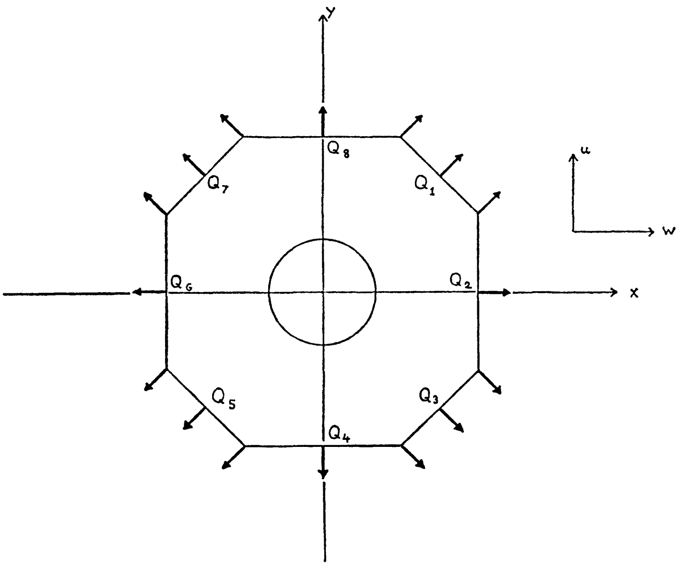

FIGURE 5

as described in the fifth attempt of Section 2. We therefore replace the circle by an octagon bounded by the lines $x=\pm 2$, $y=\pm 2$, $x+y=\pm 3$, $x-y=\pm 3$, as shown in Figure 5.

We replace the radius function by

$$(3.1)\quad \psi(x, y)=(2-x)(2+x)(2-y)(2+y)(3+x+y)(3-x-y)(3-x+y)(3+x-y)\,.$$

We define

$$D=\{(x, y)\colon -2 \leqq x \leqq 2,\ -2 \leqq y \leqq 2,\ -3 \leqq x+y \leqq 3,\ -3 \leqq x-y \leqq 3\}\,.$$

On int D, $\psi > 0$.

LEMMA 3.2. *Let R be any interval with one endpoint at 0 and the other on the boundary of D. If R is parametrized linearly, then ψ restricted to R has non-zero derivative, except possibly at the endpoints. In particular, $d\psi \neq 0$ on $\operatorname{int} D - \{0\}$. If $0 < r < 1296$, then $\psi^{-1}(r) \cap D$ is a simple closed curve.*

Proof.

$$\psi(x, y)=(4-x^2)(4-y^2)\big(9-(x+y)^2\big)\big(9-(x-y)^2\big)\,.$$

Differentiating along R we obtain four non-positive terms, with at most one being zero. The lemma follows.

We define A to be the annulus

$$A=D \cap \{(x, y)\colon \psi(x, y) \leqq 1\}\,.$$

The spinning operations are performed as follows. We work in $\mathbf{R}^6$, with variables $(x, y, u_1, u_2, w_1, w_2)$, and we consider the subset of $\mathbf{R}^6$ consisting of points $(x, y, u_1, u_2, w_1, w_2)$ with $(x, y) \in A$ and satisfying the equations

$$(3.3)\qquad u_1^2+u_2^2=4-x^2$$

and

$$(3.4)\qquad w_1^2+w_2^2=4-y^2\,.$$

The projection of this subset onto A has the properties required. Namely, the inverse image of a point of Q_2 or Q_6 is a circle in the w-plane; the inverse image of a point in Q_4 or Q_8 is a circle in the u-plane; and the inverse image of any other point of A is a torus.

We now perform the doubling operation. We add another variable z, so that we are working in $\mathbf{R}^7$, subject to the additional condition

$$(3.5)\qquad z^2=\rho(x, y)\,,$$

where

$$(3.6)\qquad \rho(x, y)=(1-\psi)(3-x-y)(3-x+y)(3+x-y)(3+x+y)\,.$$

The effect of these equations is that every point of our subset in $\mathbf{R}^6$ gives rise to two distinct points in $\mathbf{R}^7$, corresponding to the two solutions for z, except that if $\psi(x, y) = 1$, or if $(x, y) \in Q_1 \cup Q_3 \cup Q_5 \cup Q_7$, we obtain only one point in $\mathbf{R}^7$.

4. Rigorous description of the 4-manifold

Let $\xi = (x, y, u_1, u_2, w_1, w_2, z) \in \mathbf{R}^7$. We define $F\colon \mathbf{R}^7 \to \mathbf{R}^3$ as the polynomial function given by the formulas

$$F_1(\xi) = u_1^2 + u_2^2 - 4 + x^2 ,$$
$$F_2(\xi) = w_1^2 + w_2^2 - 4 + y^2 ,$$
$$F_3(\xi) = z^2 - \rho(x, y) ,$$

where $F(\xi) = \big(F_1(\xi), F_2(\xi), F_3(\xi)\big) \in \mathbf{R}^3$. We define $M = F^{-1}(0)$. Then M is the subset of $\mathbf{R}^7$ satisfying (3.3), (3.4), and (3.5). We will prove that $F|M$ is regular and that M is compact, so that M is a compact 4-manifold smoothly embedded in $\mathbf{R}^7$, and a non-singular real algebraic variety.

LEMMA 4.1. *The projection of $\mathbf{R}^7$ onto the first two coordinates maps M onto A.*

Proof. Let $\xi = (x, y, u_1, u_2, w_1, w_2, z) \in M$. Since $F_1 = 0$, we have $4 - x^2 \geqq 0$. Since $F_2 = 0$, we have $4 - y^2 \geqq 0$. Hence $(4 - x^2)(4 - y^2) \geqq 0$. Since $F_3 = 0$, we have $\rho(x, y) \geqq 0$. Hence

$$0 \leqq \rho(x, y)(4 - x^2)(4 - y^2) = (1 - \psi)\psi .$$

Therefore $0 \leqq \psi \leqq 1$.

We now suppose that $(x, y) \notin A$, and obtain a contradiction. By applying the symmetries $x \to -x$, and $y \to -y$, which preserve both ψ and ρ, we may suppose that $x + y > 3$. Since we already know that $-2 \leqq x \leqq 2$ and $-2 \leqq y \leqq 2$, we must have $1 \leqq x \leqq 2$ and $1 \leqq y \leqq 2$. Hence $-1 \leqq x - y \leqq 1$ and so

$$\big(9-(x + y)^2\big)\big(9-(x - y)^2\big) < 0 .$$

Since $\rho \geqq 0$, (3.6) shows that $\psi \geqq 1$. Hence $\psi = 1$. But then

$$1 = \psi = (4 - x^2)(4 - y^2)\big(9-(x + y)^2\big)\big(9-(x - y)^2\big) \leqq 0 ,$$

which is a contradiction.

Conversely, if $(x, y) \in A$, we can clearly solve for u_1, u_2, w_1, w_2, and z, so that $\xi \in M$.

LEMMA 4.2. *$F\colon \mathbf{R}^7 \to \mathbf{R}^3$ is regular on $M = F^{-1}(0)$, so that M is a 4-dimensional manifold.*

Proof. The Jacobian of F is

$$\begin{bmatrix} 2x & 0 & 2u_1 & 2u_2 & 0 & 0 & 0 \\ 0 & 2y & 0 & 0 & 2w_1 & 2w_2 & 0 \\ -\partial\rho/\partial x & -\partial\rho/\partial y & 0 & 0 & 0 & 0 & 2z \end{bmatrix}.$$

Neither of the first two rows can be zero on M, so the matrix has rank 2 or 3. We will suppose it has rank 2 and deduce a contradiction.

If the matrix has rank 2, then the third row must be α times the first row plus β time the second row, where $\alpha, \beta \in \mathbf{R}$. Therefore $z = 0$, and so $\rho = 0$ by (3.5).

If $(9-(x+y)^2)(9-(x-y)^2) \neq 0$, then $\psi = 1$ by 3.6, and so

$$d\rho = -d\psi(9-(x+y)^2)(9-(x-y)^2),$$

which is non-zero by Lemma 3.2. But since $\psi = 1$, $u_1^2 + u_2^2 \neq 0$ and $w_1^2 + w_2^2 \neq 0$ by (3.3) and (3.4). Hence $\alpha = \beta = 0$, so that $\partial\rho/\partial x = 0 = \partial\rho/\partial y$. This shows that if the rank is 2, then $(9-(x+y)^2)(9-(x-y)^2) = 0$.

By applying the symmetries $x \to -x$ and $y \to -y$, we may assume that $x + y = 3$. Then $\psi = 0$ and

$$d\rho = 6(dx + dy)((x-y)^2 - 9)$$

which is non-zero since $(x, y) \in A$. Therefore $\partial\rho/\partial x = \partial\rho/\partial y \neq 0$ and so both α and β are non-zero. But then $u_1 = u_2 = w_1 = w_2 = 0$, which is impossible by (3.3), (3.4), and Lemma 4.1.

LEMMA 4.3. *M is compact.*

Proof. We know that M is closed. That M is bounded follows from Lemma 4.1, and equations (3.3), (3.4), (3.5).

5. Finding the equations for the vector field

From Section 2, we see that the vector field should preserve the level surfaces of ψ. We have

$$\dot\psi = (\partial\psi/\partial x)\dot x + (\partial\psi/\partial y)\dot y .$$

So we get $\dot\psi = 0$ by making $(\dot x, \dot y)$ proportional to $(\partial\psi/\partial y, -\partial\psi/\partial x)$. Since we also want $(\dot x, \dot y)$ to be zero when $\psi = 0$, we define

$$\dot x = \psi\partial\psi/\partial y \quad\text{and}\quad \dot y = -\psi\partial\psi/\partial x . \tag{5.1}$$

Then ψ is constant along orbits. By Lemma 3.2, $(\dot x, \dot y)$ is zero on M, if and only if $\psi = 0$. The orbits in the (x, y)-plane are simple closed curves in A, except when $\psi = 0$.

To find out what z should be, we differentiate (3.5), obtaining

$$2z\dot z = (\partial\rho/\partial x)\dot x + (\partial\rho/\partial y)\dot y = \psi(\partial\psi/\partial y)(\partial\rho/\partial x) - \psi(\partial\psi/\partial x)(\partial\rho/\partial y) . \tag{5.2}$$

Differentiating the logarithm of

$$\rho(x, y) = (1 - \psi)(9-(x + y)^2)(9-(x - y)^2) ,$$

we see that the right-hand side of (5.2) becomes $2\rho(x, y)\sigma(x, y)$ where

$$(5.3) \qquad \sigma(x, y) = \left(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\right)\left(\frac{(x + y)\psi}{9-(x + y)^2}\right) + \left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\right)\left(\frac{(y - x)\psi}{9-(x - y)^2}\right) .$$

Thus $\sigma(x, y)$ is a polynomial with integer coefficients. Since $z^2 = \rho(x, y)$, we set

$$(5.4) \qquad \dot{z} = z\sigma(x, y) .$$

We now wish to define u_1, u_2, w_1, and w_2 so as to achieve the situation shown in Figure 5. Restricting our attention for the moment to the u-plane, note that we would like to have a circular motion about the center. Such a flow is given by

$$\dot{u}_1 = -pu_2 \quad \text{and} \quad \dot{u}_2 = pu_1$$

where $p = \dot{\theta}$ is the angular velocity. Note that it is the angular velocity which is depicted by the vectors in Figures 2, 3, 4, and 5. (Look first at Figure 2 to see this.)

Simultaneouly the radius of the circle in the u-plane will change with time, since (3.3) shows that

$$d(u_1^2 + u_2^2)/dt = -2x\dot{x} = -2x\psi\partial\psi/\partial y .$$

The equations $\dot{u}_1 = Ku_1 - pu_2$ and $\dot{u}_2 = pu_1 + Ku_2$ combine both an angular and a radial velocity. This gives rise on M to

$$d(u_1^2 + u_2^2)/dt = 2K(u_1^2 + u_2^2) = 2K(4 - x^2) .$$

So we define

$$(5.5) \qquad K(x, y) = -x(\partial\psi/\partial y)(4 - y^2)(9-(x + y)^2)(9-(x - y)^2) .$$

Similarly, putting

$$\dot{w}_1 = Lw_1 - qw_2 \quad \text{and} \quad \dot{w}_2 = qw_1 + Lw_2 ,$$

we are led to define

$$(5.6) \qquad L(x, y) = y(\partial\psi/\partial x)(4 - x^2)(9-(x + y)^2)(9-(x - y)^2) .$$

In order to define p and q, we look again at Figure 5. We see that we want to have angular velocity functions p and q with the following properties:

$$\begin{array}{lll} (5.7) \quad \text{i)} & p(x, y) > 0 \quad \text{if} \quad 0 < y \leqq 2 ; \\ \quad\quad\;\; \text{ii)} & q(x, y) > 0 \quad \text{if} \quad 0 < x \leqq 2 ; \\ \quad\quad\;\; \text{iii)} & p(x, y) = p(-x, y) = -p(x, -y) ; \end{array}$$

iv) $$q(x, y) = -q(-x, y) = q(x, -y)\ ;$$
v) $$p = q \quad \text{on} \quad x + y = 3\ .$$

If we define

$$p(x, y) = (9 + x^2 - y^2)y \quad \text{and} \quad q(x, y) = (9 - x^2 + y^2)x\ , \tag{5.8}$$

we see that these conditions are satisfied.

6. Rigorous treatment of the vector field

We define a vector field X on $\mathbf{R}^7$ as follows. Let $\xi = (x, y, u_1, u_2, w_1, w_2, z)$. Then

$$\begin{aligned} X_\xi = \psi(\partial\psi/\partial y)\partial/\partial x - \psi(\partial\psi/\partial x)\partial/\partial y + (Ku_1 - pu_2)\partial/\partial u_1 + (pu_1 + Ku_2)\partial/\partial u_2 \\ +(Lw_1 - qw_2)\partial/\partial w_1 + (qw_1 + Lw_2)\partial/\partial w_2 + z\sigma(x, y)\partial/\partial z \end{aligned}$$

where K is defined in (5.5), L in (5.6), p and q in (5.8) and σ in (5.3).

LEMMA 6.1. *If $\xi \in M$, then X_ξ is tangent to M.*

Proof. Since $F\colon \mathbf{R}^7 \to \mathbf{R}^3$ is regular on M, we need only show that $(DF)X_\xi = 0$, and this is a trivial calculation.

LEMMA 6.2. *X is never zero on M.*

Proof. We suppose $X_\xi = 0$ for $\xi = (x, y, u_1, u_2, w_1, w_2, z) \in M$ and deduce a contradiction. We have

$$0 = \dot{u}_2 u_1 - \dot{u}_1 u_2 = p(x, y)(u_1^2 + u_2^2)$$

and similarly $q(x, y)(w_1^2 + w_2^2) = 0$. If $u_1^2 + u_2^2 \neq 0$ then $p(x, y) = 0$, and so $y = 0$ by (5.7). By (3.4) $w_1^2 + w_2^2 \neq 0$, and so $q(x, y) = 0$, which implies that $x = 0$ by (5.7). But $(x, y) = (0, 0)$ is not in A, which contradicts Lemma 4.1.

It follows that $u_1^2 + u_2^2 = 0$, and, by (3.3), that $x^2 = 4$. Therefore $q(x, y) \neq 0$. Hence $w_1^2 + w_2^2 = 0$ and so $y^2 = 4$. Once again $(x, y) \notin A$, a contradiction.

LEMMA 6.3. *The functions ψ, ρ, σ, and F and the vector field X are invariant under rigid rotations about the origin in either the u-plane or the w-plane.*

Proof. This is clear for ψ, ρ, σ, and F. It follows for X since a matrix of the form $\begin{bmatrix} K & -\rho \\ \rho & K \end{bmatrix}$ is a scalar multiple of a rigid rotation of $\mathbf{R}^2$ about the origin. Hence the matrix commutes with such rotations. The lemma follows.

LEMMA 6.4. *Let $T\colon \mathbf{R}^7 \to \mathbf{R}^7$ be the involution defined by*

$$T(x, y, u_1, u_2, w_1, w_2, z) = (-x, -y, u_2, u_1, w_2, w_1, z)\ .$$

Then ψ, ρ, F, M, *and* X *are invariant under* T.

Proof. If T transforms an object such as a function, a differential form or a vector field to its negative, we will call that object *anti-invariant*. Clearly ψ, ρ, and F are invariant, and hence $d\psi$ is invariant. Since dx and dy are anti-invariant, we have

$$\begin{aligned}(\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy &= d\psi \\ &= T^*d\psi \\ &= -(\partial\psi/\partial x \circ T)dx - (\partial\psi/\partial y \circ T)dy \ .\end{aligned}$$

Hence $\partial\psi/\partial x$ and $\partial\psi/\partial y$ are anti-invariant, and the same goes for $\partial\rho/\partial x$ and $\partial\rho/\partial y$. From (5.3), σ is invariant. The polynomials p and q are anti-invariant, and K and L are invariant. Hence the definition of X shows that it is invariant.

Recall that $\dot\psi = 0$ by (5.1). Hence ψ is constant on each orbit.

LEMMA 6.5. *On* M, *if* $\psi > 0$, *then each orbit is (diffeomorphic to) a circle. As* ψ *tends to zero, the time of first return tends to infinity.*

Proof. Let

$$\xi(t) = \big(x(t),\, y(t),\, u_1(t),\, u_2(t),\, w_1(t),\, w_2(t),\, z(t)\big) \in M$$

be a point moving under the flow at time t. Since $(\dot x, \dot y) \neq 0$, by Lemma (3.2), and since the level curves of ψ in A are simple closed curves, we see that $\big(x(t), y(t)\big)$ traverses such a level curve, always moving in the same direction (counter-clockwise). Let $2\lambda > 0$ be the time of first return of $\big(x(t), y(t)\big)$.

It is clear, by the symmetry of ψ, that $x(\lambda) = -x(0)$ and $y(\lambda) = -y(0)$. If $z(0) = 0$, then by (3.5) and (3.6) $\psi = 1$ (since we are assuming that $\psi > 0$). It follows from (3.6) that $z(t) = 0$ for all t. Therefore, if $z(0) > 0$, then $z(t) > 0$ for all t, and if $z(0) < 0$, then $z(t) < 0$ for all t.

By (3.5), z is determined, apart from its sign, by x and y. Hence $z(0) = z(\lambda)$. By (3.3) and (3.4),

$$u_1(\lambda)^2 + u_2(\lambda)^2 = u_1(0)^2 + u_2(0)^2$$

and

$$w_1(\lambda)^2 + w_2(\lambda)^2 = w_1(0)^2 + w_2(0)^2 \ .$$

Therefore there is a rotation R in the u-plane and a rotation S in the v-plane such that

$$\xi(\lambda) = RST\xi(0) \ .$$

Of course, R and S depend on $\xi(0)$. By Lemmas (6.3) and (6.4), $T_*X = R_*X = S_*X = X$. It follows that $\xi(t + \lambda) = RST\xi(t)$ for all t. Also T

conjugates both R and S to their inverse, so that $RSTRST$ is the identity. By putting $t = \lambda$, we deduce that $\xi(2\lambda) = \xi(0)$. Hence the orbit is a simple closed curve with time of first return equal to 2λ.

As ψ tends to zero, $(\dot{x}, \dot{y})$ tends to zero. But the length of the orbit of (x, y) in A does not tend to zero. (In fact it tends to $8 + 4\sqrt{2}$.) Hence λ tends to infinity as ψ tends to zero.

LEMMA 6.6. *If $x + y = \pm 3$ or $x - y = \pm 3$ for a point in M, then the orbit through this point is a circle.*

Proof. We have $\psi = \rho = 0$. Therefore $x(t)$ and $y(t)$ are independent of t. By (3.5), $z(t) = 0$ for all t. By (5.5) and (5.6), $K(x(t), y(t)) = L(x(t), y(t)) = 0$ for all t. Therefore, throughout the orbit, the following equations are satisfied

$$\dot{x} = \dot{y} = \dot{z} = 0\,, \quad \dot{u}_1 = -pu_2\,, \quad \dot{u}_2 = pu_1\,, \quad \dot{w}_1 = -qw_2\,, \quad \dot{w}_2 = qw_1\,.$$

By (5.7) $p = q$ or $p = -q$ throughout the orbit. The angular speeds in the u-plane and in the w-plane are equal to $|p|$, which is non-zero by (5.7). Therefore, after a time $2\pi/|p|$, the point returns to its starting position, and this is also the time of first return.

LEMMA 6.7. *If $x = \pm 2$ or $y = \pm 2$ for a point in M, then the orbit through this point is a circle.*

Proof. Suppose for example that $x = 2$. Then $u_1 = u_2 = 0$ by (3.3). Also $\dot{x} = \dot{y} = 0$ since $\psi = 0$. Therefore $z(t)$ is constant by (3.5) and so $\dot{z} = 0$. Hence $\dot{x} = \dot{y} = \dot{z} = \dot{u}_1 = \dot{u}_2 = 0$ and $L(x, y) = 0$ throughout the orbit. So, on the orbit, we have the equation $\dot{w}_1 = -qw_2$, $\dot{w}_2 = qw_1$, where $w_1^2 + w_2^2 = 4 - y^2 \neq 0$. The time of first return is $2\pi/|q| = \pi/5 + y^2$.

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