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A TOPOLOGICAL INVARIANT OF FLOWS ON 1-DIMENSIONAL SPACES

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To A $k \times k$ matrix A of non-negative integers whose sum is s one can associate a shift invariant subset X_A (see below) of the countable product of $\{1, \ldots, s\}$ with itself. The shift on this subset is a transformation T_A which can be naturally studied in set theory [3], probability theory [3] and [4], and topology [1]. These points of view merge in the dynamical study of diffeomorphisms.

In this note we consider the flow with cross section X_A and Poincaré map T_A and show that $\det(1 - A)$ is a topological invariant of the underlying space of the flow. We compare this invariant to the classification of the transformation T_A itself in the three contexts above. The argument here is purely topological but a related result was first discovered in [5] using variational principles for the entropy of flows. This invariant $\det(1 - A)$ allows us to answer anew Bowen's question [2] on the non-triviality of the equivalence relation between non-negative matrices induced by the topological classification of the corresponding flows (see also [5]).

Finally, it is an interesting problem to give a direct topological description of this invariant of flows.

Now we go into more detail. If $A = (a_{ij})$ X_A can be described as a certain set of infinite paths in the complex K_A made up of vertices $\{1, \ldots, k\} \times \{\text{integers}\}$ with a_{ij} edges between (i, n) and $(j, n+1)$ for each $1 \le i, j \le k$ and $-\infty < n < \infty$. X_A consists of those infinite paths in K_A which map isomorphically to the real line under the natural projection $K_A \to$ real line. There are s intervals over $[n, n+1]$ so X_A is naturally a subset of the countable product of $\{1, \ldots, s\}$ with itself. We will assume that A is irreducible in the sense that for any $1 \le i, j \le k$ there is a path in K_A connecting $(i, 0)$ with (j, m) for some m. Then X_A is either finite or uncountable. We will dispense with the first case where A is just a permutation matrix and study the transformation T_A induced on X_A by the shift $(\cdot, n) \to (\cdot, n-1)$ of K_A in the case where X_A is uncountable.

In *set theory* T_A is determined up to isomorphism by the orbit structure. It is easy to see that T_A has a finite number N_n of periodic points of order n (in fact, $N_n = \text{trace } A^n$). So T_A has uncountably many non-periodic points and is determined as a set transformation by the sequence of integers N_n. These can be neatly catalogued in the Weil zeta function of T_A, $\zeta(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} t^n\right)$ which is just the rational function $1/\det(1 - tA)$, [3].

Now X_A has a natural topology homeomorphic to the Cantor set. Open sets of paths in X_A are obtained by restricting the position of the path at a finite number of places. In this topology the periodic points of the homeomorphism T_A are dense (again easy to see) and there is a Baire set of points whose orbits are dense [1]. As a *topological transformation* T_A is determined by the class of the matrix A under the equivalence relation (shift equivalence) generated by $A \sim A'$ if $A = XY$ and $A' = YX$ where X and Y are (not necessarily square) matrices of non-negative integers (see [7] and the errata [8]).

T_A can be treated measure theoretically as an "intrinsic Markov chain" (see [4]). There is a unique invariant measure which maximizes entropy. As a *measure preserving transformation* T_A is determined by this entropy (which is just the log of the maximum eigenvalue of A; see [1] and [6]) and the periodicity of A.†

Now we turn to the flow with cross section X_A and Poincaré map T_A. The underlying space of the flow F_A is $X_A \times \{\text{real line}\}$ modulo the action of $T_A \times$ unit translation. We will say that the

†If A has periodicity 1, $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ has periodicity two.

homeomorphisms T_A and T_B (or the matrices A and B) are *flow equivalent* if there is a homeomorphism between F_A and F_B carrying flow lines to flow lines with the proper direction. Since X_A is totally disconnected, the arc components of F_A are just the flow lines and flow equivalence is just "oriented homeomorphism". (A remark of Charles Pugh).

Bowen asked "what is the equivalence relation on non-negative integral matrices induced by flow equivalence?" To answer this consider the operation on matrices $A \to \tilde{A}$ defined by

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} 0 & a_{11} & \dots & a_{1n} \\ 1 & 0 & \dots & 0 \\ 0 & a_{21} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix} = \tilde{A}.$$

Our result is the

THEOREM. *The 'flow equivalence relation' on square non-negative integral matrices is generated by the relations*

(i) $XY \sim YX$ (*X, Y not necessarily square*)

(ii) $A \sim \tilde{A}$.

COROLLARY. *If A and B are flow equivalent then* $\det(1-A) = \det(1-B)$.†

Example. For every $n \neq m$, the full n-shift is not flow equivalent to the full m-shift (this also follows from the result in [5]).

We summarize all these statements about the transformation T_A and induced flow in the classification table

Context	Classifying invariant
T_A as a transformation of sets	the zeta function, $1/\det(1-tA)$
T_A as a measure preserving transformation	the entropy, the log of the maximum eigenvalue of A and the periodicity of A
T_A as a topological transformation	the shift equivalence class of A, the relation being generated by $XY \sim YX$
The topological study of the flow associated to T_A	the equivalence class of A under the relation generated by (i) $XY \sim YX$ (ii) $A \sim \tilde{A}$

We remark that the problem of interpreting the invariant $\det(1-A)$ topologically is just the problem of interpreting the value of the zeta function at 1. As noted above the zeta function near $t = 0$ is determined by the periodic point structure of T_A which is somewhat lost in the flow picture. The problem is complicated by the fact that 1 is beyond the radius of convergence of this power series for $\zeta(t)$.

Now we discuss the proof of the theorem.

Consider self-homeomorphisms of Cantor sets under the relation of flow equivalence, i.e. oriented homeomorphisms of the corresponding flow spaces. Given one such $X \xrightarrow{f} X$ we define its *expansion* $\tilde{X} \xrightarrow{\tilde{f}} \tilde{X}$ given a decomposition $X = A \cup B$ into disjoint open and closed sets. Let $\tilde{B} \xrightarrow{i} B$ be an identification between B and another copy $\tilde{B}$, let $\tilde{X} = A \cup B \cup \tilde{B}$, and define $\tilde{f}\colon \tilde{X} \to \tilde{X}$ by $\tilde{f} = i^{-1} \circ f$ on $f^{-1}B$, $\tilde{f} = i$ on $\tilde{B}$, $\tilde{f} = f$ on $f^{-1}A$. (See picture below). Then we have the

PROPOSITION. *The flow equivalence relation on homeomorphisms of Cantor sets is generated by topological conjugacy and expansion.*

Proof of Proposition. First, that f and $\tilde{f}$ are flow equivalent is clear from the picture

†Bob Williams pointed out that one could remove our original absolute value signs (which arose from the fact that $\det(A-I) = -\det(\tilde{A}-I)$).

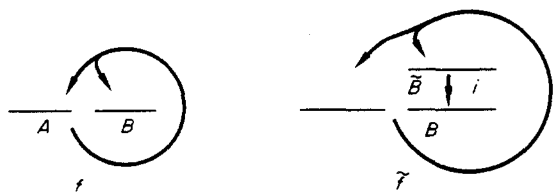

Fig. 1.

Conversely, if in one flow space we have two Cantor set cross sections X and Y, we can easily find a third Cantor set cross section Z so that the three are arrayed as in the figure

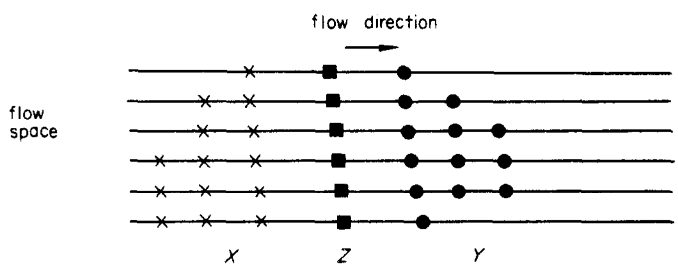

Fig. 2.

In fact, Z is a forward translate of that subset of X which meets Y before X under the flow. Clearly, the homeomorphisms of X and Y induced by the flow are multiple expansions of the Poincaré map of Z. This proves the proposition.

This proposition is given in an equivalent form in [5] in terms of discrete rational reparametrizations of flows.

The proof of the theorem is completed by the following remarks:

(i) By changing A in its shift equivalence class we can assume A is a $k \times k$ matrix of 0's and 1's and the open and closed set for an expansion is just that subset of $\prod_{-\infty}^{\infty} \{1, \dots, k\}$ in which the first coordinate is restricted to a subset of $\{1, \dots, k\}$ (see [7]).

(ii) If we expand T_A along the subset $\{\cdot \cdots .1 \cdots \cdot\}$ we obtain $T_{\tilde{A}}$.

(iii) Expanding along a disjoint union is equivalent to a consecutive expansion on each piece.

Q.E.D.

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