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GALOIS SYMMETRY IN MANIFOLD THEORY AT THE PRIMES

by Dennis SULLIVAN

Let $\mathfrak{M}$ denote the category generated by compact simply connected manifolds(1) and homeomorphisms. In this note we consider certain formal manifold categories related to $\mathfrak{M}$. We have the profinite category $\hat{\mathfrak{M}}$, the rational category $\mathfrak{M}_Q$ and the adele category $\mathfrak{M}_A$. The objects in these categories are CW complexes whose homotopy groups are modules over the ground ring of the category ($\hat{Z} = \varprojlim Z/n$, Q, and $A = Q \otimes \hat{Z}$), and which have certain additional manifold structure.

From these formal manifold categories we can reconstruct $\mathfrak{M}$ up to equivalence. For example a classical manifold M corresponds to a profinite manifold $\hat{M}$, a rational manifold M_Q, and an equivalence between the images of $\hat{M}$ and M_Q in $\mathfrak{M}_A$. In fact $\mathfrak{M}$ is the fibre product of $\hat{\mathfrak{M}}$ and $\mathfrak{M}_Q$ over $\mathfrak{M}_A$.

Thus we can study $\mathfrak{M}$ by studying these related categories. Here we find certain advantages.

– the structure of $\mathfrak{M}$ finds natural expression in the related categories.

– these categories are larger and admit more examples – manifolds with certain singularities and more algebraic entities than topological spaces.

– there is a pattern of symmetry not directly observable in $\mathfrak{M}$.

For the last point consider the collection of all non-singular algebraic variétiés over C. The Galois group of C over Q permutes these variétiés (by conjugating the coefficients of the defining relations) and provides certain (discontinuous) self maps when these coefficients are fixed.

As far as geometric topology is concerned we can restrict attention to the field of algebraic numbers $\bar{Q}$ (for coefficients) and its Galois group Gal $(\bar{Q}/Q)$. Conjugate variétiés have the same profinite homotopy type (canonically) so Gal $(\bar{Q}/Q)$ permutes a set of smooth manifold structures on one of these profinite homotopy types. [S3]

If we pass to the topological category $\hat{\mathfrak{M}}$ we find this galois action is abelian and extends to a natural group of symmetries on the category of profinite manifolds;

$$\begin{matrix}\text{abelianized} \\ \text{Gal }(\bar{Q}/Q)\end{matrix} \simeq \left\{\begin{matrix}\text{group of} \\ \text{units} \\ \text{of } \hat{Z}\end{matrix}\right\} \text{ acts on } \hat{\mathfrak{M}}.$$

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(1) The case $\pi_1 \neq 0$ can be treated to a considerable extent using families (see S3).

First description.

The possibility of defining formal manifold categories arises from the viewpoint begun by Browder and Novikov. For example, Browder observes that

(i) a manifold has an underlying homotopy type satisfying Poincaré duality.

(ii) there is a Euclidean bundle over this homotopy type – the normal bundle in R^n (which is classified by a map into some universal space B)

(iii) the one point compactification of the bundle has a certain homotopy theoretical property – a degree one map from the sphere S^n. (the normal invariant)

Novikov used the invariant of (iii) to classify manifolds with a fixed homotopy type and tangent bundle, while Browder constructed manifolds from the ingredients of (i), (ii), and (iii).

We propose tensoring such a homotopy theoretical description of a simply connected manifold with a ring R. For appropriate R we will obtain formal manifold categories $\mathfrak{M}_R$.

To have such a description of $\mathfrak{M}_R$ we assume there is a natural construction in homotopy theory $Y \to Y_R$ which tensors the homotopy groups with R (under the appropriate hypotheses) and that a map $X \xrightarrow{\nu} B_R$ has an associated sphere fibration (sphere $= S^n_R$).

There are such constructions for R any of the subrings of Q, Z_l, any of the non-Archimedean completions of Q, Q_p, the arithmetic completions of Z, $\hat{Z}_p$ and $\hat{Z}$, the finite Adeles $Q \otimes \hat{Z}$ (see $S1$).

The fibre product statement

$$\mathfrak{M} \sim \mathfrak{M}_Q \times_{\mathfrak{M}_A} \hat{\mathfrak{M}} \qquad (\hat{\mathfrak{M}} = \mathfrak{M}_{\hat{Z}})$$

follows from the Browder surgery theorem and analogous decomposition of ordinary (simply connected) homotopy theory (see [B] and [$S1$] chapter 3).

Second description.

If one pursues the study of Browder's description of classical manifolds in a more intrinsic manner – internal to the manifolds studied – certain transversality invariants occur in a natural way. These signature and arf invariants of quadratic forms on submanifolds control the situation and the structure which accrues can be expressed in the formal manifold categories see [$S2$] and [$S3$].

In the "rational manifold theory", a manifold is just a rational homotopy type satisfying homological duality over Q together with a preferred characteristic class

$$l_n + l_{n-4} + \ldots + l_{n-4i} + \ldots = l_X \in H_{n-4*}(X, Q)\ , \ n = \dim X.$$

Here l_n is an orientation class and l_0 is the signature of X (if $n \equiv 0\ (4)$).

To pursue a more precise discussion we should regard X as a specific CW complex endowed with specific chains representing the characteristic class.

Then a homotopy equivalence $X \xrightarrow{f} Y$ between two such complexes and a chain ω_f so that $f_\# l_X - l_Y = \partial\omega_f$ determines a "homeomorphism" up to concordance.

There is an analogous "homological" description for $\mathfrak{M}_A$ if we replace Q by $A = Q \otimes \hat{Z}$, (or by any field of characteristic zero).

The profinite manifold theory has a more intricate structure. First of all there is a complete splitting into p-adic components

$$\hat{\mathfrak{M}} \sim \prod_p \hat{\mathfrak{M}}_p$$

where the product is taken over the set of prime numbers and $\hat{\mathfrak{M}}_p$ is the formal manifold category based on the ring $R = \hat{Z}_p$, the p-adic integers.

For the odd primes we have a uniform structure. Let (k_*, k^*) denote the cohomology theory constructed from the p-adic completion of real K-theory by converting the filtration into a grading. Then the p-adic manifolds are just the k-duality spaces at the prime p. That is, we have a CW complex X (with p-adic homotopy groups) and a k-homology class.

$$\mu_X \in k_m(X) \qquad m = \dim X \text{ (defn)}$$

so that forming cap products with the orientation class gives the Poincaré duality

$$k^i(X) \sim k_{m-i}(X)(^1)$$

The homeomorphisms in $\hat{\mathfrak{M}}_p$ correspond to the maps $X \to Y$ giving an isomorphism of this natural duality in k-theory

$$\begin{tikzcd} k_* X \arrow[r, "\simeq"] & k_* Y \\ k^* Y \arrow[u, "\cap \mu_X", "\simeq"'] & k^* Y, \arrow[l, "\simeq"'] \arrow[u, "\cap \mu_Y", "\simeq"'] \end{tikzcd} \qquad \begin{array}{c}\text{"homeomorphism}\\ \text{condition"}\end{array}$$

ie f is a homotopy equivalence and $f_* \mu_X = \mu_Y$.

Again a more precise discussion (determining a concordance class of homeomorphisms. . .) requires the use of cycles (analogous to the chains above) and a specific homology producing the relation $f_* \mu_X = \mu_Y$.

Note that $K(X)^*$, the group of units in $k^0(X)$, acts bijectively on the set of all orientations of X. Thus the set of all manifold structures (up to equivalence) on the underlying homotopy type of X is parametrized exactly by this group of units.

Also note that a homotopy type occurs as that of a p-adic manifold precisely when there is a k-duality in the homotopy type : (see [S2] and [S3]).

At the prime 2 the manifold category is not as clear. To be sure the 2-adic manifolds have underlying homotopy types satisfying homological duality (coefficients $\hat{Z}_2$). Thus we have the natural (mod 2) characteristic classes of *Wu*.

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(1) We could reformulate this definition of a k-duality space at the prime p in terms of homological Poincaré duality and the existence of an orientation class in "periodic" K-homology. Using the connective k-theory seems more elegant and there is a natural cycle interpretation of k_* in terms of manifolds with signature free singularities. (see [S2])

We also note that the Pontryagin character of μ_X would be compatible with the rational characteristic class of a classical manifold determining X.

$$\nu_X = \nu_1 + \nu_2 + \ldots + \nu_i + \ldots \qquad i \leqslant \frac{\dim X}{2}$$

where $\nu_i \in H^i(X, Z/2)$ is defined by duality and the Steenrod operations

$$\nu_i \cup x = S_q^i x \qquad \dim x + i = \dim X.$$

The square of this class only has terms in dimensions congruent to zero mod 4

$$\nu_2^2 + \nu_4^2 + \nu_6^2 + \ldots$$

and the "manifold structure" on X defines a lifting of this class to $\hat{Z}_2$ coefficients.

(1) $$\mathcal{L}_X = l_1 + l_2 + \ldots + l_i + \ldots \qquad \text{in } H^{4*}(X, \hat{Z}_2),$$

The possible manifold structures on the homotopy type of X are acted on bijectively by a group constructed from the cohomology algebra of X. We take inhomogeneous cohomology classes,

$$u = u_2 + u_4 + u_6 + \ldots + u_{2i} + \ldots$$

using $Z/2$ or $\hat{Z}_2$ coefficients in dimensions congruent to 2 or 0 mod 4 respectively. We form a group G from such classes by calculating in the cohomology ring using the law,

$$u \cdot \nu = u + \nu + 8\, u\nu.$$

Note that G is the product of the various vector spaces of (mod 2) cohomology (in dimensions $4i + 2$) and the subgroup G_8 generated by inhomogeneous classes of $H^{4*}(X, \hat{Z}_2)$.

If we operate on the manifold structure of X by the element u in G_8 the characteristic class changes by the formula

$$\mathcal{L}_{Xu} = \mathcal{L}_X + 8\, u\,(1 + \mathcal{L}_X).$$

For example, the characterictic class mod 8 is a homotopy invariant.(see $S3$)

Local Categories

If l is a set of primes, we can form a local manifold category $\mathfrak{M}_l$, by constructing the fibre product

$$\mathfrak{M}_l \equiv \mathfrak{M}_Q \times_{\mathfrak{M}_A} \Big(\prod_{p \in l} \hat{\mathfrak{M}}_p \Big)$$

The objects in $\mathfrak{M}_l$ satisfy duality for homology over Z_l plus the additional manifold condition imposed at each prime in l and at Q.

For example we can form $\mathfrak{M}_2$ and $\mathfrak{M}_\theta$ the local categories corresponding to $l = \{2\}$ and $l = \theta = \{\text{odd primes}\}$. Then our original manifold category $\mathfrak{M}$ satisfies

$$\mathfrak{M} \cong \mathfrak{M}_2 \times_{\mathfrak{M}_Q} \mathfrak{M}_\theta$$

and we can say

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(1) Again, the Poincaré dual of $\mathcal{L}_X$ would be compatible with the rational characteristic class of a classical manifold determining X.

$\mathfrak{M}$ is built from $\mathfrak{M}_2$ and $\mathfrak{M}_\theta$ with coherences in $\mathfrak{M}_Q$

$\mathfrak{M}_2$ is defined by homological duality spaces over $Z_{(2)}$ satisfying certain homological conditions and having homological invariants (at 2).

$\mathfrak{M}_\theta$ is defined by homological duality spaces over $Z[1/2]$ with the extra structure of a $KO \otimes Z[1/2]$ orientation.

$\mathfrak{M}_Q$ is defined by homological duality spaces over Q with a rational charactéristic class.

Examples

(1) Let V be a polyhedron with the local homology properties of an oriented manifold with R coefficients. Then V satisfies homological duality for R coefficients.

If $R = Q$, the rational characteristic class can be constructed by transversality. (Thom) and we have a rational manifold

$$V \in \mathfrak{M}_Q$$

The Thom construction can be refined to give more information. The characteristic class l_V satisfies a canonical integrality condition. At 2 l_V can be lifted to an integral class. At $p > 2$ l_V can be lifted (via the Chern character) to a canonical K-homology class [S1].

So if V also satisfica Z/p – duality (p > 2) we have a k-duality space and a local manifold at odd primes, $V \in \mathfrak{M}_\theta$.

If V satisfies $Z/2$ – duality we have a good candidate for a manifold at 2. ($V \in \mathfrak{M}_2$?)

Note that such polyhedra are readily constructed by taking the orbit space of an action of a finite group π on a space W(1). For example if the transformations of π are orientation preserving then W/π is a Z/p homology manifold if W is and p is prime to the order of π. W/π is a Q homology manifold if W is.

(2) Now let V be a non-singular algebraic variety over an algebraically closed field k of characteristic p. Then the complete etale type of V determines a q-adic homological duality space at each prime q not equal to p (See [AM] and [S1]).

V has an algebraic tangent bundle T. Using the etale realization of the projective bundle of T one can construct a complex K-duality for V. To make this construction we have only to choose a generator μ_k of

$$H^1(k - \{0\}, \hat{Z}_q) \simeq \hat{Z}_q$$

This K-duality is transformed using the action of the Galois group to the appropriate (signature) duality in real K-theory, $q > 2$. If $\pi_1 V = 0$, we obtain a q-adic manifold for each $q \neq 2$ or $q \neq p$

$$[V] \in \hat{\mathfrak{M}}_q \quad (2)$$

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(1) More generally with finite isotropy groups.

(2) The prime 2 can also be treated. [S3].

Now suppose that V is the reduction mod p of a variety in characteristic zero. Let $V_{\mathbf{C}}$ denote the manifold of complex points for some embedding of the new ground ring into $\mathbf{C}$

Of course $V_{\mathbf{C}}$ determines q-adic manifolds for each q, $[V_{\mathbf{C}}] \in \hat{\mathfrak{M}}_q$.

We have the following comparison. If μ_k corresponds to the natural generator of $H^1(\mathbf{C} - 0, \hat{Z}_q)$ then

$$[V] \simeq [V_{\mathbf{C}}] \qquad \text{in } \hat{\mathfrak{M}}_q.$$

The Galois symmetry

To construct the symmetry in the profinite manifold category $\mathfrak{M}$ we consider the primes separately.

For $p > 2$ we have the natural symmetry of the p-adic units $\hat{Z}_p^*$ in isomorphism classes in $\hat{\mathfrak{M}}_p$. If M is defined by the homotopy type X with k-orientation μ_X, define M^α by X and the k-orientation μ_X^α using the Galois action of $\alpha \in \hat{Z}_p^*$ on k-theory.($q \in Z_p^*$ acts by the Adams operation ψ^q when q is an ordinary integer). Note that M and M^α have the same underlying homotopy type(1).

For $p = 2$ we proceed less directly. Let M be a manifold in $\hat{\mathfrak{M}}_2$ with characteristic class $\mathcal{L}_M = l_1 + l_2 + \ldots$. If $\alpha \in Z_2^*$ define $u_\alpha \in G_8(M)$ by the formula

$$1 + 8\,u_\alpha = \frac{1 + \alpha^2 l_1 + \alpha^4 l_2 + \ldots}{1 + l_1 + l_2 + \ldots}$$

ie $$u_\alpha = \left(\frac{\alpha^2 - 1}{8}\right) l_1 + \left(\frac{\alpha^4 - 1}{8} l_2 + \frac{1 - \alpha^2}{8} l_1^2\right) + \ldots$$

Define M^α by letting u_α act on the manifold structure of M. An interesting calculation shows that we have an action of $\hat{Z}_2^*$ on the isomorphism classes of 2-adic manifolds – again the underlying homotopy type stays fixed(1)

We have shown the

THEOREM. – *The profinite manifold category $\mathfrak{M}$ possesses the symmetry of the subfield of* $\mathbf{C}$ *generated by the roots of unity.*

The compatibility of this action of $\hat{Z}^*$ on $\mathfrak{M}$ with the Galois action on complex variéties discussed above is clear at $p > 2$, and at $p = 2$ up to the action of elements of order 8 in the underlying cohomology rings of the homotopy types. We hope to make the more precise calculation in [S3].

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(1) This connection proves the Adams conjecture for vector bundles ([S1] chapters 4 and 5), an extension to topological euclidean bundles (chapter 6), and finally an analogue in manifold theory [S3].

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