Similarity and diagonalization
(Section 4.4 in the book)

A few examples of matrix transformations

Recall that if A is an $m \times n$ matrix, then $T_A(v) = Av$ is a matrix linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

If AB is a product of matrices, then

$$T_{AB}(v) = ABv = T_A(T_B(v)).$$

If A is an invertible matrix, then

$$T_A^{-1}(v) = A^{-1}v = T_{A^{-1}}(v).$$

Example. Find a matrix P such that

$$T_P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T_P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

We will often write it as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{T_P} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{T_P} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Solution. Suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix},$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Therefore, $P = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$. □

Example. Find a matrix Q such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{T_Q} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{T_Q} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Answer: $Q = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. $\square$

Example. Find a matrix S such that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \xmapsto{T_S} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{T_S} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution. Since $T_S^{-1} = T_{S^{-1}}$, we may reformulate the problem as

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \xleftarrow{T_{S^{-1}}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xleftarrow{T_{S^{-1}}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, $S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. $\square$

Example. Find a matrix R such that

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} \xmapsto{T_R} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{T_R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution. As in the previous example:

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} \xleftarrow{T_{R^{-1}}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xleftarrow{T_{R^{-1}}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, $R^{-1} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \end{bmatrix}$. $\square$

Example. Find a matrix L such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{T_L} 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{T_L} -6 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Answer: $L = \begin{bmatrix} 3 & 0 \\ 0 & -6 \end{bmatrix}$. $\square$

Example. Find a matrix N such that

$$2\begin{bmatrix}1\\0\end{bmatrix} \xmapsto{T_N} \begin{bmatrix}1\\0\end{bmatrix}$$

$$4\begin{bmatrix}0\\1\end{bmatrix} \xmapsto{T_N} \begin{bmatrix}0\\1\end{bmatrix}.$$

Answer: $N^{-1} = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$ and $N = \begin{bmatrix}1/2 & 0\\0 & 1/4\end{bmatrix}$. $\square$

In the last two examples the matrices L and N are **diagonal**.

Similarity and diagonalization

Let A, B be $n \times n$ matrices. We say that A is **similar** to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we **write** $A \sim B$.

An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to B – that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Theorem (4.23 in the book). Let A be an $n \times n$ matrix. Then A is diagonal if and only if A has n linearly independent eigenvectors.

More precisely, there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entrices of D are eigenvalues of A corresponding to the eigenvectors in P in the same order.

Let us illustrate the above theorem in the following **example**. Consider a matrix $A = \begin{bmatrix}5 & -1\\2 & 2\end{bmatrix}$. Using the previous lectures, we can find eigenvalues of A and bases for the corresponding eigenspaces. The matrix A has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ corresponding to eigenvalues $4, 3$ respectively:

$$T_A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5 & -1\\2 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5-1\\2+2\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix}$$

$$T_A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5 & -1\\2 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5-2\\2+4\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix},$$

where $T_A(v) = Av$. We may write:

$$T_A\left(x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix}\right) = 4x\begin{bmatrix}1\\1\end{bmatrix} + 3y\begin{bmatrix}1\\2\end{bmatrix} \tag{1}$$

for every x, y.

Set $D = \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}$. We have:

$$T_D\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix}$$

$$T_D\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix} = 3\begin{bmatrix}0\\1\end{bmatrix},$$

and we may also write

$$T_D\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = 4x\begin{bmatrix}1\\0\end{bmatrix} + 3y\begin{bmatrix}0\\1\end{bmatrix} \tag{2}$$

for every x, y.

Observe the "similarity" between Equations (1) and (2). The theorem states that A and D are indeed similar:

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}5 & -1\\2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix},$$

or $P^{-1}AP = D$, where $P = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ is a matrix whose columns are linearly independent eigenvectors.

Explanation why $P^{-1}AP$ is diagonal. Observe first that $P = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ is constructed so that

$$\begin{bmatrix}1\\0\end{bmatrix} \xmapsto{T_P} \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \xmapsto{T_P} \begin{bmatrix}1\\2\end{bmatrix};$$

in other words, T_P maps $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ to eigenvectors of A.

Let us calculate $P^{-1}AP\begin{bmatrix}1\\0\end{bmatrix}$ and $P^{-1}AP\begin{bmatrix}0\\1\end{bmatrix}$:

$$T_{P^{-1}AP}\begin{bmatrix}1\\0\end{bmatrix} = P^{-1}AP\begin{bmatrix}1\\0\end{bmatrix} = P^{-1}A\begin{bmatrix}1\\1\end{bmatrix} = P^{-1}\left(4\begin{bmatrix}1\\1\end{bmatrix}\right) = 4\begin{bmatrix}1\\0\end{bmatrix}$$

$$T_{P^{-1}AP}\begin{bmatrix}0\\1\end{bmatrix} = P^{-1}AP\begin{bmatrix}0\\1\end{bmatrix} = P^{-1}A\begin{bmatrix}1\\2\end{bmatrix} = P^{-1}\left(3\begin{bmatrix}1\\2\end{bmatrix}\right) = 3\begin{bmatrix}0\\1\end{bmatrix}.$$

We may illustrate the above calculations by the following picture

$$\begin{tikzcd}
\begin{bmatrix}1\\1\end{bmatrix} \arrow[rr, mapsto, bend left, "T_A"] & \begin{bmatrix}1\\2\end{bmatrix} \arrow[rr, mapsto, bend left, "T_A"] & 4\begin{bmatrix}1\\1\end{bmatrix} \arrow[d, mapsto, "T_{P^{-1}}"'] & 3\begin{bmatrix}1\\2\end{bmatrix} \arrow[d, mapsto, "T_{P^{-1}}"'] \\
\begin{bmatrix}1\\0\end{bmatrix} \arrow[u, mapsto, "T_P"] & \begin{bmatrix}0\\1\end{bmatrix} \arrow[u, mapsto, "T_P"] & 4\begin{bmatrix}1\\0\end{bmatrix} & 3\begin{bmatrix}0\\1\end{bmatrix}
\end{tikzcd}$$

Since $T_{P^{-1}AP}\begin{bmatrix}1\\0\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix}$ and $T_{P^{-1}AP}\begin{bmatrix}0\\1\end{bmatrix} = 3\begin{bmatrix}0\\1\end{bmatrix}$, the matrix $P^{-1}AP$ must be $D = \begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}$. This explains $P^{-1}AP = D$.

Look at more examples in the book.