MAT 211: Linear Algebra

Practice problems

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Problem 1.
Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}.$$

Find the standard matrix of T.

Solution. We have:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(-1/3\begin{bmatrix}1\\2\end{bmatrix} + 2/3\begin{bmatrix}2\\1\end{bmatrix}\right) = -1/3\begin{bmatrix}1\\2\end{bmatrix} + 2/3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1/3\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(2/3\begin{bmatrix}1\\2\end{bmatrix} - 1/3\begin{bmatrix}2\\1\end{bmatrix}\right) = 2/3\begin{bmatrix}1\\2\end{bmatrix} - 1/3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2/3\\1\end{bmatrix}$$

$\square$

Answer: $\begin{bmatrix}-1/3 & 2/3\\0 & 1\end{bmatrix}$. $\square$

Problem 2.
Consider the matrix

$$A = \begin{bmatrix}1 & 0 & 0 & 2\\0 & 0 & 2 & 0\\0 & 2 & 0 & 0\\2 & 0 & 0 & k\end{bmatrix},$$

where k is a real parameter.

1. Find the determinant of A and say for which values of k the matrix A is invertible.
2. Find the dimensions of null(A) and col(A) as k varies.
3. For $k = 4$,
   - find the eigenvalues of A;
   - find an eigenvector corresponding to the eigenvalue $\lambda = 5$.

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Solution. 1. We have

$$\det\begin{bmatrix}1 & 0 & 0 & 2\\0 & 0 & 2 & 0\\0 & 2 & 0 & 0\\2 & 0 & 0 & k\end{bmatrix} = 16 - 4k.$$

The matrix A is invertible if and only if $k \neq 4$.

2. If $k \neq 4$, then A is invertible, thus the dimension of $\mathrm{col}(A)$ is 4 and the dimension of $\mathrm{null}(A)$ is 0.

   Consider the case $k = 4$. Then

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}$$

   and we can check that the dimension of $\mathrm{col}(A)$ is 3 and the dimension of $\mathrm{null}(A)$ is 1.

3. For $k = 4$, we calculate the characteristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & 0 \\ 2 & 0 & 0 & 4-\lambda \end{bmatrix} = (\lambda^2 - 5\lambda)(\lambda^2 - 4).$$

   Therefore, A has eigenvalues $5, 0, -2, 2$.

   For $\lambda = 5$, we have

$$A - \lambda I = \begin{bmatrix} -4 & 0 & 0 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & -5 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix},$$

   and we calculate that $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 5$.

$\square$

Problem 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

1. Find the eigenvalues and corresponding eigenspaces of A. Conclude that A is diagonalizable.

2. Write down a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ consisting of eigenvectors of A. Using this, find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

3. Find the coordinates of $\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$ with respect to the basis $\mathcal{B}$; i.e. write $\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$ as a linear combination of v_1, v_2, and v_3.

4. Compute $A^{456} \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$.

Solution. 1. The eigenvalues of A are 0 and 3;

- span $\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$ is the eigenspace corresponding to 0, and $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$ are two linearly independent eigenvectors corresponding to 0,
- span $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ is the eigenspace corresponding to 3, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector corresponding to 3.

2. $\mathcal{B} = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a basis consisting of eigenvectors of A. Set

$$S = \begin{bmatrix}1 & 1 & 1\\-1 & 0 & 1\\0 & -1 & 1\end{bmatrix},$$

then $S^{-1}AS$ is a diagonal matrix.

3. Solving the system

$$\begin{bmatrix}6\\-1\\-2\end{bmatrix} = c_1 \begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3 \begin{bmatrix}1\\1\\1\end{bmatrix}$$

we obtain $c_1 = 2, c_2 = 3, c_3 = 1$; i.e.:

$$\begin{bmatrix}6\\-1\\-2\end{bmatrix} = 2 \begin{bmatrix}1\\-1\\0\end{bmatrix} + 3 \begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

4. We have:

$$A^{456} \begin{bmatrix}6\\-1\\-2\end{bmatrix} = A^{456} \left(2 \begin{bmatrix}1\\-1\\0\end{bmatrix} + 3 \begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}\right) =$$

$$0^{456} \times 2 \begin{bmatrix}1\\-1\\0\end{bmatrix} + 0^{456} \times 3 \begin{bmatrix}1\\0\\-1\end{bmatrix} + 3^{456} \begin{bmatrix}1\\1\\1\end{bmatrix} = 3^{456} \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

□

Problem 4.

Write the matrix representing the linear transformation T on $\mathbb{R}^2$ that reflects vectors about the line $y = x$. Is it invertible? What about diagonalizability?

Solution. We can compute that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}.$$

Therefore, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix of T.

The transformation T is invertible.

The transformation T has two linearly independent eigenvectors, thus T is diagonalizable. ☐

Problem 5.

Consider the vector subspace $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$. Find the projection of $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ onto W and onto $W^{\perp}$.

Solution. We have:

$$\operatorname{proj}_W \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \frac{\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{4+4+6}{1+4+9} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and:

$$\operatorname{proj}_{W^{\perp}} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \operatorname{perp}_W \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \operatorname{proj}_W \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

☐

Problem 6.

Find all a, b, c such that the following matrices are simultaneously non-invertible

$$\begin{bmatrix} a-4 & -2 \\ b & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 4-a-c & a+b \end{bmatrix}, \quad \begin{bmatrix} 1 & \frac{1}{2}-c \\ 2 & a+b \end{bmatrix}.$$

Answer: $a = 2, b = 1, c = -1$. ☐

Solution. The matrices are non-invertible if and only if their determinants are zero. We need to solve the system:

$$\det \begin{bmatrix} a-4 & -2 \\ b & 1 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & 1 \\ 4-a-c & a+b \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & \frac{1}{2}-c \\ 2 & a+b \end{bmatrix} = 0$$

or:

$$a - 4 - (-2)b = 0$$

$$a + b - (4 - a - c) = 0$$

$$a+b-2\left(\frac{1}{2}-c\right)=0$$

or:

$$a+2b=4$$

$$2a+b+c=4$$

$$a+b+2c=1$$

or:

$$\left[\begin{array}{ccc|c}1&2&0&4\\2&1&1&4\\1&1&2&1\end{array}\right]\xrightarrow{\substack{R_2-2R_1\\R_3-R_1}}\left[\begin{array}{ccc|c}1&2&0&4\\0&-3&1&-4\\0&-1&2&-3\end{array}\right]\xrightarrow{R_2-3R_3}\left[\begin{array}{ccc|c}1&2&0&4\\0&0&-5&5\\0&-1&2&-3\end{array}\right]$$

$$\xrightarrow{\substack{R_2/(-5)\\-R_3}}\left[\begin{array}{ccc|c}1&2&0&4\\0&0&1&-1\\0&1&-2&3\end{array}\right]\xrightarrow{R_3+2R_2}\left[\begin{array}{ccc|c}1&2&0&4\\0&0&1&-1\\0&1&0&1\end{array}\right]\xrightarrow{R_1-2R_3}\left[\begin{array}{ccc|c}1&0&0&2\\0&0&1&-1\\0&1&0&1\end{array}\right].$$

Therefore, $a=2, b=1, c=-1$. □

Problem 7.
Suppose v_1, v_2, v_3 are linearly independent vectors.

1. Find all scalars a and b such that

$$2av_1-v_2=v_1+bv_2.$$

2. Find all scalars k such that the vectors

$$v_1+3v_3,\quad 2v_1+kv_3,\quad 2v_2$$

are linearly dependent.

Solution. 1. We have:

$$(2a-1)v_1+(-1-b)v_2=0.$$

Since v_1, v_2 are linearly independent, we obtain $2a-1=0$ and $-1-b=0$. Answer: $a=\frac{1}{2}, b=-1$.

2. The vectors $v_1+3v_3,\ \ 2v_1+kv_3,\ \ 2v_2$ are linearly independent if and only if there are scalars c_1, c_2, c_3, at least one of which is not zero, such that

$$c_1(v_1+3v_3)+c_2(2v_1+kv_3)+c_3(2v_2)=0,$$

or:

$$(c_1+2c_2)v_1+(2c_3)v_2+(3c_1+kc_2)v_3=0.$$

Since v_1, v_2, v_3 are linearly independent, $(c_1+2c_2)v_1+(2c_3)v_2+(3c_1+kc_2)v_3$ is equal to 0 if and only if

$$c_1+2c_2=0$$

$$2c_3 = 0$$

$$3c_1 + kc_2 = 0.$$

We need to find all k such that the last system has a non-zero solution. Using the determinant test, we have:

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 3 & k & 0 \end{bmatrix} = 0,$$

we obtain that $k = 6$.

□

Problem 8. Find an orthogonal basis of $\operatorname{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$.

Solution. Since $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ are linearly independent, $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ form a basis of $W = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$.

The vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}$ also form a basis for W, and we need to find t so that $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}$ are orthogonal vectors.

We have:

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \left(\begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 0,$$

or: $3 - 2t = 0$. Thus $t = 3/2$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1/3\\0\\1/3\end{bmatrix}$ is an orthogonal basis for W. □