MAT 211: Linear Algebra
Practice Midterm 2

Stony Brook University — Fall 2021
Dzmitry Dudko

A few remarks:

Theorem. Let A be an $n \times n$ matrix. The following statements are equivalent:

1. A is invertible.
2. $Ax = b$ has a unique solution for every vector $b \in \mathbb{R}^n$.
3. $Ax = 0$ has only the trivial solution.
4. $\operatorname{rank}(A) = n$.
5. $\det(A) \neq 0$.
6. 0 is not an eigenvalue of A.

Recall that

$$\operatorname{rank}(A) = \dim \operatorname{col}(A) = \dim \operatorname{row}(A) = n - \dim \operatorname{null}(A).$$

If A is an $n \times n$ matrix, then $\operatorname{rank}(A) = n$ if and only if the row vectors of A form a basis for $\mathbb{R}^n$, or, equivalently, the column vectors of A form a basis for $\mathbb{R}^n$. More details in **this Short Notes.**

Remember that

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[]{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

is a composition of elementary row operations, because we may rewrite:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right],$$

but

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[]{\substack{R_3 - R_2 \\ R_2 - R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is **not** a composition of elementary row operations; this is not allowed in the elimination method.

We also recall that $\det(AB) = \det(A)\det(B)$.

Problem 1. Solve the following system of linear equations

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right].$$

Answer: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. □

Solution. Using the elimination method:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 2 & 5 & 9 & 16 \\ 0 & 3 & 9 & 19 & 31 \end{array}\right] \xrightarrow{\substack{R_3 - 2R_2 \\ R_4 - 3R_2}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 10 & 13 \end{array}\right]$$

$$\xrightarrow{R_4 - 3R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\substack{R_1 - R_4 \\ R_2 - 3R_4 \\ R_3 - 3R_4}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\substack{R_1 - R_3 \\ R_2 - 2R_3}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

□

Problem 2. Give bases for $\text{row}(A), \text{col}(A), \text{null}(A)$, where

1) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$,

2) $A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$.

Solution. Using elementary row operations:

1) $$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3/(-2)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

- $\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ is a basis for row(A).
- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis for col $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$
  (because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for col $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$); and
- $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for null(A).

The rank of A is 3.

2)
$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 3 \\ 2 & -4 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 + R_1 \\ R_3 - 2R_1}}$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & -2 & -6 & -7 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2/2}$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

- $\begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 7/2 \end{bmatrix}$ is a basis for row(A);
- $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a basis for col $\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$
  (because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for col $\begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$);
- $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}$ is a basis for null(A).

$\square$

Problem 3. Find all possible values of rank(A) as a varies

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}.$$

Answer.

- if $a = -1$, then rank$(A) = 1$;
- if $a = 2$, then rank$(A) = 2$;
- otherwise rank$(A) = 3$.

$\square$

Solution. Using the elimination method, we obtain:

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow[]{\substack{R_2 + 2R_1 \\ R_3 - aR_1}} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} = B$$

Let us consider two cases.

Case 1: $a = -1$. Then the matrix B is equal to

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, B (and hence A) has rank 1.

Case 2: $a \neq -1$. Then we divide the second and the third rows of B by $4a + 4$ and $-2 - 2a$ respectively:

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \xrightarrow[]{\substack{R_2/(4a+4) \\ R_3/(-2-a)}} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 1 & \frac{(1-a)(1+a)}{-2-2a} \end{bmatrix} = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 1 & \frac{a-1}{2} \end{bmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = C.$$

Let us again consider two cases.

Case 2a: $a = 2$. Then

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2.

Case 2b: $a \neq 2$. Then

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} \xrightarrow{R_3/\frac{a-2}{2}} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The last matrix has rank 3. □

Problem 4. Find all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Answer: c and d are any numbers, while $b = c$ and $a = c/2$. In other words, all matrices have the form

$$\begin{bmatrix} c/2 & c \\ c & d \end{bmatrix}.$$

□

Solution. Evaluating the products, we obtain:

$$\begin{bmatrix} a & 2a \\ c & 2c \end{bmatrix} = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}.$$

We need to solve the system of linear equations:

$$\begin{aligned} a &= a \\ c &= 2a \\ 2a &= b \\ 2c &= 2b \end{aligned},$$

or:

$$\begin{aligned} 0 &= 0 \\ -2a + c &= 0 \\ 2a - b &= 0 \\ -2b + 2c &= 0 \end{aligned},$$

or:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{array}\right].$$

Solving the system, we obtain that c, d can be taken to be any numbers, and $b = c$, $a = c/2$. □

Problem 5. **(a)** Find a basis for the minimal subspace in $\mathbb{R}^4$ containing the points $(1, -1, 0, 0)$, $(0, 1, 0, -1)$, $(0, 0, -1, 1)$, $(-1, 0, 1, 0)$.

(b) Find a basis for the minimal subspace in $\mathbb{R}^3$ containing the point $(0, 1, 1)$ and the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) *Answer:* $\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}$. □

Solution. We need to find a basis for

$$\operatorname{span}\left(\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\right) = \operatorname{row} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Using the elimination method:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 + R_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 + R_2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}$ is a basis. □

(b) *Answer:* any basis of $\mathbb{R}^3$. For example:

- $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; or
- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

□

Solution. The subspace is equal to the span of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ – these are linearly independent vectors. □

Problem 6. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}.$$

Find the standard matrix of T.

Answer: $\begin{bmatrix}-6/5 & 9/5\\1 & -1\end{bmatrix}$. □

Solution 1. Recall that we write a matrix linear transformation as $T_C(v) = Cv$. We need to find a matrix C such that

$$\begin{bmatrix}-1\\1\end{bmatrix} \xmapsto{T_C} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \xmapsto{T_C} \begin{bmatrix}0\\1\end{bmatrix}.$$

Let us find matrices A, B such that

$$\begin{bmatrix}-1\\1\end{bmatrix} \xmapsto{T_A} \begin{bmatrix}1\\0\end{bmatrix} \xmapsto{T_B} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \xmapsto{T_A} \begin{bmatrix}0\\1\end{bmatrix} \xmapsto{T_B} \begin{bmatrix}0\\1\end{bmatrix}.$$

Then $T_C(v) = T_B(T_A(v))) = BAv$ and the matrix C is equal to BA; more details in Syllabus, April 11 Lecture.

Since

$$\begin{bmatrix}1\\0\end{bmatrix} \xmapsto{T_B} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \xmapsto{T_B} \begin{bmatrix}0\\1\end{bmatrix},$$

$B = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$. Since

$$\begin{bmatrix}-1\\1\end{bmatrix} \xleftarrow{T_{A^{-1}}} \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \xleftarrow{T_{A^{-1}}} \begin{bmatrix}0\\1\end{bmatrix},$$

$A^{-1} = \begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}$ and $A = \begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}^{-1}$.

We have $C = BA = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}-2/5 & 3/5\\1/5 & 1/5\end{bmatrix} = \begin{bmatrix}-6/5 & 9/5\\1 & -1\end{bmatrix}$.

□

Solution 2. Let us present $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ as linear combinations of $\begin{bmatrix}-1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$.

Write

$$\begin{bmatrix}1\\0\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\end{bmatrix}.$$

Solving this system we obtain $c_1 = -2/5$, $c_2 = 1/5$; i.e.

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{-2}{5}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}$$

and we have

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\frac{-2}{5}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}\right) = \frac{-2}{5}T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) + \frac{1}{5}T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) =$$

$$-2/5\begin{bmatrix}3\\-2\end{bmatrix} + 1/5\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-6/5\\1\end{bmatrix}.$$

Similarly, write

$$\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\end{bmatrix}.$$

Solving this system we obtain $c_1 = 3/5$, $c_2 = 1/5$; i.e.

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{3}{5}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}$$

and we have

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\frac{3}{5}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}\right) = \frac{3}{5}T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) + \frac{1}{5}T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) =$$

$$3/5\begin{bmatrix}3\\-2\end{bmatrix} + 1/5\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}9/5\\-1\end{bmatrix}.$$

Since

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-6/5\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}9/5\\-1\end{bmatrix},$$

$\begin{bmatrix}-6/5 & 9/5\\1 & -1\end{bmatrix}$ is the standard matrix of T. $\square$

Problem 7. Let u, v be a basis for $\mathbb{R}^2$. Show that

1) $u+v, u+v$ is not a basis for $\mathbb{R}^2$;

2) $u+v, v$ is a basis for $\mathbb{R}^2$;

3) $u+v, u-v$ is a basis for $\mathbb{R}^2$.

Solution: 1) Since $(u+v) - 1(u+v) = 0$, the vectors $u+v, u+v$ are linearly dependent; thus they do not form a basis.

2) Let us show that $u+v, v$ are linearly independent. Suppose

$$c_1(u+v) + c_2 v = 0.$$

Then

$$c_1 u + (c_1 + c_2)v = 0.$$

Since u and v are linearly independent, we obtain that $c_1 = 0$ and $c_1 + c_2 = 0$. This implies that $c_1 = c_2 = 0$, which shows that u, v are linearly independent.

3) Let us show that $u+v, u-v$ are linearly independent. Suppose

$$c_1(u+v) + c_2(u-v) = 0.$$

Then

$$(c_1 + c_2)u + (c_1 - c_2)v = 0.$$

Since u and v are linearly independent, we obtain

$$c_1 + c_2 = 0$$

$$c_1 - c_2 = 0.$$

Solving the last system of linear equations, we obtain $c_1 = c_2 = 0$.

Remark: vectors u, v are linearly independent if and only if $u + cv, v$ are linearly independent for every c (the same argument as in 2)). Similarly, u, v, w are linearly independent if and only if $u + cv, v, w$ are linearly independent for every c; and so on. □

Problem 8. Are the following transformations linear?

1) $T \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} x - y \\ 3 \end{bmatrix},$

2) $K \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 + y \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix},$

3) $S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ |x| \end{bmatrix}.$

Answer: 1) no, 2) no, 3) no. □

Solution. 1) $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 21 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, hence T is not a linear transformation. (If T was a linear transformation, then we would have $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

2) $K \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix} \neq 2K \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$

3) $S\begin{bmatrix}1\\0\end{bmatrix} + S\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$. (If S was a liner transformation, then we would have $S\begin{bmatrix}1\\0\end{bmatrix} + S\begin{bmatrix}-1\\0\end{bmatrix} = S\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.)

□

An **example** of a linear transformation:

$$N\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + 7\begin{bmatrix}x-y\\y\end{bmatrix}$$

because

$$N\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8 & -7\\2 & 7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

(the standard form).

Problem 9. Let F be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that F reflects a vector in the x-axis. Compute the standard matrix of F.

Solution. Reflecting a vector in the x-axis means negating the y and z-coordinates. So

$$F\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\-y\\-z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

□

Problem 10. Compute the determinant of

$$A = \begin{bmatrix}1 & -1 & 0 & 3\\2 & 5 & 2 & 6\\0 & 1 & 0 & 0\\1 & 4 & 2 & 1\end{bmatrix}.$$

Answer: 4. □

Solution 1. Expanding along the second row, and then along the first row we obtain:

$$\det(A) = \begin{vmatrix}1 & -1 & 0 & 3\\2 & 5 & 2 & 6\\0 & 1 & 0 & 0\\1 & 4 & 2 & 1\end{vmatrix} = -1\begin{vmatrix}1 & 0 & 3\\2 & 2 & 6\\1 & 2 & 1\end{vmatrix} = -1\begin{vmatrix}2 & 6\\2 & 1\end{vmatrix} - 3\begin{vmatrix}2 & 2\\1 & 2\end{vmatrix} =$$

$$-(2-12) - 3(4-2) = 10 - 6 = 4$$

□

Solution 2. Using the elimination method:

$$\det(A) = \begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} \overset{\substack{R_2-2R_1 \\ R_4-R_1}}{=} \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 7 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 2 & -2 \end{vmatrix} \overset{\substack{R_1+R_3 \\ R_2-7R_3 \\ R_4-5R_3}}{=}$$

$$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} \overset{\substack{R_2/2 \\ R_4/2}}{=} 4\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_4-R_2}{=}$$

$$4\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \overset{\substack{R_1+3R_4 \\ R_2\leftrightarrow R_3}}{=} -4\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

□

Problem 11. Is the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

invertible? If yes, compute the inverse of A.

Solution. We have:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-2R_1}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_2-2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 2 & -2 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{\substack{R_1+2R_2 \\ R_3-4R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & -3 & 4 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \xrightarrow{R_1+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right].$$

Therefore, $A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$.

The fact that A is invertible also follows from $\det(A) = 1$.

$\square$

Problem 12. Find all a such that the matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$$

is invertible.

Solution 1. Note that $\det(A) = a^3$. Therefore, A is invertible if and only if $a \neq 0$. $\square$

Solution 2. If $a = 0$, then A is clearly not invertible.

If $a \neq 0$, then

$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_1/a \\ R_2/a \\ R_3/a}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 1/a & 1 & 0 & 0 & 1/a & 0 \\ 0 & 1/a & 1 & 0 & 0 & 1/a \end{array}\right] \xrightarrow{R_2 - aR_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 1/a & 1 & 0 & 0 & 1/a \end{array}\right] \xrightarrow{R_3 - aR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 0 & 1 & 1/a^3 & -1/a^2 & 1/a \end{array}\right],$$

and $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}$. $\square$