

**MAT 211: Linear Algebra**  
Practice Midterm 1

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**Problem 1.** Check if  $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix}$  are orthogonal vectors.

*Answer:* Yes.

*Solution.* The vectors are orthogonal because their dot product is zero:

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix} = 1 \cdot 0 + 2 \cdot 6 + 4 \cdot (-3) = 0.$$

**Problem 2.** In the following problems compute  $u \cdot v$ . □

1)  $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

2)  $u = \begin{bmatrix} 1 \\ x \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 3 \\ y \end{bmatrix}$

3)  $u = \begin{bmatrix} x \\ 2 \\ -3 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ 4 \\ x \end{bmatrix}$

*Solution:* 1)  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot (-3) + 3 \cdot 1 = -1$

2)  $\begin{bmatrix} 1 \\ x \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ y \end{bmatrix} = 1 \cdot 2 + x \cdot 3 + 3 \cdot y = 2 + 3x + 3y$

3)  $u = \begin{bmatrix} x \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ x \end{bmatrix} = x \cdot 3 + 2 \cdot 4 + (-3) \cdot x = 8.$  □

**Problem 3.** In the following problems find all  $k$  such that  $u$  and  $v$  are parallel vectors.

$$1) u = \begin{bmatrix} k \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$2) u = \begin{bmatrix} k \\ 1 \\ k \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$3) u = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

*Answer:* 1)  $k = 4$ , 2) no solution, 3)  $k$  is any.

*Solution.* Since  $u, v$  are non-zero,  $u, v$  are parallel if and only if  $u = cv$  for some scalar  $c$ , equivalently if the coordinates of  $u$  and  $v$  are proportional. We have:

$$1) k/2 = 2/1, \text{ or } k = 4$$

$$2) k/1 = 1/2 = k/3 - \text{no solution.}$$

$$3) k/1 = -k/(-1); \text{ i.e. } k \text{ is any.}$$

□

**Problem 4.**

- 1) Find the general and parametric equations of the line passing through the points  $(3, 1)$  and  $(1, 0)$ .
- 2) Are the points  $(2, 1)$ ,  $(1, 2)$ , and  $(4, -1)$  on the same line? If yes, find the general and parametric equations of the line passing through these points.
- 3) Find the parametric equation of the line passing through  $(1, 1)$  and  $(0, x)$ .

*Solution.* 1) Write  $A = (1, 0)$  and  $B = (3, 1)$ , and  $O = (0, 0)$  – the origin. Then  $\vec{AB} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is a direction vector of the line. Also  $\vec{OB} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

The point  $X = (x, y)$  is on the line if and only if  $\vec{BX}$  is parallel to  $\vec{AB}$ ; equivalently if  $\vec{BX} = t\vec{AB}$ . Since  $\vec{BX} = \vec{OX} - \vec{OB}$ , we have

$$\vec{OX} = \vec{OB} + t\vec{AB}$$

or:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This is the vector form of the equation of the line. The parametric equation:

$$\begin{aligned} x &= 3 + 2t \\ y &= 1 + t. \end{aligned}$$

Observe that  $n = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$  is a normal vector, because  $n$  is orthogonal to  $\overrightarrow{AB}$ :

$$n \cdot \overrightarrow{AB} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-1) \cdot 2 + 2 \cdot 1 = 0.$$

*Remark:*  $\begin{bmatrix} -b \\ a \end{bmatrix}$  is always orthogonal to  $\begin{bmatrix} a \\ b \end{bmatrix}$  :

$$\begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = -ba + ab = 0.$$

Therefore,  $X = (x, y)$  is on the line if and only if  $\overrightarrow{BX}$  is orthogonal to  $n$ , equivalently:

$$n \cdot \underbrace{(\overrightarrow{OX} - \overrightarrow{OB})}_{\overrightarrow{BX}} = 0 \quad \text{or} \quad n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OB},$$

or

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This is the normal form of the equation of the line. The general form of the equation of the line is obtained by evaluating the dot product:

$$-x + 2y = -1.$$

- 2) Write  $A = (2, 1)$ ,  $B = (1, 2)$ ,  $C = (4, -1)$ , then  $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  and  $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ . Since  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are parallel (because  $-1/2 = 1/(-2)$ ), the points  $A$ ,  $B$ , and  $C$  are on the same line.

Since  $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  is a direction vector, the vector form of the equation is

$$\overrightarrow{OX} = \overrightarrow{OA} + t\overrightarrow{AB} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The parametric equation:

$$\begin{aligned} x &= 2 - t \\ y &= 1 + t. \end{aligned}$$

Observe that  $n = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a normal vector because  $n \cdot \overrightarrow{AB} = 0$ . The general form of the equation is obtained by evaluating the dot product in  $n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OB}$ :

$$x + y = 3.$$

(We can double check that  $C = (4, -1)$  is on the line passing through  $A$  and  $B$ :  $4 + (-1) = 3$ .)

3) Using  $x_1, x_2$ -axes, we have:

$$\begin{aligned}x_1 &= 0 + t \\x_2 &= x + t(1 - x).\end{aligned}$$

□

**Problem 5.** Find the general and parametric equations of the plane passing through the point  $(1, 1, 1)$  and orthogonal to the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

*Solution.* Write  $A = (1, 1, 1)$ . Since the vector  $n = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is normal, the general form of the equation is obtained by evaluating the dot product in  $n \cdot \vec{OX} = n \cdot \vec{OA}$ :

$$x + y + z = 3.$$

To find the parametric equation, it is sufficient to solve the equation  $x + y + z = 3$ . We set  $z = t, y = s$ , then  $x = 3 - s - t$ ; i.e.:

$$\begin{aligned}x &= 3 - s - t \\y &= s \\z &= t.\end{aligned}$$

□

**Problem 6.** Find the general and parametric equations of the plane passing through the points  $(0, 1, 2)$ ,  $(1, 0, 1)$ , and  $(2, 1, 4)$ . Does the plane also pass through the origin  $(0, 0, 0)$ ?

*Solution.* Write  $A = (0, 1, 2)$ ,  $B = (1, 0, 1)$ ,  $C = (2, 1, 4)$ , and  $O = (0, 0, 0)$  – the origin.

Then  $\vec{AB} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$  and  $\vec{AC} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$  are two non-parallel direction vectors of the plane. Also

$\vec{OB} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ . The vector form of the equation is

$$\vec{OX} = \vec{OB} + s\vec{AB} + t\vec{AC},$$

or:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

The parametric equation:

$$\begin{aligned}x &= 1 + s + 2t \\y &= -s \\z &= 1 - s + 2t.\end{aligned}$$

Next we need to find a normal vector  $n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$  of the plane. Such vector  $n$  is orthogonal to the direction vectors  $\vec{AB}$  and  $\vec{AC}$ . Therefore,  $n$  satisfies  $n \cdot \vec{AB} = 0$  and  $n \cdot \vec{AC} = 0$ , or

$$n_1 - n_2 - n_3 = 0$$

$$2n_1 + 2n_3 = 0$$

This system has infinitely many solutions,  $n_1 = 1, n_2 = 2, n_3 = -1$  is one of non-zero solutions. The normal form of the equation is obtained by evaluating the dot product in  $n \cdot \vec{OX} = n \cdot \vec{OB}$ :

$$x + 2y - z = 0.$$

Since  $(0, 0, 0)$  satisfies  $x + 2y - z = 0$ , the plane passes through the origin  $O = (0, 0, 0)$ .  $\square$

**Problem 7.** Solve the following system of linear equations:

$$x + 2y - 3z = 9,$$

$$2x - y + z = 0,$$

$$4x - y + z = 4.$$

*Answer*  $x = 2, y = 5, z = 1$ .

*Solution 1.* We have:

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array} \right] \xrightarrow{R_2/(-5)}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & -9 & 13 & -32 \end{array} \right] \xrightarrow{\substack{R_1 - 2R_2 \\ R_3 + 9R_2}} \left[ \begin{array}{ccc|c} 1 & 0 & -0.2 & 1.8 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & 0 & 0.4 & 0.4 \end{array} \right] \xrightarrow{R_3/0.4} \left[ \begin{array}{ccc|c} 1 & 0 & -0.2 & 1.8 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{\substack{R_1 + 0.2R_3 \\ R_2 + 1.4R_3}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

$\square$

*Solution 2.* To simplify calculations, let us first subtract the second equation from the last equation:

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array} \right] \xrightarrow{R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 2 & 0 & 0 & 4 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{aligned}
\left[ \begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -3 & 9 \end{array} \right] &\xrightarrow{R_1/2} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -3 & 9 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 2 & -3 & 7 \end{array} \right] \\
&\xrightarrow{-R_2} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 2 & -3 & 7 \end{array} \right] \xrightarrow{R_3 - 2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & -1 \end{array} \right] \xrightarrow{-R_3} \\
&\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{R_2 + R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array} \right].
\end{aligned}$$

□

**Problem 8.** Solve the following system of linear equations:

$$\left[ \begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array} \right]$$

*Solution.* We have:

$$\begin{aligned}
\left[ \begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array} \right] &\xrightarrow{-R_1} \left[ \begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \\
\left[ \begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array} \right] &\xrightarrow{\substack{R_2/(-3) \\ R_3/2}} \left[ \begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{array} \right] \xrightarrow{\substack{R_1 - 2R_2 \\ R_3 - R_2}} \\
&\left[ \begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

The solution:  $x_1 = -2 + 3t$ ,  $x_2 = t$ ,  $x_3 = 1 + 2s$ ,  $x_4 = s$ .

□

**Problem 9.** Determine if the following vectors are linearly independent

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}.$$

*Solution.* We need to check if the system

$$c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution; i.e. when at least one of  $c_1, c_2, c_3$  is not zero. We have:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array} \right] & \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 1 & -5 & 0 \\ 2 & 3 & 1 & 0 \end{array} \right] & \xrightarrow{\begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}} \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -1 & -3 & 0 \end{array} \right] & \xrightarrow{R_2/(-3)} \\ & & & & \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array} \right] & \xrightarrow{\begin{array}{l} R_1 - 2R_2 \\ R_3 + R_2 \end{array}} \left[ \begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

□

Therefore, the system has a non-trivial solution, for example  $c_1 = 4, c_2 = -3, c_3 = 1$ :

$$4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Problem 10.** Find all  $k$  such that the following vectors are linearly independent

$$\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

*Answer:*  $k \neq 1/2$ .

*Solution.* Two vectors  $\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  are linearly independent if and only if they are not parallel.

The vectors  $\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  are parallel if and only if  $k = 1/2$ .

□

**Problem 11.** Check if the span of the following vectors is  $\mathbb{R}^3$ .

$$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix}.$$

*Answer:* yes.

*Solution.* The span of three vectors is  $\mathbb{R}^3$  if and only if the vectors are linearly independent. Let us check if the vectors are linearly independent.

We need to check if the system

$$c_1 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution; i.e. when at least one of  $c_1, c_2, c_3$  is not zero.

We have:

$$\left[ \begin{array}{ccc|c} 2 & 1 & 7 & 0 \\ 3 & -4 & 6 & 0 \\ 3 & -2 & 0 & 0 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[ \begin{array}{ccc|c} 2 & 1 & 7 & 0 \\ 1 & -5 & -1 & 0 \\ 1 & -3 & -7 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 1 & -5 & -1 & 0 \\ 2 & 1 & 7 & 0 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - 2R_1}} \left[ \begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 7 & 21 & 0 \end{array} \right] \xrightarrow{R_2/(-2)} \left[ \begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 7 & 21 & 0 \end{array} \right]$$

$$\xrightarrow{\substack{R_1 + 3R_2 \\ R_3 - 7R_2}} \left[ \begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 42 & 0 \end{array} \right] \xrightarrow{R_3/42} \left[ \begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 + 16R_3 \\ R_2 + 3R_3}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right].$$

The last system has only the trivial solution:  $c_1 = 0, c_2 = 0, c_3 = 0$ . □

**Problem 12.** Calculate the product

$$\begin{bmatrix} 2k & 1 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

*Solution.*

$$\begin{bmatrix} 2k & 1 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2k - 1 & -2k + 1 \\ 1 - k & -1 + k \end{bmatrix}.$$

□