1 Basis and Dimension

Recall that S is a **subspace** of $\mathbb{R}^n$ if:

1) S contains 0;

2+3) a linear combination of vectors from S is in S.

Equivalently, S is a subspace if and only if $S = \text{span}(v_1, v_2, \ldots, v_k)$. If $v_1, v_2, \ldots, v_k$ are linearly independent, then $v_1, v_2, \ldots, v_k$ is a **basis** for S. In this case k is the **dimension** of S.

Example: lines and planes through the origin are subspaces.

The dimension of a subspace is independent of the choice of a basis because:

Theorem. Any two bases for a subspace have the same number of vectors.

1.1 row(A)

Consider the following **problem**: find a basis for

$$\text{span}\begin{pmatrix} \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \\ \begin{bmatrix}1 & 1 & 1\end{bmatrix}, \\ \begin{bmatrix}3 & 1 & 3\end{bmatrix} \end{pmatrix}$$

Note that $\begin{bmatrix}3 & 1 & 3\end{bmatrix} = 2\begin{bmatrix}1 & 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 1 & 1\end{bmatrix}$. By definition, $\text{span}\begin{pmatrix} \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \\ \begin{bmatrix}1 & 1 & 1\end{bmatrix}, \\ \begin{bmatrix}3 & 1 & 3\end{bmatrix} \end{pmatrix}$ is

$$\begin{gathered} c_1\begin{bmatrix}1 & 0 & 1\end{bmatrix} + c_2\begin{bmatrix}1 & 1 & 1\end{bmatrix} + c_3\begin{bmatrix}3 & 1 & 3\end{bmatrix} = \\ c_1\begin{bmatrix}1 & 0 & 1\end{bmatrix} + c_2\begin{bmatrix}1 & 1 & 1\end{bmatrix} + c_3(2\begin{bmatrix}1 & 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 1 & 1\end{bmatrix}) = \\ (c_1 + 2c_3)\begin{bmatrix}1 & 0 & 1\end{bmatrix} + (c_2 + c_3)\begin{bmatrix}1 & 1 & 1\end{bmatrix}. \end{gathered}$$

Since c_1, c_2, c_3 are any numbers, $c_1 + 2c_3$ and $c_2 + c_3$ are also any numbers. Therefore,

$$(c_1 + 2c_3)\begin{bmatrix}1 & 0 & 1\end{bmatrix} + (c_2 + c_3)\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

is

$$\text{span}\begin{pmatrix} \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \\ \begin{bmatrix}1 & 1 & 1\end{bmatrix} \end{pmatrix}$$

Clearly, $\begin{bmatrix}1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\end{bmatrix}$ are linearly independent (they are not parallel). Thus, $\begin{bmatrix}1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\end{bmatrix}$ is a basis for $\text{span}\begin{pmatrix} \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \\ \begin{bmatrix}1 & 1 & 1\end{bmatrix}, \\ \begin{bmatrix}3 & 1 & 3\end{bmatrix} \end{pmatrix}$, and this subspace has dimension 2.

Recall next that $\text{span}\begin{pmatrix} \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \\ \begin{bmatrix}1 & 1 & 1\end{bmatrix}, \\ \begin{bmatrix}3 & 1 & 3\end{bmatrix} \end{pmatrix}$ is $\text{row}\left(\begin{bmatrix}1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3\end{bmatrix}\right)$. Hence $\begin{bmatrix}1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\end{bmatrix}$ is also a basis for $\text{row}\left(\begin{bmatrix}1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3\end{bmatrix}\right)$.

Remark. Let A be a matrix. Then elementary row operations do not change row(A).

In particular, we can remake the previous example using the elementary row operations as follows:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

thus

$$\text{row} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} = \text{row} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is a basis for row $\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \right)$.

We can also continue

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eliminating the first 1 in the second row. We obtain a matrix in reduced echelon form. The dimension of row $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is clearly two.

1.2 null(A) and col(A)

Recall that elementary row operations of a matrix do not change solutions of the associated system of linear equations. Therefore, if A is a matrix, then **elementary row operations of A do not change null(A).**

In our example:

$$\text{null} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the last matrix, the corresponding system is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} x \quad + z &= 0 \\ y \quad &= 0. \end{aligned}$$

We conclude that null(A), or the set of solutions, is $t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or span $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

We can also double-check that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a solution for the original system of equations

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{we have:} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Remark. Elementary row operations change col(A) but **do not change the dimension of col**(A).

In our example, we see that the first two columns of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are linearly independent, therefore the first two columns of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ are also linearly independent. On the other hand, the last column of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is equal to its first column. Similarly, the last column of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ is equal to its first column.

1.3 Rank.

Definition. The **rank** of a matrix is the dimension of its row space. The rank of a matrix is also equal to the dimension of its column space. The rank of a matrix is equal to the number of non-zero rows in its echelon form. The **nullity** of a matrix is the dimension of its null space.

The Rank Theorem. If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Explanation. Consider a system of linear equations $Ax = 0$, where x is a column vector. Assume that A is in row echelon form. Then rank(A) is equal to the number of non-free variables, while nullity(A) is equal to the number of free variables. The number of all variables is n.

In our example,

$$\text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2$$

and

$$\text{nullity} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} = \text{nullity} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1.$$

The rank theorem takes form:

$$2 + 1 = 3.$$

1.4 Another example

Question: Find bases for $\text{row}(A), \text{null}(A), \text{col}(A)$, and find $\text{rank}(A)$, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solution. Applying the elimination method, we obtain:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow[R_3/(-1)]{R_2/(-4)} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\text{row} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \text{row} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ is a basis for $\text{row} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

We can also continue

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(reduced echelon form is more convenient than just echelon form). We see that $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ is also a basis for $\text{row} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

We have $\text{null} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and we need to solve

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have

$$\text{null} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left(\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right).$$

Since x and y are leading (non-free) variables, the first two columns of A form a basis for $\text{col}(A)$. In particular,

$$\text{col}(A) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right).$$

Finally, $\text{rank} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 2$. The Rank Theorem "$\text{rank}(A) + \text{nullity}(A) = n$" takes form $2 + 1 = 3$. $\square$