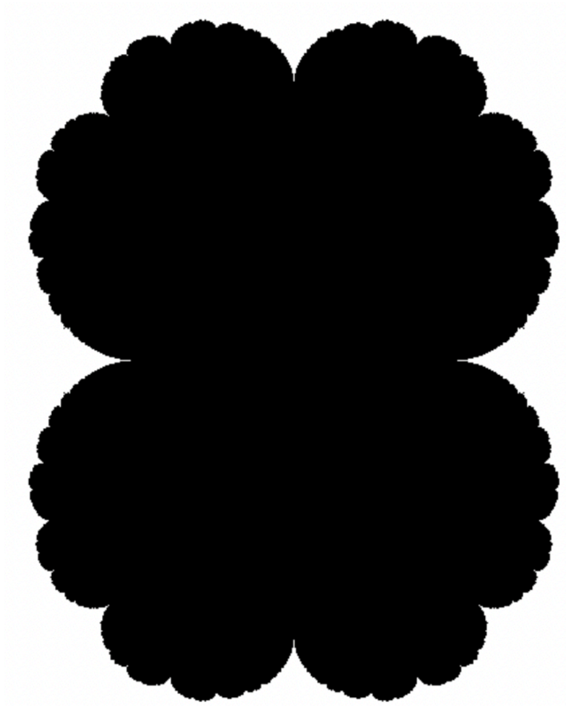
Unification of near-Parabolic and Siegel Renormalizations

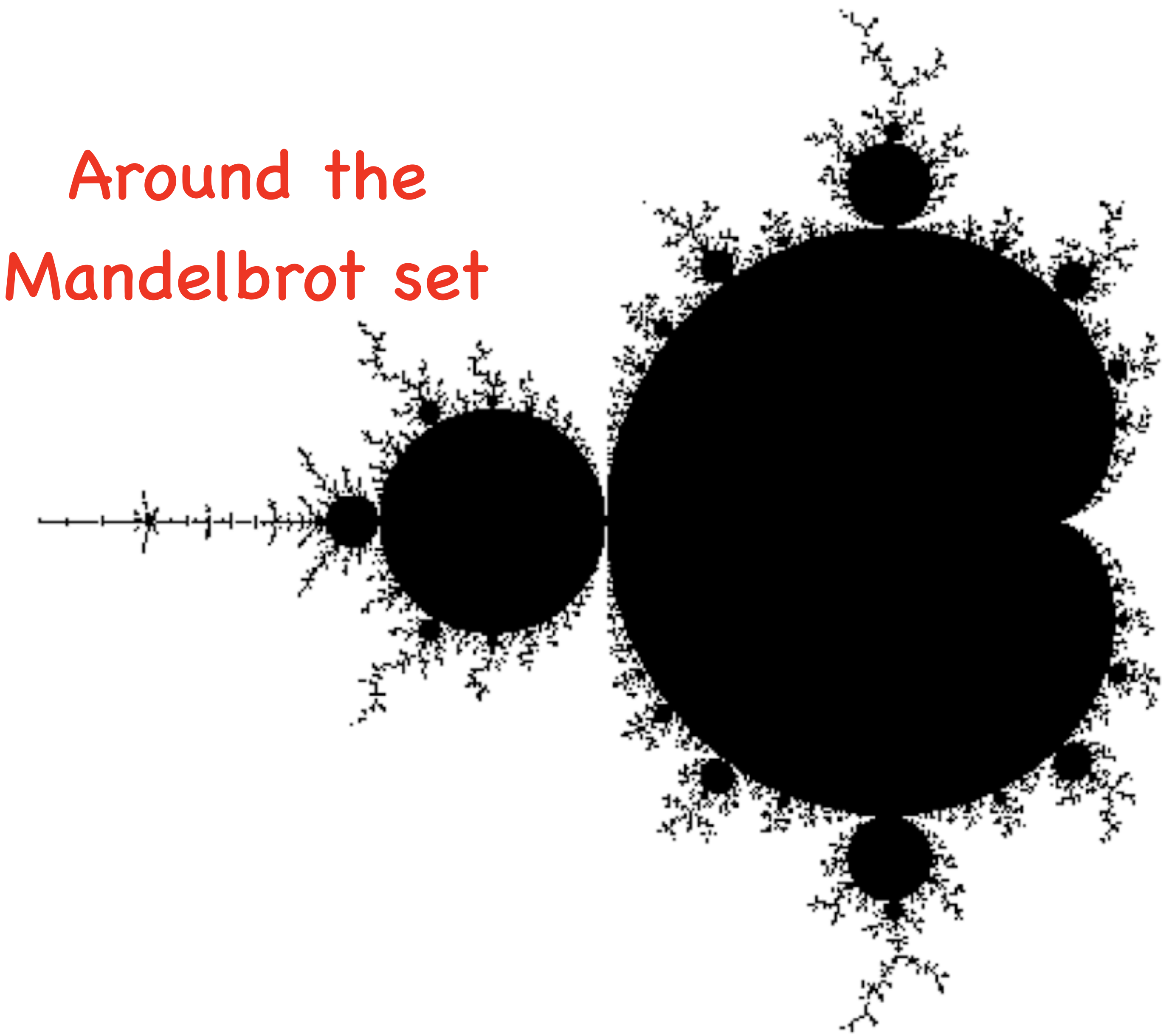
A conference celebrating the 60th birthday of

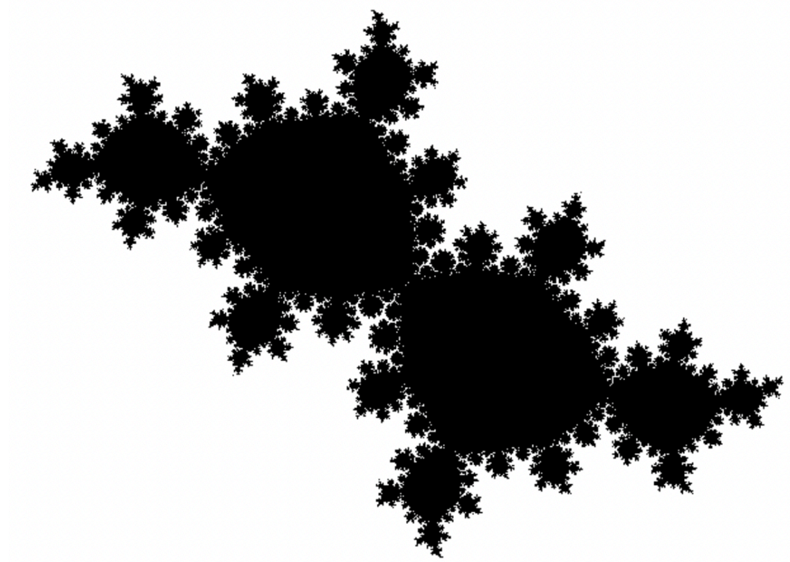
Mitsuhiro Shishikura

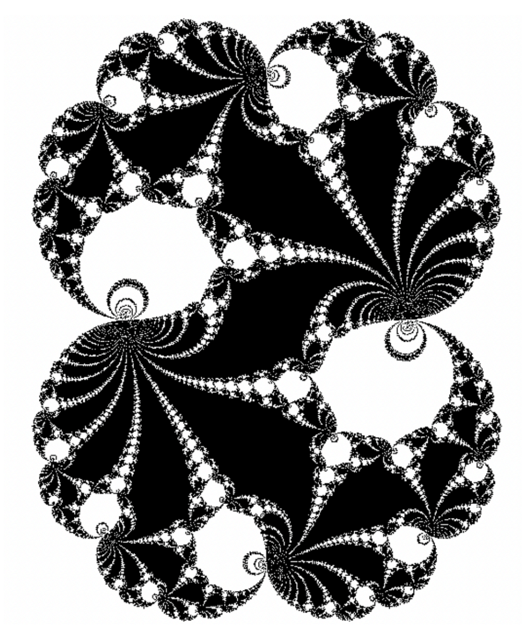
RIMS, May 30, 2023

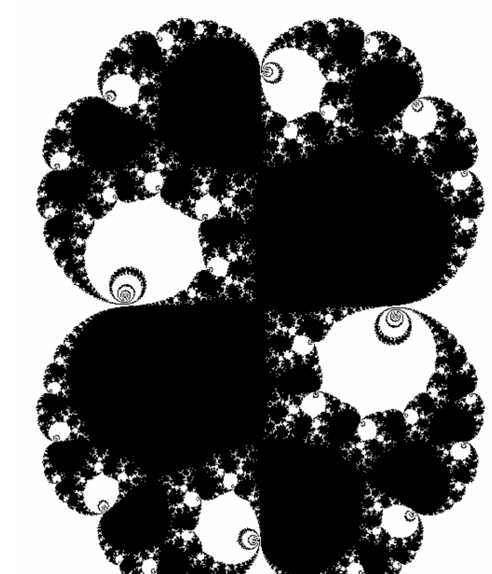


Around the Mandelbrot set





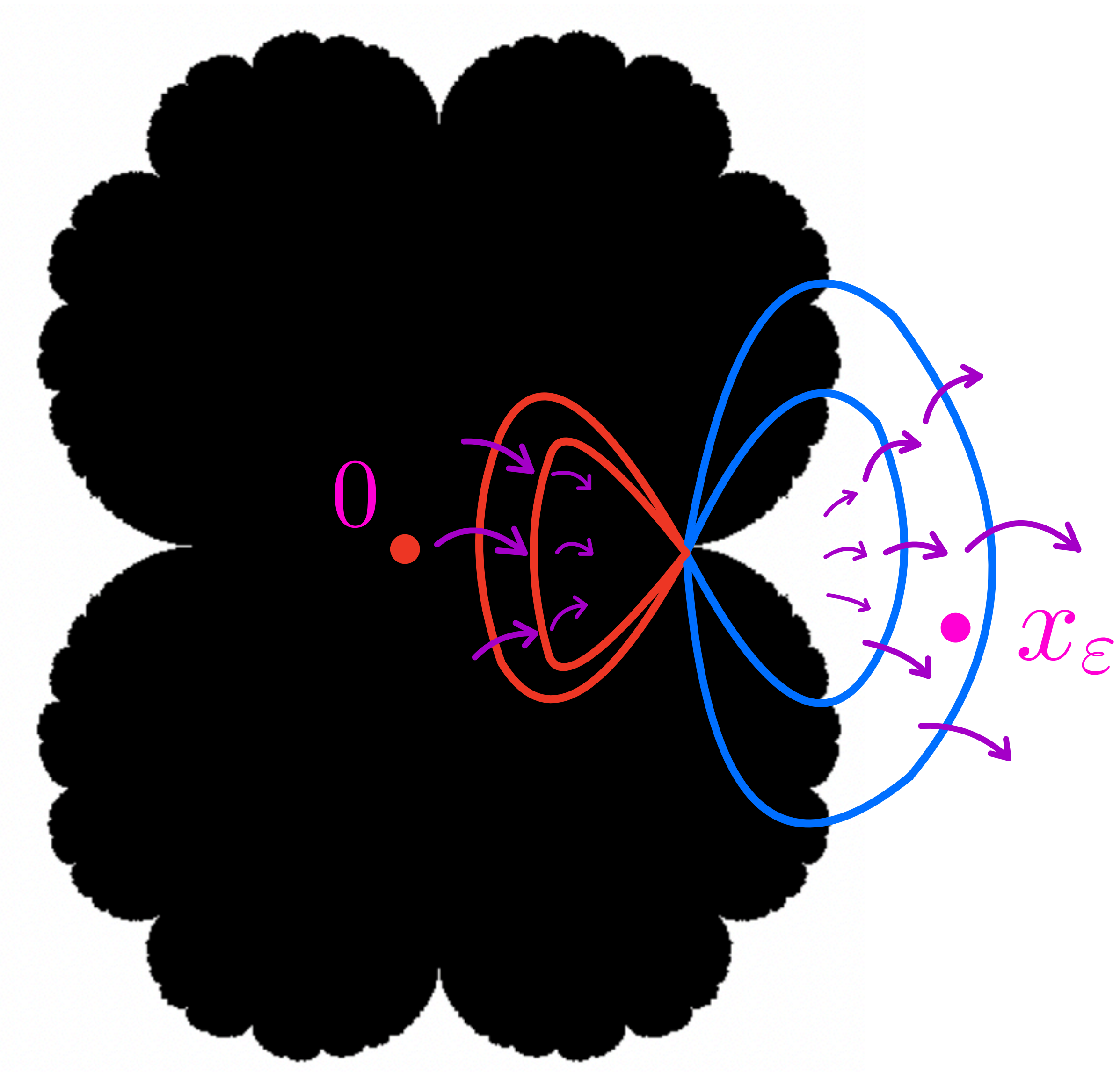




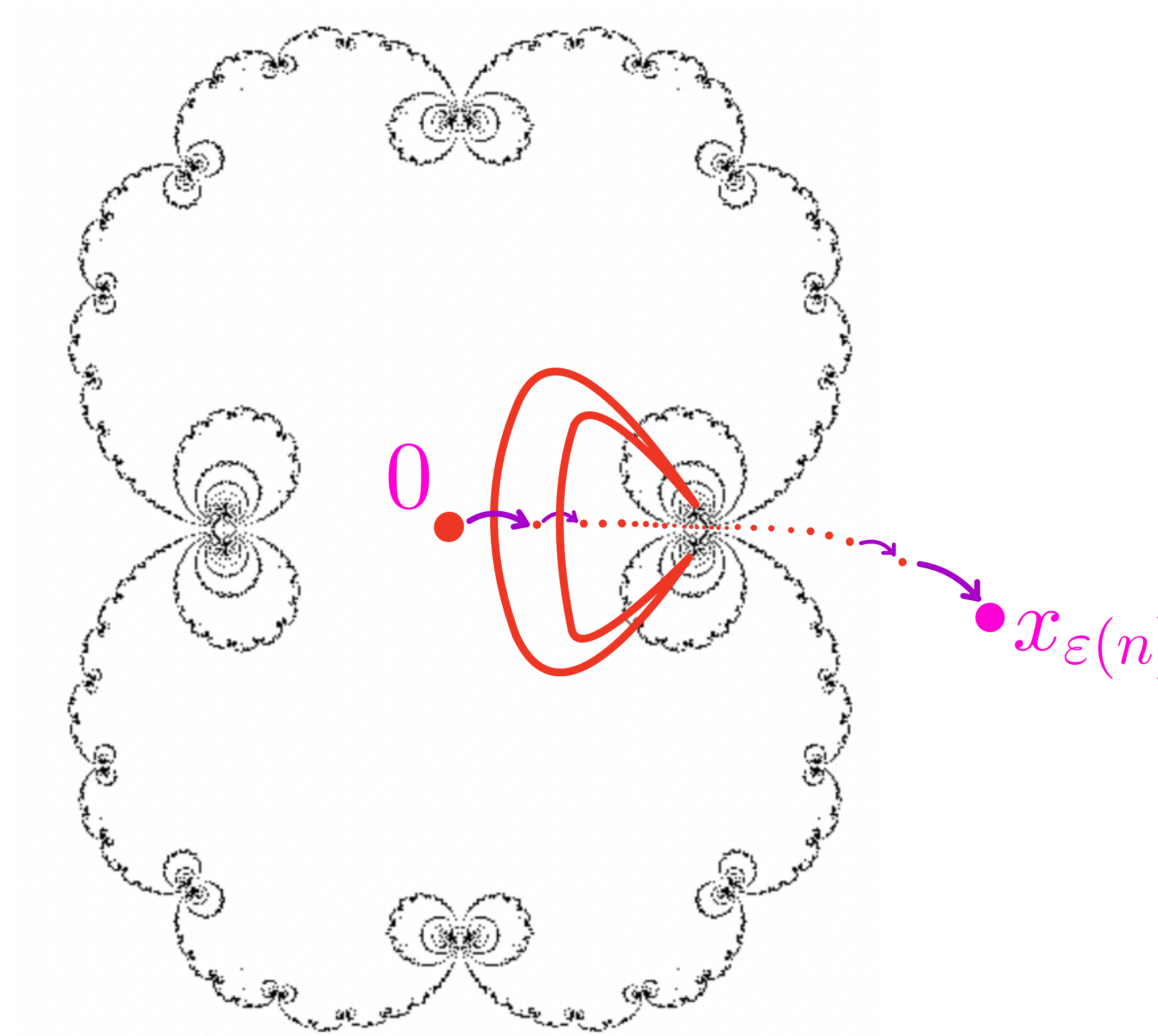
Dzmitry Dudko

Stony Brook University

$$f_0(z) = z^2 + \frac{1}{4}$$



$$f_\varepsilon(z) = z^2 + \frac{1}{4} + \varepsilon$$



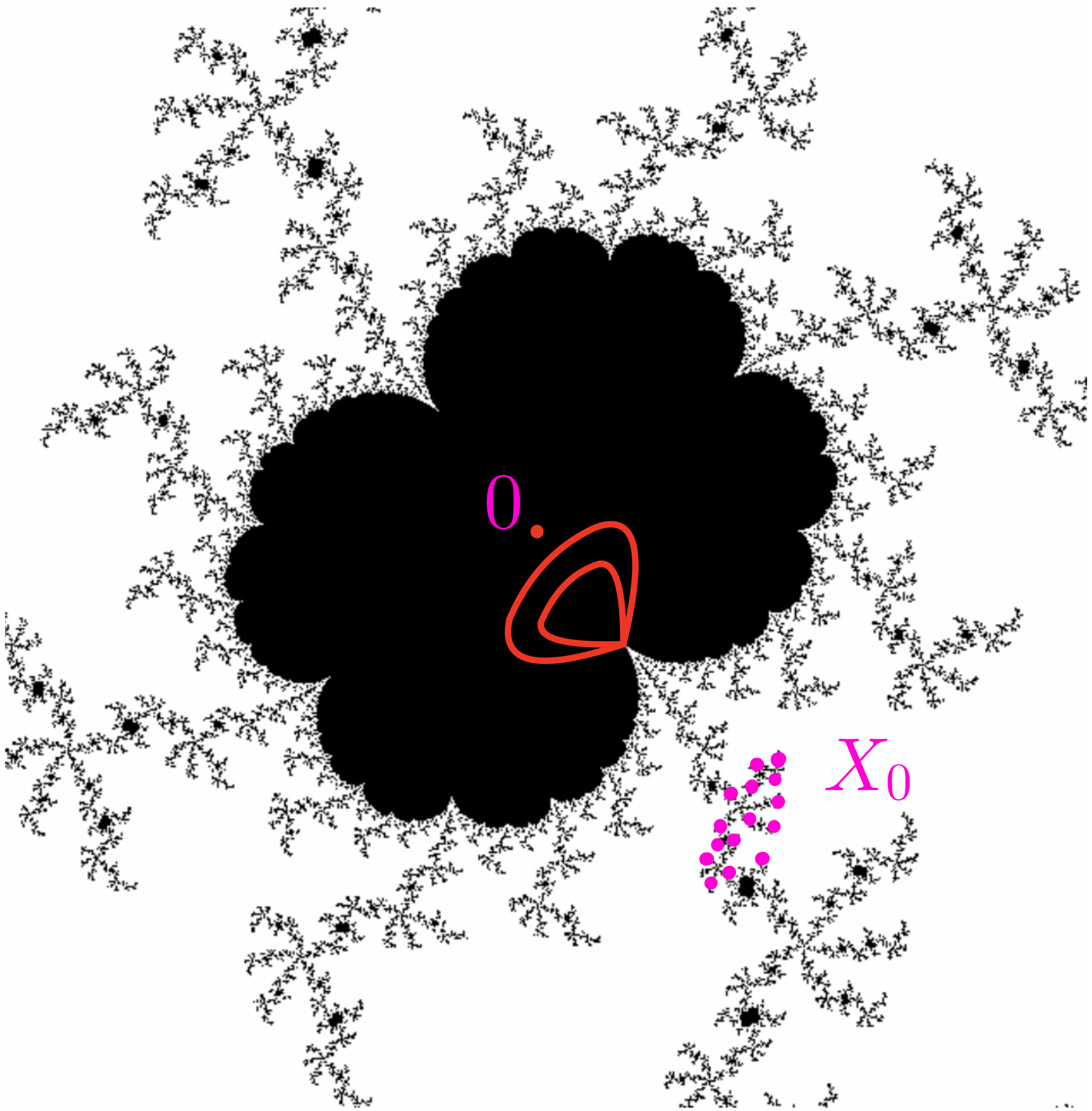
near-Parabolic Dynamics:

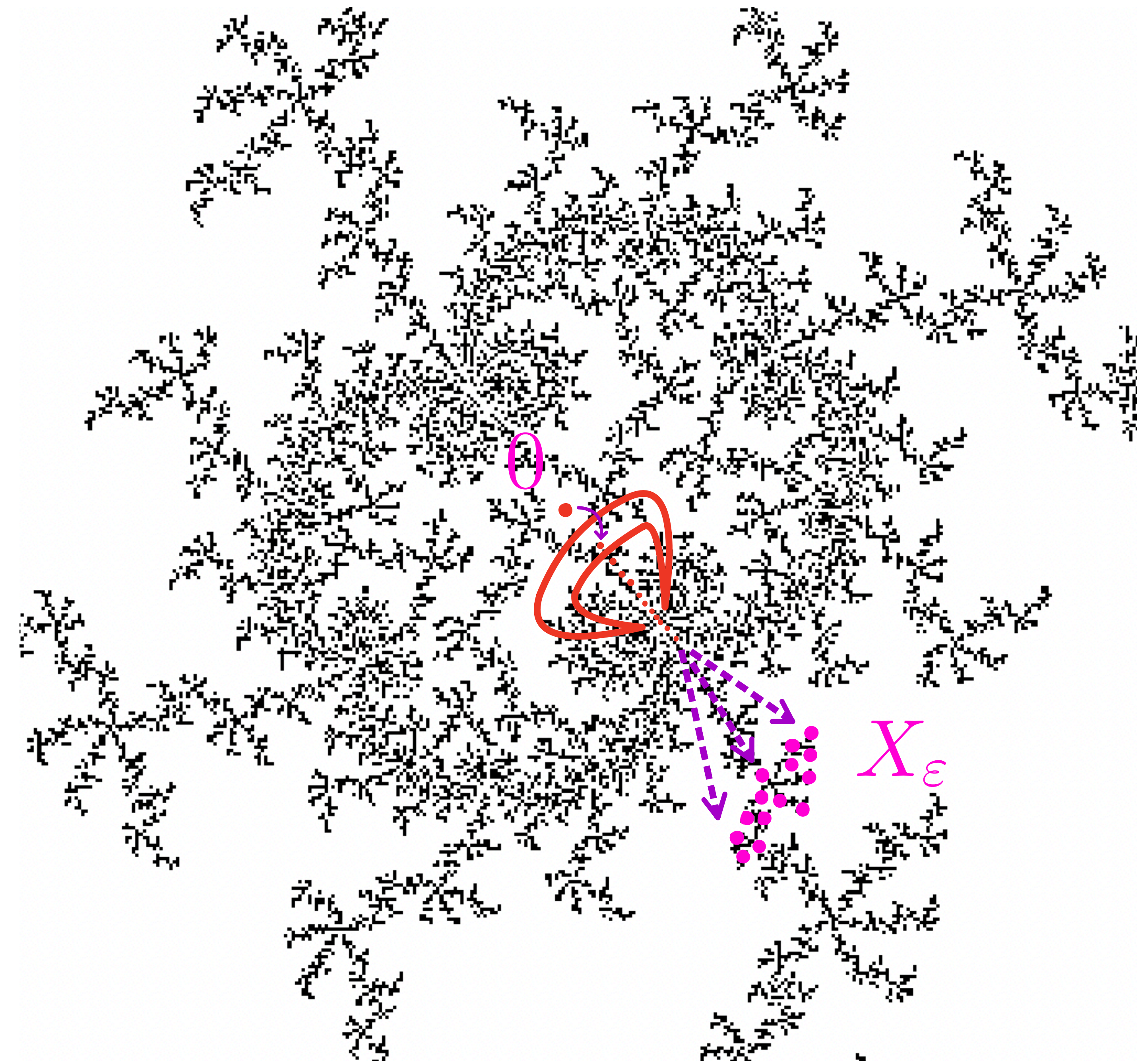
Douady, Lavaurs, Shishikura,
Epstein, Petersen, Inou,
Buff, Cheritat, Hubbard,
Cheraghi, ...

Shooting Argument:

by shooting 0 at x_ε ,
it becomes visible in the parameter plane:

$$\forall n \gg 1 \ \exists! \ \varepsilon(n) \ \text{s. t.} \ f^n_{\varepsilon(n)}(0) = x_{\varepsilon(n)}$$





Shishikura: $\exists X_\varepsilon \subset \mathcal{J}_{g_\varepsilon}$ with $\dim_{\mathrm{H}}(X_\varepsilon) \approx 2 - 0$

(1991) (double parabolic implosion)

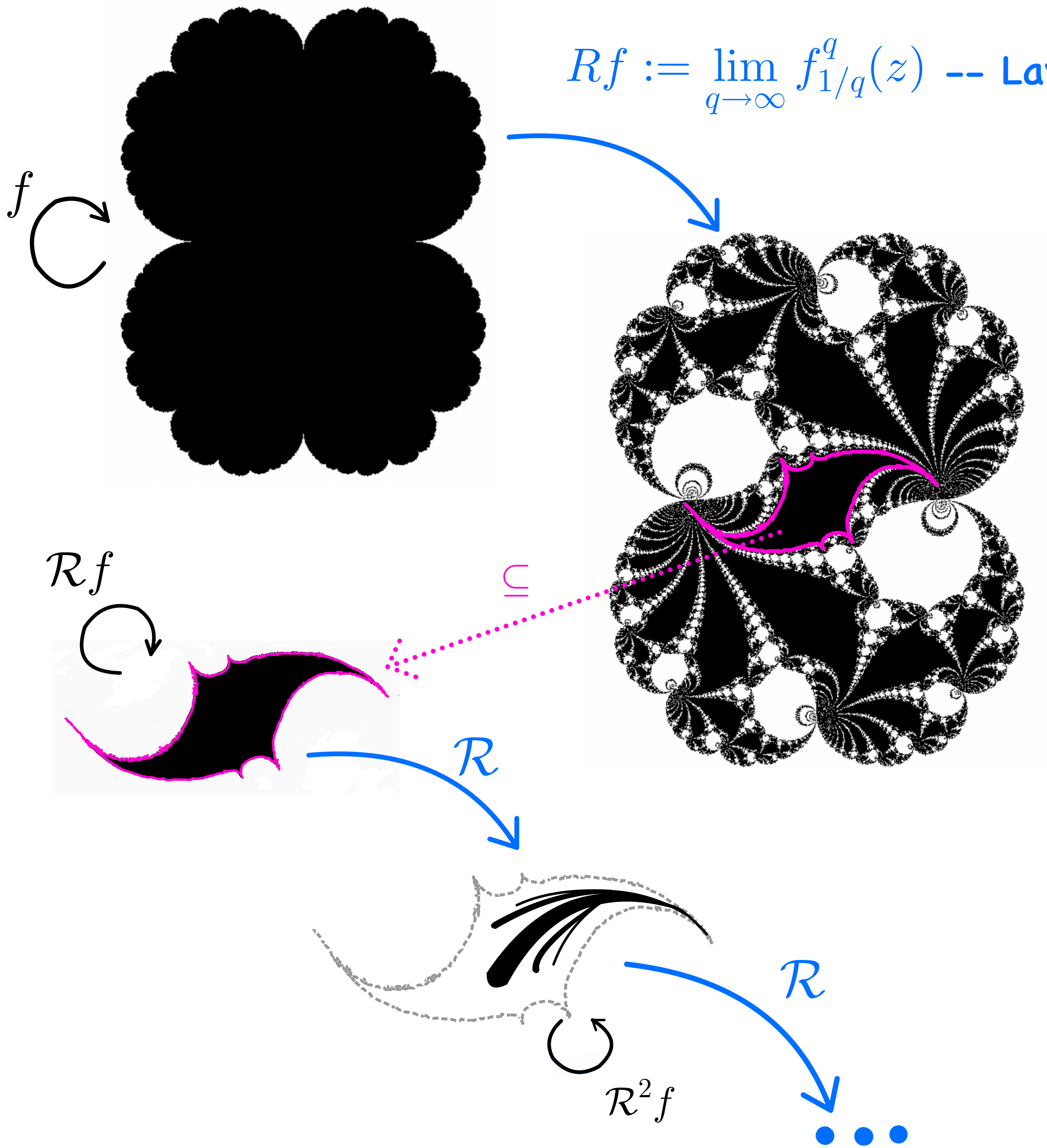
(similarly, the landing of parameter rays can be justified -- useful for non-polynomial)

Shooting 0 at X_ε we obtain

non-recurrent parameters $\rightarrow \mathcal{X} \subset \partial\mathcal{M}$ with $\dim_H(\mathcal{X}) \approx 2 - 0$

i.e., $\dim_H(\partial\mathcal{M}) = 2$

$Rf := \lim_{q\to\infty} f_{1/q}^{q}(z)$ -- Lavaurs map, $f_{1/q}$ is the fat 1/q Rabbit



Inou-Shishikura (2006):
near-Parabolic Renormalization
is hyperbolic; roughly the lim

$$f_* = \lim_{n\to\infty} [\mathcal{R}^n f]^{\text{linear rescaling}}$$

exists (sort of)

and extra properties hold;
the theory is applicable to
nearby maps (high type
combinatorics)

proven in
the context of
Cylinder
Renormalization:

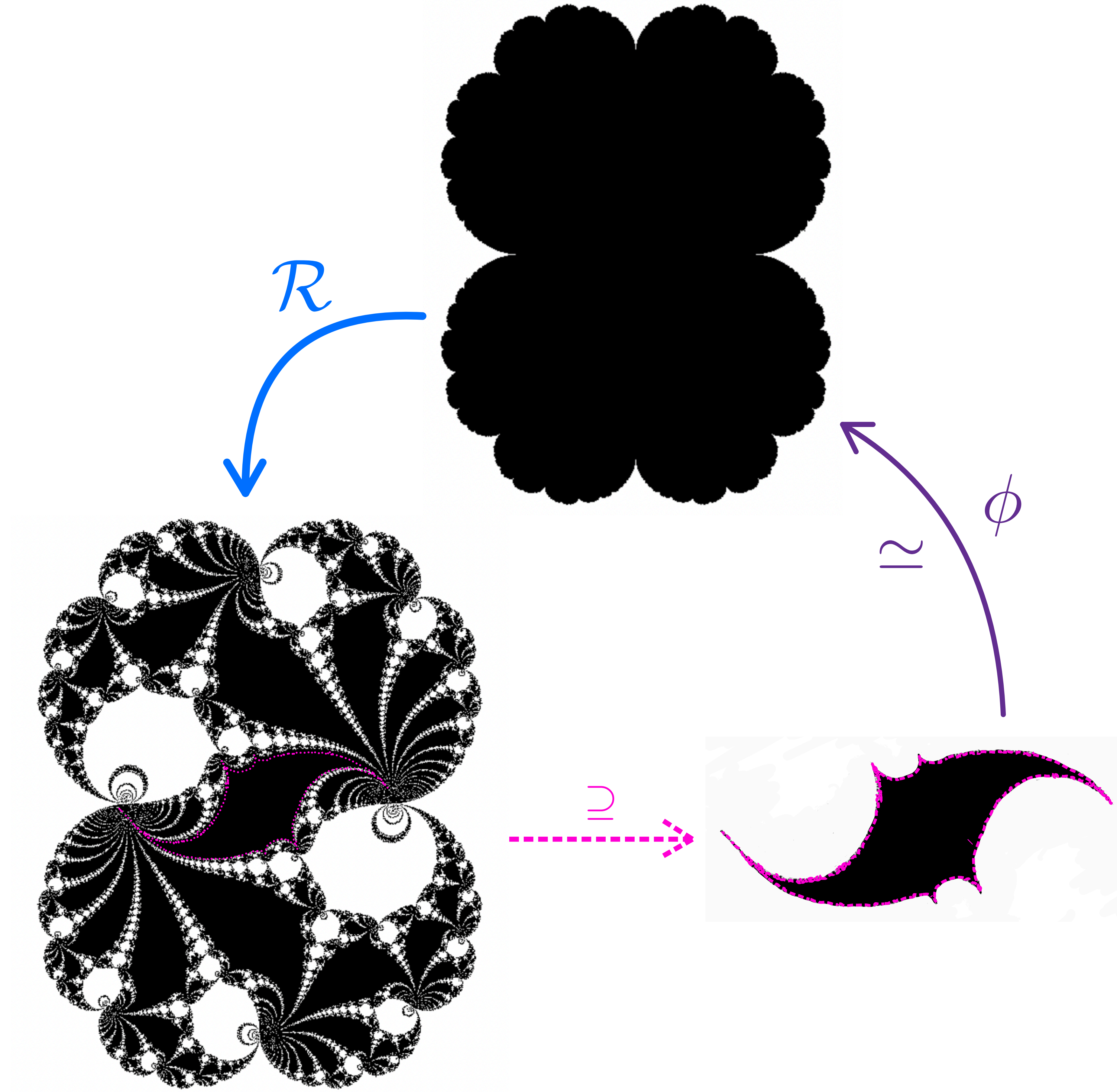
In particular, there is a renormalization fixed point

with the pattern:

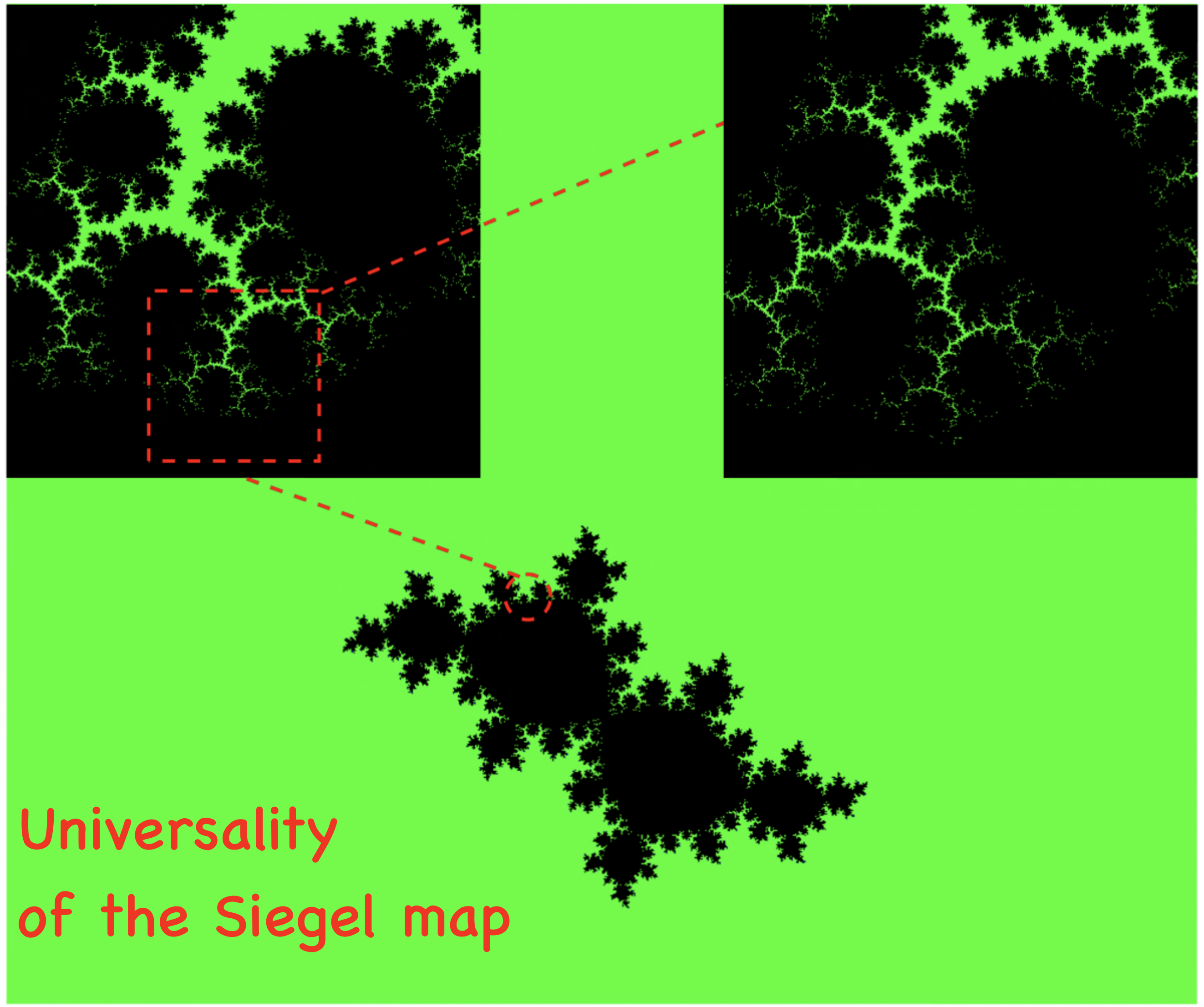
Applications of the IS-theory:

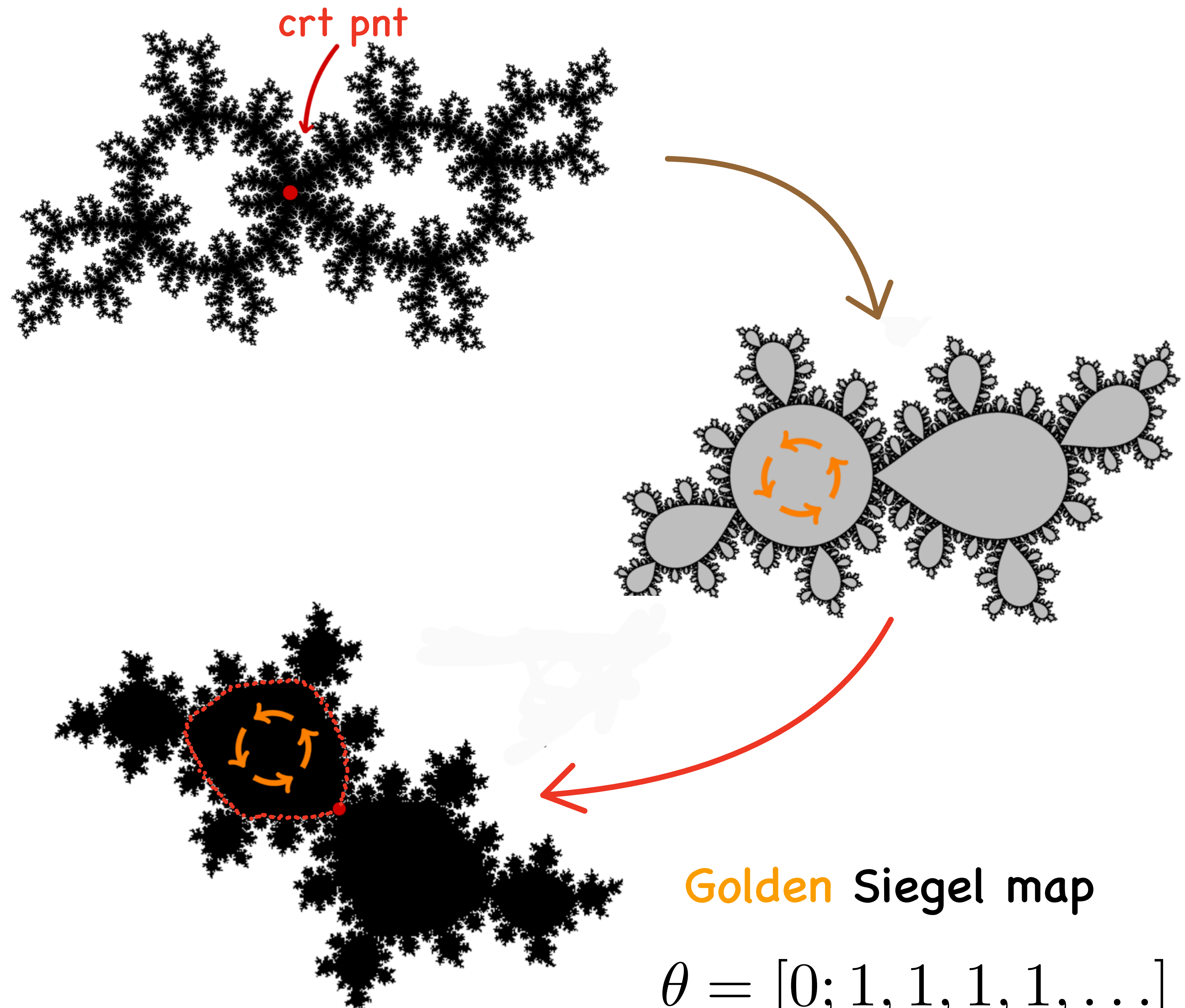
Area Problem,
Cremer/Siegel dynamics,
failure of local connectivity,
MLC, ...

Shishikura, Inou, Buff,
Cheritat, Cheraghi, Avila,
Lyubich, Yang, ...



antirenormalization change of variables ϕ^{-1} is contracting





Siegel Renormalization:

Douady, Ghys, McMullen, Peterson, Yampolsky, Zakery, Avila, Lyubich, Gaidashev, ...

it is similar to the renormalization theory of critical circle maps

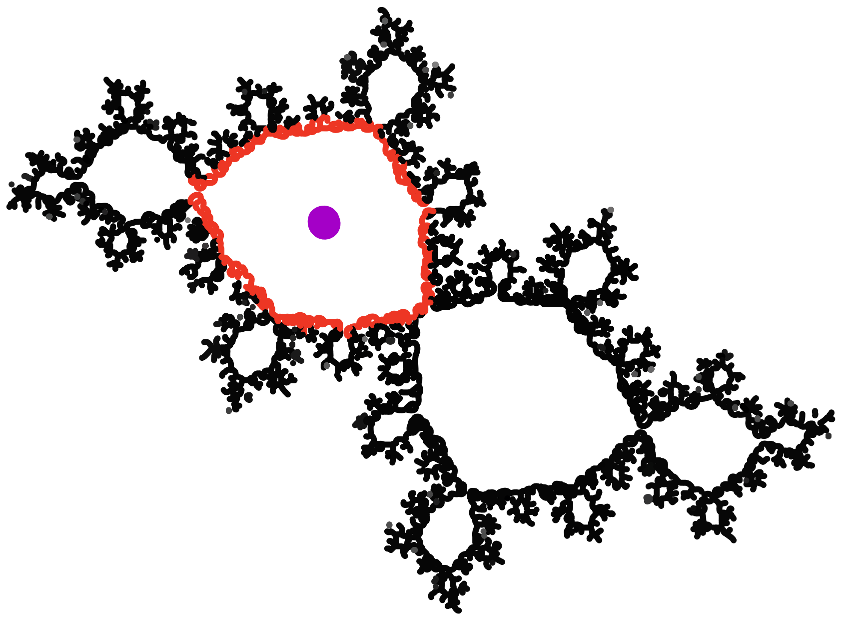
$\overline{Z}$ is a qc disk

Golden Siegel map

$$\theta = [0; 1, 1, 1, 1, \dots]$$

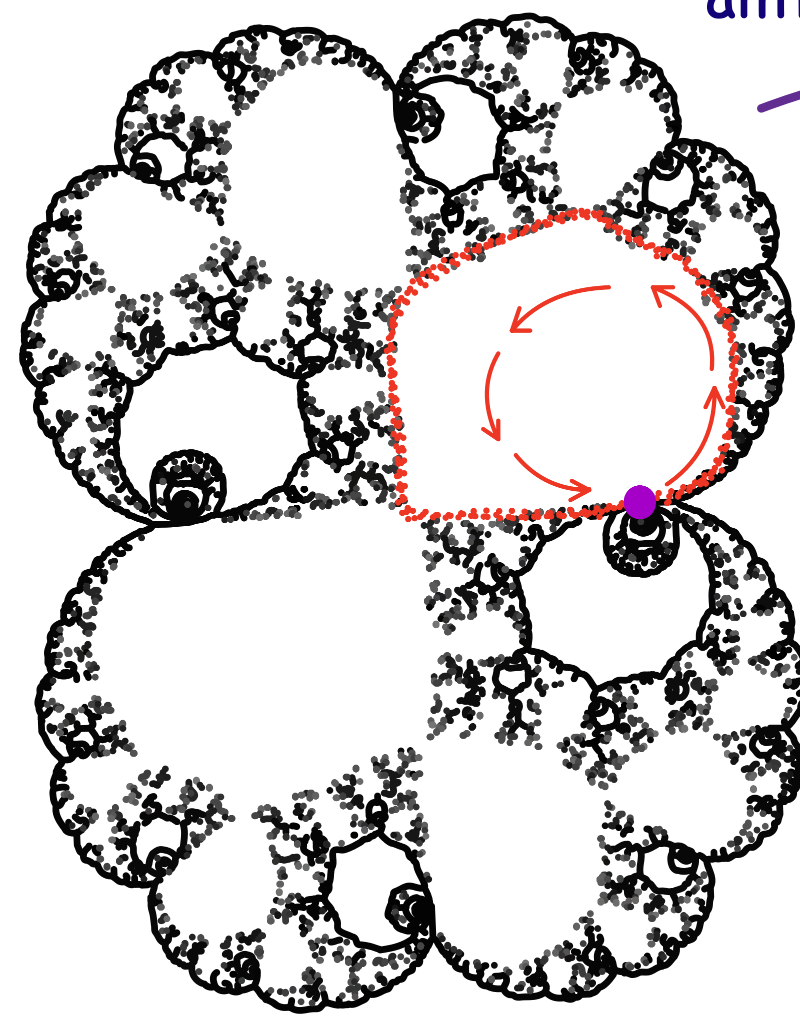
Lyubich, Selinger, DD (2017): Siegel renormalization (bounded type) is hyperbolic

Parabolic degenerations of (bounded type) Siegel disks:



$$\theta = [0; 1, 1, 1, 1, \dots]$$

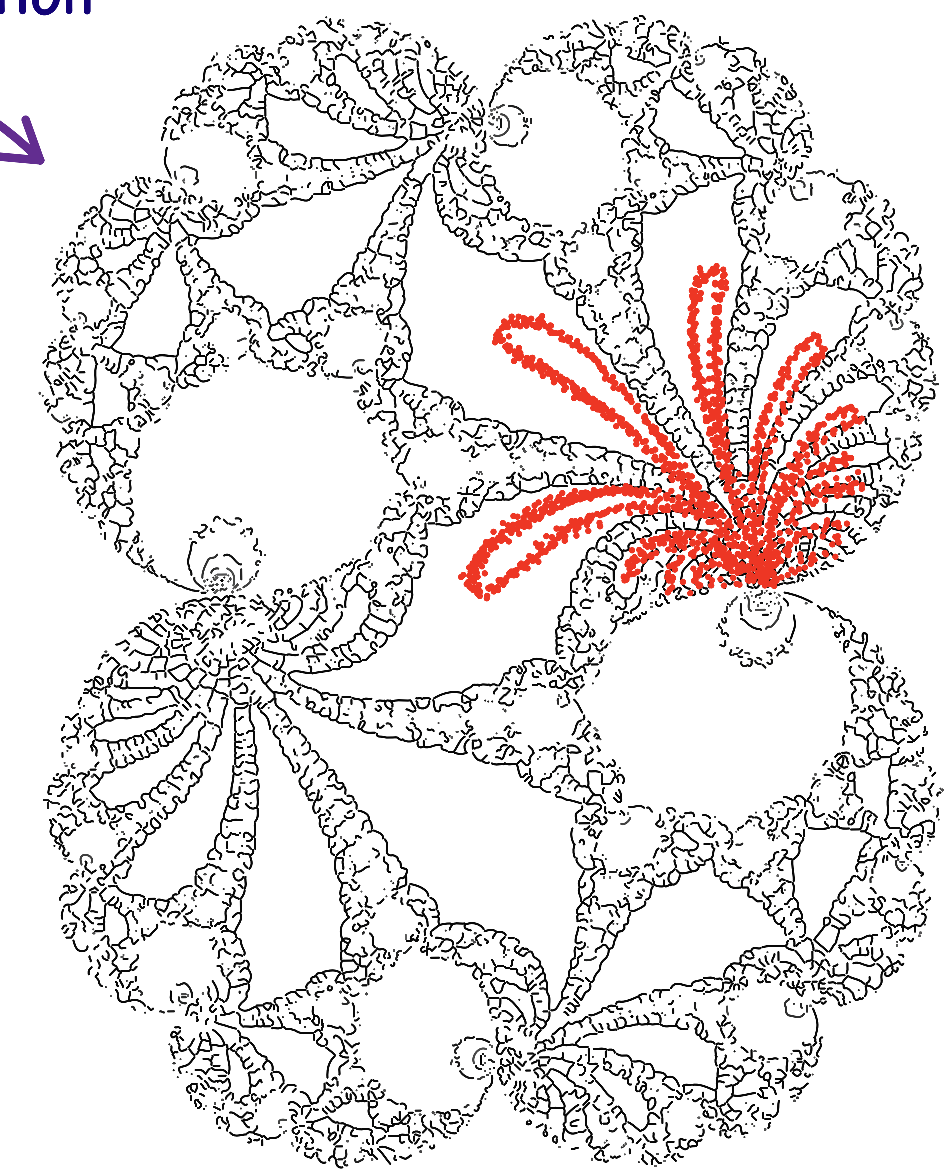
qc disk



$$\theta = [0; N, 1, 1, 1, \dots]$$

still a qc disk but its rotational center is on the boundary

antirenormalization

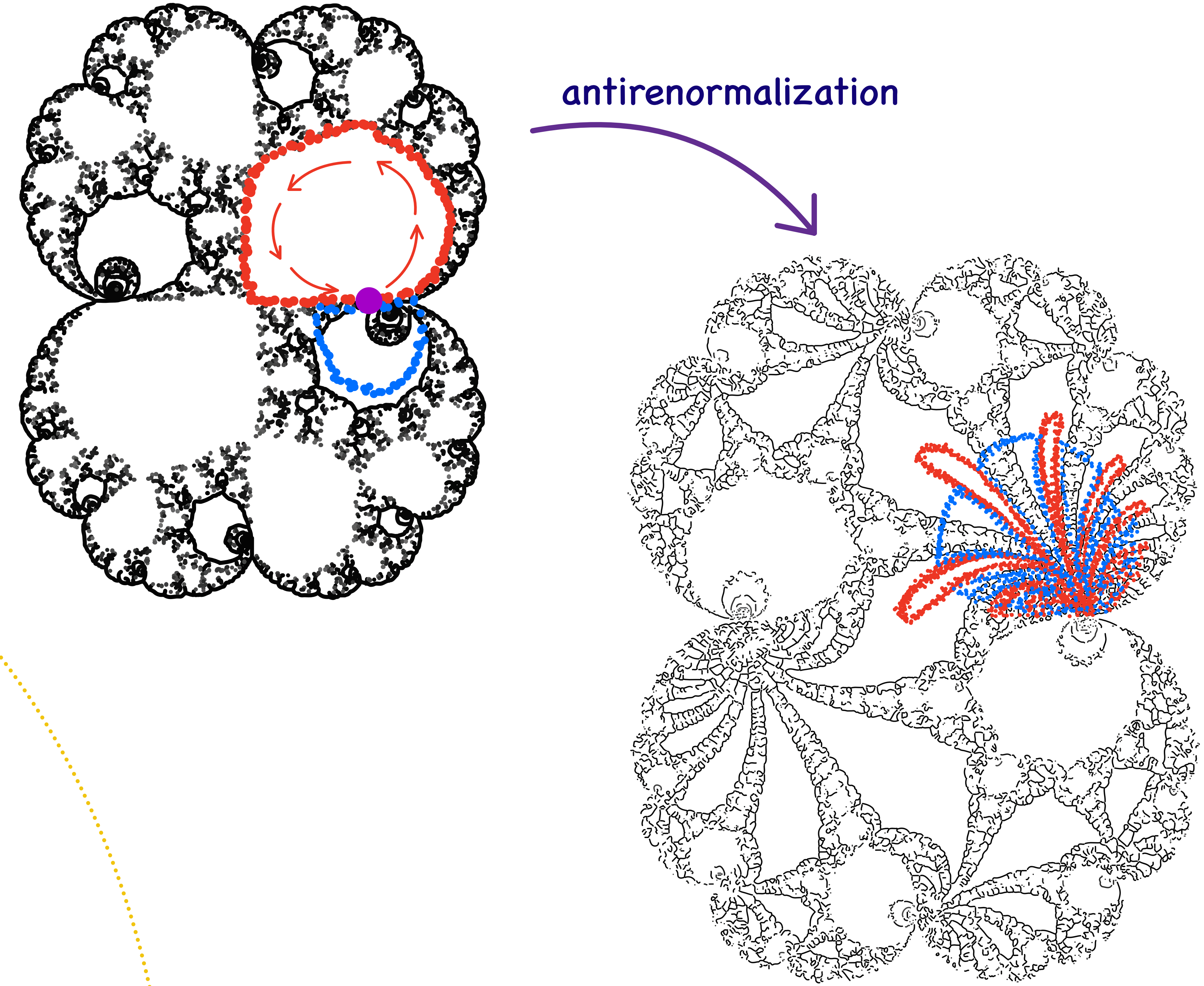


$$\theta = [0; M, N, 1, 1, \dots]$$

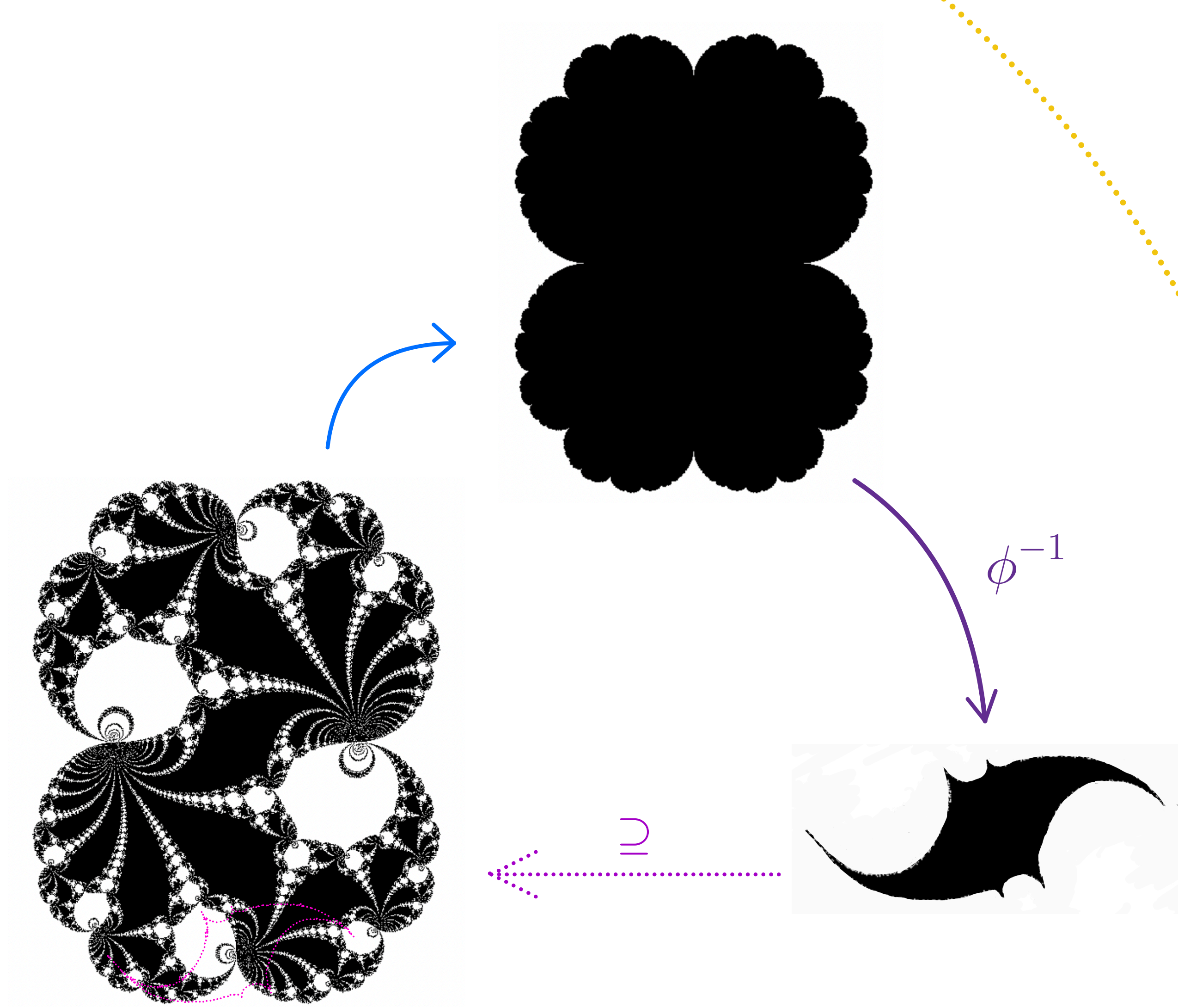
not a uniform qc disk

$$M \gg N \gg 1$$

to construct uniform qc disks
let's add parabolic fjords:



following:



to construct uniform qc disks

let's add parabolic fjords:

pseudo-Siegel disks:

$$\widehat{Z}^m := \overline{Z} \bigcup_{n \geq m} \left\{ \begin{matrix} \text{level-}n \\ \text{parabolic} \\ \text{fjords} \end{matrix} \right\}$$

$f^i \mid \widehat{Z}^m$ is injective (and almost rotation)

$i \leq q_{m+1}$

Lyubich, DD, uniform bounds:

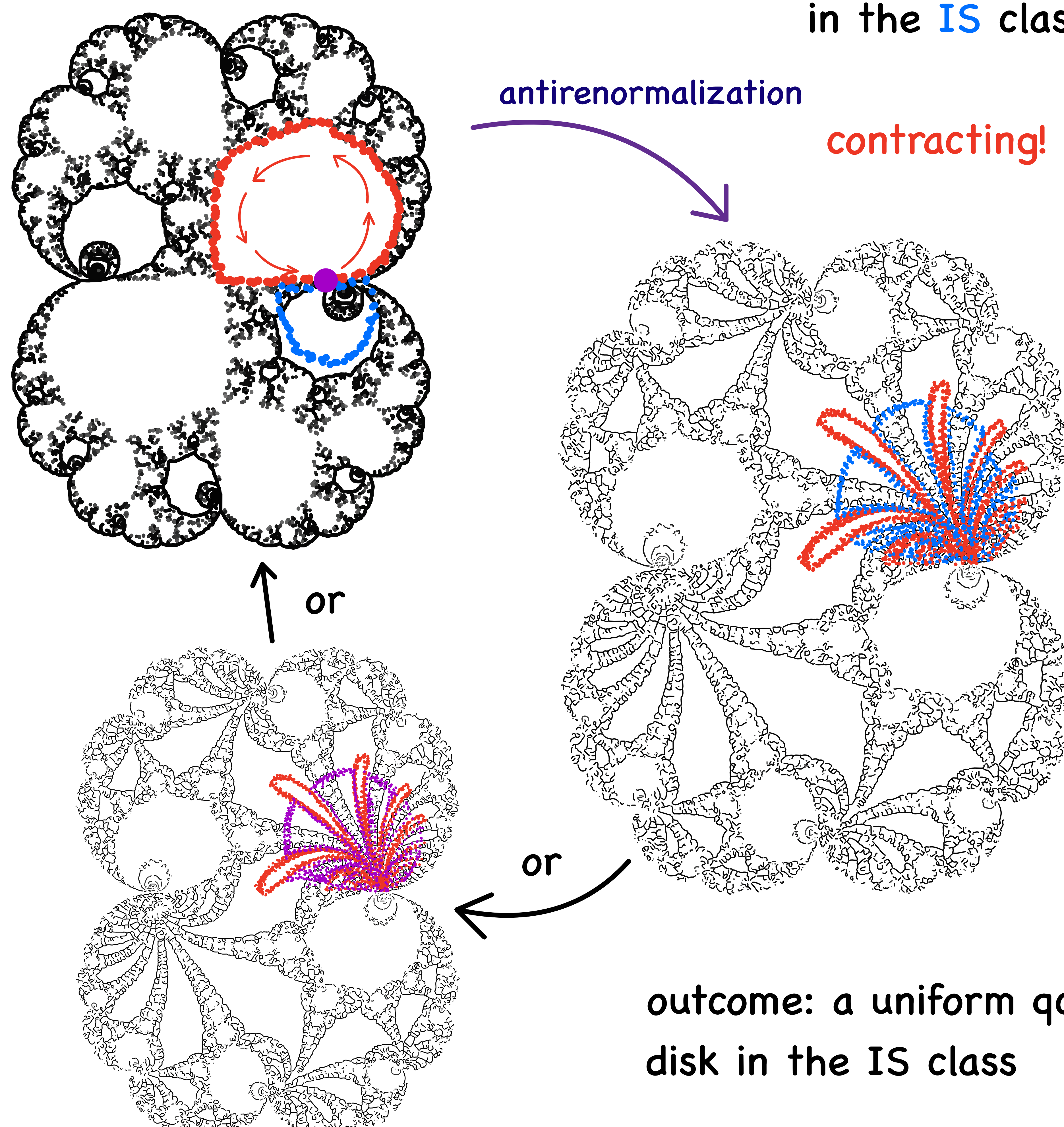
$\widehat{Z} = \widehat{Z}^{-1}$ is a uniformly qc disk

$f \mid \widehat{Z}$ is injective

$\widehat{Z} \supset \mathrm{PostCrit}$

for all neutral quadratics

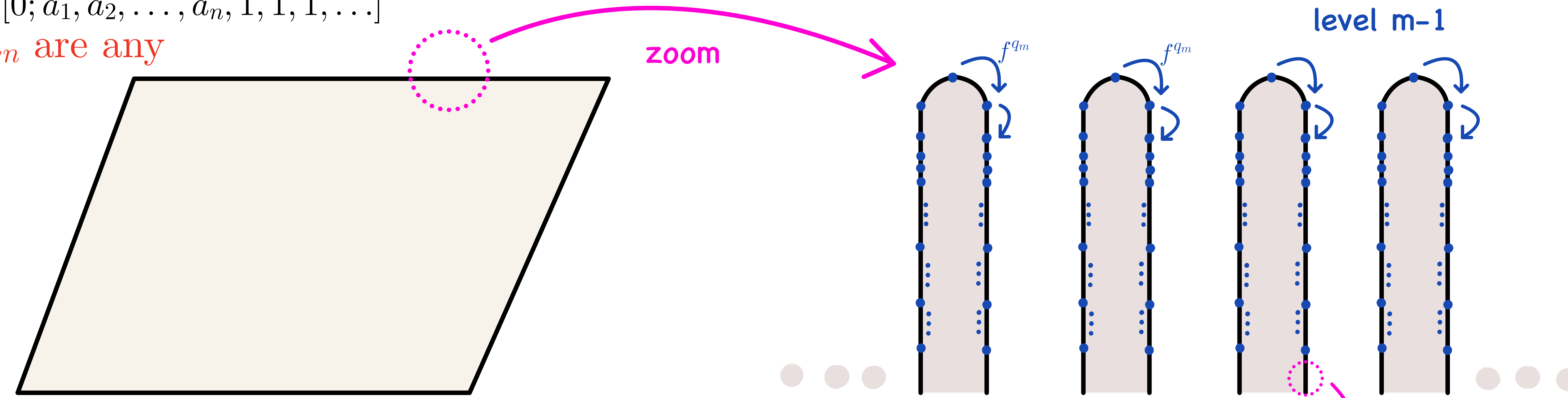
in the IS class:



outcome: a uniform qc disk in the IS class

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any



Thm: $\widehat{Z}^{-1}$ is **uniformly** qc disk

where, by induction on $m \in \{n+1, n, \ldots, 0\}$

if $\quad \frac{q_{m+1}}{q_m} \asymp 1$ then $\quad \widehat{Z}^{m-1} := \widehat{Z}^m$

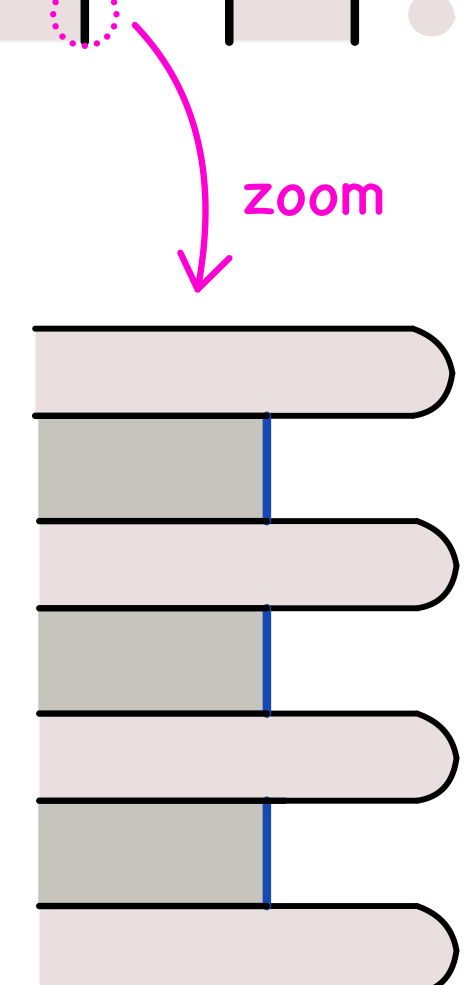
(bounded type)

if $\quad \frac{q_{m+1}}{q_m} \gg 1$ then $\quad \widehat{Z}^{m-1} = \widehat{Z}^m \cup \left\{ \begin{matrix} \text{level } m-1 \\ \text{parabolic} \\ \text{fjords} \end{matrix} \right\}$

(high type)

$\left(\widehat{Z}^m = \overline{Z} \ \text{ for } \ m > n\right)$

level $\quad n \geq m$



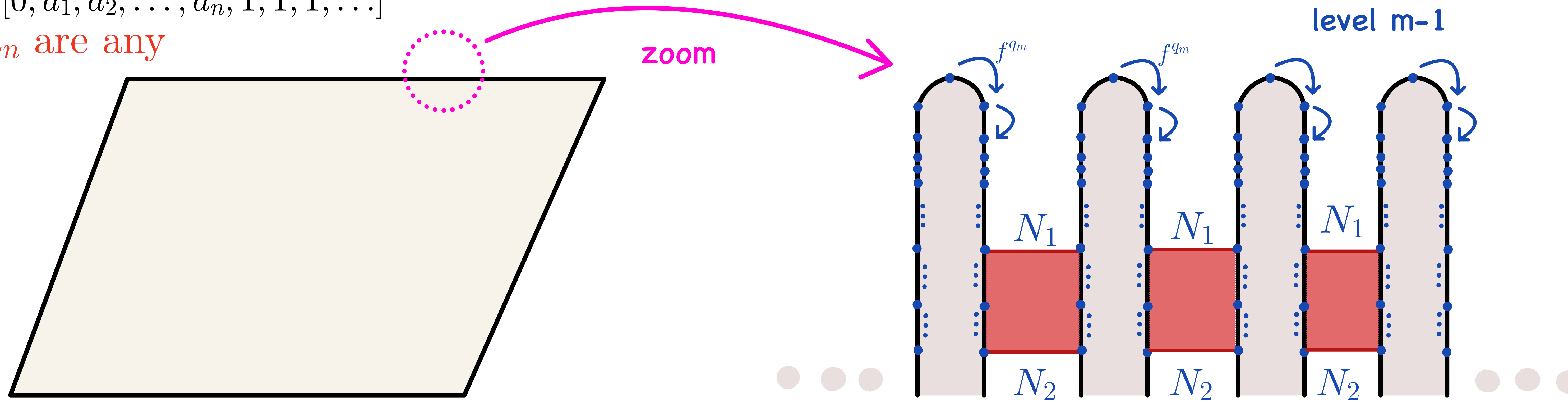
assume all $\widehat{Z}^n, \ n \geq m$

are constructed

let's construct $\quad \widehat{Z}^{m-1}$

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any



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where, by induction on $m \in \{n+1, n, \ldots, 0\}$

if $\frac{q_{m+1}}{q_m} \asymp 1$ then $\widehat{Z}^{m-1} := \widehat{Z}^m$

(bounded type)

if $\frac{q_{m+1}}{q_m} \gg 1$ then $\widehat{Z}^{m-1} = \widehat{Z}^m \cup \left\{ \begin{array}{c} \text{level } m-1 \\ \text{parabolic} \\ \text{fjords} \end{array} \right\}$

(high type)

$\left(\widehat{Z}^m = \overline{Z} \text{ for } m > n \right)$

in level m-1 parabolic fjords consider rectangles on depth between N_1 and N_2

have width $\asymp \ln \frac{N_2}{N_1}$ and hence are almost invariant

$N_1 \gg \frac{N_2}{N_1} \gg 1$ are fixed

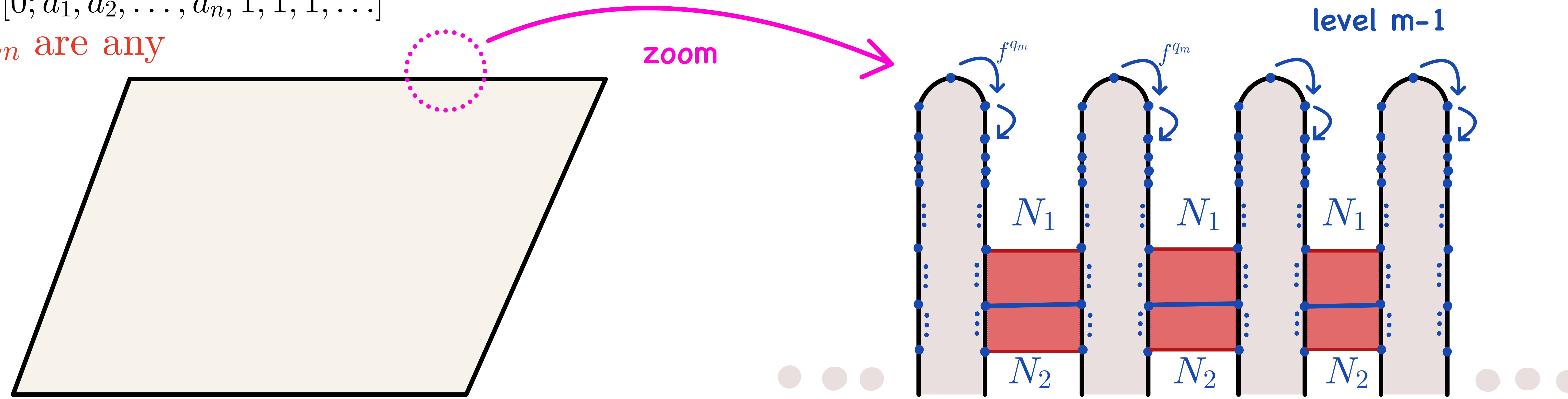
(critical points of f^{q_m} are on top of peninsulas)

assume all $\widehat{Z}^n$, $n \geq m$ are constructed

let's construct $\widehat{Z}^{m-1}$

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any



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if $\dfrac{q_{m+1}}{q_m} \asymp 1$ then $\widehat{Z}^{m-1} := \widehat{Z}^m$

(bounded type)

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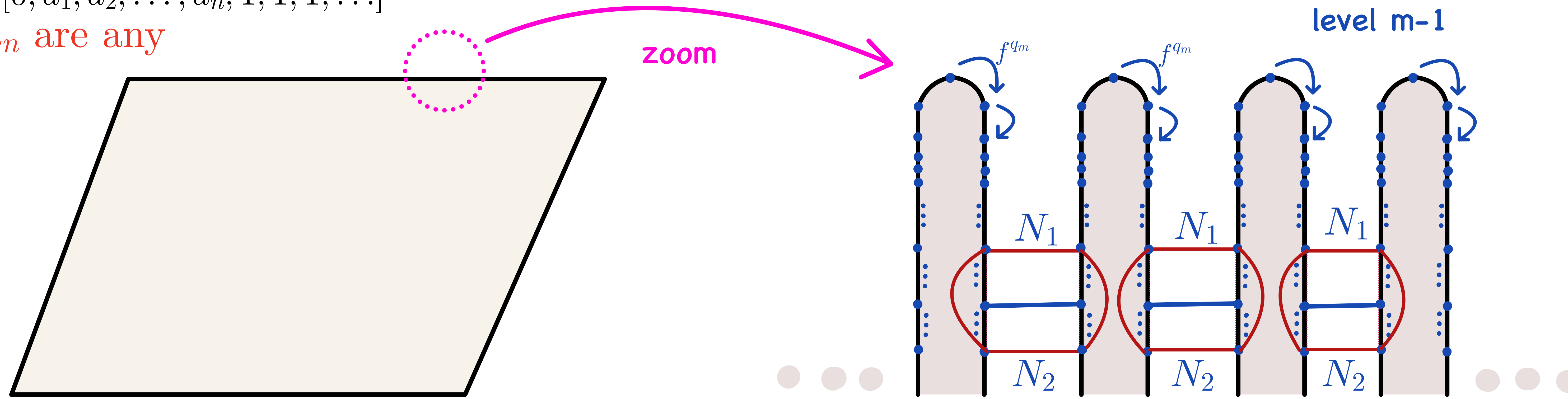
$N_1 \gg \dfrac{N_2}{N_1} \gg 1$ are fixed

dams are mid-curves of the rectangles

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any



$N_1 \gg \frac{N_2}{N_1} \gg 1$

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where, by induction on $m \in \{n+1, n, \ldots, 0\}$

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(bounded type)

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(high type)

$\left(\widehat{Z}^m = \overline{Z} \text{ for } m > n\right)$

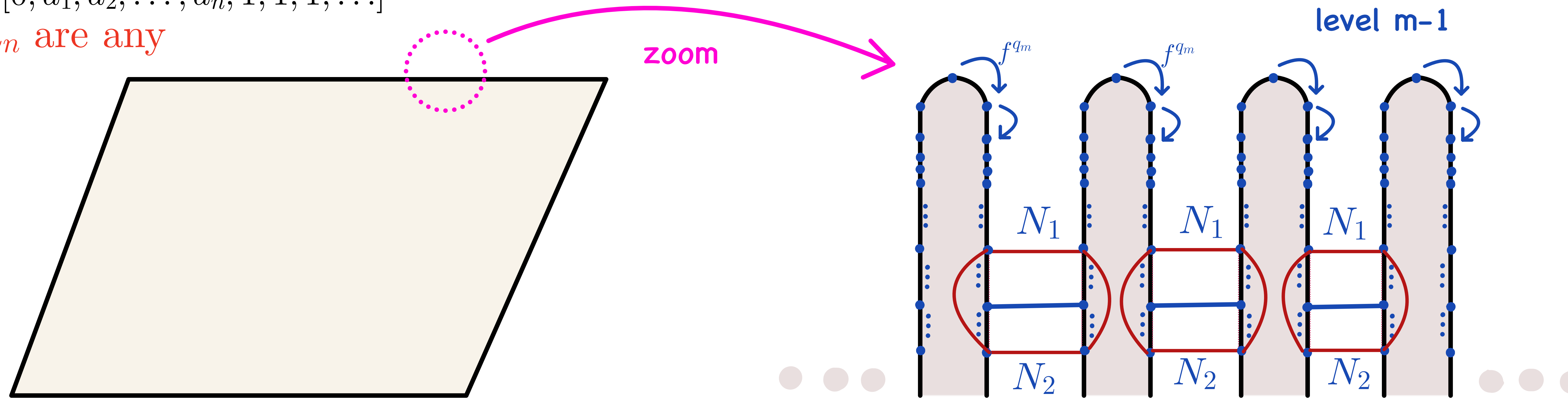
dams are mid-curves of the rectangles

annuli around dams have positive moduli and control almost invariance of $f^i \mid \widehat{Z}^{m-1}$

$i \leq q_m$

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any



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(high type)

$\left(\widehat{Z}^m = \overline{Z} \text{ for } m > n\right)$

annuli around dams have positive moduli and control almost invariance of $f^i \mid \widehat{Z}^{m-1}$

beau bounds: $i \leq q_m$

on level m-1, $\widehat{Z}^m$ is much more invariant than on level m (because $q_{m+1} \gg q_m$)

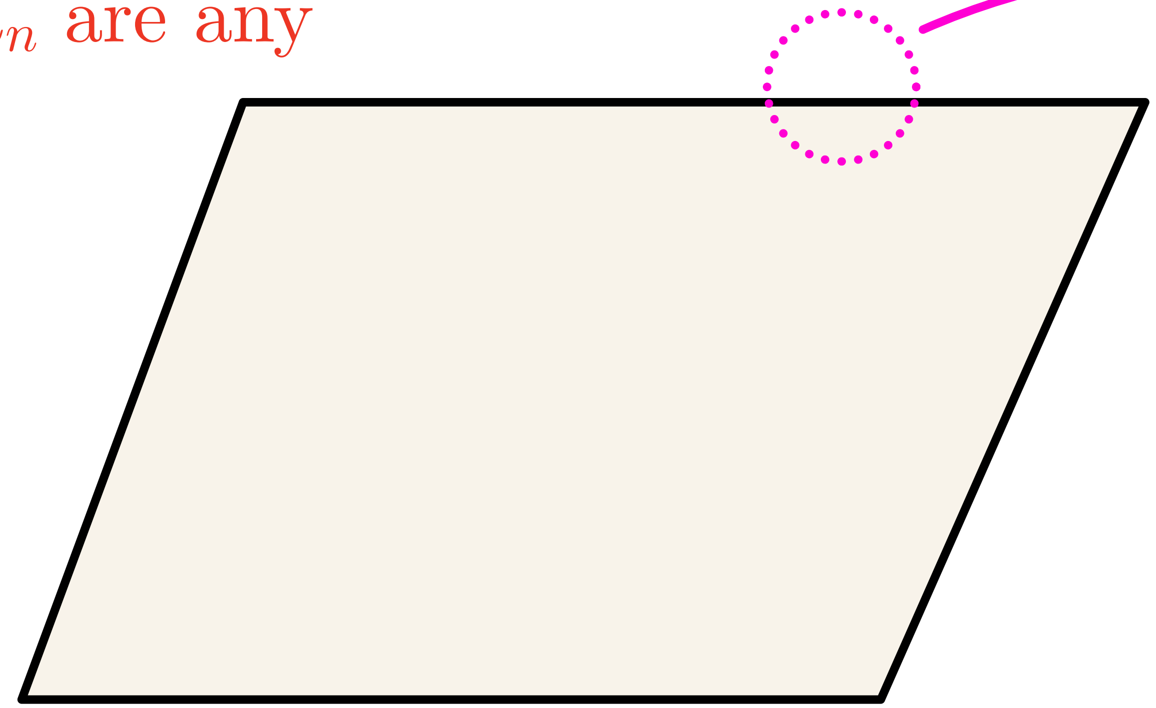
i.e., the errors do not accumulate

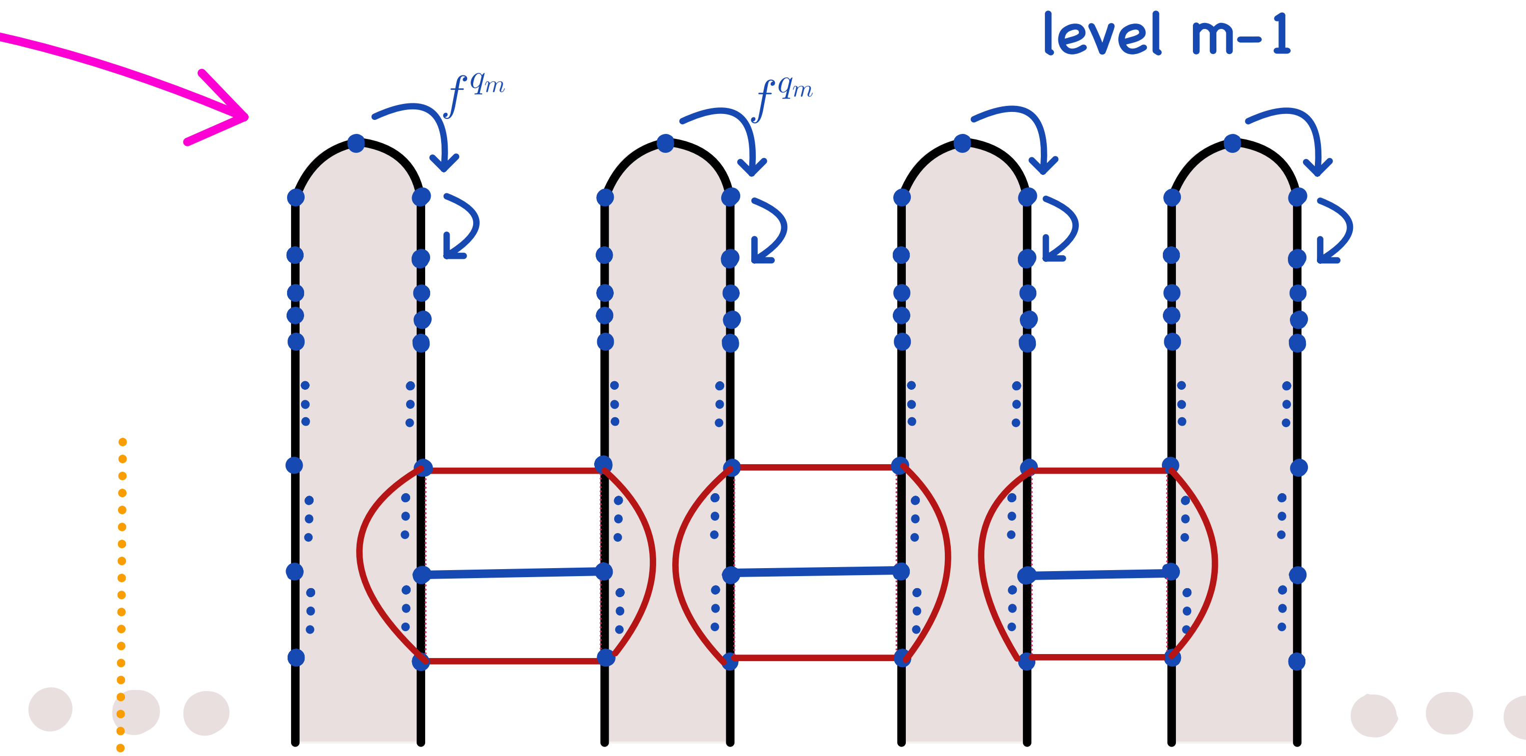
all is justified in the near-degenerate regime (B. Thurston, Kahn, Lyubich,...)

roughly: the Covering Lemma [KL] controls the outside geometry of the pseudo-Siegel disks while the Shift Argument control the geometry inside

$\theta = [0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots]$

n, a_n are any





a priori, no control of the geometry inside $\widehat{Z}^{-1}$

but one can define the sector renormalization:

$$f_n := \mathcal{R}^n_{\text{sector}} f$$
$$\widehat{Z}^{-1}[f_n] = \mathcal{R}^n_{\text{sector}}\left(\widehat{Z}^{n-1}[f_n]\right)$$

Then all $\widehat{Z}^{-1}[f_n]$ are uniformly qc disks

hopefully, the results from the IS-class on neutral dynamics (Shishikura-Yang, Cheraghi,...) can be applicable with these bounds

work in progress:

checking if uniform hyperbolicity of $\mathcal{R}_{\text{sector}}$ can be established following [DLS]

a key step:

dim(unstable manifold) ≤ 1
because it is formed by a
Transcendental (σ-proper)
Family of maps
with a single critical orbit

Happy Birthday, Mitsu!!