EXAM

Practice Final

Math 132

May 15, 2004

ANSWERS

Problem 1. Find the volume of a hemisphere two ways:

(a) Use the disc method to find the volume of the "eastern" hemisphere formed by rotating the region under $y = \sqrt{1-x^2}$ from $x = 0$ to $x = 1$ around the x axis.

Answer:

$$V = \int_0^1 \pi \left(\sqrt{1-x^2}\right)^2 dx \quad (1)$$

$$= \int_0^1 \pi \left(1-x^2\right) dx \quad (2)$$

$$= \pi \left(x - \frac{x^3}{3}\right)\Big]_0^1 \quad (3)$$

$$= \frac{2\pi}{3} \quad (4)$$

(b) Use the shell method to find the volume of the "northern" hemisphere formed by rotating the region under $y = \sqrt{1-x^2}$ from $x = 0$ to $x = 1$ around the y axis.

Answer:

$$V = \int_0^1 2\pi x \sqrt{1-x^2} dx \quad (5)$$

$$= -\pi \frac{2}{3} \left(1-x^2\right)^{\frac{3}{2}}\Big]_0^1 \quad (6)$$

$$= \frac{2\pi}{3} \quad (7)$$

It may be helpful to recall that the curve $y = \sqrt{1-x^2}$ is a semicircle of radius 1 centered at the origin.

Problem 2. Compute the arclength of curve $y = \sqrt{1-x^2}$ from $x = 0$ to $x = 1$.

Answer:

The formula for arclength involves $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, so we compute

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(-\frac{x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

So, the arclength equals $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$. We compute:

$$L = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \, dx$$

use the subsitution $x = \sin(t)$:

$$\begin{aligned} x &= \sin(t) \\ dx &= \cos(t) dt \end{aligned} \qquad \text{and} \qquad \begin{aligned} x = 0 &\Rightarrow t = 0 \\ x = 1 &\Rightarrow t = \tfrac{\pi}{2} \end{aligned}$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin(t)^2}} \cos(t) \, dt \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos(t)^2}} \cos(t) \, dt \\ &= \int_0^{\frac{\pi}{2}} dt \\ &= t \Big]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}. \end{aligned}$$

Problem 3. It is easy to check that $\dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1}$. Use this fact to

(a) Compute $\displaystyle\int_1^\infty \frac{dx}{x^2+x}$

Answer:

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2+x} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+x} \\
&= \lim_{b\to\infty} \int_1^b \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\
&= \lim_{b\to\infty} (\ln(x) - \ln(x+1))\Big]_1^b \\
&= \lim_{b\to\infty} \ln(b) - \ln(b+1) - (\ln(1) - \ln(2)) \\
&= \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) + \ln(2) \\
&= \ln(2).
\end{aligned}$$

(b) Compute $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$

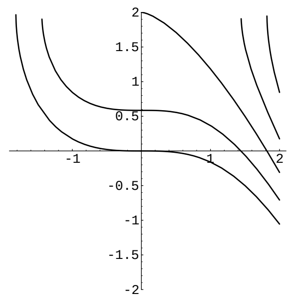
$$\begin{aligned}
\sum_{n=1}^\infty \frac{1}{n^2+n} &= \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots
\end{aligned}$$

Written this way, one see that a lot of terms cancel giving the n^{th} partial sum $1 - \dfrac{1}{n}$ hence

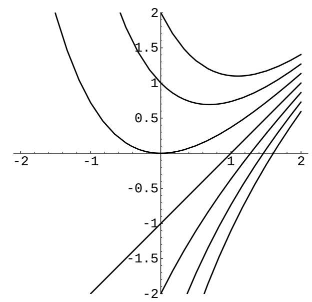
$$\sum_{n=1}^\infty \frac{1}{n^2+n} = 1.$$

Problem 4.

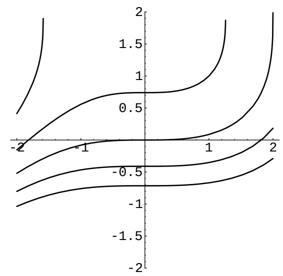
(a) Which plot shows solution curves for the differential equation $y' = x - y$?



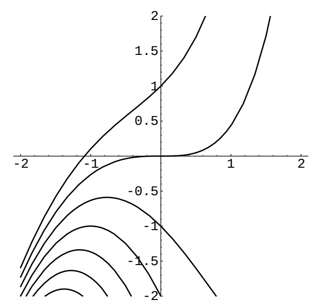
(i)



(iii)



(ii)



(iv)

Answer:

The answer is (iii), the picture in the upper right hand corner. There are many ways to see this, but for one, look in the first quadrant where x and y are positive. Since, $y' = x - y$ will be both positive and negative, the solution curves must both increase and decrease there. In fact, when the curves lie below the straight line $y = x$, $y' > 0$ so the solution curves must increase, and they must decrease when they lie above $y = x$.

(b) Which is a solution to the differential equation $y' = x - y$?

(i) $y = x + \dfrac{1}{e^x} - 1$

(ii) $y = \dfrac{x^2}{2} - x + 1$

(iii) $y = \sin(x)$

(iv) $y = e^x \cos(x)$

Answer:

The answer is (i). A quick computation shows that if $y = x + \frac{1}{e^x} - 1$, then $y' = 1 - \frac{1}{e^x}$, and $x - y = x + \frac{1}{e^x} - 1 = y'$.

Problem 5. Use separation of variables to find the solution to

$$\frac{dy}{dx} = xe^y, \quad y(1) = 0.$$

Answer:
We have

$$\begin{aligned}\frac{dy}{dx} = xe^y &\Rightarrow e^{-y}dy = xdx \\ &\Rightarrow \int e^{-y}dy = \int xdx \\ &\Rightarrow -e^{-y} = \frac{x^2}{2} + C\end{aligned}$$

We use $x = 1$ and $y = 0$ to find that $-1 = C$. So, we have

$$\begin{aligned}-e^{-y} = \frac{x^2}{2} + C &\Rightarrow -e^{-y} = \frac{x^2}{2} - 1 \\ &\Rightarrow e^{-y} = 1 - \frac{x^2}{2} \\ &\Rightarrow -y = \ln\left(1 - \frac{x^2}{2}\right) \\ &\Rightarrow y = -\ln\left(1 - \frac{x^2}{2}\right).\end{aligned}$$

Problem 6. Essay Question. Compare the exponential and logistic models for population growth.
A full analysis will include a discussion of direction fields, sensitivity to initial conditions, asymptotic behavior, and the analytic solutions.

Answer:
See chapter seven of the textbook.

Problem 7. Determine whether the following converge or diverge. Justify your answers completely.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

Answer:

This integral converges: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} 2\sqrt{x}\Big]_b^1 = \lim_{b\to 0^+} 2\sqrt{b} = 2.$

(b) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$

Answer:

This integral diverges: $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\Big]_0^b = \lim_{b\to\infty} 2\sqrt{b} = \infty.$

(c) $\displaystyle\sum_{k=1}^\infty \frac{k!}{k^k}$

Answer:

This series converges by the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}\right| = \left|\frac{k+1}{(k+1)^{k+1}} \frac{k^k}{1}\right| = \left|\frac{k^k}{(k+1)^k}\right| = \left|\left(\frac{k}{k+1}\right)^k\right| \overset{k\to\infty}{\longrightarrow} \frac{1}{e} < 1.$$

(d) $\displaystyle\sum_{n=1}^\infty \frac{\sin(n)}{n^2}$

Answer:

First, we note that $0 < \left|\frac{\sin(n)}{n^2}\right| < \frac{1}{n^2}$, so the series $\displaystyle\sum_{n=1}^\infty \left|\frac{\sin(n)}{n^2}\right|$ converges by an ordinary comparison with the convergent p series $\displaystyle\sum \frac{1}{n^2}$ (here $p = 2 > 1$). Therefore, $\displaystyle\sum_{n=1}^\infty \frac{\sin(n)}{n^2}$ converges too.

Problem 7. (Continued)

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer:

This series converges by the alternating series test. The $(-1)^n$ makes the terms alternate sign. We check that $\frac{1}{\sqrt{n}}$ decreases and tends to zero as $n \to \infty$.

(f) $\sum_{k=1}^{\infty} \frac{3n}{n^2+1}$

Answer:

A limit comparison test with the divergent harmonic series is conclusive: Let $b_n = \frac{1}{n}$ and $a_n = \frac{3n}{n^2+1}$. Note that $a_n > 0$ and $b_n > 0$. We have

$$\frac{a_n}{b_n} = \frac{3n^2}{n^2+1} \to 3.$$

SInce 3 is finite and nonzero, the limit comparison test says that the series $\sum_{k=1}^{\infty} \frac{3n}{n^2+1}$ and $\sum_{k=1}^{\infty} \frac{1}{n}$ do the same thing, which is diverge.

(g) $\int_1^{\infty} \frac{x}{e^x}\, dx$

Answer:

We compute using integration by parts with $u = x$ and $dv = \frac{dx}{e^x}$ (which gives $du = dx$ and $v = -e^{-x}$).

$$\begin{aligned}
\int_1^{\infty} \frac{x}{e^x}\, dx &= \lim_{b\to\infty} \int_1^{b} \frac{x}{e^x}\, dx \\
&= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big]_1^b \\
&= \lim_{b\to\infty} -\frac{b}{e^b} - \frac{1}{e^b} - \left(-\frac{1}{e} - \frac{1}{e}\right) \\
&= \frac{2}{e}.
\end{aligned}$$

(One can use L'Hôpital's rule to see that $\lim_{b\to\infty} -\frac{b}{e^b} = 0$.)

Problem 8. Use Euler's method with a step size of $\frac{1}{3}$ to approximate $y\left(\frac{2}{3}\right)$ if y satisfies the differential equation

$$y' = y\left(2 - \frac{1}{2}y^2\right), \quad y(0) = 1.$$

Answer:
We compute when $x = 0$, $y = 1$, so

$$y'(0) = 1\left(2 - \frac{1}{2}\right) = \frac{3}{2}.$$

Then

$$y\left(\frac{1}{3}\right) \approx y(0) + \frac{1}{3}y'(0) \approx 1 + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \frac{3}{2}.$$

When $x = \frac{1}{3}$, $y \approx \frac{3}{2}$, so

$$y'\left(\frac{1}{3}\right) \approx \frac{3}{2}\left(2 - \frac{1}{2}\left(\frac{3}{2}\right)^2\right) = \frac{21}{16}.$$

Then

$$y\left(\frac{2}{3}\right) \approx y\left(\frac{1}{3}\right) + \frac{1}{3}y'\left(\frac{1}{3}\right) \approx \frac{3}{2} + \left(\frac{1}{3}\right)\left(\frac{21}{16}\right) = \frac{31}{16}.$$

Problem 9. True or False?

(a) $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$.

Answer:

True. Use the fact that $\int \frac{dx}{1+x^2} = \arctan(x)$.

(b) For any constant c, $p = \dfrac{1}{1+(c-1)\,e^{-t}}$ is a solution to $p' = p(1-p)$.

Answer:

True. You can see this if you're familiar with the logistic differential equation, or just check it directly.

(c) $\dfrac{1}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - + \cdots$

Answer:

True. Write the equation $\frac{1}{e^x} = e^{-x}$ in power series.

(d) If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges but $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ converges.

Answer:

True. $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges by the ratio test. In addition, the ratio test applied to $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ yields $\frac{a_{n+1}}{3^{n+1}} \cdot \frac{3^n}{a_n} = \frac{a_{n+1}}{a_n} \cdot \frac{1}{3} \to \frac{2}{3} < 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ converges.

Problem 9. (Continued.)

(e) If $\sum_{k=1}^{\infty} |a_k|$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges also.

Answer:

This is false. For example, $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ converges (by the alternating series test) and $\sum_{k=1}^{\infty} \left|(-1)^k \frac{1}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (by the p series test).

(f) Suppose $a_n > 0$ for all n and $\lim_{n\to\infty} na_n = 3$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

Answer:

False. If $\lim_{n\to\infty} na_n = 3$ then $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = 3$ which implies, by the limit comparison theorem, that $\sum a_n$ and the series $\sum \frac{1}{n}$ do the same thing, which is diverge.

(g) $\int f(x)g(x)dx = \left(\int f(x)dx\right)\left(\int g(x)dx\right)$.

Answer:

False. Check with almost any example to see it.

Problem 10. Sometimes it is possible to find the sum of a convergent series precisely by comparing it to a familiar power series specialized to a particular value of x. Find the sum:

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$

Answer:

$= \arctan(1) = \frac{\pi}{4}$

(b) $\frac{\pi}{2} - \frac{\pi^3}{3! \cdot 2^3} + \frac{\pi^5}{5! \cdot 2^5} - + \cdots$

Answer:

$= \cos\left(\frac{\pi}{2}\right) = 0.$

(c) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

Answer:

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, we have $e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$, so the answer to the problem is $e - 1$.

(d) $1 + 6\left(-\frac{1}{2}\right) + 15\left(-\frac{1}{2}\right)^2 + 20\left(-\frac{1}{2}\right)^3 + 15\left(-\frac{1}{2}\right)^4 + 6\left(-\frac{1}{2}\right)^5 + \left(-\frac{1}{2}\right)^6$

Answer:

For any x, $(1+x)^6 = 1 + 6x + \frac{(6)(5)}{2!}x^2 + \frac{(6)(5)(4)}{3!}x^3 + \cdots + 6x^5 + x^6$. So,

$$1+6\left(-\frac{1}{2}\right)+15\left(-\frac{1}{2}\right)^2+20\left(-\frac{1}{2}\right)^3+15\left(-\frac{1}{2}\right)^4+6\left(-\frac{1}{2}\right)^5+\left(-\frac{1}{2}\right)^6 = \left(1+\left(-\frac{1}{2}\right)\right)^6 = \frac{1}{64}.$$

Problem 11. Use power series to approximate

$$\int_0^1 x^2 \cos\left(x^{\frac{3}{2}}\right)\, dx$$

with an error less than $\frac{1}{(12)(720)}$.

Answer:
Begin with

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - + \cdots$$

. Then,

$$\cos\left(x^{\frac{3}{2}}\right) = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \frac{x^9}{6!} + - \cdots$$

and

$$x^2 \cos\left(x^{\frac{3}{2}}\right) = x^2 - \frac{x^5}{2!} + \frac{x^8}{4!} - \frac{x^11}{6!} + - \cdots .$$

Now, we integrate

$$\begin{aligned}\int_0^1 x^2 \cos\left(x^{\frac{3}{2}}\right)\, dx &= \int_0^1 x^2 - \frac{x^5}{2!} + \frac{x^8}{4!} - \frac{x^11}{6!} + - \cdots \\ &= \frac{x^3}{3} - \frac{x^6}{6 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^12}{12 \cdot 6!} + - \cdots \Big]_0^1 \\ &= \frac{1}{3} - \frac{1}{6 \cdot 2!} + \frac{1}{9 \cdot 4!} - \frac{1}{12 \cdot 6!} + - \cdots \\ &\approx \frac{1}{3} - \frac{1}{6 \cdot 2!} + \frac{1}{9 \cdot 4!} \\ &= \frac{55}{216}.\end{aligned}$$

Since the series

$$\frac{1}{3} - \frac{1}{6 \cdot 2!} + \frac{1}{9 \cdot 4!} - \frac{1}{12 \cdot 6!} + - \cdots$$

converges by the alternating series test, the error in made by approximating the sum with the first three terms is smaller than the fourth term, which is $\frac{1}{12 \cdot 6!} = \frac{1}{(12)(720)}$.

Problem 12. Let $f(x) = \dfrac{x^2}{e^{2x}}$. Use power series to find $f^{(5)}(0)$, the fifth derivative of f at $x = 0$.

Answer:
Start with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

to get

$$e^{-2x} = 1 - 2x + 4\frac{x^2}{2} - 8\frac{x^3}{6} + \cdots$$

and

$$x^2e^{-2x} = x^2 - 2x^3 + 2x^4 - \frac{4}{3}x^5 + \cdots .$$

We know that the coefficient of x^5 in this expansion equals $\frac{f^{(5)}(0)}{5!}$. Therefore

$$-\frac{4}{3} = \frac{f^{(5)}(0)}{5!} \Rightarrow f^{(5)}(0) = -5!\left(\frac{4}{3}\right) = -160.$$