Lebesgue-Stieltjes Integrals

When we defined the measure of an open set $U \subseteq \mathbf{R}$ as

$$m(U) = \sum_{I \in \mathcal{I}_U} v(I), \tag{1}$$

we mentioned that it would be how much U weighed, in grams, if it were made out of material of linear density 1 g/cm. What if the density were not 1 g/cm, and was not even uniform? And what if some points had nonzero weight?

As we know, if $f : \mathbf{R} \to [0, \infty)$ is Lebesgue measurable, we can define a new measure on $\mathbf{R}$ by $\mu(E) = \int_E f dx$. (Here we assume E is Lebesgue measurable.) If $\int f = 1$, ϕ is the probability measure with "density" f. We could also interpret f as linear density in the usual sense of the word, and so $\mu(U)$ provides a partial answer to our question. However, whatever f is, in this situation, the weight of any point would be zero.

Let us then look at the question differently. *Can we describe all the inner regular volume functions μ on $\mathbf{R}$?* Any such volume function is completely determined once we have specified $\mu(I)$ for any open interval I, since any open set U is the countable disjoint union of the intervals in $\mathcal{I}_U$.

To see how to do this, let us *assume* we have such an inner regular volume function and see what we can find out about it. We know that there is an associated measure μ (it will be called a *Lebesgue-Stieltjes measure*).

Fix $a \in \mathbf{R}$, and, for $x > a$, define $g_a(x)$ to be $\mu((a, x))$. Of course $g_a(x)$ must be a *nondecreasing function* of x. Note that if $x > a$ and $x_n \nearrow x$ (by which we mean all $x_n < x$, $x_n \to x$ and $x_n \nearrow$) , then

$$\cup_n (a, x_n) = (a, x), \tag{2}$$

so that we must have $g_a(x_n) \nearrow g_a(x)$. This easily implies that g_a must be *continuous from the left*. On the other hand, if instead $x_n \searrow x$, we have only

$$\cap_n (a, x_n) = (a, x], \tag{3}$$

so that $g_a(x_n) \searrow \mu((a, x])$, which may not be $g_a(x)$ (and is not, if $\mu(\{x\}) > 0$).

It is therefore important for us to think about nondecreasing functions more carefully. One has:

Proposition 0.1 *Suppose $F : (a, \infty) \to \mathbf{R}$ is nondecreasing. Then:*
(a) F has at most countably many discontinuites, and each is a jump discontinuity (i.e., for any

$c > a$, $F(c^-)$ and $F(c^+)$ exist. (Here $F(c^-) = \lim_{x \to c^-} F(x)$.) Of course, $F(c^-) \le F(c^+)$.
(b) Define $G : (a, \infty) \to \mathbf{R}$ *by* $G(x) = F(x^-)$. *Then* G *is nondecreasing and continuous from the left. Moreover, for any* $c > a$, $G(c^+) = F(c^+)$.
(c) Similarly, if we define $H : (a, \infty) \to \mathbf{R}$ *by* $H(x) = F(x^+)$. *Then* H *is nondecreasing and continuous from the right. Moreover, for any* $c > a$, $H(c^-) = F(c^-)$.
(d) In addition, for any $c > a$, $\lim_{x \to c^-} F(x^+) = F(c^-)$, *and* $\lim_{x \to c^+} F(x^-) = F(c^+)$.

Proof For (a), say $c > a$. Take any sequence $\{c_n\}$ with all $c_n > a$, such that $c_n \nearrow c$. The sequence $\{F(c_n)\}$ is nondecreasing and bounded above (by $F(c)$), so it has a limit, say L. We claim that $F(c^-) = L$. Indeed, given $\epsilon > 0$, we may select m with $F(c_m) > L - \epsilon$. If $c_m < x < c$, we must have $c_m < x < c_n$ for some n, so $F(c_m) < F(x) < F(c_n)$, so $L - \epsilon \le F(x) \le L$, as desired.

Thus all the discontinuities are jump discontinuities. To see that there are only countably many of them, it suffices to show that there are only countably many in any finite interval (a, b). At a point of discontinuity c, let us say that the "jump" is $F(c^+) - F(c^-)$. For each positive integer n, let J_n denote the set of points of discontinuity in (a, b) where the jump exceeds $1/n$. It suffices to show that each J_n is a finite set (since the set of all points of discontinuity in (a, b) is the union of the J_n). But in fact, J_n cannot have more than $n[F(b) - F(a)]$ elements. Indeed, if there were elements $c_1 < c_2 < \ldots < c_N$ in J_n, where $N > n[F(b) - F(a)]$, then we would have the contradiction

$$\begin{aligned}
F(b) - F(a) &\ge F(c_N^+) - F(c_1^-) \\
&= [F(c_N^+) - F(c_N^-)] + [F(c_N^-) - F(c_{N-1}^+)] + [F(c_{N-1}^+) - F(c_{N-1}^-)] + \ldots [F(c_1^+) - F(c_1^-)] \\
&\ge \sum_{k=1}^{N-1} [F(c_{k+1}^+) - F(c_k^-)] \\
&\ge N/n.
\end{aligned}$$

For (b), it is evident that G is nondecreasing, since if $a < b < c$, we may take sequences $b_n \nearrow b$ and $c_n \nearrow c$ with $b_n \in (a, b)$ for all n and $c_n \in (b, c)$ for all n. Thus $b_n < c_n$ for all n, which implies that $G(b) = F(b^-) \le F(c^-) = G(c)$.

To see that G is continuous from the left, we need only note that G equals F except at at most countably many points. Thus, if $c > a$, we can find a sequence $c_n \nearrow c$ such that $G(c_n) = F(c_n)$ for all n. Accordingly

$$G(c^-) = \lim_{n \to \infty} G(c_n) = \lim_{n \to \infty} F(c_n) = F(c^-) = G(c),$$

Similarly, by choosing a sequence $c_n \searrow c$ such that $G(c_n) = F(c_n)$ for n, we find that $G(c^+) = F(c^+)$, as desired.

The proof of (c) is similar to that of (b). For (d), we note that if $x < c$, then $F(x^-) \le F(x^+) \le F(c^-)$. Thus $\lim_{x \to c^-} F(x^+) = F(c^-)$ follows from the squeeze rule and (b). Similarly, using (c), we see that $\lim_{x \to c^+} F(x^-) = F(c^+)$. This completes the proof.

Continuing now with our analysis from before the proposition: say $a < b < x$. We have

$$g_b(x) = \mu((b, x)) = \mu((a, x)) - \mu((a, b]) = g_a(x) - g_a(b^+).$$

This shows in particular that the functions g_a and g_b differ only by a constant on (b,∞) (the intersection of their domains). Say now $a < b < 0$. Then if $a < b < x$ we have $g_b(x) - g_a(x) = g_b(0) - g_a(0)$, whence $g_b(x) - g_b(0) = g_a(x) - g_a(0)$. For any $x \in \mathbf{R}$ we may therefore define

$$\psi(x) = g_a(x) - g_a(0)$$

for any $a < \min(0,x)$; the definition does not depend on the choice of a. We find:
(i) ψ is a nondecreasing function on $\mathbf{R}$, which is continuous from the left; and
(ii) whenever $b < x$, $\mu((b,x)) = \psi(x) - \psi(b^+)$.

This describes, then, conditions which our assumed inner regular volume function μ must satisfy. We claim that these conditions are both necessary and sufficient. That is:

Suppose $\psi : \mathbf{R} \to \mathbf{R}$ satisfies (i); then there is a unique inner regular volume function μ satisfying $\mu((b,x)) = \psi(x) - \psi(b^+)$ whenever $b < x$.

Associated to this μ, then, is a measure μ, which is called the *Lebesgue-Stieltjes measure* associated to ψ. Equivalently:

Theorem 0.2 *Suppose that $\phi : \mathbf{R} \to \mathbf{R}$ is nondecreasing. Then there exists a unique inner regular volume function μ such that $\mu((b,x)) = \phi(x^-) - \phi(b^+)$ whenever $-\infty < b < x < \infty$.*

The measure associated to this μ is called the Lebesgue-Stieltjes measure *associated to ϕ.*

This appears more general than (*), but it really is not. If we knew (*), then to prove the theorem we need only define ψ by $\psi(x) = \phi(x^-)$, and let μ be the Lebesgue-Stieltjes measure associated to ψ. (We used Proposition 0.1.)

To prove Theorem 0.2, we proceed as follows. If $I = (a,b) \subseteq [-\infty,\infty]$ is an open interval, we define $w(I) = \phi(b^-) - \phi(a^+)$. (This will have to turn out to be in the end to be $\mu(I)$, even if a or b is not finite. For instance, if a is finite, $\mu((a,\infty))$ will have to be $\lim_{N\to\infty}[\phi(N^-) - \phi(a^+)] = \phi(\infty-) - \phi(a^+)$, since for any N, $\phi(N-1) \le \phi(N^-) \le \phi(\infty^-)$ and we can use the squeeze rule.)

If $J \subseteq \mathbf{R}$ is any interval (not necessarily open), we set

$$w(J) = \inf w(I) \tag{4}$$

where the inf is taken over all open intervals containing I. (This will have to turn out to be $\mu(J)$, since μ is to be outer regular. (In fact, $\mu(J)$ will be $\inf \mu(U)$, where the inf is taken over all open sets containing J; but if $U \supseteq J$ then some interval I in $\mathcal{I}_U$ must contain J, since J is connected; so in fact we can take the inf over all open intervals containing J and get the same answer, by the majorization principle for infs).

By the results of Proposition 0.1, we see that $w([a,b)) = \phi(b^-) - \phi(a^-)$, $w((a,b]) = \phi(b^+) - \phi(a^+)$, and $w([a,b]) = \phi(b^+) - \phi(a^-)$, (the latter holding even if $a = b \in \mathbf{R}$, in which case $[a,b]$ means $\{a\}$.) (To see these facts, note that, for instance, if $I \supseteq [a,b]$ is open, then $w(I) \ge \phi(b^+) - \phi(a^-)$ always, and, for I suitable, is as close to $\phi(b^+) - \phi(a^-)$ as desired.) The results of Proposition 0.1 now show that, if I is any interval, then

$$w(I) = \sup w(J), \tag{5}$$

the sup being taken over all compact intervals $J \subseteq I$.

If $U \subseteq \mathbf{R}$ is open, we set (and, in fact, must set)

$$\mu(U) = \sum_{I \in \mathcal{I}_U} w(I).$$

In order to show that this is an inner regular volume function, we simply imitate our proof that m is an inner regular volume function:
1. We define the Riemann-Stieltjes integral of a step function on a closed interval R, showing it is independent of choice of partition.
2. We use this to prove the Rectangle Lemma with w in place of v. This then shows, by the arguments of Lemma 4.2.12, that μ is a volume function.
3. We show that this volume function is inner regular.

For #1, if s is a step function on R for the partition $P = (x_0, \ldots, x_n)$, and $s = a_k$ on the interval $I_k = (x_{k-1}, x_k)$ of P, we define

$$\mathcal{I}_\phi(s, P) = \sum_{k=1}^{n} a_i w(I_k) + \sum_{j=0}^{n} s(x_j) w(\{x_j\}). \tag{6}$$

(If we extend s outside R to be zero there, this is what $\int s d\mu$ would have to be.) We need to show that this is independent of the choice of partition. As in the proof of Proposition 4.1.2, we only need show that $\mathcal{I}_\phi(s, P) = \mathcal{I}_\phi(s, Q)$ for a partition Q of R which has exactly one more point than P. Say that point is t, and it lies in the intervsl I_m. Then, to evaluate $\mathcal{I}_\phi(s, Q)$, we need only replace the term $a_m w(I_m)$ in the right side of (6) by $a_m[w((x_{m-1}, t)) + w(\{t\}) + w((t, x_m))]$; but this is

$$a_m([\phi(x_m^-) - \phi(t^+)] + [\phi(t^+) - \phi(t^-)] + [\phi(t^+) - \phi(x_{m-1}^+)]) = a_m[\phi(x_m^-) - \phi(x_{m-1}^+)] = a_m w(I_m)$$

as desired. We then define $\int_R s d\phi = \mathcal{I}_\phi(s, P)$, and note, that for step functions s, t on R, $\int_R (s + t) d\phi = \int_R s d\phi + \int_R t d\phi$, and if $s \leq t$, then $\int_R s d\phi \leq \int_R t d\phi$.

For #2, in this situation we shall also need to allow our rectangles (i.e. real intervals) to be half-infinite or infinite in extent. We modify the second paragraph of the proof of the rectangle lemma as follows: For each k, we let I_k^ϵ be any compact interval contained in I_k with $w(I_k^\epsilon) \geq (1 - \epsilon) w(I_k)$ (here we use (5)). Choose a finite open interval $(A, B) \supseteq \cup_{k=1}^N I_k^\epsilon$. For each l, we let K_l^ϵ be an open interval containing J_l with $w(K_l^\epsilon) \leq (1 + \epsilon) w(J_l)$, and let $J_l^\epsilon = K_l^\epsilon \cap (A, B)$. Then $w(J_l^\epsilon) \leq (1 + \epsilon) w(J_l)$, and $\cup_{k=1}^N I_k^\epsilon \subseteq \cup_l J_l^\epsilon$. The proof is now concluded as for the usual rectangle lemma, except that we use w in place of v and the $\int_R d\phi$ in place of $\int_R dx$; here $R = [A, B]$. The fact that μ is a volume function now follows just as in the proof of Lemma 4.2.12, except that we use the $\mathcal{I}_U$ instead of the $\mathcal{D}_U$.

For #3, we note that, for any interval J, $\mu_e(J) = w(J)$, by (4) and the connectedness of J. Thus, by (5), every open interval is μ-inner regular. This shows, by Lemma 4.4.2, that every open set is μ-inner regular, since it is a countable disjoint union of open intervals. Thus the volume function μ is inner regular, as desired.

Lebesgue-Stieltjes integrals occur frequently in probability and in physics.