A DERIVED CATEGORY APPROACH TO GENERIC VANISHING

CHRISTOPHER D. HACON

Abstract. We prove a Generic Vanishing Theorem for coherent sheaves on an abelian variety over an algebraically closed field k. When $k=\mathbb{C}$ this implies a conjecture of Green and Lazarsfeld.

1. Introduction

In [GL1] and [GL2], Green and Lazarsfeld prove the following result which is an essential tool in the study of irregular varieties:

Theorem 1.1. *(Generic Vanishing Theorem.) Let X be a smooth complex projective variety. Then then every irreducible component of*

$$V^i(\omega_X) := \{P \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes P) \neq 0\}$$

is a translate of a subtorous of $\mathrm{Pic}^0(X)$ *of codimension at least*

$$i - (\dim(X) - \dim(\mathrm{a}_X(X))).$$

If $\dim(X) = \dim \mathrm{a}_X(X)$ *then there are inclusions:*

$$V^0(\omega_X) \supset V^1(\omega_X) \supset ... \supset V^{\dim(X)}(\omega_X) = \{\mathcal{O}_X\}.$$

The proof of this theorem (and of its generalizations) relies heavily on the use of Hodge theory. It is a natural question to try and understand to what extent these results depend on Hodge theory, and what aspects of the proof can be replaced by an algebraic approach (cf. [EV] 13.13.d). Since the above theorem can also be understood in terms of the sheaves $R^i\mathrm{a}_{X,*}(\omega_X)$, it is also natural to ask to what extent the Generic Vanishing Theorem generalizes to coherent sheaves on an abelian variety. It is surprising that both questions can be answered via simple algebraic methods. (Transcendantal methods are of course used in the proofs of Kollár's theorems on higher direct images of dualizing sheaves. See however [EV] for a discussion of algebraic approaches to vanishing theorems.)

We begin by fixing some notation. Let A be an abelian variety over an algebraically closed field k, $\hat{A} = \mathrm{Pic}^0(A)$ the dual abelian variety and $\mathcal{L}$ be the normalized Poincaré line bundle. We will denote by $p_A, p_{\hat{A}}$ the projections of $A \times \hat{A}$ on to $A, \hat{A}$. For any ample line bundle L on $\hat{A}$, the isogeny $\phi_L : \hat{A} \longrightarrow A$ is defined by $\phi_L(\hat{a}) = t^*_{\hat{a}} L^\vee \otimes L$. Let $\hat{L}$ be the vector bundle on A defined by

$$\hat{L} := p_{A,*}(p^*_{\hat{A}} L \otimes \mathcal{L}).$$

One has that

$$\phi^*_L\left(\hat{L}^\vee\right) \cong \bigoplus_{h^0(L)} L.$$

We will prove the following:

Theorem 1.2. *Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. The following are equivalent:*

(1) *For any sufficiently ample line bundle L on $\hat{A}$,*

$$H^i(A, \mathcal{F}\otimes\hat{L}^\vee) = 0 \quad \forall\ i > 0;$$

(2) *There is an isomorphism*

$$Rp_{\hat{A},*}(p_A^* D_A(\mathcal{F})\otimes\mathcal{L}) \cong R^0 p_{\hat{A},*}(p_A^* D_A(\mathcal{F})\otimes\mathcal{L}).$$

We refer to this result as a Generic Vanishing Theorem as it implies in particular that every irreducible component of

$$V^i(A, \mathcal{F}\otimes P) := \{P \in \hat{A} \mid h^i(A, \mathcal{F}\otimes P) \neq 0\}$$

has codimension at least i in A (cf. Corollary 3.2) and that one has inclusions

$$V^0(\mathcal{F}) \supset V^1(\mathcal{F}) \supset ... \supset V^n(\mathcal{F}).$$

When X is a smooth complex projective variety and $\mathcal{F} = R^i \mathrm{a}_{X,*}\omega_X$, one recovers a generalization of the results of Green and Lazarsfeld. It is worthwhile to point out that contrary to what one might expect from the results of Green and Lazarsfeld, the support of the sheaves $R^0 p_{\hat{A},*}(p_A^* D_A(\mathcal{F})\otimes\mathcal{L})$ are not necessarily reduced subvarieties of $\hat{A}$. More precisely, we have:

Example 1.3. *Consider a nontrivial extension*

$$0 \longrightarrow \mathcal{O}_A \longrightarrow V \longrightarrow \mathcal{O}_A \longrightarrow 0.$$

Let $\mathcal{F} = V$, then $\mathcal{F}$ satisfies the hypothesis of Theorem 1.2, and one has that

$$R^0 p_{\hat{A},*}(p_A^* D_A(V)\otimes\mathcal{L})$$

is an Artinian $\mathcal{O}_{\hat{A},\hat{0}}$ module of length 2.

It is also not the case that the loci

$$V^i(\mathcal{F}) := \{P \in \hat{A}\ s.t.\ H^i(\mathcal{F}\otimes P) \neq 0\}$$

(or equivalently that the sheaves $R^i\hat{\mathcal{S}}(\mathcal{F}) := R^i p_{\hat{A},*}(p_A^*\mathcal{F}\otimes\mathcal{L})$) should be supported on subtori of $\hat{A}$ as is illustrated by the following:

Example 1.4. *Let (A, Θ) be a principally polarized abelian variety over $\mathbb{C}$, then for any ample line bundle L on $\hat{A}$, one has that $\mathcal{O}_A(\Theta)\otimes\hat{L}^\vee$ is globally generated (cf.* [PP]*) and has vanishing higher cohomology groups. Fix any point $x \in A$ and let $\mathcal{F} := \mathcal{O}_A(\Theta)\otimes\mathcal{I}_x$. From the short exact sequence*

$$0 \longrightarrow \mathcal{F}\otimes\hat{L}^\vee \longrightarrow \mathcal{O}_A(\Theta)\otimes\hat{L}^\vee \longrightarrow \mathcal{O}_A(\Theta)\otimes\hat{L}^\vee\otimes\mathbb{C}(x) \longrightarrow 0,$$

one sees that $H^i(A, \mathcal{F}\otimes\hat{L}^\vee) = 0$ for all $i > 0$. One has $R^i\hat{\mathcal{S}}(\mathcal{F}) = 0$ for all $i \geq 2$ and there is an exact sequence

$$0 \longrightarrow R^0\hat{\mathcal{S}}(\mathcal{F}) \longrightarrow \widehat{\mathcal{O}_A(\Theta)} \longrightarrow P_x \longrightarrow R^1\hat{\mathcal{S}}(\mathcal{F}) \longrightarrow 0,$$

where P_x is the topologically trivial line bundle on $\hat{A}$ determined by the point $x \in \hat{\hat{A}} = A$ and $\widehat{\mathcal{O}_A(\Theta)} := R^0\hat{\mathcal{S}}(\mathcal{O}_A(\Theta))$ is identified with $\mathcal{O}_A(\Theta)^\vee$ under the isomorphism $\phi_\Theta : \hat{A} \longrightarrow A$. Since the homomorphism $\widehat{\mathcal{O}_A(\Theta)} \longrightarrow P_x$ is non zero, one has that it is an inclusion of line bundles. Therefore, $R^0\hat{\mathcal{S}}(\mathcal{F}) = 0$ and the sheaf $R^1\hat{\mathcal{S}}(\mathcal{F})$ is supported on a divisor in $|\Theta\otimes P_x|$ and hence not on a subtorous of $\hat{A}$.

The statement of Theorem 1.2, appears to be very technical, however it seems to have concrete applications to the birational geometry of complex projective varieties. In particular, we prove Theorem 4.1, (a more general version) of a conjecture of Green and Lazarsfeld (cf. Problem 6.2 [GL2]):

Theorem 1.5. *If X is a smooth complex projective variety with* $\dim(\mathrm{a}_X(X)) = \dim(X)$*, then for the universal family of topological trivial line bundles* $\mathcal{P} \longrightarrow X \times \mathrm{Pic}^0(X)$*, one has that*

$$R^i\pi_{\hat{A},*}(\mathcal{P}) = 0$$

for all $i < \dim(X)$.

It should be noted that the results of Green and Lazarsfeld hold for X a Kähler manifold. The methods of this paper only apply to the projective setting and so the above theorem does not completely answer Problem 6.2 of [GL2]. It should however be possible to answer Problem 6.2 using results of K. Takegoshi [Ta] by using the same methods of this paper. We do not pursue it in this paper.

In section 5, we give an application to the higher direct images of tensor powers of the canonical line bundle under a surjective morphism to an abelian variety. To be more precise, let $a : X \longrightarrow A$ be a surjective morphism with connected fibers from a smooth complex projective variety with $\kappa(X) = 0$ to an abelian variety. In [CH2] it was shown that if $P_1(X) \neq 0$, then $a_*(\omega_X) = \mathcal{O}_A$. Here we extend this result as follows: *For all $N > 0$ there exists a unipotent vector bundle V_N and an inclusion $V_N \hookrightarrow a_*(\omega_X^{\otimes N})$ which is generically an isomorphism and induces an isomorphism on global sections.* In particular, the sheaves $a_*(\omega_X^{\otimes N})$ are semi-positive (cf. [V1]). It would be interesting to see if one can show that the inclusion is an isomorphism in codimension 1, and if this result can be used to give bounds on $\kappa(F_{X/A})$ (compare [Ka2]).

The techniques used to prove Theorem 1.2 are standard results in the theory of derived categories for which we refer to [Ha2]. It is surprising that one is able to obtain such a generalization of the results of Green and Lazarsfeld by means of such a simple argument. It is our hope that these methods might also give interesting applications to other Fourier-Mukai isomorphisms.

Acknowledgments. The author would like to thank A. Craw for useful suggestions. The author was partially supported by NSA grant no: MDA904-03-1-0101 and by a grant from the Sloan Foundation. Shortly after the submission of this paper, the author learned that G. Pareschi had also obtained a proof of Problem 6.2 of [GL2] by a completely different approach.

1.1. **Notation and conventions.** We will work over k an algebraically closed field. On a smooth variety, we will identify Cartier divisors and line bundles, and we will use the additive and multiplicative notation interchangeably. For any normal variety, with structure map $f : X \longrightarrow Spec(k)$, one has a dualizing complex $\omega^{\cdot}_X := f^!k$ (cf. [Ha2]). If X is a smooth projective variety, then K_X will be a canonical divisor and $\omega_X = \mathcal{O}_X(K_X)$, and we denote by $\kappa(X)$ the Kodaira dimension, by $q(X) := h^1(\mathcal{O}_X)$ the *irregularity* and by $P_m(X) := h^0(\omega_X^{\otimes m})$ the *$m-th$ plurigenus*. If $f\colon X \to Y$ is a morphism of smooth projective varieties, we write $K_{X/Y} := K_X - f^*K_Y$. We denote by $\mathrm{a}\colon X \to A$ the Albanese map and by $\hat{A} = \mathrm{Pic}^0(A)$ the dual abelian variety to A which parameterizes all topologically trivial line bundles on A. Recall that $\mathrm{Pic}^0(A) = \mathrm{Pic}^0(X)_{red}$. For a $\mathbb{Q}-$divisor D we let $\lfloor D \rfloor$ be the integral part and $\{D\}$ the fractional part. Numerical equivalence is denoted by $\equiv$ and we write $D \prec E$ if $E - D$ is an effective divisor. If L is a Cartier divisor, $|L|$ denotes the complete linear series associated to L. The rest of the notation is standard in algebraic geometry.

2. Preliminaries

2.1. **Derived Categories.** For a covariant left exact functor $F : A \longrightarrow B$ of abelian categories A, B, $RF : D(A) \longrightarrow D(B)$ will denote the right derived functor between the corresponding derived categories. In particular for any $X \in Ob(A)$, the i-th cohomology group of $RF(X)$ is just $R^iF(X)$. For any scheme X, $D(X)$ is the derived category of the category of $\mathcal{O}_X$-modules $\mathrm{Mod}(X)$. $D_c(X), D_{qc}(X)$ denote the full subcategories of $D(X)$ consisting of complexes whose cohomologies are coherent, quasi-coherent. $D^-(X), D^+(X), D^b(X)$ denote the full subcategories of $D(X)$ consisting of complexes bounded above, below, on both sides. $F[n]$ denotes the complex obtained by shifting the complex n places to the left. For the convenience of the reader we recall the following facts:

(1) **Projection formula (P.F.).** (cf. [Ha2] §II) Let $f : X \longrightarrow Y$ be a proper morphism of quasi-projective varieties. Then there is a functorial isomorphism:
$$Rf_*(F)\otimes_Y G \longrightarrow Rf_*(F\otimes_X Lf^*G)$$
for $F \in D^-(X)$ and $G \in D^-_{qc}(Y)$.

(2) **Grothendieck Duality (G.D.).** (cf. [Ha2] §VII) For an n-dimensional variety X, the dualizing functor is defined by $D_X(?) = R\mathcal{H}om(?, \omega^{\cdot}_X[n])$. In particular $\mathcal{O}_X \cong D_X(\omega^{\cdot}_X[n])$. Let $f : X \longrightarrow Y$ a morphism of projective varieties, $F \in D^b_{qc}(X)$ then
$$Rf_*D_X(F) \cong D_Y Rf_*(F).$$
In particular one has that
$$R\Gamma(D_X F) \cong D_k(R\Gamma(F)),$$
(where $R\Gamma = R\gamma_*$ and $\gamma : X \longrightarrow k$ is the structure morphism).

2.2. **Higher direct images of dualizing sheaves.** Recall the following results

Theorem 2.1. *(*[Ko1], [Ko2]*) Let X, Y be complex projective varieties of dimension n, k with X smooth. Let $f : X \longrightarrow Y$ be a surjective map and L an ample line bundle on Y. Then*

(1) *$R^if_*\omega_X$ is torsion free for $i \geq 0$,*
(2) *$H^j(Y, L\otimes R^if_*\omega_X) = 0$ for $j > 0$,*
(3) *$Rf_*\omega_X \cong \sum R^if_*\omega_X[-i]$.*
(4) *$Rf_*\mathcal{O}_X \cong \sum {}^dR^if_*\mathcal{O}_X$, where ${}^dR^if_*\mathcal{O}_X = D_Y(R^{n-k-i}f_*\omega_X[k+i])$.*

Theorem 2.2. *(*[Ko3] *§10) Let $f\colon X \to Y$ be a surjective map of projective varieties, X smooth, Y normal. Let M be a line bundle on X such that $M \equiv N + f^*L + \Delta$, where N is a $\mathbb{Q}$-divisor on X which is either nef and big or numerically trivial, L is a $\mathbb{Q}-$divisor on Y and Δ has normal crossing support with $\lfloor \Delta \rfloor = 0$. Then*

(1) *$R^jf_*(\omega_X\otimes M)$ is torsion free for $j \geq 0$;*
(2) *If L is nef and big, and $g : Y \longrightarrow Z$ is any morphism with Z projective, then $H^i(Y, R^jf_*(\omega_X\otimes M)) = 0$ for $i > 0$, $j \geq 0$ and*
$$H^i(Y, R^jf_*(K_X + M)) = H^i(Z, g_*(R^jf_*(K_X + M))), \quad i, j \geq 0.$$

2.3. **Sheaves on abelian varieties.** Let A be a g-dimensional abelian variety and $\hat{A} = \mathrm{Pic}^0(A)$ its dual abelian variety. We will denote by $p_A, p_{\hat{A}}$ the projections of $A \times \hat{A}$ on to $A, \hat{A}$. Let $\mathcal{L}$ be the normalized Poincaré line bundle on $A \times \hat{A}$. The main tool in understanding sheaves on an abelian variety is the Theory of Fourier-Mukai transforms. Recall that Mukai defines the functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by
$$\hat{\mathcal{S}}(M) = p_{\hat{A},*}(\mathcal{L}\otimes p_A^*M).$$
Similarly one defines $\mathcal{S}(N) = p_{A,*}(\mathcal{L}\otimes p_{\hat{A}}^*N)$. By [Mu], one has:

Theorem 2.3. Mukai *There exists isomorphisms of functors*

$$R\hat{\mathcal{S}}\circ R\mathcal{S}\cong(-1_{\hat{A}})^*[-g]$$

and

$$R\mathcal{S}\circ R\hat{\mathcal{S}}\cong(-1_A)^*[-g].$$

We will need the following results analogous to cohomology and base change ([Ha1] III.12 and [EGA] §7):

Lemma 2.4. *Let $F\in D^b_c(A)$, and $P_{\hat{a}}$ be the topologically trivial line bundle corresponding to the point $\hat{a}\in\hat{A}$. If $R^i\Gamma(F\otimes P_{\hat{a}})=0$, then $R^i\hat{\mathcal{S}}(F)\otimes k(\hat{a})=0$ and the natural map*

$$\varphi^{i-1}(\hat{a}):R^{i-1}\hat{\mathcal{S}}(F)\otimes k(\hat{a})\longrightarrow R^{i-1}\Gamma(F\otimes P_{\hat{a}})$$

is surjective.

We recall for future reference the following example of Mukai (cf. [Mu] Example 2.9).

Example 2.5. (Mukai) *A vector bundle U on A is unipotent if there exists a filtration*

$$0=U_0\subset U_1\subset\ldots\subset U_{n-1}\subset U_n=U$$

such that $U_i/U_{i-1}\cong\mathcal{O}_A$ for all $1\leq i\leq n$. One has that $R\hat{\mathcal{S}}(U)=R^g\hat{\mathcal{S}}(U)$ and $R^g\hat{\mathcal{S}}(U)$ is supported on $\hat{0}\in\hat{A}$. This gives an equivalence between the category of unipotent vector bundles on A and the category of coherent sheaves on $\hat{A}$ supported on $\hat{0}$ i.e. the category of Artinian $\mathcal{O}_{\hat{A},\hat{0}}$-modules.

3. Main result

We now proceed to prove Theorem 1.2:

Proof. By Grothendieck Duality (G.D.) and the projection formula (P.F.) it follows that

$$D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))\cong^{G.D.}R\Gamma(D_A(\mathcal{F}\otimes\hat{L}^\vee))\cong$$
$$R\Gamma(D_A(\mathcal{F})\otimes\hat{L})\cong R\Gamma(D_A(\mathcal{F})\otimes p_{A,*}(\mathcal{L}\otimes p^*_{\hat{A}}L))\cong^{P.F.}$$
$$R\Gamma(Lp^*_A D_A(\mathcal{F})\otimes\mathcal{L}\otimes p^*_{\hat{A}}L)\cong^{P.F.}R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L).$$

In particular, $D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))$ is a sheaf if and only if $R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L)$ is a sheaf. The theorem now follows from the following remarks:

(1) Notice that there is an isomorphism

$$H^0(A,\mathcal{F}\otimes\hat{L}^\vee)^\vee\cong D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))$$

if and only if

$$H^i(A,\mathcal{F}\otimes\hat{L}^\vee)=0\ for\ all\ i>0.$$

(2) For L sufficiently ample, the coherent sheaves $R^j\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L$ are globally generated and have vanishing higher cohomology. Consider the spectral sequence

$$E_2^{i,j}=R^i\Gamma(R^j\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L)\Rightarrow R^{i+j}\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L).$$

Since $E_2^{i,j}=0$ for all $i\neq 0$, this sequence degenerates at the E_2 level and hence

$$H^0(\hat{A},R^j\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L)\cong R^j\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L).$$

Therefore,

$$R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L)\cong R^0\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F}))\otimes L)$$

if and only if

$$R\hat{\mathcal{S}}(D_A(\mathcal{F})) \cong R^0\hat{\mathcal{S}}(D_A(\mathcal{F})).$$

□

Corollary 3.1. *Let $\mathcal{F}$ be a coherent sheaf as above. For all $i > 0$, the support of $R^i\hat{\mathcal{S}}(\mathcal{F})$ has codimension at least i in $\hat{A}$.*

Proof. By [Mu] (3.8), one has that

$$R\hat{\mathcal{S}}(\mathcal{F}) \cong D_{\hat{A}}\left((-1^*_{\hat{A}} \circ R^0\hat{\mathcal{S}} \circ D_A)(\mathcal{F})[g]\right) \cong R\mathcal{H}om\left(\mathcal{G}, \mathcal{O}_{\hat{A}}\right),$$

where $\mathcal{G} := -1^*_{\hat{A}}(R^0\hat{\mathcal{S}}(D_A(\mathcal{F})))$ is a sheaf. It suffices therefore to show that for any coherent sheaf $\mathcal{G}$ on a smooth affine variety $Y = Spec(A)$, one has that the sheaves $R^i\mathcal{H}om(\mathcal{G}, \mathcal{O}_Y)$ are supported in codimension at least i. Let $M = \Gamma(Y, \mathcal{G})$ and W be an irreducible component of the support of $R^i\mathcal{H}om(\mathcal{G}, \mathcal{O}_Y) = Ext^i(M, A)^{\frown}$. Let $P \in Spec(A)$ such that $W = Spec(A/P)$. Since localization is exact, one has that

$$0 \neq Ext^i(M, A) \otimes A_P \cong Ext^i(M_P, A_P).$$

Since A is regular, it follows that A_P is regular and hence that

$$i \leq \dim A_P = \dim(Y) - \dim(W)$$

(cf. [Ha1] §III.6). □

Corollary 3.2. *Let $\mathcal{F}$ be a coherent sheaf as above, and $P \in \hat{A}$, then:*

(1) *If $i \geq 0$ and $H^i(A, \mathcal{F} \otimes P) = 0$, then $H^{i+1}(A, \mathcal{F} \otimes P) = 0$.*
(2) *For all $i > 0$ any irreducible component of the loci*

$$V^i(\mathcal{F}) := \{P \in \mathrm{Pic}^0(A) \mid h^i(\mathcal{F} \otimes P) \neq 0\}$$

has codimension at least i in $\hat{A}$.
(3) *If $H^0(A, \mathcal{F} \otimes P) = 0$ then $R^0\hat{\mathcal{S}}(D_A(\mathcal{F})) \otimes k(P^\vee) = 0$.*
(4) *If $H^0(A, \mathcal{F} \otimes P) = 0$ for all $P \neq \mathcal{O}_A$, then $\mathcal{F}$ is a unipotent vector bundle.*

Proof. (1) Since

$$0 = H^i(A, \mathcal{F} \otimes P)^\vee \cong H^{-i}(D_{\mathbb{C}}R\Gamma(\mathcal{F} \otimes P)) \cong R^{-i}\Gamma(D_A(\mathcal{F}) \otimes P^\vee),$$

by Cohomology and Base Change, one sees that the natural homomorphism

$$0 = R^{-i-1}\hat{\mathcal{S}}(D_A(\mathcal{F})) \otimes k(P^\vee) \longrightarrow R^{-i-1}\Gamma(D_A(\mathcal{F}) \otimes P^\vee)$$

is surjective and hence that $H^{i+1}(A, \mathcal{F} \otimes P) = 0$.

(2) Let W be an irreducible component of $V^i(\mathcal{F})$ and P a general point in W. Let j be the largest integer such that $W \subset V^j(\mathcal{F})$. By Cohomology and Base Change (for sheaves), one has that $R^j\hat{\mathcal{S}}(\mathcal{F}) \otimes k(P) \neq 0$ and by the previous corollary, one has that W has codimension at least j.

(3) This follows immediately from Cohomology and Base Change.

(4) By (3), we have that $R^0\hat{\mathcal{S}}(D_A(\mathcal{F}))$ is supported on the closed point $\hat{0} \in \hat{A}$ corresponding to $\mathcal{O}_A$. By Example 2.5, $R^0\hat{\mathcal{S}}(D_A(\mathcal{F}))$ is an Artinian module and hence

$$D_A(\mathcal{F}) \cong (-1_A)^* R\mathcal{S}R\hat{\mathcal{S}}(D_A(\mathcal{F}))$$

is (the shift of) a unipotent vector bundle say V. Finally it is easy to see that

$$\mathcal{F} = D_A(D_A(\mathcal{F})) = V^\vee$$

is also a unipotent vector bundle. □

4. A conjecture of Green and Lazarsfeld

When X is a smooth complex projective variety and $\mathrm{a} : X \longrightarrow A$ is the Albanese morphism, one recovers a generalization of the conjecture of Green and Lazarsfeld mentioned in the introduction. To fix the notation, let $\mathcal{P} = (\mathrm{a}, id_{\hat{A}})^*\mathcal{L}$ and let $\pi_{\hat{A}}$ denote the projection of $X \times \hat{A}$ onto $\hat{A}$.

Theorem 4.1. *Let X be a smooth complex projective variety of dimension n, and Albanese dimension $k := \dim \mathrm{a}(X)$. Then*

$$R\pi_{\hat{A},*}(\mathcal{P}) \cong \sum_{i=0}^{n-k} R^{k+i}\hat{\mathcal{S}}({}^dR^i\mathrm{a}_*\mathcal{O}_X).$$

In particular, when X is of maximal Albanese dimension (i.e. $k = n$), one has that $R\pi_{\hat{A},*}(\mathcal{P}) \cong R^n\hat{\mathcal{S}}({}^dR^0\mathrm{a}_*\mathcal{O}_X)$ is a sheaf.

Proof. Let $Y := a(X)$. We have that

$$R\pi_{\hat{A},*}(\mathcal{P}) \cong R\pi_{\hat{A},*}((\mathrm{a} \times id_{\hat{A}})^*\mathcal{L}) \cong^{P.F.} Rp_{\hat{A},*}(Lp_A^*(R\mathrm{a}_*\mathcal{O}_X)\otimes\mathcal{L}) \cong$$

$$Rp_{\hat{A},*}(Lp_A^*(\sum_{i=0}^{n-k} {}^dR^i\mathrm{a}_*\mathcal{O}_X)\otimes\mathcal{L}) \cong \sum_{i=0}^{n-k} R\hat{\mathcal{S}}({}^dR^i\mathrm{a}_*\mathcal{O}_X).$$

Since

$${}^dR^i a_*\mathcal{O}_X \cong D_Y(R^{n-k-i}\mathrm{a}_*\omega_X[k+i]) \cong D_A(R^{n-k-i}\mathrm{a}_*\omega_X[k+i])$$

it suffices to show that the sheaves $\mathcal{F}^{n-k-i} := R^{n-k-i}\mathrm{a}_*\omega_X$ satisfy the hypothesis of Theorem 1.2. This is however an immediate consequence of Theorem 2.1 and the observations that follow:

Let $\mathcal{F} = \mathcal{F}^{n-k-i}$. As mentioned in the Introduction, $\hat{L} := R^0\mathcal{S}(L) \cong R\mathcal{S}(L)$ is a vector bundle on A of rank $h^0(L)$ and

$$\phi_L^*\left(\hat{L}^\vee\right) \cong (L)^{\oplus h^0(L)}.$$

We have that $H^i(A, \mathcal{F}\otimes\hat{L}^\vee)$ is a direct summand of

$$H^i(A, \phi_{L,*}(\mathcal{O}_{\hat{A}})\otimes\mathcal{F}\otimes\hat{L}^\vee) \cong^{P.F.} H^i(\hat{A}, \phi_L^*(\mathcal{F}\otimes\hat{L}^\vee)) \cong \bigoplus_{i=1}^{h^0(L)} H^i(\hat{A}, \phi_L^*(\mathcal{F})\otimes L).$$

So it suffices to show that $H^i(\hat{A}, \phi_L^*(\mathcal{F})\otimes L) = 0$ for all $i > 0$. Let $X' := X \times_A \hat{A}$ and $\mathrm{a}' : X' \longrightarrow \hat{A}$ be the induced morphism. By flat base change, $\phi_L^*(\mathcal{F}) \cong R^{n-k-i}\mathrm{a}'_*\omega_{X'}$, and so by Theorem 2.1, we have the required vanishing:

$$h^i(\hat{A}, R^{n-k-i}\mathrm{a}'_*\omega'_X\otimes L) = 0 \textit{ for all } i > 0.$$

$\square$

Corollary 4.2. *Let $\mathrm{a} : X \longrightarrow A$ be as in the preceding theorem, then*

1. *Every irreducible component of the loci $V^i(R^ja_*\omega_X)$ has codimension at least i in $\hat{A}$.*
2. *For all $i < \dim \mathrm{a}(X)$ and general $P \in \mathrm{Pic}^0(X)$, one has that $H^i(X, P) = 0$.*
3. *If $H^j(R^i\mathrm{a}_*\omega_X\otimes P) = 0$, then $H^l(R^i\mathrm{a}_*\omega_X\otimes P) = 0$ for all $l > j \geq 0$ and $0 \leq i \leq k$.*

Proof. This is analogous to Corollary 3.2. $\square$

Remark 4.3. *Following results of Deligne, Illusie and Raynaud (cf.* [EV] §*11), one can recover some similar statements for X a proper smooth scheme over a perfect field k admitting a lifting $\tilde{X}$ to $W_2(k)$ and $char(k) > \dim(X)$.*

5. Higher direct images of pluricanonical line bundles

Recall the following cf. [Mo]:

Conjecture K (Ueno) *Let X be a smooth complex projective variety with $\kappa(X) = 0$, and let $\mathrm{a} : X \longrightarrow A$ be its Albanese morphism. Then*

(1) a *is surjective and has connected fibers, i.e.* a *is an algebraic fiber space;*
(2) *if F is the general fiber of* a*, then $\kappa(F) = 0$;*
(3) *there is an étale covering $B \longrightarrow A$ such that $X \times_A B$ is birationally equivalent to $F \times B$ over B.*

In [Ka1], Kawamata proved (1) above. In this section, we will give some evidence towards (2). Ideally, one would like to show that if $\kappa(X) = 0$ and $P_1(X) = 1$, then for all $N > 0$, there is an isomorphism $\mathcal{O}_A \cong \mathrm{a}_*(\omega_X^{\otimes N})^{\vee\vee}$. This would imply (2) above and give convincing evidence towards (3). In [CH2] this was established for $N = 1$. We will show that for $N \geq 2$ there is a unipotent vector bundle V_N and an inclusion $V_N \hookrightarrow \mathrm{a}_*(\omega_X^{\otimes N})$ which is a generic isomorphism.

We begin by recalling some well known properties of multiplier ideal sheaves. We refer to [La] for a more complete treatment.

Let X be a smooth projective variety and $|L|$ a (non-empty) linear series on X, let $\nu : \tilde{X} \longrightarrow X$ be log resolution of $|L|$ i.e. a proper birational morphism from a smooth projective variety such that $\nu^*|L| = |M| + F$ where $|M|$ is base point free and $F \cup \{\textit{Exceptional locous of} \nu\}$ has normal crossings support. For any rational number $c > 0$, define the multiplier ideal sheaf associated to c and $|L|$ by

$$\mathcal{I}(X, c \cdot |L|) := \nu_*(K_{\tilde{X}/X} - \lfloor cF \rfloor).$$

When $c = 1$ and $D = |L|$ this is denoted by $\mathcal{I}(D)$. For every $k \geq 1$, one has that

$$\mathcal{I}(c \cdot |L|) \subset \mathcal{I}(\frac{c}{k} \cdot |kL|).$$

Therefore, the family of ideals

$$\{\mathcal{I}\left(\frac{c}{k} \cdot |kL|\right)\}_{k \geq 0}$$

has a unique maximal element which we denote by $\mathcal{I}(c \cdot ||L||)$. For all $k \gg 0$ one has

$$\mathcal{I}(c \cdot ||L||) = \mathcal{I}(\frac{c}{k} \cdot |kL|).$$

Consider now $f : X \longrightarrow Y$ a surjective morphism of projective varieties, with connected fibers and X smooth. Fix H an ample line bundle on Y, and L a line bundle on X with $\kappa(L) \geq 0$. For all $t \geq 0$ let

$$\mathcal{I}_t := \mathcal{I}\left(\frac{1}{t} \cdot ||tL + f^*H||\right).$$

For any sufficiently big and divisible integer k, one has that (after replacing k by kt)

$$\mathcal{I}_t = \mathcal{I}\left(\frac{1}{k} \cdot |kL + \frac{k}{t} f^*H|\right)$$

and similarly for $\mathcal{I}_{t+1}$. Let $\nu : \tilde{X} \longrightarrow X$ be a log resolution of $|kL + \frac{k}{t} f^*H|$ and of $|kL + \frac{k}{t+1} f^*H|$, ($k$ sufficiently big and divisible) so that

$$\nu^*|kL + \frac{k}{t} f^*H| = |M| + F, \qquad \nu^*|kL + \frac{k}{t+1} f^*H| = |M'| + F'$$

with $|M|, |M'|$ base point free and F, F' with simple normal crossings support. From the inclusion of linear series

$$\nu^*|kL + \frac{k}{t+1} f^*H| \times \nu^* f^* |\frac{k}{t(t+1)} H| \longrightarrow \nu^*|kL + \frac{k}{t} f^*H|$$

one sees that $F \prec F'$ and therefore

$$\mathcal{I}_{t+1} = \nu_*(K_{\tilde{X}/X} - \lfloor \frac{1}{k} F' \rfloor) \subset \nu_*(K_{\tilde{X}/X} - \lfloor \frac{1}{k} F \rfloor) = \mathcal{I}_t.$$

Proposition 5.1. *There exists an integer t_0 such that for all $t \geq t_0$, one has*

$$f_*(\omega_X \otimes L \otimes \mathcal{I}_t) = f_*(\omega_X \otimes L \otimes \mathcal{I}_{t_0}).$$

Proof. For all $m \geq 1$ and $t \geq 2$,

$$K_{\tilde{X}} + \nu^* L - \lfloor \frac{1}{k} F \rfloor + m\nu^* f^* H \equiv K_{\tilde{X}} + \frac{1}{k} M + \{\frac{1}{k} F\} + (m - \frac{1}{t})\nu^* f^* H$$

with $\frac{1}{k}M$ nef and big, $(m - \frac{1}{t})H$ ample, $\{(1/k)F\}$ has normal crossings support and $\lfloor \{(1/k)F\} \rfloor = 0$. By Theorem 2.2, one has that

$$H^i(Y, f_*(\omega_X \otimes L \otimes f^* H^{\otimes m} \otimes \mathcal{I}_t)) =$$

$$H^i(Y, (f \circ \nu)_*(\omega_{\tilde{X}} \otimes \nu^* L \otimes \nu^* f^* H^{\otimes m}(-\lfloor \frac{1}{k} F \rfloor))) = 0.$$

Consider now the exact sequence of coherent sheaves on Y

$$0 \longrightarrow f_*(K_X \otimes L \otimes \mathcal{I}_{t+1}) \otimes H^{\otimes m} \longrightarrow f_*(K_X \otimes L \otimes \mathcal{I}_t) \otimes H^{\otimes m} \longrightarrow \mathcal{Q}_t \otimes H^{\otimes m} \longrightarrow 0,$$

The coherent sheaves $f_*(\omega_X \otimes L \otimes f^* H^{\otimes m} \otimes \mathcal{I}_t)$, $f_*(\omega_X \otimes L \otimes f^* H^{\otimes m} \otimes \mathcal{I}_{t+1})$, have vanishing higher cohomology groups for $m \geq 1$ and therefore, $\mathcal{Q}_t \otimes H^{\otimes m}$ also has vanishing higher cohomology groups for all $m \geq 1$. We may assume that H is very ample. By Mumford regularity, $\mathcal{Q}_t \otimes H^{\otimes m}$ is globally generated for all $m \geq \dim(Y) + 1$. It follows that if $f_*(K_X \otimes L \otimes \mathcal{I}_t)$ and $f_*(K_X \otimes L \otimes \mathcal{I}_{t+1})$ are not isomorphic (i.e. if $\mathcal{Q}_t$ is non zero), then $H^0(\mathcal{Q}_t \otimes H^{\otimes m}) \neq 0$. Therefore,

$$h^0(f_*(K_X \otimes L \otimes \mathcal{I}_{t+1}) \otimes H^{\otimes m}) < h^0(f_*(K_X \otimes L \otimes \mathcal{I}_t) \otimes H^{\otimes m}).$$

In particular, for fixed $m = \dim(Y)+1$, and $t \geq 2$, one has that $f_*(K_X \otimes L \otimes \mathcal{I}_t) \otimes H^{\otimes m}$ and $f_*(K_X \otimes L \otimes \mathcal{I}_{t+1}) \otimes H^{\otimes m}$ are not isomorphic for at most finitely many values of t. The Proposition now follows. □

Corollary 5.2. *Let $e : \tilde{Y} \longrightarrow Y$ be any étale morphism. For all $t \gg 0$, $i > 0$ H, M ample on $Y, \tilde{Y}$, one has*

$$H^i(\tilde{Y}, e^* f_*(\omega_X \otimes L \otimes \mathcal{I}(||L + \frac{1}{t} f^* H||)) \otimes M) = 0.$$

Proof. For $t \gg 0$, one has that $M \otimes (1/t) e^* H^\vee$ is ample. Let $\tilde{X} = X \times_Y \tilde{Y}$ and $\epsilon : \tilde{X} \longrightarrow X$ the induced étale morphism. Proceeding as above, one sees that for $g : \tilde{X} \longrightarrow \tilde{Y}$,

$$H^i(\tilde{Y}, e^* f_*(\omega_X \otimes L \otimes \mathcal{I}_t) \otimes M) = H^i(\tilde{Y}, g_* \epsilon^*(\omega_X \otimes L \otimes \mathcal{I}_t) \otimes M) =$$

$$H^i(\tilde{Y}, g_*(\omega_{\tilde{X}} \otimes \epsilon^* L \otimes g^* M \otimes \epsilon^* \mathcal{I}_t)).$$

It follows that these cohomology groups vanish for all $t \gg 0$. □

Corollary 5.3. *If $L = \omega_X^{\otimes(N-1)}$ for some $N \geq 2$, then for all $t \gg 0$ one has that:*

1. $H^0(Y, f_*(\omega_X \otimes L \otimes \mathcal{I}_t)) \cong H^0(Y, f_*(\omega_X^{\otimes N}))$;
2. *For general $y \in Y$, the rank of $f_*(\omega_X \otimes L \otimes \mathcal{I}_t)$ at y is $P_N(F)$.*

Proof. (1) Let $\nu : \tilde{X} \longrightarrow X$ be a log resolution of $|NK_X|$, $|k(N-1)K_X|$ and $|k(N-1)K_X + (k/t)f^*H|$ (for some $t \gg k \gg 0$ sufficiently big and divisible). The corresponding linear series have moving parts $|M_N|, |M_{k(N-1)}|, |M|$ which are base point free and the corresponding fixed components $F_N, F_{k(N-1)}, F$ that have simple normal crossings support. It is easy to see that

$$\frac{F}{k} \prec \frac{F_{k(N-1)}}{k} \prec \frac{N-1}{N} F_N \prec F_N.$$

In particular we have inclusions

$$|N\nu^*K_X - F_N| \longrightarrow |N\nu^*K_X - \lfloor\frac{F}{k}\rfloor| \longrightarrow |N\nu^*K_X| \longrightarrow |NK_{\tilde{X}}|.$$

Since $h^0(\nu^*\omega_X^{\otimes N}(-F_N)) = h^0(\nu^*\omega_X^{\otimes N}) = h^0(\omega_{\tilde{X}}^{\otimes N})$, these are all isomorphisms. Finally, we have that

$$h^0(f_*(\omega_X\otimes L\otimes\mathcal{I}_t)) = h^0(\omega_{\tilde{X}}\otimes\nu^*\omega_X^{\otimes(N-1)}(-\lfloor\frac{1}{k}F\rfloor)) = h^0(\omega_{\tilde{X}}^{\otimes N}) = P_N(X).$$

To see the (2) recall that for fixed $k \gg 0$ and for some $t \gg 0$, one has that

$$\mathcal{F} := a_*\nu_*(\omega_{\tilde{X}}\otimes\nu^*\omega_X^{\otimes(N-1)} - \lfloor\frac{1}{k}F\rfloor)$$

where $\nu : \tilde{X} \longrightarrow X$ is a log resolution of $|k(N-1)K_X + (k/t)a^*H|$ and so

$$|k(N-1)K_X + \frac{k}{t}\nu^*a^*H| = |M| + F$$

with $|M|$ base point free and F with simple normal crossings support. We may assume that ν also induces a log resolution of $|NK_{X_y}|$ i.e. that

$$|NK_{\tilde{X}_y}| = |M'| + F'$$

with $|M'|$ base point free and F' a divisor with simple normal crossings support. By a result of Viehweg (c.f. [V1], [V2], see also [Ko4] §10), we have that

$$\lfloor\frac{F}{k}\rfloor \prec F'.$$

In particular, at a generic point $y \in Y$, one has that

$$\mathcal{F}\otimes\mathbb{C}(y) \cong H^0(\mathcal{F}\otimes\mathbb{C}(y)) \cong H^0(X_y, \mathcal{O}_{X_y}(NK_{\tilde{X}} - \lfloor\frac{F}{k}\rfloor)) \cong$$

$$H^0(X_y, \mathcal{O}_{X_y}(NK_{\tilde{X}_y})) \cong \mathbb{C}^{P_N(X_y)}.$$

□

Proposition 5.4. *Let X be a smooth complex projective variety with $\kappa(X) = 0$ and $P_1(X) = 1$. Then for all $N > 0$, there exists a unipotent vector bundle V_N of rank $P_N(F)$ and an inclusion $V_N \hookrightarrow \mathrm{a}_*(\omega_X^{\otimes N})$.*

Proof. Let X be a smooth projective variety with $\dim(X) = n$, q(X)=q, and $\kappa(X) = 0$. By the above mentioned result of Kawamata, the Albanese morphism $\mathrm{a} : X \longrightarrow A$ is an algebraic fiber space. For a fixed ample divisor H on A and any $t \gg 0$ let

$$\mathcal{F} := \mathrm{a}_*\left(\omega_X^{\otimes N}\otimes\mathcal{I}(||(N-1)K_X + \frac{1}{t}\mathrm{a}^*H||)\right).$$

Since, $\kappa(X) = 0$, it is easy to see that $h^0(\omega_X^{\otimes N}\otimes P) = 0$ for all $P \neq \mathcal{O}_X$ and $h^0(\omega_X^{\otimes N}) = 1$ (cf. [CH2] Lemma 3.1). In particular, as $\mathcal{F}$ is a non-zero subsheaf of $\mathrm{a}_*(\omega_X^{\otimes N})$, one has $h^0(\mathcal{F}\otimes P) = 0$ for all $P \neq \mathcal{O}_X$ and $h^0(\mathcal{F}) \leq 1$. By Corollary 5.3, it follows that

$$\mathcal{F} \cong V_N,$$

where V_N is an unipotent vector bundle of rank $P_N(F)$. □

References

[CH1] Chen, J. A. and Hacon, C. D.: Characterization of abelian varieties, Invent. Math. **143**, 435-447 (2001)

[CH2] Chen, J. A. and Hacon, C. D.: On algebraic fiber spaces over varieties of maximal albanese dimension, Duke Math. J., Vol. 111, No. 1, (2002)

[EGA] Grothendieck, A. and Dieudonné, J.: Eléments de Géométrie Algébrique III. Étude cohomologique des faisceaux cohérents, Publ. Math. IHES **17**, 1963

[EV] Esnault, H. and Viehweg W. Lectures on Vanishing Theorems, DMV Seminar, 20. Birkhäuser Verlag, 1992

[GL1] Green, M. and Lazarsfeld, R.: Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. Inventiones Math. **90** (1987), 389-407

[GL2] Green, M. and Lazarsfeld, R.: Higher obstructions to deforming cohomology groups of line bundles, Jour. Amer. Math. Soc. vol. 4, **1**, 1991, 87-103

[Ha1] Hartshorne, R.: Algebraic Geometry, Graduate texts in Math., 1977, 52 Springer Verlag

[Ha2] Hartshorne, R.: Residues and Duality, L.N.M. **20**, 1996, Springer Verlag

[Ka1] Kawamata, Y.: Characterization of abelian varieties, Compositio Math. **43**, 1981, 253-276

[Ka2] Kawamata, Y.: Kodaira Dimension of Algebraic Fiber Spaces Over Curves, Invent. Math. **66**, 57-71 (1982)

[Ko1] Kollár, J.: Higher direct images of dualizing sheaves I, Ann. Math. **123**, 1986, 11–42

[Ko2] Kollár, J.: Higher direct images of dualizing sheaves II, Ann. Math. **124**, 1986, 171–202

[Ko3] Kollár, J.: Shafarevich maps and automorphic forms, M. B. Porter Lectures, Princeton Univ. Press, Princeton, 1995

[Ko4] Kollár, J.: Shafarevic maps and plurigenera of algebraic varieties, Invent. Math **113**, 1993, 177-215

[La] Lazarsfeld, R.: Positivity in Algebraic Geometry, Preprint

[Mo] Mori, S.: Classification of higher dimensional varieties, Algebraic Geometry (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. **46**, Amer. Math. Soc., Providence, 1987, 269-331

[Mu] Mukai, S.: Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya math. J. **81**, 1981, 153-175

[PP] Pareschi, G. and Popa, M.: Regularity of abelian varieties I. arXiv:math.AG/0110003, 30 Sep 2001

[Ta] Takegoshi, K.: Higher direct images of canonical sheaves tensorized with Nakano semi-positive vector bundles by proper Kähler morphisms, Math. Ann. **303**, 389-416 (1995)

[V1] Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension of certain fiber spaces. Adv. Studies in Pure Math 1, 1983, 329-353

[V2] Viehweg, E.: Quasi-projective moduli for polarized manifolds, Ergenbisse der Mathematik 30 (Springer, 1995)

Christopher D. Hacon
Department of Mathematics, University of Utah
155 South 1400 East, JWB 233
Salt Lake City, UT 84112-0090 - USA
email: hacon@math.utah.edu
fax: 801-581-4148