

Complex analytic Néron models for arbitrary families of intermediate Jacobians

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Dedicated to Herb Clemens on the occasion of his 70th birthday

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Abstract Given a family of intermediate Jacobians (for a polarizable variation of integral Hodge structure of odd weight) on a Zariski-open subset of a complex manifold, we construct an analytic space that naturally extends the family. Its two main properties are: (a) the horizontal and holomorphic sections are precisely the admissible normal functions without singularities; (b) the graph of any admissible normal function has an analytic closure inside our space. As a consequence, we obtain a new proof for the zero locus conjecture of M. Green and P. Griffiths. The construction uses filtered \mathcal{D} -modules and M. Saito's theory of mixed Hodge modules; it is functorial, and does not require normal crossing or unipotent monodromy assumptions.

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1 Overview

1.1 Introduction

In February 2008, during a lecture at the Institute for Pure and Applied Mathematics, P. Griffiths posed the problem of constructing Néron models for arbitrary families of intermediate Jacobians. In other words, given a family of intermediate Jacobians over a Zariski-open subset X of a complex manifold \bar{X} , one should construct a space that extends the family to all of \bar{X} . This has to

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be done in such a way that normal functions extend to sections of the Néron model.

It is known that two additional conditions need to be imposed to make this into a reasonable question. Firstly, the family of intermediate Jacobians should come from a *polarizable* variation of Hodge structure \mathcal{H} , which we may normalize to be of weight -1 . Secondly, one should consider only *admissible* normal functions. The Néron model is then expected to have the following structure: (1) Over each point of \bar{X} , its fiber should be a countable union of complex Lie groups. (2) The components over a point $x \in \bar{X} - X$ where the variation degenerates should be indexed by a countable group, whose elements are the possible values for the *singularity* at x of admissible normal functions—an invariant introduced by M. Green and P. Griffiths [15] that measures whether the cohomology class of a normal function is trivial in a neighborhood of x . (3) The horizontal sections of the identity component of the Néron model should be the admissible normal functions without singularities.

The existence of Néron models with good properties has useful consequences, for instance, a proof of the following conjecture by M. Green and P. Griffiths:

Conjecture 1.1 *Let ν be an admissible normal function on an algebraic variety X . Then the zero locus $Z(\nu)$ is an algebraic subvariety of X .*

By Chow's Theorem, it suffices to show that the closure of $Z(\nu)$ inside a projective compactification \bar{X} remains analytic; this is almost automatic once ν has been extended to a section of a Néron model over \bar{X} with good properties. M. Saito has established Conjecture 1.1 for $\dim X = 1$ by this method [35]; an entirely different approach has been pursued by P. Brosnan and G. Pearlstein [4, 5], who have announced a full proof in the summer of 2009 [6].

In this paper, we largely solve P. Griffiths' problem, by constructing an analytic space $\bar{J}(\mathcal{H})$ that has all the properties expected for the identity component of the Néron model—in particular, its horizontal and holomorphic sections are precisely the admissible normal functions without singularities. We also show that the graph of any admissible normal function has an analytic closure inside our space; one consequence is a new proof for Conjecture 1.1. Lastly, we describe the construction of an analytic Néron model for admissible normal functions with torsion singularities. Based on some examples, we argue that this is the most general setting in which a Néron model exists as an analytic space or even as a Hausdorff space.

The construction that is proposed here is very natural and suitably functorial; it is motivated by unpublished work of H. Clemens on the family of hypersurface sections of a smooth projective variety (briefly reviewed in

Sect. 1.4 below). An important point is that no assumptions on the singularities of $D = \bar{X} - X$, or on the local monodromy of the variation of Hodge structure are needed. This is in contrast to the traditional approach, which would be to make D into a divisor with normal crossings by using resolution of singularities, and then to pass to a finite cover to get unipotent monodromy. We accomplish this generality with the help of M. Saito's theory of mixed Hodge modules [32].

Two other solutions to the problem have been given recently. One is by P. Brosnan, G. Pearlstein, and M. Saito [7], whose Néron model is a topological space to which admissible normal functions extend as continuous sections. They also show that the base manifold \bar{X} can be stratified in such a way that, over each stratum, their space is a family of complex Lie groups, and the extended normal function a holomorphic section. Unfortunately, it is not clear from the construction whether the resulting space is Hausdorff; and when the local monodromy is not unipotent, the fibers of their Néron model can be too small, even in one-parameter degenerations of abelian varieties. We address both issues in Sect. 4.6 below, by showing that there is always a continuous and surjective mapping from the analytic space constructed here to the identity component of their Néron model.

A second solution is contained in a series of papers by K. Kato, C. Nakayama, and S. Usui [24, 25], who use classifying spaces of pure and mixed nilpotent orbits to define a Néron model in the category of log manifolds. When I first wrote this paper, their construction was only available for $\dim X = 1$; in the two years since then, they have extended their construction to the case when D is a normal crossing divisor and $\mathcal{H}_{\mathbb{Z}}$ has unipotent monodromy, and have used it to give a third proof of Conjecture 1.1. It seems likely that there will be a connection between the identity component of their Néron model and the subset of $\bar{J}(\mathcal{H})$ defined by the horizontality condition in Sect. 4.2. This question is currently under investigation by T. Hayama, who has proved a similar result in the case $\dim X = 1$ [19].

1.2 Conventions

(i) In dealing with filtrations, we index increasing filtrations (such as weight filtrations, or Hodge filtrations on \mathcal{D} -modules) by lower indices, and decreasing filtrations (such as Hodge filtrations on vector spaces, or V -filtrations on left \mathcal{D} -modules) by upper indices. We may pass from one to the other by the convention that $F^{\bullet} = F_{-\bullet}$. To be consistent, shifts in the filtration thus have different effects in the two cases:

$$F[1]^{\bullet} = F^{\bullet+1}, \quad \text{while } F[1]_{\bullet} = F_{\bullet-1}.$$

This convention agrees with the notation used in M. Saito's papers. In the case of weight filtrations, we also define $W[1]_{\bullet} = W_{\bullet-1}$; the reader should be aware that this is different from the convention used in [10].

(ii) Unless stated otherwise, all (mixed) Hodge structures are assumed to be defined over \mathbb{Z} . When dealing with mixed Hodge modules and mixed Hodge structures (or variations of mixed Hodge structure) at the same time, we usually consider the Hodge filtrations on the latter as increasing filtrations.

(iii) Throughout the paper, we work exclusively with *left* \mathcal{D} -modules, and the term “ \mathcal{D} -module” shall always mean “left \mathcal{D} -module” (in contrast to [32], where right \mathcal{D} -modules are used most of the time).

(iv) When M is a mixed Hodge module, the effect of a Tate twist $M(k)$ on the underlying filtered \mathcal{D} -module (\mathcal{M}, F) is as follows:

$$(\mathcal{M}, F)(k) = (\mathcal{M}, F[k]) = (\mathcal{M}, F_{\bullet-k}).$$

(v) For a mixed Hodge module M on a complex manifold X , we let $\mathbf{D}_X(M)$ denote the dual; its underlying holonomic \mathcal{D} -module is $\mathcal{E}xt_{\mathcal{D}}^{d_X}(\mathcal{M}, \mathcal{D}_X \otimes \omega_X^{-1})$, where $d_X = \dim X$. When M is pure of weight w , a polarization of M is an isomorphism $\mathbf{D}_X(M) \simeq M(w)$ in the category of mixed Hodge modules on X .

(vi) The dual of a complex vector space V will be denoted by $V^\vee = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Similar notation is used for mixed Hodge structures and for coherent sheaves.

1.3 Summary of the principal results

We now describe the construction of the analytic space $\bar{J}(\mathcal{H})$, and summarize the main results of the paper. Throughout, we let \mathcal{H} be a polarizable variation of integral Hodge structure of weight -1 , defined on a Zariski-open subset X of a complex manifold \bar{X} . We denote the corresponding family of intermediate Jacobians by $J(\mathcal{H}) \rightarrow X$.

To begin with, let us consider a single integral Hodge structure H of weight -1 , with underlying abelian group $H_{\mathbb{Z}}$. To emphasize the analogy with what comes later, we shall view the Hodge filtration as an increasing filtration $F_{\bullet}H_{\mathbb{C}}$ by setting $F_p H_{\mathbb{C}} = F^{-p} H_{\mathbb{C}}$. Since H has weight -1 , the (normalized) dual Hodge structure $\check{H} = \text{Hom}_{\text{MHS}}(H, \mathbb{Z}(1))$ is again integral of weight -1 , and we have an isomorphism $H_{\mathbb{C}}/F_0 H_{\mathbb{C}} \simeq (F_0 \check{H}_{\mathbb{C}})^\vee$; this justifies defining the *intermediate Jacobian* as

$$J(H) = (F_0 \check{H}_{\mathbb{C}})^\vee / H_{\mathbb{Z}},$$

where the map $H_{\mathbb{Z}} \hookrightarrow (F_0 \check{H}_{\mathbb{C}})^\vee$ is induced by the natural action of $H_{\mathbb{C}}$ on $\check{H}_{\mathbb{C}}$. The reader can find a comparison with the usual definition in Sect. 2.1.

The advantage of this point of view is that an extension of mixed Hodge structures

$$0 \rightarrow H \rightarrow V \rightarrow \mathbb{Z}(0) \rightarrow 0$$

of “normal function type” determines a point in $J(H)$ with only one choice: after dualizing the extension, one has $F_0\check{V}_{\mathbb{C}} \simeq F_0\check{H}_{\mathbb{C}}$; now any element $v_{\mathbb{Z}} \in V_{\mathbb{Z}}$ lifting $1 \in \mathbb{Z}$ defines a linear functional on $F_0\check{H}_{\mathbb{C}}$, and hence a point in $J(H)$.

Similarly, the sheaf of sections of the family $J(\mathcal{H}) \rightarrow X$ is given by $(F_0\check{\mathcal{H}}_{\mathcal{O}})^{\vee} / \mathcal{H}_{\mathbb{Z}}$, where $\mathcal{H}_{\mathbb{Z}}$ is the local system underlying the variation, and $F_{\bullet}\check{\mathcal{H}}_{\mathcal{O}}$ the Hodge filtration on the flat vector bundle underlying $\check{\mathcal{H}} = \mathcal{H}^{\vee}(1)$. To extend this formula in a natural way to \bar{X} , we view \mathcal{H} as a Hodge module on X ; according to M. Saito’s theory, it can be extended in a canonical manner to a polarizable Hodge module $M = j_{!*}\mathcal{H}[\dim X]$ on \bar{X} . The holonomic \mathcal{D} -module \mathcal{M} underlying M is always the *minimal extension* of the flat vector bundle $(\mathcal{H}_{\mathcal{O}}, \nabla)$; in particular, its de Rham complex $\text{DR}(\mathcal{M})$ is isomorphic to the intersection complex of $\mathcal{H}_{\mathbb{C}}$. The \mathcal{D} -module comes with a good filtration $F = F_{\bullet}\mathcal{M}$ by $\mathcal{O}_{\bar{X}}$ -coherent subsheaves; $F_p\mathcal{M}$ is difficult to describe in general, but may be viewed as a natural (if somewhat mysterious) extension of the Hodge bundle $F_p\mathcal{H}_{\mathcal{O}}$.

Guided by the above, we let $\check{M} = \mathbf{D}_{\bar{X}}(M)(1 - n)$ be the (normalized) dual of M in the category of Hodge modules on \bar{X} , and denote by $(\check{\mathcal{M}}, F)$ its underlying filtered \mathcal{D} -module. We then define the space $\bar{J}(\mathcal{H})$ in such a way that its sheaf of holomorphic sections is $(F_0\check{\mathcal{M}})^{\vee} / j_*\mathcal{H}_{\mathbb{Z}}$. Namely, we let $T(F_0\check{\mathcal{M}})$ be the analytic spectrum of the symmetric algebra of $F_0\check{\mathcal{M}}$ (see Sect. 2.3), and $T_{\mathbb{Z}}$ the étalé space of the sheaf $j_*\mathcal{H}_{\mathbb{Z}}$. Using a basic result about holonomic \mathcal{D} -modules, we show that there is a holomorphic mapping $\varepsilon : T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$ (see Sect. 2.6). The main technical result of the paper is that the image of ε is a closed analytic subset of $T(F_0\check{\mathcal{M}})$.

Theorem A *The mapping $\varepsilon : T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$ is a proper holomorphic embedding. Consequently, the fiberwise quotient space $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space, and in particular Hausdorff.*

In fact, the second statement follows from the first by simple topological arguments (see Sect. 2.5). We now define $\bar{J}(\mathcal{H}) = T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$; this is an analytic space over \bar{X} that naturally extends the family of intermediate Jacobians. Further evidence that it is a good candidate for the identity component of the Néron model is given by the following list of properties:

- (1) Every normal function on X that is admissible (relative to \bar{X}) and without singularities extends to a holomorphic section of $\bar{J}(\mathcal{H})$ (Proposition 4.2).

In fact, the process that gives the extension is analogous to the one for a single Hodge structure, explained above.

- (2) There is a notion of horizontality for sections of $\bar{J}(\mathcal{H})$, and the holomorphic and horizontal sections are precisely the admissible normal functions without singularities (Proposition 4.4).
- (3) The construction is functorial, in the following sense: Given a holomorphic mapping $f: \bar{Y} \rightarrow \bar{X}$ such that $Y = f^{-1}(X)$ is dense in \bar{Y} , let $f^*\mathcal{H}$ denote the pullback of the variation of Hodge structure to Y . Then there is a canonical holomorphic mapping

$$\bar{Y} \times_{\bar{X}} \bar{J}(\mathcal{H}) \rightarrow \bar{J}(f^*\mathcal{H}),$$

compatible with normal functions (Proposition 2.22).

- (4) There is a continuous and surjective mapping from $\bar{J}(\mathcal{H})$ to the identity component of the Néron model defined in [7], compatible with normal functions, and which partially contracts of certain fibers (Lemma 4.13).

A few words about the proof of Theorem A. We use results from M. Saito’s theory, in particular nearby and vanishing cycle functors and their description in terms of the V -filtration of M. Kashiwara and B. Malgrange, to reduce the general problem to the case where $\bar{X} = \Delta^n$, $X = (\Delta^*)^n$, and the local system $\mathcal{H}_{\mathbb{Z}}$ has unipotent monodromy (see Sect. 2.11). In that case, there is an explicit description of the sheaf $F_0\check{\mathcal{M}}$ in terms of P. Deligne’s canonical extension of $(\check{\mathcal{H}}_{\mathcal{O}}, \nabla)$: for every $k \geq 0$, $F_0\check{\mathcal{M}}$ contains all k -th derivatives of sections in $F_{-k}\check{\mathcal{H}}_{\mathcal{O}}^e = F^k\check{\mathcal{H}}_{\mathcal{O}}^e$. In particular, we have a holomorphic mapping $T(F_0\check{\mathcal{M}}) \rightarrow T(F_0\check{\mathcal{H}}_{\mathcal{O}}^e)$. In general, the image of $T_{\mathbb{Z}}$ in $T(F_0\check{\mathcal{H}}_{\mathcal{O}}^e)$ is badly behaved, which stems from the fact that sections of $F_0\check{\mathcal{H}}_{\mathcal{O}}^e$ are not sufficient to separate sections of $\mathcal{H}_{\mathbb{Z}}$ “in the limit”. The following result shows that $F_0\check{\mathcal{M}}$ has enough additional sections to overcome this problem.

Theorem B *Let $H_{\mathbb{C}}$ denote the space of sections of $\mathcal{H}_{\mathbb{C}}$ on the universal covering space \mathbb{H}^n , and let N_1, \dots, N_n be the logarithms of the monodromy operators. Also let $\sigma_1, \dots, \sigma_r$ be a collection of holomorphic sections that generate $F_0\check{\mathcal{M}}$ on Δ^n , and write Q for the natural pairing between $H_{\mathbb{C}}$ and sections of $\check{\mathcal{M}}$. Then there are constants $C > 0$ and $\alpha > 0$, such that for every $z = (z_1, \dots, z_n) \in \mathbb{H}^n$ and every real vector $h \in H_{\mathbb{R}}$,*

$$\max_{k \geq 0} \left\| (y_1 N_1 + \dots + y_n N_n)^k h \right\| \leq C \cdot \max_{1 \leq j \leq r} \left| Q(h, \sigma_j(z)) \right|, \tag{1.1}$$

provided that $y_j = \text{Im } z_j \geq \alpha$ and $0 \leq \text{Re } z_j \leq 1$ for all $j = 1, \dots, n$.

The estimate (1.1), which will be proved in Sect. 3.7 below, quickly leads to the proof of Theorem A in the normal crossing case. We obtain it essentially

by linear algebra methods, using only familiar consequences of the SL_2 -Orbit Theorem [10].

Perhaps surprisingly, the space $\bar{J}(\mathcal{H})$ is also useful for the study of normal functions with nontrivial singularities. Of course, such normal functions cannot be extended to holomorphic sections; nevertheless, the following is true (see Sect. 4.3).

Theorem C *Let $\nu: X \rightarrow J(\mathcal{H})$ be a normal function, admissible relative to \bar{X} . Then the topological closure of its graph inside $\bar{J}(\mathcal{H})$ is a closed analytic subset.*

This result clearly implies that the closure of the zero locus of ν is an analytic subset of \bar{X} , and leads to a different proof for Conjecture 1.1. To prove Theorem C in the normal crossing case (see Sect. 5.2), we use one consequence of the SL_2 -Orbit Theorem of [23], namely the boundedness of the canonical splitting in mixed nilpotent orbits. The rest of the argument is elementary linear algebra. The reader who is mainly interested in the proof of Conjecture 1.1 should focus on Parts 3 and 5 of the paper, where mixed Hodge modules play no role.

For admissible normal functions with *torsion* singularities, it turns out (in Proposition 4.8) that there is always a maximal extension whose graph is closed inside of $\bar{J}(\mathcal{H})$. As a consequence, it is possible to construct a Néron model for this class of normal functions by a gluing construction similar to the one used in [7] (see Sect. 4.4).

Theorem D *There is an analytic space $\bar{J}_{\text{tor}}(\mathcal{H}) \rightarrow \bar{X}$ whose holomorphic and horizontal sections are the admissible normal functions with torsion singularities. It contains $\bar{J}(\mathcal{H})$ as the identity component, and has similar functoriality properties.*

Unfortunately, it appears that admissible normal functions with torsion singularities are the biggest class for which there exists a Néron model that is an analytic space (or a Hausdorff space). The reason is the following: Over a point in \bar{X} where an admissible normal function has a non-torsion singularity, the closure of its graph may have a fiber of positive dimension. This happens even in very simple examples, such as two-parameter families of elliptic curves (see Sect. 6.3). As we argue in Sect. 4.5 below, it is therefore unlikely that there can be a Néron model that (a) graphs all admissible normal functions, (b) has a reasonable identity component, and (c) is Hausdorff as a topological space. Nevertheless, the result of Theorem C in itself is probably sufficient to study singularities of normal functions in the way proposed in [16], without having recourse to such a more general Néron model.

Three examples are described in Part 6, to illustrate different aspects of the construction. On the other hand, given the length of the paper, we have not

included any background on mixed Hodge modules, degenerations of variations of Hodge structure, or admissible normal functions. Here, the reader is advised to consult the following sources: (1) for mixed Hodge modules, the nice survey paper [33]; (2) for degenerations of variations of Hodge structure, the paper [10]; (3) for a discussion of admissibility, the papers [22] and [34].

1.4 Background for the construction

The idea for constructing the analytic space $\bar{J}(\mathcal{H})$ goes back to unpublished work of H. Clemens, for the case of hypersurface sections of an even-dimensional variety. Since this has, unfortunately, never appeared in print, we shall give a brief description here.

Let W be a smooth projective variety of dimension $2m$, and consider the family of its hypersurface sections of large degree, parametrized by the projective space $\bar{P} = |\mathcal{O}_W(d)|$ with $d \gg 0$. Denote by $D \subseteq \bar{P}$ the dual variety; then $P = \bar{P} - D$ parametrizes the nonsingular hypersurfaces. Let $\pi : \mathcal{X} \rightarrow P$ be the universal family, and let $R^{2m-1}\pi_*\mathbb{Z}_{\mathcal{X}}(m)$ be the variation of Hodge structure on the cohomology of the fibers, normalized to be of weight -1 .

Consider now a smooth hypersurface section X of W . A basic fact, due to P. Griffiths in the case of projective space, and to M. Green [14] in general, is that the variable part of the cohomology of X is generated by residues of meromorphic forms; moreover, the Hodge filtration is essentially the filtration by pole order. In particular, for $d \gg 0$, the residue map $\text{Res} : H^0(W, \Omega_W^{2m}(mX)) \rightarrow F^m H_{\text{var}}^{2m-1}(X, \mathbb{C})$ is surjective. H. Clemens observed that, consequently, the intermediate Jacobian

$$J_{\text{var}}(X) = \frac{(F^m H_{\text{var}}^{2m-1}(X, \mathbb{C}))^\vee}{H_{\text{var}}^{2m-1}(X, \mathbb{Z})}$$

is a subspace of the bigger object

$$K_{\text{var}}(X) = \frac{H^0(W, \Omega_W^{2m}(mX))^\vee}{H_{\text{var}}^{2m-1}(X, \mathbb{Z})}.$$

The original motivation for introducing $K_{\text{var}}(X)$ was to extend the Abel-Jacobi map to certain “topological cycles”, and to obtain a form of Jacobi inversion for such cycles. But if we observe that $H^0(W, \Omega_W^{2m}(mX))$ is isomorphic to the space of sections of the line bundle $\Omega_W^{2m} \otimes \mathcal{O}_W(m)$, then the numerator in the definition of $K_{\text{var}}(X)$ is essentially independent of X , and makes sense even when X becomes singular. This suggests that residues might be useful in extending the family of intermediate Jacobians from P to \bar{P} .

Let $\mathcal{H} \subseteq R^{2m-1}\pi_{*\mathbb{Z},\mathcal{X}}(m)$ be the variation of Hodge structure on the variable part of the cohomology, and $(\mathcal{H}_\mathcal{O}, \nabla)$ the corresponding flat vector bundle. It can be shown that the residue calculus extends to the family of all hypersurface sections, including the singular ones, in the following way: Let $j: P \hookrightarrow \bar{P}$ be the inclusion, and define subsheaves $F_p\mathcal{M}$ of $j_*\mathcal{H}_\mathcal{O}$ by the condition that a section in $H^0(U \cap P, \mathcal{H}_\mathcal{O})$ belongs to $H^0(U, F_p\mathcal{M})$ iff it is the residue of a meromorphic $2m$ -form on $U \times W$ with a pole of order at most $m + p$ along the incidence variety. Let \mathcal{M} be the union of the $F_p\mathcal{M}$; then \mathcal{M} is a holonomic \mathcal{D} -module on \bar{P} , extending the flat vector bundle $\mathcal{H}_\mathcal{O}$, and $F_\bullet\mathcal{M}$ is a good filtration. It was proved in [38, 39] that (\mathcal{M}, F) underlies a polarized Hodge module on \bar{P} , namely the intermediate extension $M = j_{!*}\mathcal{H}[\dim P]$ of the variation of Hodge structure. This is how filtered \mathcal{D} -modules and M. Saito's theory introduce themselves into the problem.

This very important example is also the reason why we construct the Néron model without resolving singularities and without passing to a finite cover. The geometry of the family of hypersurfaces is beautifully simple: \bar{P} is a projective space, and \mathcal{X} is a projective bundle over X . Any attempt to resolve the singularities of $\bar{P} - P$ would destroy this nice picture. Moreover, it can be shown that the sheaf $F_p\mathcal{M}$ is a quotient of $H^0(W, \Omega_W^{2m}(m+p)) \otimes \mathcal{O}_{\bar{P}}(m+p)$, and therefore ample. This circumstance gives our Néron model many good properties that will be explained in a separate article; it may also place restrictions on global holomorphic sections of $\bar{J}(\mathcal{H})$ —that is, on normal functions without singularities. This is of interest because M. Green and P. Griffiths have related the existence of singularities of normal functions to the Hodge conjecture [15, 16].

1.5 History of Néron models

Néron models have their origin in a construction for abelian varieties due to A. Néron [29]. Let A be an abelian variety, defined over the field of functions K of a Dedekind domain R . Then the *Néron model* for A is a smooth and commutative group scheme \mathcal{A} over R , such that $\mathcal{A}(S) = \mathcal{A}(S \times_R K)$ for any smooth morphism $S \rightarrow R$; more details can be found in the book [3]. The definition means that \mathcal{A} is the natural extension of A from the open subset $\text{Spec } K$ to all of $\text{Spec } R$.

In the complex-analytic setting, a family of abelian varieties is a special case of a polarizable variation of Hodge structure of weight -1 . After P. Griffiths popularized the use of normal functions in Hodge theory, Néron models for one-parameter degenerations of more general variations were constructed by several people. S. Zucker [41] introduced a generalized intermediate Jacobian for hypersurfaces with one ordinary double point, and used it to define

the identity component of a Néron model in Lefschetz pencils. H. Clemens [12] extended this to the construction of a Néron model for one-parameter degenerations with certain restrictions on the local monodromy. In his paper on admissible normal functions, M. Saito [34] generalized both constructions to arbitrary one-parameter degenerations, and also constructed a compactification of the “Zucker extension” (which, however, is usually not Hausdorff).

The recent interest in Néron models stems from the work by M. Green, P. Griffiths, and M. Kerr [17], who observed that a subspace of the Zucker extension is sufficient to graph admissible normal functions without singularities. Briefly summarized, their construction works as follows: Let \bar{X} be a smooth curve, and \mathcal{H} a polarizable variation of Hodge structure with unipotent monodromy, defined on a Zariski-open subset X . At each of the points $x \in \bar{X} - X$, a choice of local coordinate determines an asymptotic mixed Hodge structure; the monodromy-invariant part $H = \ker(T - \text{id})$ is independent of that choice. The identity component of the Néron model in [17] has the generalized intermediate Jacobian $J(H) = H_{\mathbb{C}} / (F^0 H_{\mathbb{C}} + H_{\mathbb{Z}})$ as its fiber over x . M. Green, P. Griffiths, and M. Kerr also defined the full Néron model that graphs arbitrary admissible normal functions, and computed its finite group of components at each point of $\bar{X} - X$. Their construction produces a so-called “slit” analytic space; M. Saito [35] has shown that the resulting topological space is now Hausdorff.

As mentioned above, a construction of a Néron model for \bar{X} of arbitrary dimension has been proposed by P. Brosnan, G. Pearlstein, and M. Saito [7]. They observe that, at each point $x \in \bar{X}$, the stalk H_x of the sheaf $R^1 j_* \mathcal{H}_{\mathbb{Z}}$ carries a mixed Hodge structure of weight ≤ -1 , and therefore defines a generalized intermediate Jacobian $J(H_x) = \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H_x)$. The identity component of their Néron model is the disjoint union of the complex Lie groups $J(H_x)$, topologized in a rather tricky way by reduction to the normal-crossing case. The full Néron model is then obtained by gluing. As pointed out in [7], the construction does not seem to work very well in the case of non-unipotent local monodromy.

The most recent work, also alluded to above, is by K. Kato, C. Nakayama, and S. Usui [24], in the case when $\bar{X} - X$ is a divisor with normal crossings. The variation of Hodge structure \mathcal{H} determines a period map $\Phi: X \rightarrow \Gamma \backslash D$, and if the local monodromy is unipotent, then according to the general theory in [26], the period map extends to $\bar{\Phi}: \bar{X} \rightarrow \Gamma \backslash D_{\Sigma}$, where D_{Σ} is a space of nilpotent orbits. They show that there is a good choice of a compatible weak fan Σ' , such that an admissible normal function defines a map from \bar{X} into a space $D'_{\Sigma'}$ of nilpotent orbits of normal function type. The Néron model can

then be constructed as the fiber product

$$\begin{array}{ccc}
 \bar{J}_{\Sigma'}(\mathcal{H}) & \longrightarrow & \Gamma' \backslash D'_{\Sigma'} \\
 \downarrow & & \downarrow \\
 \bar{X} & \xrightarrow{\bar{\Phi}} & \Gamma \backslash D_{\Sigma}
 \end{array}$$

in the category $\mathcal{B}(\log)$ of log manifolds. The advantage of this construction is that the Néron model becomes a moduli space for certain log mixed Hodge structures. On the other hand, the assumption that $\bar{X} - X$ be a normal crossing divisor is essential for applying the methods of log geometry.

For families of complex abelian varieties, there is a complete construction of a Néron model in the unpublished Ph.D. thesis of A. Young [40]. His construction uses toric geometry, and is therefore again restricted to the case when $\bar{X} - X$ is a divisor with normal crossings. The identity component of his model agrees with an older construction by Y. Namikawa [28] for degenerations of abelian varieties (and, therefore, with the model that is proposed in this paper), and is in particular a complex manifold. When all components are considered together, the space is however not Hausdorff.

2 The construction of the analytic space

2.1 Intermediate Jacobians

The intermediate Jacobian is a complex torus associated to a Hodge structure of weight -1 . It turns out that a nonstandard definition is best suited for the purpose of constructing Néron models, but we begin with a brief review of the standard definition, following the conventions in [18, 2.2]. Let H be an integral Hodge structure of weight -1 . We shall always assume that the underlying abelian group $H_{\mathbb{Z}}$ is torsion-free, and for consistency with later sections, we write the Hodge filtration on the underlying complex vector space $H_{\mathbb{C}}$ as an increasing filtration, according to the convention $F_p H_{\mathbb{C}} = F^{-p} H_{\mathbb{C}}$.

The *intermediate Jacobian* of H is usually defined as

$$J(H) = \frac{H_{\mathbb{C}}}{F_0 H_{\mathbb{C}} + H_{\mathbb{Z}}}.$$

There is a natural bijection between $J(H)$ and the group $\text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H)$ of (equivalence classes of) extensions of mixed Hodge structure of the form

$$0 \rightarrow H \rightarrow V \rightarrow \mathbb{Z}(0) \rightarrow 0, \tag{2.1}$$

where V is an integral mixed Hodge structure with $W_{-1}V = H$ and $\mathrm{Gr}_0^W V = \mathbb{Z}(0)$. Given an extension as in (2.1), one can find an element $v_{\mathbb{Z}} \in V_{\mathbb{Z}}$ lifting $1 \in \mathbb{Z}$, as well as an element $v_F \in F_0 V_{\mathbb{C}}$ lifting $1 \in \mathbb{C}$, and the difference $v_{\mathbb{Z}} - v_F$ determines a well-defined point of $J(H)$. Conversely, a point $h + F_0 H_{\mathbb{C}} + H_{\mathbb{Z}} \in J(H)$ corresponds to the mixed Hodge structure whose underlying bi-filtered vector space is $H_{\mathbb{C}} \oplus \mathbb{C}$, and whose underlying abelian group is the image of

$$H_{\mathbb{Z}} \oplus \mathbb{Z} \hookrightarrow H_{\mathbb{C}} \oplus \mathbb{C}, \quad (x, n) \mapsto (x + n \cdot h, n).$$

It turns out that a definition based on duality is much better suited for the purpose of constructing Néron models. Recall that $\mathbb{Z}(1)$ denotes the Hodge structure of weight -2 whose underlying complex vector space is \mathbb{C} , and whose underlying abelian group is $2\pi i \cdot \mathbb{Z}$. Let

$$\check{H} = H^{\vee}(1) = \mathrm{Hom}_{\mathrm{MHS}}(H, \mathbb{Z}(1))$$

be the (normalized) dual of H , again an integral Hodge structure of weight -1 . Its Hodge filtration is given by

$$F_p \check{H}_{\mathbb{C}} = \{ \psi \in \mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}, \mathbb{C}) \mid \psi(F_{-p} H_{\mathbb{C}}) = 0 \},$$

and its underlying abelian group is

$$\check{H}_{\mathbb{Z}} = \{ \psi \in \mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}, \mathbb{C}) \mid \psi(H_{\mathbb{Z}}) \subseteq 2\pi i \cdot \mathbb{Z} \}.$$

It follows that $H_{\mathbb{C}}/F_0 H_{\mathbb{C}} \simeq (F_0 \check{H}_{\mathbb{C}})^{\vee}$, and so the quotient $(F_0 \check{H}_{\mathbb{C}})^{\vee}/H_{\mathbb{Z}}$ is isomorphic to $J(H)$. One theme of this paper is that this is, in fact, the correct way to define the intermediate Jacobian.

Definition 2.1 Let H be an integral Hodge structure of weight -1 , and define $\check{H} = \mathrm{Hom}_{\mathrm{MHS}}(H, \mathbb{Z}(1))$. The *intermediate Jacobian* of H is the complex torus

$$J(H) = (F_0 \check{H}_{\mathbb{C}})^{\vee}/H_{\mathbb{Z}},$$

where the homomorphism $H_{\mathbb{Z}} \hookrightarrow (F_0 \check{H}_{\mathbb{C}})^{\vee}$ is given by evaluation.

To motivate what follows, let us briefly discuss the correspondence between extensions of mixed Hodge structure as in (2.1) and points of $J(H)$. Given such an extension, the underlying sequence of \mathbb{Z} -modules

$$0 \rightarrow H_{\mathbb{Z}} \rightarrow V_{\mathbb{Z}} \rightarrow \mathbb{Z} \rightarrow 0$$

splits non-canonically, and so we can find $v_{\mathbb{Z}} \in V_{\mathbb{Z}}$ mapping to $1 \in \mathbb{Z}$. After dualizing (2.1), we obtain a second exact sequence

$$0 \rightarrow \mathbb{Z}(1) \rightarrow \check{V} \rightarrow \check{H} \rightarrow 0,$$

and the strictness of morphisms of Hodge structure gives $F_0\check{V}_{\mathbb{C}} \simeq F_0\check{H}_{\mathbb{C}}$. Now $v_{\mathbb{Z}}$ defines a linear operator on $\check{V}_{\mathbb{C}}$, and hence on $F_0\check{V}_{\mathbb{C}}$; taking the ambiguity in choosing $v_{\mathbb{Z}}$ into account, we therefore get a well-defined point in the quotient

$$J(H) = (F_0\check{H}_{\mathbb{C}})^{\vee} / H_{\mathbb{Z}}.$$

The main advantage of this construction is that it removes the need to choose an additional lifting (namely v_F).

Lemma 2.2 *Under the natural isomorphism*

$$\frac{H_{\mathbb{C}}}{F_0H_{\mathbb{C}} + H_{\mathbb{Z}}} \simeq \frac{(F_0\check{H}_{\mathbb{C}})^{\vee}}{H_{\mathbb{Z}}},$$

the above construction gives rise to the same point as the usual one.

Proof Recall that if A and B are two mixed Hodge structures, $F_p \text{Hom}(A_{\mathbb{C}}, B_{\mathbb{C}})$ consists of all linear maps $f: A_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$ with $f(F_k A_{\mathbb{C}}) \subseteq F_{k+p} B_{\mathbb{C}}$. Therefore

$$F_0\check{H}_{\mathbb{C}} = \{ \psi: H_{\mathbb{C}} \rightarrow \mathbb{C} \mid \psi(F_0H_{\mathbb{C}}) = 0 \},$$

and the isomorphism $H_{\mathbb{C}}/F_0H_{\mathbb{C}} \rightarrow (F_0\check{H}_{\mathbb{C}})^{\vee}$ takes a point $h + F_0H_{\mathbb{C}}$ to the linear functional $\psi \mapsto \psi(h)$. Similarly, we have

$$F_0\check{V}_{\mathbb{C}} = \{ \phi: V_{\mathbb{C}} \rightarrow \mathbb{C} \mid \phi(F_0V_{\mathbb{C}}) = 0 \}.$$

Now let $v_F \in F_0V_{\mathbb{C}}$ be a lifting of $1 \in \mathbb{C}$, and consider the composition

$$H_{\mathbb{C}}/F_0H_{\mathbb{C}} \rightarrow (F_0\check{H}_{\mathbb{C}})^{\vee} \rightarrow (F_0\check{V}_{\mathbb{C}})^{\vee}.$$

It takes the element $(v_{\mathbb{Z}} - v_F) + F_0H_{\mathbb{C}}$ to the linear functional $\phi \mapsto \phi(v_{\mathbb{Z}} - v_F) = \phi(v_{\mathbb{Z}})$, and this shows that both constructions define the same point in the intermediate Jacobian, as asserted. \square

Note Suppose that the Hodge structure H is polarized, and let $Q: H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \rightarrow \mathbb{Z}$ be the alternating and nondegenerate pairing that underlies the polarization. Then

$$H \rightarrow \check{H}, \quad h \mapsto 2\pi i \cdot Q(h, -),$$

is an isomorphism of rational Hodge structures; it is an isomorphism of integral Hodge structures if the polarization is principal.

For the local study of the Néron model, we need a small generalization of the intermediate Jacobian; it already appears in the paper [7].

Definition 2.3 Let H be an integral mixed Hodge structure of weight ≤ -1 . The *generalized intermediate Jacobian* of H is the complex Lie group

$$J(H) = (F_0\check{H}_{\mathbb{C}})^{\vee} / H_{\mathbb{Z}},$$

where $\check{H} = \text{Hom}_{\text{MHS}}(H, \mathbb{Z}(1))$ is an integral mixed Hodge structure of weight ≥ -1 .

For the same reason as before, we have

$$\text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H) \simeq \frac{H_{\mathbb{C}}}{F_0H_{\mathbb{C}} + H_{\mathbb{Z}}} \simeq J(H),$$

and (equivalence classes of) extensions of H by $\mathbb{Z}(0)$ are therefore still classified by the points of $J(H)$.

2.2 Outline of the construction

Our Néron model is an analytic space, obtained by taking a certain quotient similar to $J(H) = (F_0\check{H}_{\mathbb{C}})^{\vee} / H_{\mathbb{Z}}$. We now introduce the relevant objects, and give an outline of how the analytic space is constructed.

Let \bar{X} be a complex manifold of dimension n , and let $X = \bar{X} - D$ be the complement of a closed analytic subset. Let $\mathcal{H} = (\mathcal{H}_{\mathcal{O}}, \nabla, F_{\bullet}\mathcal{H}_{\mathcal{O}}, \mathcal{H}_{\mathbb{Z}})$ be a polarizable variation of Hodge structure of weight -1 on X . To introduce some notation, we recall that this means the following: $\mathcal{H}_{\mathcal{O}}$ is a holomorphic vector bundle with a flat connection $\nabla: \mathcal{H}_{\mathcal{O}} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{H}_{\mathcal{O}}$, and $\mathcal{H}_{\mathbb{Z}}$ is a local system of free \mathbb{Z} -modules such that $\ker \nabla \simeq \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. The Hodge bundles $F_p\mathcal{H}_{\mathcal{O}}$ are holomorphic subbundles of $\mathcal{H}_{\mathcal{O}}$ that satisfy Griffiths' transversality condition $\nabla(F_p\mathcal{H}_{\mathcal{O}}) \subseteq \Omega_X^1 \otimes F_{p+1}\mathcal{H}_{\mathcal{O}}$. Finally, the condition that \mathcal{H} is polarizable means that there should exist an alternating and nondegenerate pairing $Q: \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \rightarrow \mathbb{Z}_X$ that is flat with respect to the connection ∇ , and satisfies $Q(F_p\mathcal{H}_{\mathcal{O}}, F_q\mathcal{H}_{\mathcal{O}}) = 0$ if $p + q \leq 0$. The specific choice of Q plays no role in the construction.

Note Here and in what follows, we consider the flat vector bundle $(\mathcal{H}_{\mathcal{O}}, \nabla)$ as a special case of a left \mathcal{D} -module; it is then more natural to write the Hodge filtration as an increasing filtration, by setting $F_p\mathcal{H}_{\mathcal{O}} = F^{-p}\mathcal{H}_{\mathcal{O}}$.

Each Hodge structure in the variation has its associated intermediate Jacobian (defined as in Sect. 2.1); they fit together into a holomorphic fiber bundle that we denote by $J(\mathcal{H}) \rightarrow X$. By definition, its sheaf of holomorphic sections is given by $(F_0\check{\mathcal{H}}_{\mathcal{O}})^\vee/\mathcal{H}_{\mathbb{Z}}$, where $\check{\mathcal{H}} = \mathcal{H}^\vee(1)$ again denotes the (normalized) dual variation.

Now let M be the Hodge module on \bar{X} , obtained from the variation \mathcal{H} by intermediate extension via the inclusion map $j: X \hookrightarrow \bar{X}$ [32, Theorem 3.21]. Then M is a polarizable Hodge module of weight $n - 1$ with strict support equal to all of \bar{X} . Its underlying perverse sheaf $\text{rat } M$ is simply the intersection complex of the local system $\mathcal{H}_{\mathbb{Z}} \otimes \mathbb{Q}$. Let (\mathcal{M}, F) be the filtered left \mathcal{D} -module underlying M . This means that \mathcal{M} is a filtered holonomic $\mathcal{D}_{\bar{X}}$ -module, and $F = F_\bullet \mathcal{M}$ is an increasing filtration of \mathcal{M} by $\mathcal{O}_{\bar{X}}$ -coherent subsheaves that is good in the sense of [2, Chap. II, Sect. 4]. The condition on the strict support implies that \mathcal{M} is the minimal extension of the flat vector bundle $(\mathcal{H}_{\mathcal{O}}, \nabla)$ from X to \bar{X} . The coherent sheaves $F_p \mathcal{M}$ are natural extensions of the Hodge bundles, because $j^*(F_p \mathcal{M}) = F_p \mathcal{H}_{\mathcal{O}}$.

To extend $J(\mathcal{H})$ to an analytic space over \bar{X} , we attempt to generalize the formula $J(H) = (F_0\check{H}_{\mathbb{C}})^\vee/H_{\mathbb{Z}}$, again by using duality. So let

$$\check{M} = \mathbf{D}_{\bar{X}}(M)(1 - n)$$

be the (normalized) dual Hodge module; it is the intermediate extension of $\check{\mathcal{H}}$ and again has weight $n - 1$. Let $(\check{\mathcal{M}}, F)$ be the underlying filtered \mathcal{D} -module. Then $T(F_0\check{\mathcal{M}})$, constructed in Sect. 2.3 below, is an analytic space whose sheaf of holomorphic sections is $(F_0\check{\mathcal{M}})^\vee$. On the other hand, we may let $T_{\mathbb{Z}}$ be the étalé space of the sheaf $j_*\mathcal{H}_{\mathbb{Z}}$; this is an analytic space over \bar{X} with sheaf of sections $j_*\mathcal{H}_{\mathbb{Z}}$. It is then very natural to try to define the desired extension of $J(\mathcal{H})$ as the quotient

$$\bar{J}(\mathcal{H}) = T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}.$$

To make this idea work, we have to do several things. Firstly, we construct in Sect. 2.4 a holomorphic mapping $\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$ that generalizes the embedding of the local system $\mathcal{H}_{\mathbb{Z}}$ into the vector bundle $T(F_0\check{\mathcal{H}}_{\mathcal{O}})$. Secondly, we prove that the ε is a closed embedding, and that the fiberwise quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space (in particular, Hausdorff), provided that the following condition is satisfied.

Condition 2.4 *The mapping $\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$ is injective, and $\varepsilon(T_{\mathbb{Z}})$ is a closed analytic subset of $T(F_0\check{\mathcal{M}})$.*

Thirdly—and most importantly—we show that Condition 2.4 is always true. We reduce the general problem to the case when D is a normal crossing

divisor and $\mathcal{H}_{\mathbb{Z}}$ has unipotent local monodromy, using methods from Saito’s theory (in particular, an analysis of how the Hodge filtration on a mixed Hodge module behaves under pullback). Finally, we establish Condition 2.4 by a local analysis, using results and methods from the theory of degenerating variations of Hodge structure [10]. A valuable consequence is that the construction of $\bar{J}(\mathcal{H})$ works without any assumptions on the divisor $D = \bar{X} - X$ or on the local monodromy of $\mathcal{H}_{\mathbb{Z}}$.

2.3 The analytic space associated to a coherent sheaf

Let X be an analytic space, and \mathcal{F} a coherent analytic sheaf on X . In this section, we describe how to associate to \mathcal{F} an analytic space $T(\mathcal{F}) \rightarrow X$, relatively Stein, whose sheaf of holomorphic sections is $\mathcal{F}^{\vee} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. The construction of $T(\mathcal{F})$ is extremely simple: let $\text{Sym}_{\mathcal{O}_X}(\mathcal{F})$ be the symmetric algebra in \mathcal{F} , and define

$$T(\mathcal{F}) = \text{Spec}_X(\text{Sym}_{\mathcal{O}_X}(\mathcal{F}))$$

as the analytic spectrum of this sheaf of algebras. When $\mathcal{F} = \mathcal{O}_X(E)$ is the sheaf of sections of a holomorphic vector bundle $E \rightarrow X$, we recover the dual vector bundle since $T(\mathcal{F}) = E^*$. This leads to the following more concrete description of $T(\mathcal{F})$. Let $j: U \hookrightarrow X$ be any open subset of X that is Stein. Then $j^*\mathcal{F}$ can be written as a quotient of locally free sheaves on U ,

$$\mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_0 \rightarrow j^*\mathcal{F} \rightarrow 0.$$

Let $E_i^* \rightarrow U$ be the holomorphic vector bundle whose sheaf of sections is \mathcal{E}_i^{\vee} . Then φ induces a map of vector bundles $E_0^* \rightarrow E_1^*$, and $T(j^*\mathcal{F}) \subseteq E_0^*$ is the preimage of the zero section. The reason is that $\text{Sym}_{\mathcal{O}_X}(j^*\mathcal{F})$ is the quotient of $\text{Sym}_{\mathcal{O}_X}(\mathcal{E}_0)$ by the ideal generated by $\varphi(\mathcal{E}_1)$.

From the local description, it follows that $T(\mathcal{F}) \rightarrow X$ is relatively Stein, meaning that the preimage of every Stein open subset is again Stein; moreover, every fiber is a linear space, and over any analytic subset of X where the fiber dimension is constant, $T(\mathcal{F})$ is a holomorphic vector bundle. As an analytic space, $T(\mathcal{F})$ has the following universal property.

Lemma 2.5 *For any holomorphic mapping $f: Y \rightarrow X$ from an analytic space Y ,*

$$\text{Map}_X(Y, T(\mathcal{F})) \simeq \text{Hom}_{\mathcal{O}_Y}(f^*\mathcal{F}, \mathcal{O}_Y).$$

Proof Holomorphic mappings $Y \rightarrow T(\mathcal{F})$ over X are in one-to-one correspondence with morphisms of \mathcal{O}_X -algebras $\text{Sym}_{\mathcal{O}_X}(\mathcal{F}) \rightarrow f_*\mathcal{O}_Y$, hence with morphisms of \mathcal{O}_X -modules $\mathcal{F} \rightarrow f_*\mathcal{O}_Y$, and finally with morphisms of \mathcal{O}_Y -modules $f^*\mathcal{F} \rightarrow \mathcal{O}_Y$. □

In particular, the sheaf of global holomorphic sections of $T(\mathcal{F}) \rightarrow X$ is precisely the sheaf \mathcal{F}^\vee . The next lemma shows that the construction of $T(\mathcal{F})$ behaves well under pullback by arbitrary holomorphic mappings. It follows that the fiber over a point $x \in X$ is the dual of the vector space $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Lemma 2.6 *For any holomorphic mapping $f : Y \rightarrow X$, we have*

$$Y \times_X T(\mathcal{F}) \simeq T(f^* \mathcal{F}).$$

Proof This is true because $f^* \text{Sym}_{\mathcal{O}_X}(\mathcal{F}) \simeq \text{Sym}_{\mathcal{O}_Y}(f^* \mathcal{F})$, by the universal property of the symmetric algebra. □

Lemma 2.7 *Let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ be a surjective map of coherent sheaves. Then the induced map $T(\mathcal{G}) \rightarrow T(\mathcal{F})$ is a closed embedding.*

Proof The statement is local on X , and so we may assume without loss of generality that X is a Stein manifold. By writing \mathcal{F} as the quotient of a locally free sheaf \mathcal{E}_0 , we can find compatible presentations

$$\begin{array}{ccccccc} \mathcal{E}_1 & \xrightarrow{\varphi} & \mathcal{E}_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \parallel & & \downarrow & & \\ \mathcal{E}_2 & \xrightarrow{\psi} & \mathcal{E}_0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Obviously, we now have $T(\mathcal{G}) \subseteq T(\mathcal{F}) \subseteq E_0^*$, proving the assertion. □

Note Another analytic space with sheaf of sections \mathcal{F}^\vee would be $T(\mathcal{F}^{\vee\vee})$, obtained by replacing \mathcal{F} by its double dual. Since the sheaf $\mathcal{F}^{\vee\vee}$ is reflexive, this may seem a more natural choice at first glance. But the problem is that taking the dual does not commute with pullbacks by non-flat maps; this second choice of space is therefore not sufficiently functorial for our purposes.

2.4 A lemma about holonomic modules

Here we recall a general result about holonomic \mathcal{D} -modules. It is used twice in the paper: to construct the morphism $j_* \mathcal{H}_{\mathbb{Z}} \hookrightarrow (F_0 \check{\mathcal{M}})^\vee$; and to study extensions of admissible normal functions.

Let \mathcal{M} be a holonomic left \mathcal{D} -module on a complex manifold X of dimension n . By Kashiwara’s theorem, the holomorphic de Rham complex (in degrees $-n, \dots, 0$)

$$\text{DR}(\mathcal{N}) = [\mathcal{N} \rightarrow \Omega_X^1 \otimes \mathcal{N} \rightarrow \Omega_X^2 \otimes \mathcal{N} \rightarrow \dots \rightarrow \Omega_X^n \otimes \mathcal{N}][n]$$

is constructible, and we let $\mathcal{H}^k(\mathrm{DR}(\mathcal{N}))$ be its cohomology sheaf in degree k , a constructible sheaf of complex vector spaces. Let

$$\check{\mathcal{N}} = \mathbf{D}_X(\mathcal{N}) = \mathrm{Ext}_{\mathcal{D}_X}^n(\mathcal{N}, \mathcal{D}_X \otimes \omega_X^{-1})$$

be the left \mathcal{D} -module dual to \mathcal{N} . The following result is certainly well-known, but we include a proof for the sake of completeness.

Lemma 2.8 *Let \mathcal{N} be a holonomic \mathcal{D} -module on a complex manifold X .*

(1) *We have $\mathrm{Ext}_{\mathcal{D}_X}^k(\check{\mathcal{N}}, \mathcal{O}_X) \simeq \mathcal{H}^{k-d_X}(\mathrm{DR}(\mathcal{N}))$ for $k \in \mathbb{Z}$, and in particular,*

$$\mathrm{Hom}_{\mathcal{D}_X}(\check{\mathcal{N}}, \mathcal{O}_X) \simeq \ker(\nabla: \mathcal{N} \rightarrow \Omega_X^1 \otimes \mathcal{N}).$$

(2) *There is a canonical injective morphism*

$$j_* j^{-1} \mathcal{H}^{-n}(\mathrm{DR}(\mathcal{N})) \hookrightarrow \mathrm{Hom}_{\mathcal{D}_X}(\check{\mathcal{N}}, \mathcal{O}_X)$$

for any open subset $j: U \hookrightarrow X$ such that $j^{-1}\mathcal{N}$ is a flat vector bundle.

Proof Let $n = \dim X$. The Spencer complex $\mathrm{Sp}_X(\mathcal{D}_X)$, with terms

$$\mathrm{Sp}_X^\bullet(\mathcal{D}_X) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{-\bullet} \Theta_X,$$

gives a natural resolution for \mathcal{O}_X as a left \mathcal{D}_X -module. A simple spectral sequence argument, together with the vanishing of $\mathrm{Ext}_{\mathcal{D}_X}^k(\check{\mathcal{N}}, \mathcal{D}_X)$ for $k \neq n$, then shows that we must have

$$\mathrm{Ext}_{\mathcal{D}_X}^k(\check{\mathcal{N}}, \mathcal{O}_X) \simeq \mathcal{H}^{k-n}(\mathrm{Ext}_{\mathcal{D}_X}^n(\check{\mathcal{N}}, \mathrm{Sp}_X(\mathcal{D}_X))).$$

On the other hand, we compute that

$$\mathrm{Ext}_{\mathcal{D}_X}^n(\check{\mathcal{N}}, \mathrm{Sp}_X^\bullet(\mathcal{D}_X)) \simeq \mathrm{Ext}_{\mathcal{D}_X}^n(\check{\mathcal{N}}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{-\bullet} \Theta_X \simeq \mathrm{Sp}_X^\bullet(\mathcal{N} \otimes \omega_X).$$

It is well-known that the Spencer complex of a right \mathcal{D} -module is isomorphic to the de Rham complex of the associated left \mathcal{D} -module; therefore, $\mathrm{Sp}_X(\mathcal{N} \otimes \omega_X) \simeq \mathrm{DR}(\mathcal{N})$, and so we obtain the desired isomorphism. All the remaining assertions are simple consequences. □

2.5 Quotients of certain complex manifolds

In this section, we explain how conditions analogous to Condition 2.4 allow one to prove that certain quotient spaces of holomorphic vector bundles are again complex manifolds. The reasoning uses only very basic topology, but I

have not been able to find a source where the results that we need are stated in exactly this form.

In any case, situation that we are concerned with is the following. Let $p: E \rightarrow X$ be a holomorphic vector bundle on a complex manifold X . Let \mathcal{G} be a sheaf of finitely generated abelian groups on X , and suppose that we have a morphism of sheaves $\mathcal{G} \rightarrow \mathcal{O}_X(E)$. It defines a holomorphic mapping $\varepsilon: G \rightarrow E$, where G is the étalé space of the sheaf \mathcal{G} . We shall assume that the following two conditions hold:

- (i) The image $\varepsilon(G) \subseteq E$ is a closed analytic subset of E .
- (ii) The mapping ε is injective.

For a point $x \in X$, we let $E_x = p^{-1}(x)$ and $G_x = \varepsilon^{-1}(E_x)$ be the fibers. The second condition is equivalent to the injectivity of the individual maps $G_x \rightarrow E_x$; note that $\varepsilon(G_x)$ is then automatically a discrete subset of E , being both closed analytic and countable.

Lemma 2.9 *For any point $g \in G$, there is an open neighborhood of $\varepsilon(g) \in E$ whose intersection with $\varepsilon(G)$ is the image of a local section of G .*

Proof As an analytic subset, $\varepsilon(G)$ has a decomposition into (countably many) irreducible components, and there is a small open neighborhood of $e = \varepsilon(g)$ that meets only finitely many of them. Shrinking that neighborhood, if necessary, we can find an open set U containing e , such that $\varepsilon(G) \cap U$ has finitely many irreducible components, each passing through the point e . Since ε is injective by (ii), there can be only one such component Z ; noting that $\varepsilon(G_x)$ is discrete in E , we may further shrink U and assume that $Z \cap E_x = \{e\}$. For dimension reasons, we then have $\dim Z = \dim X$. Now G is the étalé space of the sheaf \mathcal{G} , and so we can find a local section of G , defined in a suitable neighborhood V of the point $x = p(e) \in X$, with the property that $\gamma(x) = g$. It follows that $Z = \varepsilon(\gamma(V))$, as claimed. □

Lemma 2.10 *The mapping $\varepsilon: G \rightarrow E$ is a closed embedding.*

Proof First of all, ε is a proper map. To see this, let $g_n \in G$ be any sequence of points in G such that $\varepsilon(g_n)$ converges to a point $e \in E$. By (i), the limit is of the form $e = \varepsilon(g)$ for some $g \in G$. By the preceding lemma, there is an open neighborhood U containing e , and a local section $\gamma: V \rightarrow G$, such that $U \cap \varepsilon(G) = \varepsilon(\gamma(V))$ and $g = \gamma(x)$. We conclude that $g_n = \gamma(x_n)$ for some choice of $x_n \in V$. But now $x_n = p(\varepsilon(g_n)) \rightarrow x$, and therefore $g_n \rightarrow g$; this establishes the properness of ε . Lemma 2.9 also shows that $\varepsilon: G \rightarrow \varepsilon(G)$ is a local isomorphism. Since ε is in addition proper and injective, it has to be a closed embedding. □

The lemma justifies identifying G with its image in E ; from now on, we regard G as a closed submanifold of E . We are then interested in taking the fiberwise quotient of E by G . Let \sim be the equivalence relation on E defined by

$$e \sim e' \quad \text{if and only if } p(e) = p(e') \text{ and } e - e' \in G.$$

Let $q: E \rightarrow E/\sim$ be the map to the quotient, endowed with the quotient topology.

Lemma 2.11 *The mapping q is open.*

Proof Let $U \subseteq E$ be any open set; we need to verify that $q^{-1}(q(U))$ is again open. It suffices to show that for any sequence of points e_n that converges to some $e \in q^{-1}(q(U))$, all but finitely many of the e_n also belong to $q^{-1}(q(U))$. Since $q(e) \in q(U)$, there exists $e' \in U$ with $e \sim e'$, hence $e' - e \in G$. Let $\gamma: V \rightarrow G$ be a local section such that $e' = e + \gamma(p(e))$. If we put $e'_n = e_n + \gamma(p(e_n))$, then $e'_n \rightarrow e'$, and so $e'_n \in U$ for large n . But then $e_n \sim e'_n$ also belongs to $q^{-1}(q(U))$. □

Lemma 2.12 *The quotient E/\sim is Hausdorff.*

Proof Since q is open, the quotient E/\sim is Hausdorff if and only if the equivalence relation \sim is closed in $E \times E$. Suppose that we have a sequence of points (e_n, e'_n) with $e_n \sim e'_n$, such that $(e_n, e'_n) \rightarrow (e, e') \in E \times E$. Since p is continuous, we deduce that $p(e) = p(e')$. But then $e'_n - e_n \in G$ converges to $e' - e$, and because G is closed, it follows that $e' - e \in G$, and so $e' \sim e$. This proves that \sim is indeed a closed subset of $E \times E$. □

Proposition 2.13 *If the two conditions in 2.5(i) and 2.5(ii) are satisfied, then the quotient space E/\sim is a complex manifold, and the mapping q is holomorphic.*

Proof From Lemma 2.9 and the fact that q is open, it follows that any sufficiently small open set in E is mapped homeomorphically onto its image in E/\sim , and thus can serve as a local chart for the quotient. Being Hausdorff, E/\sim is then a complex manifold, and the quotient map q is holomorphic by construction. □

2.6 The construction of the quotient

In this section, we define the holomorphic mapping $\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$, and prove that the quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space, provided that Condition 2.4 is satisfied.

Let $T_{\mathbb{Z}} \rightarrow \bar{X}$ denote the étalé space of the sheaf $j_*\mathcal{H}_{\mathbb{Z}}$; as a set, $T_{\mathbb{Z}}$ is the union of all the stalks of the sheaf, topologized to make every section continuous. For every point in $T_{\mathbb{Z}}$, there is a unique local section of $j_*\mathcal{H}_{\mathbb{Z}}$ that passes through that point. By using such local sections as charts, $T_{\mathbb{Z}}$ acquires the structure of a complex manifold, making the projection map and every section of the sheaf holomorphic. Note that the map $p_{\mathbb{Z}}: T_{\mathbb{Z}} \rightarrow \bar{X}$ is locally an isomorphism, and therefore flat.

We now explain how to embed $T_{\mathbb{Z}}$ into the analytic space $T(F_0\check{\mathcal{M}})$. On X , where we have a variation of Hodge structure of weight -1 , it is clear how to do this. To extend the embedding to all of \bar{X} , we need to know that sections of $j_*\mathcal{H}_{\mathbb{Z}}$ can act on arbitrary sections of the \mathcal{D} -module $\check{\mathcal{M}}$. Recall that M is the intermediate extension of the variation of Hodge structure \mathcal{H} . The underlying \mathcal{D} -module \mathcal{M} is thus the minimal extension of the flat vector bundle $\mathcal{H}_{\mathcal{O}} = \mathcal{H}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_X$, which implies that

$$j_*\mathcal{H}_{\mathbb{C}} \simeq \ker(\nabla: \mathcal{M} \rightarrow \Omega_{\bar{X}}^1 \otimes \mathcal{M}).$$

On the other hand, Lemma 2.8 provides us with an isomorphism

$$\ker(\nabla: \mathcal{M} \rightarrow \Omega_{\bar{X}}^1 \otimes \mathcal{M}) \simeq \mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\check{\mathcal{M}}, \mathcal{O}_{\bar{X}}).$$

Now $j_*\mathcal{H}_{\mathbb{Z}}$ is a subsheaf of $j_*\mathcal{H}_{\mathbb{C}}$, while $F_0\check{\mathcal{M}}$ is a subsheaf of $\check{\mathcal{M}}$; after restriction, we therefore get a canonical injective morphism

$$j_*\mathcal{H}_{\mathbb{Z}} \hookrightarrow (F_0\check{\mathcal{M}})^{\vee}.$$

Since the projection $p_{\mathbb{Z}}: T_{\mathbb{Z}} \rightarrow \bar{X}$ is flat, the morphism determines a holomorphic section of $p_{\mathbb{Z}}^*(F_0\check{\mathcal{M}})^{\vee} \simeq (p_{\mathbb{Z}}^*F_0\check{\mathcal{M}})^{\vee}$ on $T_{\mathbb{Z}}$. By the universal property of $T(F_0\check{\mathcal{M}})$ in Lemma 2.6, the section gives rise to a holomorphic mapping

$$\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}}) \tag{2.2}$$

from the complex manifold $T_{\mathbb{Z}}$ to the analytic space $T(F_0\check{\mathcal{M}})$. Our next task is to show that the fiberwise quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space.

Proposition 2.14 *Assume that Condition 2.4 is satisfied. Then the holomorphic mapping $\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0\check{\mathcal{M}})$ is a closed embedding.*

Proof The question is clearly local on \bar{X} ; thus we may assume that \bar{X} is a Stein manifold. As explained in Sect. 2.3, we present $F_0\check{\mathcal{M}}$ as a quotient of locally free sheaves,

$$\mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow F_0\check{\mathcal{M}} \rightarrow 0, \tag{2.3}$$

and let $\varphi: E_0^* \rightarrow E_1^*$ be the corresponding map of vector bundles; then $T(F_0\check{\mathcal{M}}) = \varphi^{-1}(0)$ is a closed analytic subset of E_0^* . Because of Condition 2.4, the holomorphic mapping from $T_{\mathbb{Z}}$ to E_0^* satisfies the two hypotheses in Sect. 2.5; we can now apply Lemma 2.9 to conclude that $T_{\mathbb{Z}} \rightarrow E_0^*$, and therefore also ε itself, is a closed embedding. \square

From now on, we identify $T_{\mathbb{Z}}$ with its image in $T(F_0\check{\mathcal{M}})$. Next, we deduce from the general results in Sect. 2.5 that the quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space.

Proposition 2.15 *Assume that Condition 2.4 is satisfied. Then the fiberwise quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space over \bar{X} .*

Proof This is again a local problem, and so we continue to assume that \bar{X} is a Stein manifold, and that $F_0\check{\mathcal{M}}$ has a presentation as in (2.3). Let $p: E_0^* \rightarrow \bar{X}$ be the projection, and let \sim be the equivalence relation on E_0^* given by

$$e \sim e' \quad \text{if and only if } p(e) = p(e') \text{ and } e' - e \in T_{\mathbb{Z}}.$$

The quotient space $Y = E_0^*/T_{\mathbb{Z}} = E_0^*/\sim$ is a complex manifold by Proposition 2.13, and the quotient map $q: E_0^* \rightarrow Y$ is holomorphic. In particular, the quotient is a Hausdorff space (see Lemma 2.12).

The mapping $\varphi: E_0^* \rightarrow E_1^*$ takes the submanifold $T_{\mathbb{Z}}$ into the zero section of E_1^* . This implies that we have a factorization $\varphi = \psi \circ q$, with $\psi: Y \rightarrow E_1^*$ holomorphic. Remembering that $T(F_0\check{\mathcal{M}}) = \varphi^{-1}(0)$, we see that the quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ may be naturally identified with the closed subset $\psi^{-1}(0)$ of Y , and is therefore an analytic space as well. \square

2.7 The V-filtration and pullbacks of Hodge modules

In this section, we briefly review the V -filtration, and then study the behavior of the Hodge filtration under pullbacks of mixed Hodge modules. This will be used in Sect. 2.8 below to prove the functoriality of our construction.

Let X be a complex manifold, and $Z \subseteq X$ a submanifold of codimension one. We first look at the local setting where Z is the zero locus of a holomorphic function t ; set $\partial = \partial/\partial t$. Let $I_Z = t \cdot \mathcal{O}_X$ be the corresponding ideal sheaf. Then

$$V^0\mathcal{D}_X = \{ D \in \mathcal{D}_X \mid D \cdot I_Z \subseteq I_Z \}.$$

Now let \mathcal{M} be a left \mathcal{D} -module on X . A decreasing filtration $V = V^\bullet\mathcal{M}$, indexed by \mathbb{Q} , is called a V -filtration of \mathcal{M} relative to the closed submanifold Z if it satisfies the following six conditions:

- (i) Each $V^\alpha\mathcal{M}$ is a coherent $V^0\mathcal{D}_X$ -module.

- (ii) The filtration is exhaustive, meaning that $\mathcal{M} = \bigcup_{\alpha} V^{\alpha} \mathcal{M}$, and left continuous, meaning that $V^{\alpha} \mathcal{M} = \bigcap_{\beta < \alpha} V^{\beta} \mathcal{M}$.
- (iii) The filtration is discrete, meaning that any bounded interval contains only finitely many $\alpha \in \mathbb{Q}$ such that $\text{Gr}_V^{\alpha} \mathcal{M} = V^{\alpha} \mathcal{M} / V^{>\alpha} \mathcal{M}$ is nonzero.
- (iv) One has $t \cdot V^{\alpha} \mathcal{M} \subseteq V^{\alpha+1} \mathcal{M}$ and $\partial \cdot V^{\alpha} \mathcal{M} \subseteq V^{\alpha-1} \mathcal{M}$.
- (v) For $\alpha \gg 0$, the filtration satisfies $V^{\alpha} \mathcal{M} = t \cdot V^{\alpha-1} \mathcal{M}$.
- (vi) The operator $t\partial - \alpha + 1$ is nilpotent on $\text{Gr}_V^{\alpha} \mathcal{M}$.

Elementary reasoning shows that there can be at most one such filtration; M. Kashiwara [20] and B. Malgrange [27] have proved that, for \mathcal{M} regular and holonomic, the V -filtration always exists. It is easy to deduce from the conditions that $t: V^{\alpha-1} \mathcal{M} \rightarrow V^{\alpha} \mathcal{M}$ is an isomorphism for $\alpha > 1$, and that $\partial: \text{Gr}_V^{\alpha+1} \mathcal{M} \rightarrow \text{Gr}_V^{\alpha} \mathcal{M}$ is an isomorphism for $\alpha \neq 0$.

Now consider the case when (\mathcal{M}, F) is a filtered \mathcal{D} -module. In that case, the V -filtration is said to be *compatible with F* , and (\mathcal{M}, F) is called *quasi-unipotent and regular along Z* [31, Sect. 3.2] if, in addition to the above:

- (vii) For every p , and every $\alpha > 1$, one has $F_p V^{\alpha} \mathcal{M} = t \cdot F_p V^{\alpha-1} \mathcal{M}$.
- (viii) For every p , and every $\alpha < 0$, one has $F_p \text{Gr}_V^{\alpha} \mathcal{M} = \partial \cdot F_{p-1} \text{Gr}_V^{\alpha+1} \mathcal{M}$.

When (\mathcal{M}, F) is the filtered \mathcal{D} -module underlying a polarizable mixed Hodge module, the V -filtration exists and is compatible with F ; moreover, each $\text{Gr}_V^{\alpha} \mathcal{M}$, with the induced filtration, again underlies a mixed Hodge module on Z . In fact, this is built into M. Saito’s definition [32, Sect. 2.17] of the category of mixed Hodge modules.

The V -filtration is essential for the construction of nearby cycles, vanishing cycles, and the various pullback operations on mixed Hodge modules. Suppose that M is a mixed Hodge module on X , with underlying filtered \mathcal{D} -module (\mathcal{M}, F) . To begin with, let $i: Z \hookrightarrow X$ be the inclusion of a submanifold that is defined by a single holomorphic equation t . In this situation, one can associate to M two mixed Hodge modules on Z :

- (a) The (unipotent) nearby cycles $\psi_{t,1} M$. Their underlying filtered \mathcal{D} -module is $(\text{Gr}_V^1 \mathcal{M}, F)$, where the Hodge filtration is induced by that on \mathcal{M} .
- (b) The vanishing cycles $\phi_{t,1} M$. Their underlying filtered \mathcal{D} -module is given by $(\text{Gr}_V^0 \mathcal{M}, F[-1])$.

The two standard maps $\text{can}: \psi_{t,1} M \rightarrow \phi_{t,1} M$ and $\text{Var}: \phi_{t,1} M \rightarrow \psi_{t,1} M(-1)$ are morphisms of mixed Hodge modules; on the level of \mathcal{D} -modules, can is multiplication by ∂ , and Var multiplication by t . The axioms imply that $t\partial$ is nilpotent on $\text{Gr}_V^1 \mathcal{M}$; it corresponds to $(2\pi i)^{-1} N$, where N is the logarithm of the monodromy around Z on the nearby cycles $\psi_{t,1} M$.

The pullback $i^* M$ is an object in the derived category $D^b \text{MHM}(Z)$; by [32, Corollary 2.24], it is represented by the complex (in degrees -1 and 0)

$$i^* M = \left[\psi_{t,1} M \xrightarrow{\text{can}} \phi_{t,1} M \right][1].$$

Each cohomology module $H^k i^* M$ is again a mixed Hodge module on M , nonzero only for $k = -1, 0$. Note that pulling back does not increase weights: if M has weight $\leq w$, then $H^k i^* M$ has weight $\leq w + k$ [32, Proposition 2.26]. Analogously, $i^! M$ is represented by the complex (in degrees 0 and 1)

$$i^! M = \left[\phi_{t,1} M \xrightarrow{\text{Var}} \psi_{t,1} M(-1) \right],$$

and $H^k i^! M$ has weight $\geq w + k$ if M has weight $\geq w$.

We now describe how the operation $i^!$ interacts with the Hodge filtration on the underlying \mathcal{D} -modules, first in the case of a closed embedding, then in general.

Lemma 2.16 *Let $i : Z \hookrightarrow X$ be the inclusion of a submanifold, defined by a single holomorphic equation t . Let $i^!(\mathcal{M}, F)$ denote the complex of filtered \mathcal{D} -modules on Z underlying $i^! M \in D^b \text{MHM}(Z)$.*

(i) *For every $p \in \mathbb{Z}$, there is a canonical morphism*

$$F_{p-1} i^!(\mathcal{M}, F) \rightarrow \mathbf{L}i^*(F_p \mathcal{M})[-1]$$

in the derived category $D^b \text{Coh}(Z)$.

(ii) *When M is smooth, the morphism in (i) is an isomorphism.*

Proof According to the discussion above, the complex of \mathcal{D} -modules underlying the object $i^! M$ is

$$i^!(\mathcal{M}, F) = [\text{Gr}_V^0 \mathcal{M} \xrightarrow{t} \text{Gr}_V^1 \mathcal{M}],$$

with Hodge filtration given by

$$F_{p-1} i^!(\mathcal{M}, F) = [F_p \text{Gr}_V^0 \mathcal{M} \xrightarrow{t} F_p \text{Gr}_V^1 \mathcal{M}].$$

We regard this complex of coherent sheaves on Z as an object in the derived category $D^b \text{Coh}(Z)$. Now consider the following commutative diagram:

$$\begin{array}{ccc}
 F_p \text{Gr}_V^0 \mathcal{M} & \xrightarrow{t} & F_p \text{Gr}_V^1 \mathcal{M} \\
 \uparrow & & \uparrow \\
 F_p V^0 \mathcal{M} & \xrightarrow{t} & F_p V^1 \mathcal{M} \\
 \downarrow & & \downarrow \\
 F_p \mathcal{M} & \xrightarrow{t} & F_p \mathcal{M}
 \end{array} \tag{2.4}$$

Because V and F are compatible, the multiplication map $t: F_p V^{\alpha-1} \rightarrow F_p V^\alpha$ is an isomorphism for $\alpha > 1$; this implies that the morphism between the complexes in the first two rows of (2.4) is a quasi-isomorphism. The complex in the third row represents $\mathbf{L}i^*(F_p \mathcal{M})[-1]$, and so we obtain the desired morphism in $D^b \text{Coh}(Z)$.

Now we prove (ii). When M is smooth, and hence a variation of mixed Hodge structure on X , the underlying \mathcal{D} -module \mathcal{M} is a flat vector bundle. In that case, $V^\alpha \mathcal{M} = I_Z^{[\alpha]-1} \mathcal{M}$ is essentially the I_Z -adic filtration (with the convention that $V^\alpha \mathcal{M} = \mathcal{M}$ for $\alpha \leq 0$). In particular, $\text{Gr}_V^1 \mathcal{M} = i^* \mathcal{M}$ and $\text{Gr}_V^0 \mathcal{M} = 0$, and so $F_{p-1} i^1(\mathcal{M}, F)$ is nothing but the locally free sheaf $i^*(F_p \mathcal{M})$ in degree one. It is then immediate from the construction that the morphism is an isomorphism. \square

More generally, suppose that $i: Z \hookrightarrow X$ is the inclusion of a submanifold of codimension d . If Z is defined by holomorphic equations t_1, \dots, t_d , the functors i^* and i^1 may be obtained by iterating the construction above [32, p. 263]; thus $i^* M$ is the single complex associated to the d -fold complex of mixed Hodge modules

$$((\psi_{t_1,1} \xrightarrow{\text{can}} \phi_{t_1,1}) \circ \dots \circ (\psi_{t_d,1} \xrightarrow{\text{can}} \phi_{t_d,1})(M))[d], \tag{2.5}$$

and $i^1 M$ is the single complex associated to the d -fold complex

$$(\phi_{t_1,1} \xrightarrow{\text{Var}} \psi_{t_1,1}(-1)) \circ \dots \circ (\phi_{t_d,1} \xrightarrow{\text{Var}} \psi_{t_d,1}(-1))(M). \tag{2.6}$$

The statement of Lemma 2.16 continues to hold in this setting, and can be proved in a similar manner. It is, however, advantageous to keep the construction free of a choice of local equations for Z , and so we shall give a different argument.

Lemma 2.17 *Let $i: Z \hookrightarrow X$ be the inclusion of a submanifold of codimension d , and as before, let $i^1(\mathcal{M}, F)$ denote the complex of filtered \mathcal{D} -modules on Z underlying the object $i^1 M \in D^b \text{MHM}(Z)$.*

(i) *For every $p \in \mathbb{Z}$, there is a canonical morphism*

$$F_{p-d} i^1(\mathcal{M}, F) \rightarrow \mathbf{L}i^*(F_p \mathcal{M})[-d]$$

in the derived category $D^b \text{Coh}(Z)$.

(ii) *When M is smooth, the morphism in (i) is an isomorphism.*

Proof The functor i^1 is the right adjoint of i_* , and since i is a closed embedding, we have the adjunction morphism $i_* i^1 M \rightarrow M$ in $D^b \text{MHM}(X)$. Passing to filtered \mathcal{D} -modules, we then get a morphism $F_p(i_* i^1(\mathcal{M}, F)) \rightarrow F_p \mathcal{M}$.

Now i is a closed embedding of codimension d , and because of how the direct image functor is defined in [31, 2.3], we have a canonical morphism

$$i_*(\omega_Z \otimes F_{p-d} i^!(\mathcal{M}, F)) \rightarrow \omega_X \otimes F_p(i_* i^!(\mathcal{M}, F)).$$

Let $\omega_{Z/X} = \omega_Z \otimes i^* \omega_X^{-1}$. Composing the two morphisms, we obtain

$$i_*(\omega_{Z/X} \otimes F_{p-d} i^!(\mathcal{M}, F)) \rightarrow F_p \mathcal{M}.$$

On the level of coherent sheaves, the functor $\mathbf{L}i^! = \omega_{Z/X}[-d] \otimes \mathbf{L}i^*$ is the right adjoint of i_* ; by adjunction, we therefore get the desired morphism

$$F_{p-d} i^!(\mathcal{M}, F) \rightarrow \omega_{Z/X}^{-1} \otimes \mathbf{L}i^!(F_p \mathcal{M}) = \mathbf{L}i^*(F_p \mathcal{M})[-d]$$

in the derived category $D^b \text{Coh}(Z)$. When M is smooth, and hence a variation of mixed Hodge structure, both complexes are isomorphic to the locally free sheaf $i^*(F_p \mathcal{M})$ in degree d ; we leave the easy verification of (ii) to the care of the reader. □

Note It is not hard to show that the two different constructions in Lemma 2.16 and Lemma 2.17 give rise to the same morphism when $d = 1$; since we do not need this fact below, we omit the proof.

To conclude the discussion, we now consider the functor

$$f^! : D^b \text{MHM}(X) \rightarrow D^b \text{MHM}(Y)$$

for an arbitrary holomorphic mapping $f : Y \rightarrow X$ between complex manifolds.

Proposition 2.18 *Let $f : Y \rightarrow X$ be a holomorphic mapping between two complex manifolds, and let M be a mixed Hodge module on X with underlying filtered \mathcal{D} -module (\mathcal{M}, F) . Denote by $f^!(\mathcal{M}, F)$ the complex of filtered \mathcal{D} -modules underlying $f^! M \in D^b \text{MHM}(Y)$. Then for every $p \in \mathbb{Z}$, we have a canonical morphism*

$$F_{p+d_Y-d_X} f^!(\mathcal{M}, F) \rightarrow \mathbf{L}f^*(F_p \mathcal{M})[d_Y - d_X]$$

in $D^b \text{Coh}(Y)$, which is an isomorphism when M is smooth.

Proof Note that the argument used during the proof of Lemma 2.17 does not work here (unless f is, say, quasi-projective), because the functor f_* for mixed Hodge modules cannot be defined in this generality. Instead, we argue

by factoring f through its graph. Let $r = d_Y - d_X$. To compute $f^!M$, one factors f as

$$Y \xhookrightarrow{i} W \xrightarrow{q} X$$

with q smooth of relative dimension k , and i a closed embedding of codimension $k - r$. Then we have $f^!M = i^!N[k]$, where we set $N = q^!M[-k]$. According to the definition of the pullback in [32, 2.17],

$$N = q^!M[-k] = H^{-k}q^!M = H^kq^*M(k)$$

is a single mixed Hodge module; the underlying \mathcal{D} -module is $\mathcal{N} = q^*\mathcal{M}$, with the pullback taken in the category of quasi-coherent sheaves, and the Hodge filtration is given by $F_p\mathcal{N} = q^*(F_{p-k}\mathcal{M})$. By Lemma 2.17, we have a canonical morphism

$$F_{p+r}i^!(\mathcal{N}, F) \rightarrow \mathbf{Li}^*(F_{p+k}\mathcal{N})[r - k].$$

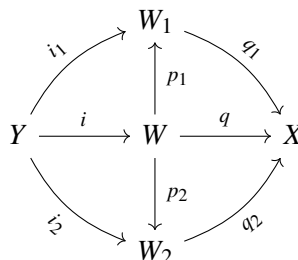
Since $\mathbf{Li}^*(F_{p+k}\mathcal{N}) = \mathbf{Li}^*q^*(F_p\mathcal{M}) = \mathbf{L}f^*(F_p\mathcal{M})$, we therefore get a morphism

$$F_{p+r}f^!(\mathcal{M}, F) = F_{p+r}i^!(\mathcal{N}, F)[k] \rightarrow \mathbf{Li}^*(F_{p+k}\mathcal{N})[r] = \mathbf{L}f^*(F_p\mathcal{M})[r],$$

as desired. For smooth M , it is an isomorphism because of the second assertion in Lemma 2.17. It remains to prove that the morphism is independent of the factorization $f = q \circ i$; this is the content of the following lemma. \square

Lemma 2.19 *Let $f = q_1 \circ i_1 = q_2 \circ i_2$ be two factorizations of $f : Y \rightarrow X$ into a closed embedding $i_j : Y \hookrightarrow W_j$ and a smooth morphism $q_j : W_j \rightarrow X$. Then the two resulting morphisms $F_{p+d_Y-d_X}f^!(\mathcal{M}, F) \rightarrow \mathbf{L}f^*(F_p\mathcal{M})[d_Y - d_X]$ are equal.*

Proof Let $W = W_1 \times_X W_2$ be the fiber product; both projections $p_j : W \rightarrow W_j$ are smooth. Because of the commutative diagram



it suffices to show that the factorizations $q_j \circ i_j$ both produce the same morphism as $q \circ i$. The construction in Proposition 2.18 is obviously insensitive to factorizing $q = q_1 \circ p_1 = q_2 \circ p_2$ since all five maps involved are smooth; this reduces the problem to considering the closed embeddings $i_j = p_j \circ i$, for which the assertion is clear by Lemma 2.17. \square

Corollary 2.20 *Under the same assumptions as in Proposition 2.18, define $N^f = H^{d_X - d_Y} f^! M(d_X - d_Y) \in \text{MHM}(Y)$, and denote the underlying filtered \mathcal{D} -module by (\mathcal{N}^f, F) . Then for every $p \in \mathbb{Z}$, we have a canonical morphism*

$$F_p \mathcal{N}^f \rightarrow f^*(F_p \mathcal{M}),$$

which is an isomorphism if M is smooth.

2.8 Functoriality

In this section, we prove that our construction of the space $\bar{J}(\mathcal{H})$ is functorial, in a sense made precise below. Let $f : \bar{Y} \rightarrow \bar{X}$ be a holomorphic mapping between two complex manifolds, such that $Y = f^{-1}(X)$ is a dense Zariski-open subset of \bar{Y} ; we also write $f : Y \rightarrow X$ for the induced mapping. As above, let \mathcal{H} be a polarizable variation of Hodge structure of weight -1 on X , let M be the Hodge module on \bar{X} obtained by intermediate extension, and $(\check{\mathcal{M}}, F)$ the filtered \mathcal{D} -module underlying $\check{M} = \mathbf{D}_{\bar{X}}(M)(1 - d_X)$. We denote the pullback of the variation of Hodge structure by $\mathcal{H}^f = f^* \mathcal{H}$, its intermediate extension to \bar{Y} by M^f , and the filtered \mathcal{D} -module underlying $\check{M}^f = \mathbf{D}_{\bar{Y}}(M^f)(1 - d_Y)$ by $(\check{\mathcal{M}}^f, F)$.

Lemma 2.21 *We have a canonical morphism of coherent sheaves*

$$F_0 \check{\mathcal{M}}^f \rightarrow f^*(F_0 \check{\mathcal{M}}),$$

whose restriction to Y is the obvious isomorphism of Hodge bundles.

Proof Let $n = d_X$ and $m = d_Y$, and note that \check{M} has weight $n - 1$. Since the functor $f^!$ does not decrease weights, the mixed Hodge module

$$N^f = H^{n-m} f^! \check{M}(n - m)$$

has weight $\geq m - 1$. The pure Hodge module $W_{m-1} N^f$ is therefore a submodule of N^f . The restriction of $W_{m-1} N^f$ to Y is canonically isomorphic to the variation of Hodge structure $\check{\mathcal{H}}^f = f^* \check{\mathcal{H}}$; in the decomposition by strict support, the component with strict support \bar{Y} has to be isomorphic to \check{M}^f . Since the decomposition is canonical, we get a canonical morphism

$\check{M}^f \hookrightarrow W_{m-1}N^f \hookrightarrow N^f$. Passing to the Hodge filtrations on the underlying \mathcal{D} -modules, we thus have a canonical morphism of coherent sheaves $F_0\check{\mathcal{M}}^f \hookrightarrow F_0\mathcal{N}^f$. We compose this with the morphism $F_0\mathcal{N}^f \rightarrow f^*(F_0\check{\mathcal{M}})$ from Corollary 2.20 to get the first half of the assertion; since $\check{\mathcal{M}}$ is smooth over the open subset X , the second half of the assertion follows from Corollary 2.20 upon restricting to Y . \square

Proposition 2.22 *Let $f : \bar{Y} \rightarrow \bar{X}$ be a morphism of complex manifolds, such that $Y = f^{-1}(X)$ remains dense in \bar{Y} . If we let $f^*\mathcal{H}$ denote the pullback of the variation of Hodge structure \mathcal{H} from X to Y , we have a canonical holomorphic mapping*

$$\bar{Y} \times_{\bar{X}} \bar{J}(\mathcal{H}) \rightarrow \bar{J}(f^*\mathcal{H})$$

over \bar{Y} , whose restriction to Y is the evident isomorphism between the two families of intermediate Jacobians.

Proof Consider the two spaces $T(F_0\check{\mathcal{M}})$ and $T(F_0\check{\mathcal{M}}^f)$ that appear in the construction of $\bar{J}(\mathcal{H})$ and $\bar{J}(\mathcal{H}^f)$. By Lemma 2.6, $\bar{Y} \times_{\bar{X}} T(F_0\check{\mathcal{M}}) \simeq T(f^*F_0\check{\mathcal{M}})$. On the other hand, Lemma 2.21 provides us with a mapping $T(f^*F_0\check{\mathcal{M}}) \rightarrow T(F_0\check{\mathcal{M}}^f)$. Composing the two, we obtain a canonical holomorphic mapping

$$\bar{Y} \times_{\bar{X}} T(F_0\check{\mathcal{M}}) \rightarrow T(F_0\check{\mathcal{M}}^f)$$

over \bar{Y} ; over Y , the left-hand side restricts to the pullback of the vector bundle associated with $(F_0\check{\mathcal{H}}_{\mathcal{O}})^\vee$, the right-hand side to the vector bundle associated with $(F_0\check{\mathcal{H}}_{\mathcal{O}}^f)^\vee$, and the map to the obvious isomorphism between them. Since $\bar{Y} \times_{\bar{X}} T_{\mathbb{Z}}$ is easily seen to map into $T_{\mathbb{Z}}^f$, we get the assertion for the quotient spaces as well. \square

2.9 Restriction to points

In this section, we describe how $T(F_0\check{\mathcal{M}})$ behaves upon restriction to points, by relating its fibers to Hodge-theoretic information. Let $i : \{x\} \hookrightarrow \bar{X}$ be the inclusion of a point. Define the rational mixed Hodge structure $H = H^{-n}i^*M$, which has weight ≤ -1 .

Lemma 2.23 *The rational mixed Hodge structure $H = H^{-n}i^*M$ is naturally defined over \mathbb{Z} , with $H_{\mathbb{Z}}$ isomorphic to the stalk of the sheaf $j_*\mathcal{H}_{\mathbb{Z}}$ at the point x . Consequently, $H_{\mathbb{Z}}$ embeds into the stalk of $\mathcal{H}_{\mathbb{Z}}$ at any nearby point $x_0 \in X$, and the quotient $\mathcal{H}_{\mathbb{Z},x_0}/H_{\mathbb{Z}}$ is torsion-free.*

Proof There is a natural linear map from the stalk of the sheaf $j_*\mathcal{H}_{\mathbb{C}}$ to $H_{\mathbb{C}}$,

$$(j_*\mathcal{H}_{\mathbb{C}})_x = \lim_{U \ni x} H^0(U \cap X, \mathcal{H}_{\mathbb{C}}) \rightarrow H_{\mathbb{C}},$$

given as follows: Let t_1, \dots, t_n be local holomorphic coordinates centered at x , and $\partial_j = \partial/\partial t_j$; then a local section of $j_*\mathcal{H}_{\mathbb{C}}$ is a section $s \in H^0(U, \mathcal{M})$ that satisfies $\partial_j s = 0$ for every $j = 1, \dots, n$. On the other hand, if V_j denotes the V -filtration relative to the divisor $t_j = 0$, then by (2.5), the vector space $H_{\mathbb{C}}$ is a subspace of $\text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_n}^1 \mathcal{M}$, consisting of those elements h for which each $\partial_j h$ is zero in $\text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_j}^0 \cdots \text{Gr}_{V_n}^1 \mathcal{M}$. It is easy to see from the axioms in Sect. 2.7 that $\partial_n s = 0$ implies $s \in V_n^1 \mathcal{M}$; after iterating this argument, s defines a vector in the subspace $H_{\mathbb{C}}$ of the vector space $\text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_n}^1 \mathcal{M}$. The resulting homomorphism $(j_*\mathcal{H}_{\mathbb{C}})_x \rightarrow H_{\mathbb{C}}$ is known to be an isomorphism; this means that the mixed Hodge structure H is defined over \mathbb{Z} , with integral lattice $H_{\mathbb{Z}}$ isomorphic to the stalk of the sheaf $j_*\mathcal{H}_{\mathbb{Z}}$.

Now let $U \subseteq \bar{X}$ be a small open ball around x , and $x_0 \in U \cap X$. The stalk of $j_*\mathcal{H}_{\mathbb{Z}}$ at x is naturally identified with the subgroup of $\mathcal{H}_{\mathbb{Z},x_0}$ consisting of classes invariant under the action by the fundamental group $\pi_1(U \cap X, x_0)$. Since $\mathcal{H}_{\mathbb{Z},x_0}$ is torsion-free, it is then easy to deduce the second assertion. \square

Recall that $\check{M} = \mathbf{D}_{\bar{X}}(M)(1-n)$; because the duality functor interchanges the two operations i^* and $i^!$, we find that

$$\mathbf{D}(i^* M) \simeq i^! \mathbf{D}_{\bar{X}}(M) \simeq i^! \check{M}(n-1),$$

and therefore

$$\check{H} = H^\vee(1) \simeq H^n i^! \check{M}(n).$$

This means that the rational mixed Hodge structure $H^n i^! \check{M}(n)$, of weight ≥ -1 , is actually defined over \mathbb{Z} as well. Using the isomorphism $\check{H} \simeq H^n i^! \check{M}(n)$, we obtain from Corollary 2.20 a linear map $\delta: F_0 \check{H}_{\mathbb{C}} \rightarrow i^*(F_0 \check{\mathcal{M}})$. On the other hand, Lemma 2.8 shows that elements of $H_{\mathbb{C}} \simeq (j_*\mathcal{H}_{\mathbb{C}})_x$ induce linear functionals on the vector space $i^*(F_0 \check{\mathcal{M}})$. The following compatibility result will be useful below.

Lemma 2.24 *The following diagram is commutative:*

$$\begin{array}{ccc} H_{\mathbb{C}} & \longrightarrow & \text{Hom}_{\mathbb{C}}(i^*(F_0 \check{\mathcal{M}}), \mathbb{C}) \\ \downarrow & & \downarrow \delta^* \\ H_{\mathbb{C}}/F_0 H_{\mathbb{C}} & \xrightarrow{\simeq} & \text{Hom}_{\mathbb{C}}(F_0 \check{H}_{\mathbb{C}}, \mathbb{C}) \end{array}$$

Proof Fix an element $h \in H_{\mathbb{C}} \simeq (j_*\mathcal{H}_{\mathbb{C}})_x$; by Lemma 2.8, it induces a morphism of \mathcal{D} -modules $\check{\mathcal{M}} \rightarrow \mathcal{O}_X$ over some open neighborhood of the point x . The linear functional $\varphi_h \in \text{Hom}_{\mathbb{C}}(i^*(F_0\check{\mathcal{M}}), \mathbb{C})$ is obtained by restricting to the subsheaf $F_0\check{\mathcal{M}}$ and applying i^* . To see what happens to this functional under δ^* , let V_j denote the V -filtration relative to the divisor t_j . Morphisms between \mathcal{D} -modules automatically respect the V -filtration, and so $h: \check{\mathcal{M}} \rightarrow \mathcal{O}_X$ induces a linear map

$$\text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_n}^1 \check{\mathcal{M}} \rightarrow \text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_n}^1 \mathcal{O}_X.$$

By (2.6), $\check{H}_{\mathbb{C}}$ is a quotient of the vector space on the left; as $\text{Gr}_{V_1}^1 \cdots \text{Gr}_{V_n}^1 \mathcal{O}_X = \mathbb{C}$, it is not hard to see that we get an induced linear functional

$$\check{H}_{\mathbb{C}} \rightarrow \mathbb{C},$$

which is of course still given by h . Now $\delta^*(\varphi_h)$ is the restriction of that functional to $F_0\check{H}_{\mathbb{C}}$, and this proves the commutativity of the diagram. \square

The following result explains how the fiber of the Néron model $\bar{J}(\mathcal{H})$ over the point $x \in \bar{X}$ is related to the mixed Hodge structure H . We are going to use it in two places: once to reduce the proof of Condition 2.4 from the general case to the normal crossing case (in Sect. 2.11); and, much later, to construct a mapping from $\bar{J}(\mathcal{H})$ to the Néron model of [7] (in Sect. 4.6).

Lemma 2.25 *Let $i: \{x\} \hookrightarrow \bar{X}$ be the inclusion of a point, and set $H = H^{-n}i^*\check{\mathcal{M}}$, which implies that $\check{H} \simeq H^n i^1\check{\mathcal{M}}(n)$. Then the linear map $F_0\check{H}_{\mathbb{C}} \rightarrow i^*(F_0\check{\mathcal{M}})$ defined in Corollary 2.20 induces a surjective mapping of complex Lie groups*

$$\bar{J}(\mathcal{H})_x \twoheadrightarrow J(H),$$

where $J(H)$ is the generalized intermediate Jacobian of H (see Sect. 2.1).

Proof Corollary 2.20 gives us a map of vector spaces

$$F_0\check{H}_{\mathbb{C}} \rightarrow i^*(F_0\check{\mathcal{M}}). \tag{2.7}$$

By Lemma 2.6, the fiber of $T(F_0\check{\mathcal{M}})$ at the point x is exactly $(i^*F_0\check{\mathcal{M}})^\vee$, and so (2.7) induces a linear map

$$T(F_0\check{\mathcal{M}})_x \simeq (i^*F_0\check{\mathcal{M}})^\vee \rightarrow (F_0\check{H}_{\mathbb{C}})^\vee.$$

We observe that this map is surjective: indeed, the composition $H_{\mathbb{C}} \twoheadrightarrow H_{\mathbb{C}}/F_0H_{\mathbb{C}} \simeq (F_0\check{H}_{\mathbb{C}})^\vee$ is obviously surjective, and by Lemma 2.24, it factors through $T(F_0\check{\mathcal{M}})_x$. If we now take the quotient by $T_{\mathbb{Z},x} \simeq H_{\mathbb{Z}}$, we arrive

at a surjective mapping

$$T(F_0\check{\mathcal{M}})_x/T_{\mathbb{Z},x} \twoheadrightarrow (F_0\check{H}_{\mathbb{C}})^\vee/H_{\mathbb{Z}}$$

between the two complex Lie groups, as asserted. □

2.10 Restriction to curves

In this section, we investigate how $T(F_0\check{\mathcal{M}})$ behaves upon restriction to curves, and use the result to show that the subset $T_{\mathbb{Z}}$ is closed under limits along analytic arcs. Throughout, we let $f: \Delta \rightarrow \bar{X}$ be a holomorphic mapping such that $f(\Delta^*) \subseteq X$, and set $x = f(0)$; the most interesting case, of course, is when $x \in \bar{X} - X$ is a boundary point.

We define \mathcal{H}^f to be the pullback of the variation of Hodge structure \mathcal{H} to Δ^* . Its intermediate extension M^f is a polarizable Hodge module of weight 0 on Δ ; as usual, we shall denote the underlying filtered \mathcal{D} -module by (\mathcal{M}^f, F) . We also let $\check{M}^f = \mathbf{D}_\Delta(M^f)$, with underlying \mathcal{D} -module $(\check{\mathcal{M}}^f, F)$. Finally, we need to introduce $N^f = H^{n-1}f^!\check{M}(n-1)$, which is a mixed Hodge module of weight ≥ 0 . As in Lemma 2.21, decomposition by strict support means that we have canonical morphisms $\check{M}^f \hookrightarrow W_0N^f \hookrightarrow N^f$, and consequently, a morphism of coherent sheaves $F_0\check{\mathcal{M}}^f \rightarrow F_0\mathcal{N}^f$. Corollary 2.20 gives us another morphism $F_0\mathcal{N}^f \rightarrow f^*(F_0\check{\mathcal{M}})$, and so we end up with two holomorphic mappings

$$\Delta \times_{\bar{X}} T(F_0\check{\mathcal{M}}) \rightarrow T(F_0\mathcal{N}^f) \rightarrow T(F_0\check{\mathcal{M}}^f) \tag{2.8}$$

of analytic spaces over Δ .

We now study the fibers of those three spaces over $0 \in \Delta$. To begin with, let $i: \{x\} \hookrightarrow \bar{X}$, and define the mixed Hodge structures $H = H^{-n}i^*M$ and $\check{H} = H^\vee(1) \simeq H^n i^! \check{M}(n)$ as in Sect. 2.9. Recall that H is of weight ≤ -1 and defined over \mathbb{Z} , with integral lattice $H_{\mathbb{Z}}$ isomorphic to the stalk of $j_*\mathcal{H}_{\mathbb{Z}}$ at the point x ; on the other hand, \check{H} is of weight ≥ -1 , and also defined over \mathbb{Z} . As in Lemma 2.25, we have a canonical mapping

$$T(F_0\check{\mathcal{M}})_x \rightarrow (F_0\check{H}_{\mathbb{C}})^\vee \simeq H_{\mathbb{C}}/F_0H_{\mathbb{C}}.$$

Similarly, let $i_0: \{0\} \hookrightarrow \Delta$, and define the two integral mixed Hodge structures $H^f = H^{-1}i_0^*M^f$ (of weight ≤ -1) and $\check{H}^f \simeq H^1 i_0^! \check{M}^f(1)$ (of weight ≥ -1); we then have a second mapping

$$T(F_0\check{\mathcal{M}}^f)_0 \rightarrow H_{\mathbb{C}}^f/F_0H_{\mathbb{C}}^f.$$

To get information about the mixed Hodge module N^f , we note that $i_0^! \circ f^! \simeq i^!$. This means that there is a spectral sequence

$$E_2^{p,q} = H^p i_0^! H^q f^! \check{M} \implies H^{p+q} i^! \check{M}.$$

Because Δ is one-dimensional, $H^p i_0^! = 0$ unless $p = 0, 1$; therefore the spectral sequence degenerates at E_2 , and we find that $H^1 i_0^! H^{n-1} f^! \check{M} \simeq H^n i^! \check{M}$, using that $H^n f^! \check{M} = 0$ for dimension reasons. Consequently,

$$H^1 i_0^! N^f(1) \simeq H^1 i_0^! H^{n-1} f^! \check{M}(n) \simeq H^n i^! \check{M}(n) = \check{H},$$

and as before, this leads to a linear map

$$T(F_0 \mathcal{N}^f)_0 \rightarrow H_{\mathbb{C}}/F_0 H_{\mathbb{C}}.$$

Since the various maps we produce are easily seen to be compatible with each other, we arrive at the following commutative diagram that relates the fibers of the analytic spaces in (2.8) to the mixed Hodge structures H and H^f :

$$\begin{CD} T(F_0 \check{\mathcal{M}})_x @>>> T(F_0 \mathcal{N}^f)_0 @>>> T(F_0 \check{\mathcal{M}}^f)_0 \\ @VVV @VVV @VVV \\ H_{\mathbb{C}}/F_0 H_{\mathbb{C}} @= H_{\mathbb{C}}/F_0 H_{\mathbb{C}} @>>> H_{\mathbb{C}}^f/F_0 H_{\mathbb{C}}^f \end{CD} \tag{2.9}$$

We can use the discussion above to show that $\varepsilon(T_{\mathbb{Z}}) \subseteq T(F_0 \check{\mathcal{M}})$ is closed under limits along analytic curves, in the following sense.

Lemma 2.26 *Let $g: \Delta \rightarrow T(F_0 \check{\mathcal{M}})$ be a holomorphic mapping with the property that $g(\Delta^*) \subseteq \varepsilon(T_{\mathbb{Z}}) \cap p^{-1}(X)$, where $p: T(F_0 \check{\mathcal{M}}) \rightarrow \bar{X}$ is the projection. Then we actually have $g(\Delta) \subseteq \varepsilon(T_{\mathbb{Z}})$.*

Proof Set $f = p \circ g$, and let $\mathcal{H}^f = f^* \mathcal{H}$ be the pullback of the variation to Δ^* ; we also use the other notation introduced above. Since $g(\Delta^*) \subseteq \varepsilon(T_{\mathbb{Z}})$, it corresponds to an integral section

$$h^f \in H_{\mathbb{Z}}^f \simeq H^0(\Delta^*, \mathcal{H}_{\mathbb{Z}}^f).$$

Over Δ^* , the two spaces $\Delta \times_{\bar{X}} T(F_0 \check{\mathcal{M}})$ and $T(F_0 \check{\mathcal{M}}^f)$ are isomorphic to the dual of a Hodge bundle; the mapping g may be viewed as a holomorphic section, which is actually given by evaluation against h^f . By Lemma 2.8, it extends to a holomorphic section of $T(F_0 \check{\mathcal{M}}^f)$ over the entire disk. The

extended section projects to a point in $H_{\mathbb{C}}^f / F_0 H_{\mathbb{C}}^f$, which is simply the image of h^f (by Lemma 2.24).

Similarly, $g(0) \in T(F_0 \check{\mathcal{M}})_x$ projects to a point in the quotient $H_{\mathbb{C}} / F_0 H_{\mathbb{C}}$; the commutativity of the diagram in (2.9) implies that

$$h^f \in F_0 H_{\mathbb{C}}^f + \text{im}(H_{\mathbb{C}} \rightarrow H_{\mathbb{C}}^f).$$

Now $H \hookrightarrow H^f$ is a morphism of mixed Hodge structures; let $H' = H^f / H$ be the quotient, still a mixed Hodge structure of weight ≤ -1 . The image of h^f in H' is both rational (because h^f is rational) and in $F_0 H'_{\mathbb{C}}$, and therefore equal to zero; consequently, $h^f \in H_{\mathbb{Q}}$. Now Lemma 2.23 implies that we automatically have $h^f \in H_{\mathbb{Z}}$: indeed, $H_{\mathbb{Z}}$ and $H_{\mathbb{Z}}^f$ are both subgroups of the stalk of $\mathcal{H}_{\mathbb{Z}}$ at some nearby point $f(t_0)$, and the quotient $\mathcal{H}_{\mathbb{Z}, f(t_0)} / H_{\mathbb{Z}}$ is torsion-free.

But then h^f defines a holomorphic section of $\varepsilon(T_{\mathbb{Z}}) \subseteq T(F_0 \check{\mathcal{M}})$ in a neighborhood of the point x . Over $f(\Delta^*)$, this section is an extension of g ; since both are holomorphic, this means that $g(\Delta) \subseteq \varepsilon(T_{\mathbb{Z}})$, as claimed. \square

Note This result by itself is not sufficient to construct the Néron model over a one-dimensional base, because in order to show that $\varepsilon(T_{\mathbb{Z}})$ is a closed analytic subset of $T(F_0 \check{\mathcal{M}})$, it is not enough to consider just limits along analytic arcs.

2.11 Reduction to the normal crossing case

This section is devoted to reducing the proof of Condition 2.4 to the following special case.

Theorem 2.27 *If Condition 2.4 is true whenever $\bar{X} - X$ is a divisor with normal crossings and $\mathcal{H}_{\mathbb{Z}}$ has unipotent local monodromy, then it is true in general.*

Evidently, the problem is local on \bar{X} , and so we may assume that $\bar{X} = \Delta^n$ is a polydisk, and that $\bar{X} - X$ is a divisor (possibly singular and with several components). Let \mathcal{H} be a polarizable variation of integral Hodge structure of weight -1 on X . Recall that $T_{\mathbb{Z}}$ is the étalé space of the sheaf $j_* \mathcal{H}_{\mathbb{Z}}$, and that we had constructed a holomorphic mapping $\varepsilon: T_{\mathbb{Z}} \rightarrow T(F_0 \check{\mathcal{M}})$ in (2.2).

We begin by showing that ε is injective. For this, it is clearly sufficient to prove that the map on fibers, $T_{\mathbb{Z}, x} \rightarrow T(F_0 \check{\mathcal{M}})_x$, is injective.

$$\begin{array}{ccc}
 T_{\mathbb{Z}} & \xrightarrow{\varepsilon} & T(F_0 \check{\mathcal{M}}) \\
 & \searrow p_{\mathbb{Z}} & \downarrow p \\
 & & \check{X}
 \end{array}$$

The results of Sect. 2.9 easily imply the injectivity on fibers, as follows.

Lemma 2.28 *For $x \in X$, let $T_{\mathbb{Z},x} = p_{\mathbb{Z}}^{-1}(x)$ and $T(F_0\check{\mathcal{M}})_x = p^{-1}(x)$ denote the fibers of $T_{\mathbb{Z}}$ and $T(F_0\check{\mathcal{M}})$, respectively. Then ε is injective, and embeds $T_{\mathbb{Z},x}$ into $T(F_0\check{\mathcal{M}})_x$ as a discrete subset.*

Proof Let $i : \{x\} \hookrightarrow X$ be the inclusion of the point, and let $H = H^{-n}i^*M$, which is an integral mixed Hodge structure of weight ≤ -1 with $H_{\mathbb{Z}} \simeq T_{\mathbb{Z},x}$. Also define the mixed Hodge structure $\check{H} = H^{\vee}(1)$ (of weight ≥ -1); it satisfies $\check{H} \simeq H^n i^! \check{M}(n)$. According to the discussion in Sect. 2.9, have a surjective linear map

$$T(F_0\check{\mathcal{M}})_x \twoheadrightarrow (F_0\check{H}_{\mathbb{C}})^{\vee} \simeq H_{\mathbb{C}}/F_0H_{\mathbb{C}}.$$

But since H has weight ≤ -1 , the set of integral points $H_{\mathbb{Z}}$ maps injectively and hence discretely into $H_{\mathbb{C}}/F_0H_{\mathbb{C}}$. Consequently, the map ε also embeds $T_{\mathbb{Z},x}$ into $T(F_0\check{\mathcal{M}})_x$ as a discrete subset, proving the assertion. \square

For the remainder of this section, we assume that Condition 2.4 is satisfied whenever $\bar{X} - X$ is a divisor with normal crossings and $\mathcal{H}_{\mathbb{Z}}$ has unipotent local monodromy. We show that it then holds in general.

Lemma 2.29 *The closure of $\varepsilon(T_{\mathbb{Z}})$ in $T(F_0\check{\mathcal{M}})$ is an analytic subset.*

Proof Since the underlying local system $\mathcal{H}_{\mathbb{Z}}$ is defined over \mathbb{Z} , the local monodromy is at least quasi-unipotent by a theorem due to A. Borel [37, Lemma 4.5]. Taking a finite branched cover, unbranched over X , and resolving singularities, we construct a proper holomorphic mapping $f : \bar{Y} \rightarrow \bar{X}$ from a complex manifold \bar{Y} of dimension n , with the following properties: $Y = f^{-1}(X)$ is dense in \bar{Y} ; the restriction of f to Y is finite and étalé; the complement $\bar{Y} - Y$ is a divisor with normal crossings; and the pullback of $\mathcal{H}_{\mathbb{Z}}$ to Y has unipotent monodromy.

Let $\mathcal{H}^f = f^*\mathcal{H}$, let M^f be its intermediate extension to a polarizable Hodge module on \bar{Y} , and $(\check{\mathcal{M}}^f, F)$ the filtered \mathcal{D} -module underlying $\check{M}^f = \mathbf{D}_{\bar{Y}}(M^f)(1-n)$. By Lemma 2.21, we have a commutative diagram of holomorphic mappings

$$\begin{array}{ccccc}
 T_{\mathbb{Z}} & \longleftarrow & \bar{Y} \times_{\bar{X}} T_{\mathbb{Z}} & \longrightarrow & T_{\mathbb{Z}}^f \\
 \downarrow \varepsilon & & \downarrow \text{id} \times \varepsilon & & \downarrow \varepsilon^f \\
 T(F_0\check{\mathcal{M}}) & \longleftarrow & \bar{Y} \times_{\bar{X}} T(F_0\check{\mathcal{M}}) & \xrightarrow{\Phi} & T(F_0\check{\mathcal{M}}^f)
 \end{array}$$

Assuming Condition 2.4 for the variation of Hodge structure \mathcal{H}^f , we know that $W = \varepsilon^f(T_{\mathbb{Z}}^f)$ is a closed analytic subset of $T(F_0\check{\mathcal{M}}^f)$. Then $\Phi^{-1}(W)$ is a closed analytic subset of $\bar{Y} \times_{\bar{X}} T(F_0\check{\mathcal{M}})$. The projection to $T(F_0\check{\mathcal{M}})$ is proper, since f is a proper mapping, and so the image of $\Phi^{-1}(W)$ in $T(F_0\check{\mathcal{M}})$ is again a closed analytic subset by Grauert's theorem. The part of it that lies over X is equal to $\varepsilon(T_{\mathbb{Z}}) \cap p^{-1}(X)$, and so the closure of $\varepsilon(T_{\mathbb{Z}})$ must be an analytic subset (and, in fact, a countable union of irreducible components of the image). \square

To conclude the reduction to the normal crossing case, we use the results about restriction to curves from Sect. 2.10 to show that taking the closure does not actually add any points to $\varepsilon(T_{\mathbb{Z}})$.

Lemma 2.30 $\varepsilon(T_{\mathbb{Z}})$ is a closed analytic subset of $T(F_0\check{\mathcal{M}})$.

Proof The restriction of $\varepsilon(T_{\mathbb{Z}})$ to X clearly has the same closure as $\varepsilon(T_{\mathbb{Z}})$ itself. Since the closure is analytic, any of its points belongs to the image of a holomorphic mapping $g: \Delta \rightarrow T(F_0\check{\mathcal{M}})$, such that $g(\Delta^*)$ is contained in $\varepsilon(T_{\mathbb{Z}}) \cap p^{-1}(X)$. Lemma 2.26 shows that $g(\Delta) \subseteq \varepsilon(T_{\mathbb{Z}})$, and this proves that $\varepsilon(T_{\mathbb{Z}})$ is itself closed. \square

3 Local analysis of the construction

3.1 Introduction

Our construction of the Néron model is based on the fact that the quotient $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$ is an analytic space. As explained in Sect. 2.6, this is the case, provided that Condition 2.4 is satisfied; in Sect. 2.11, we had further reduced the problem to the case when $\bar{X} - X$ is a divisor with normal crossings, and the local monodromy of the variation of Hodge structure \mathcal{H} is unipotent. In this part of the paper, we complete the proof by showing that Condition 2.4 is true when $\bar{X} = \Delta^n$ is a polydisk with coordinates s_1, \dots, s_n , $X = (\Delta^*)^n$ is the complement of the divisor defined by $s_1 \cdots s_n = 0$, and \mathcal{H} is a polarizable variation of Hodge structure of weight -1 on X with unipotent monodromy.

Note Both here and in Part 5, the theory of mixed Hodge modules is not used. This may be reassuring for the reader who is only interested in the case of a normal crossing divisor or in the proof of Conjecture 1.1 (which only needs to be proved in that case).

Before going into details, a brief summary of the argument may be in place. Under the above assumptions on \bar{X} and \mathcal{H} , the filtered \mathcal{D} -module $(\check{\mathcal{M}}, F)$

underlying \check{M} can be described explicitly in terms of Deligne’s canonical extension $\check{\mathcal{H}}^e_\varrho$. In particular, we have

$$F_0\check{\mathcal{M}} = \sum_{k \geq 0} F_k\mathcal{D} \cdot F_{-k}\check{\mathcal{H}}^e_\varrho,$$

and so $F_0\check{\mathcal{M}}$ consists of all sections in $F_0\check{\mathcal{H}}^e_\varrho = F^0\check{\mathcal{H}}^e_\varrho$, all first-order derivatives of sections in $F_{-1}\check{\mathcal{H}}^e_\varrho = F^1\check{\mathcal{H}}^e_\varrho$, and so on. This means that we have a natural holomorphic mapping

$$T(F_0\check{\mathcal{M}}) \rightarrow T(F_0\check{\mathcal{H}}^e_\varrho).$$

It is known that the image of $T_{\mathbb{Z}}$ in the vector bundle $T(F_0\check{\mathcal{H}}^e_\varrho)$ is not well-behaved (the quotient is the so-called “Zucker extension”, which is not generally a Hausdorff space). But we shall see that $T(F_0\check{\mathcal{M}})$, which only maps to a very restricted subset of $T(F_0\check{\mathcal{H}}^e_\varrho)$, solves this problem in a very natural way.

If we pull \mathcal{H} back to the universal covering space \mathbb{H}^n , it can be viewed as family of Hodge structures $\check{\Phi}(z)$ on a fixed vector space $H_{\mathbb{C}}$. Let $\sigma_1, \dots, \sigma_r$ be a collection of sections that generate $F_0\check{\mathcal{M}}$ over Δ^n , and let Q denote the natural pairing between $H_{\mathbb{C}}$ and sections of $\check{\mathcal{M}}$. At each point $z \in \mathbb{H}^n$, we define

$$B(z, h) = \sup_{j=1, \dots, r} |Q(h, \sigma_j(z))|,$$

noting that it gives a norm on $H_{\mathbb{R}}$ because the Hodge structures in question have weight -1 . On the other hand, we may fix a norm $\|-\|$ on $H_{\mathbb{R}}$; since $H_{\mathbb{R}}$ is finite-dimensional, the two norms can then be compared by an inequality of the form

$$\|h\| \leq C(z) \cdot B(z, h),$$

where $C(z)$ is a constant that depends only on z . The main idea is to show that, even as the imaginary parts of z_1, \dots, z_n tend to infinity, $C(z)$ remains bounded. This will allow us to control the limit of any sequence of points in $T_{\mathbb{Z}}$, and in the end, to show that $T_{\mathbb{Z}}$ is closed in $T(F_0\check{\mathcal{M}})$.

Note It is illustrative to compare this with the situation for the canonical extension. Of course, we could similarly define a quantity $B_0(z, h)$, using only sections of $F_0\check{\mathcal{H}}^e_\varrho$, and have a second inequality $B_0(z, h) \leq C_0(z) \cdot \|h\|$. It is then not hard to see that $B_0(z, h)$ is equivalent to the Hodge norm in the Hodge structure $\check{\Phi}(z)$. The norm estimates of [10] and [21] show that $C_0(z)$ need not be bounded: it will typically grow like a certain polynomial in the imaginary parts of z_1, \dots, z_n .

This leads to an interesting analogy, which was pointed out to me by H. Clemens. Namely, in the definition of $B(z, h)$, we are controlling not just the various holomorphic functions

$$f_\sigma(z) = Q(h, \sigma(z)),$$

for $\sigma \in F_0\check{\mathcal{H}}_\emptyset^e$, but also some of their derivatives. Indeed, the additional sections in $F_0\check{\mathcal{M}}$ arise precisely as derivatives of sections of $F_0\check{\mathcal{H}}_\emptyset^e$, and since h is flat, we have

$$\frac{\partial}{\partial z_j} f_\sigma(z) = Q(h, \nabla_{\partial_j} \sigma(z))$$

and so on. If we now think of the Hodge norm as an L^2 -norm, and of $B(z, h)$ as a kind of Sobolev norm, then the fact that $\|h\|$ is bounded by a fixed multiple of $B(z, h)$ resembles the classical Sobolev inequality. It would be interesting to know whether this is more than a mere analogy.

3.2 The normal form of a period map

In this section, we set up some basic notation, and describe how to represent the period map $\tilde{\Phi}(z)$ in terms of the limit mixed Hodge structure coming from the SL_2 -Orbit Theorem. All the results cited here can be found in [10, Sect. 4].

We consider a variation of polarizable Hodge structure of weight -1 on $(\Delta^*)^n$. Let $s = (s_1, \dots, s_n)$ be the standard holomorphic coordinates on Δ^n . Throughout, we shall make the assumption that the monodromy of the variation around each divisor $s_j = 0$ is unipotent. As usual, let $\mathbb{H}^n \rightarrow (\Delta^*)^n$ be the universal covering space, with $s_j = e^{2\pi i z_j}$. Let N_j be the logarithm of the monodromy transformation around the divisor $s_j = 0$.

The pullback of the variation to \mathbb{H}^n can be viewed as a varying Hodge filtration $\tilde{\Phi}: \mathbb{H}^n \rightarrow D$ on a fixed vector space $H_{\mathbb{C}}$, where D is a suitable period domain. Since the variation is integral, there is a fixed underlying integral lattice $H_{\mathbb{Z}} \subseteq H_{\mathbb{C}}$. Furthermore, we may choose once and for all a polarization, that is, a nondegenerate alternating bilinear form $Q: H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \rightarrow \mathbb{Z}$. As usual, we denote by $G_{\mathbb{R}} = \text{Aut}(H_{\mathbb{R}}, Q)$ the real Lie group determined by the pairing, and by $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. By the Nilpotent Orbit Theorem [37, Theorem 4.12], we have

$$e^{-\sum z_j N_j} \tilde{\Phi}(z) = \Psi(s),$$

with $\Psi: \Delta^n \rightarrow \check{D}$ holomorphic. Let $W^{(n)} = W(N_1, \dots, W_n)$ be the monodromy weight filtration for the cone $C^{(n)} = C(N_1, \dots, N_n)$, and set $W_{\bullet} = W_{\bullet+1}^{(n)}$. Then $(W, \Psi(0))$ is a mixed Hodge structure, polarized by Q and any

element of $C^{(n)}$, in the sense of [10, Definition 2.26]. Let $\delta \in L_{\mathbb{R}}^{-1,-1}(W, \Psi(0))$ be the unique real element for which $(W, e^{-i\delta}\Psi(0))$ is \mathbb{R} -split [10, Proposition 2.20], and define $F = e^{-i\delta}\Psi(0) \in \check{D}$. Note that δ commutes with every N_j . Let

$$I^{p,q} = I^{p,q}(W, F) = W_{p+q} \cap F^p \cap \overline{F^q}$$

be Deligne’s canonical decomposition of the \mathbb{R} -split mixed Hodge structure (W, F) .

The Lie algebra \mathfrak{g} inherits a decomposition

$$\mathfrak{g} = \bigoplus_{p,q} \mathfrak{g}^{p,q},$$

with $\mathfrak{g}^{p,q}$ consisting of those X that satisfy $XI^{a,b} \subseteq I^{a+p,b+q}$. Then we have

$$\mathfrak{g} = \mathfrak{g}^F \oplus \mathfrak{q} = \mathfrak{g}^F \oplus \bigoplus_{p < 0} \mathfrak{g}^{p,q},$$

and \mathfrak{q} is a nilpotent Lie subalgebra of \mathfrak{g} . This decomposition makes it possible to write $e^{-i\delta}\Psi(s) = e^{\Gamma(s)}F$ for a unique holomorphic map $\Gamma: \Delta^n \rightarrow \mathfrak{q}$ with $\Gamma(0) = 0$. We can therefore put the period map into the standard form

$$\tilde{\Phi}(z) = e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} F = e^{X(z)} F, \tag{3.1}$$

with $X(z) \in \mathfrak{q}$ and hence nilpotent.

The horizontality of the period map implies the following relationship between $\Gamma(s)$ and the nilpotent operators N_j . A proof can be found in [9, Proposition 2.6]; we include it here to make the discussion more concrete.

Lemma 3.1 *Let $\tilde{\Phi}(z) = e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} F$ be the normal form of a period map.*

(1) *We have*

$$d(e^{\sum z_j N_j} e^{\Gamma(s)}) = e^{\sum z_j N_j} e^{\Gamma(s)} \left(d\Gamma_{-1}(s) + \sum_{j=1}^n N_j dz_j \right).$$

(2) *For every $j = 1, \dots, n$, the commutator*

$$[N_j, e^{\Gamma(s)}] = N_j e^{\Gamma(s)} - e^{\Gamma(s)} N_j$$

vanishes along $s_j = 0$.

Proof Since $\Gamma(s)$, all the N_j , and δ belong to the nilpotent Lie algebra \mathfrak{q} , we can write

$$e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} = e^{X(z)}$$

for a unique holomorphic $X: \mathbb{H}^n \rightarrow \mathfrak{q}$. From the definition of \mathfrak{q} , we have $X(z) = X_{-1}(z) + X_{-2}(z) + \dots$, with $X_p(z) \in \bigoplus_{\mathfrak{q}} \mathfrak{g}^{p,q}$. Note that

$$X_{-1}(z) = \Gamma_{-1}(s) + \sum_{j=1}^n z_j N_j + i\delta_{-1}.$$

Horizontality of the period map $e^{X(z)} F$ is equivalent to the condition that

$$e^{-X(z)} \cdot d(e^{X(z)}) = dX_{-1}(z),$$

which gives the first assertion (because δ is constant). Writing the condition out explicitly, we get

$$e^{-\Gamma(s)} \sum_{j=1}^n N_j dz_j \cdot e^{\Gamma(s)} + e^{-\Gamma(s)} \cdot d(e^{\Gamma(s)}) = d\Gamma_{-1}(s) + \sum_{j=1}^n N_j dz_j.$$

Now $ds_j = 2\pi i s_j \cdot dz_j$; thus if we evaluate the identity on the tangent vector field $\partial/\partial z_j$, we get

$$N_j e^{\Gamma(s)} - e^{\Gamma(s)} N_j = 2\pi i s_j \cdot \left(e^{\Gamma(s)} \frac{\partial}{\partial s_j} \Gamma_{-1}(s) - \frac{\partial}{\partial s_j} e^{\Gamma(s)} \right).$$

We then obtain the second assertion by setting $s_j = 0$. □

The fact that the commutator $[N_j, e^{\Gamma(s)}]$ vanishes along the divisor $s_j = 0$ has the following highly useful consequence.

Lemma 3.2 *Let $y_j = \text{Im } z_j$, and suppose that $y_1 \geq \dots \geq y_n \geq 1$ and $0 \leq \text{Re } z_j \leq 1$. Define the nilpotent operator $N = y_1 N_1 + \dots + y_n N_n$. Then there is a constant $C > 0$ and an integer m , both independent of z , such that*

$$\|(\text{ad } N)^k e^{\Gamma(s)}\| \leq C \cdot \sum_{j=1}^n y_j^m e^{-2\pi y_j}$$

for any $k \geq 1$.

Proof Since $\Gamma(s)$ is holomorphic in $s = (s_1, \dots, s_n)$, and $\Gamma(0) = 0$, we can write

$$\begin{aligned}
 e^{\Gamma(s)} - \text{id} &= (e^{\Gamma(s_1, \dots, s_n)} - e^{\Gamma(0, s_2, \dots, s_n)}) + \dots + (e^{\Gamma(0, \dots, 0, s_n)} - \text{id}) \\
 &= s_1 B_1(s) + \dots + s_n B_n(s),
 \end{aligned}$$

where each $B_j(s)$ is an operator that depends holomorphically on s . Moreover, $B_j(s)$ commutes with N_1, \dots, N_{j-1} , and $\|B_j(s)\|$ is uniformly bounded, independent of s . We then compute that (for $k \geq 1$)

$$(\text{ad } N)^k e^{\Gamma(s)} = (\text{ad } N)^k \sum_{j=1}^n s_j B_j(s) = \sum_{j=1}^n s_j (\text{ad}(y_j N_j + \dots + y_n N_n))^k B_j(s).$$

Now each N_j is nilpotent, $y_j \geq \dots \geq y_n$, and $|s_j| = e^{-2\pi y_j}$, and so the assertion follows by taking norms. □

3.3 The minimal extension

Now let M be the intermediate extension of the variation of Hodge structure \mathcal{H} to a Hodge module on Δ^n . In this section, we review Saito’s description of the underlying filtered left \mathcal{D} -module $(\check{\mathcal{M}}, F)$. Let \mathcal{H}_θ be the holomorphic vector bundle on $(\Delta^*)^n$ underlying the variation \mathcal{H} , and let ∇ be the flat connection on \mathcal{H}_θ . Since the local monodromies are unipotent, \mathcal{H}_θ can be canonically extended to a vector bundle \mathcal{H}_θ^e on Δ^n , such that the connection has logarithmic poles along $s_1 \cdots s_n = 0$ with nilpotent residues [13, Proposition 5.2]. More explicitly, for each $v \in H_{\mathbb{C}}$, the map

$$\mathbb{H}^n \rightarrow H_{\mathbb{C}}, \quad z = (z_1, \dots, z_n) \mapsto e^{\sum z_j N_j} v$$

descends to a holomorphic section of \mathcal{H}_θ on $(\Delta^*)^n$, and \mathcal{H}_θ^e is the locally free subsheaf of $j_* \mathcal{H}_\theta$ generated by all such sections. Using the standard form of the period map in (3.1), the maps

$$\mathbb{H}^n \rightarrow H_{\mathbb{C}}, \quad z \mapsto e^{X(z)} v = e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} v \tag{3.2}$$

also induce a collection of sections that generate \mathcal{H}_θ^e . The Nilpotent Orbit Theorem implies that the Hodge bundles $F^p \mathcal{H}_\theta$ extend uniquely to holomorphic subbundles $F^p \mathcal{H}_\theta^e$ of the canonical extension. Each $F^p \mathcal{H}_\theta^e$ is generated by all those sections in (3.2) for which $v \in F^p H_{\mathbb{C}}$.

Now \mathcal{M} , the minimal extension of $(\mathcal{H}_\theta, \nabla)$ to a holonomic \mathcal{D} -module on Δ^n , is simply the \mathcal{D} -submodule of $j_* \mathcal{H}_\theta$ generated by \mathcal{H}_θ^e . Moreover, the Hodge filtration on \mathcal{M} is given by

$$F_p \mathcal{M} = \sum_{k \geq 0} F_k \mathcal{D} \cdot F^{k-p} \mathcal{H}_\theta^e.$$

It satisfies $F_k \mathcal{D} \cdot F_p \mathcal{M} \subseteq F_{k+p} \mathcal{M}$, and each $F_p \mathcal{M}$ is a coherent sheaf on Δ^n whose restriction to $(\Delta^*)^n$ is $F^{-p} \mathcal{H}_{\mathcal{O}}$. For the purposes of our construction, the important point is that $F_p \mathcal{M}$ has more sections than $F^{-p} \mathcal{H}_{\mathcal{O}}^e$; the following lemma exhibits the ones that we will use.

Lemma 3.3 *For any subset $I \subseteq \{1, \dots, n\}$ of cardinality m , and for any vector $v \in F^{-p}$, the formula*

$$\sigma_{I,v}(z) = e^{X(z)} \prod_{j \in I} \frac{N_j}{s_j} \cdot v$$

defines a holomorphic section of the coherent sheaf $F_{p+m} \mathcal{M}$ on Δ^n .

Proof We work by induction on the cardinality m of the set I . Then case $m = 0$ is clear from the definition of $F_p \mathcal{M}$. We may therefore suppose that the assertion has been proved for all subsets of cardinality at most m , and consider $I \subseteq \{1, \dots, n\}$ with $|I| = m + 1$. Let $k = \max I$ and $J = I - \{k\}$. Then

$$\sigma_{J,w}(z) = e^{X(z)} \prod_{j \in J} \frac{N_j}{s_j} \cdot w$$

is a section of $F_{p+m} \mathcal{M}$ for every $w \in F^{-p}$.

Using that $k \notin J$, the first identity in Lemma 3.1 shows that

$$\begin{aligned} \frac{\partial}{\partial s_k} \sigma_{J,v}(z) &= e^{X(z)} \left(\frac{N_k}{2\pi i s_k} + \frac{\partial \Gamma_{-1}}{\partial s_k} \right) \cdot \prod_{j \in J} \frac{N_j}{s_j} v \\ &= \frac{\sigma_{I,v}(z)}{2\pi i} + e^{X(z)} \frac{\partial \Gamma_{-1}(s)}{\partial s_k} \prod_{j \in J} \frac{N_j}{s_j} \cdot v. \end{aligned}$$

The second half of Lemma 3.1, applied recursively, implies that

$$C(s) = \left[\Gamma_{-1}(s), \prod_{j \in J} \frac{N_j}{s_j} \right] \quad \text{and} \quad \frac{\partial C(s)}{\partial s_k} = \left[\frac{\partial \Gamma_{-1}(s)}{\partial s_k}, \prod_{j \in J} \frac{N_j}{s_j} \right]$$

are holomorphic on Δ^n ; as operators, they map F^{-p} to F^{-p-m-1} . In

$$\frac{\partial}{\partial s_k} \sigma_{J,v}(z) = \frac{\sigma_{I,v}(z)}{2\pi i} + e^{X(z)} \left(\frac{\partial C(s)}{\partial s_k} v + \prod_{j \in J} \frac{N_j}{s_j} \frac{\partial \Gamma_{-1}(s)}{\partial s_k} v \right),$$

the left-hand side defines a holomorphic section of $F_{p+m+1} \mathcal{M}$; by induction, the same is true for the second term on the right-hand side. We conclude that $\sigma_{I,v}(z)$ is itself a section of $F_{p+m+1} \mathcal{M}$, thus completing the induction. \square

3.4 The main estimate

Now let $\check{\mathcal{H}} = \mathcal{H}^\vee(1)$ be the (normalized) dual variation of Hodge structure. Let $\check{\mathcal{M}}$ be its intermediate extension to a Hodge module on Δ^n , and let $(\check{\mathcal{M}}, F)$ be the underlying filtered \mathcal{D} -module. Our choice of polarization determines a morphism

$$\mathcal{H} \rightarrow \check{\mathcal{H}}, \quad h \mapsto 2\pi i \cdot Q(h, -),$$

which is an isomorphism of rational variations of Hodge structure. It gives rise to an isomorphism $(\mathcal{M}, F) \simeq (\check{\mathcal{M}}, F)$ of filtered \mathcal{D} -modules. In particular, we have $F_0\check{\mathcal{M}} \simeq F_0\mathcal{M}$, and under this isomorphism, the morphism $j_*\mathcal{H}_{\mathbb{C}} \rightarrow (F_0\check{\mathcal{M}})^\vee$ constructed in Lemma 2.8 becomes

$$j_*\mathcal{H}_{\mathbb{C}} \rightarrow (F_0\mathcal{M})^\vee, \quad h \mapsto 2\pi i \cdot Q(h, -).$$

We use the isomorphism $F_0\check{\mathcal{M}} \simeq F_0\mathcal{M}$ mostly because it simplifies the notation.

Now fix a norm $\|-\|$ on the vector space $H_{\mathbb{C}}$. Let $\sigma_1, \dots, \sigma_m$ be a collection of sections that generate the coherent sheaf $F_0\mathcal{M}$ over Δ^n . To prove that $\varepsilon(T_{\mathbb{Z}})$ is closed inside of $T(F_0\mathcal{M})$, our strategy is to show that the norm of any vector $h \in H_{\mathbb{R}}$ is bounded uniformly by the values of $Q(h, \sigma_j(z))$, provided that the imaginary parts of z_1, \dots, z_n are sufficiently large. As a matter of fact, we will prove a stronger statement that involves only the special sections $\sigma_{I,v}$ from Lemma 3.3. Given a real vector $h \in H_{\mathbb{R}}$, and a point $z \in \mathbb{H}^n$, we thus introduce the quantity

$$B(z, h) = \sup \left\{ |Q(h, \sigma_{I,v}(z))| \mid I \subseteq \{1, \dots, n\} \text{ and } v \in F^{|I|} \text{ with } \|v\| \leq 1 \right\},$$

which gives a norm on $H_{\mathbb{R}}$ for every $z \in \mathbb{H}^n$. Since we are trying to control the size of h in terms of $B(z, h)$, we also let $N = y_1N_1 + \dots + y_nN_n$, and define

$$Z(y, h) = \max_{k \geq 0} \|N^k h\|,$$

which is finite because N is nilpotent. Note that we have $\|h\| \leq Z(y, h)$ by definition. After a few preliminary results on decompositions in \mathbb{R} -split mixed Hodge structures in Sect. 3.6, the following key estimate will be proved in Sect. 3.7.

Theorem 3.4 *Let $\check{\Phi}(z) = e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} F$ be the normal form of a variation of polarizable Hodge structure of weight -1 on $(\Delta^*)^n$. Fix a norm $\|-\|$ on*

the underlying vector space $H_{\mathbb{C}}$, and a polarization Q of $H_{\mathbb{Z}}$. Then there are constants $C > 0$ and $\alpha > 0$, such that we have

$$\|h\| \leq Z(y, h) \leq C \cdot B(z, h)$$

for every $h \in H_{\mathbb{R}}$ and every $z \in \mathbb{H}^n$ with $y_j = \text{Im } z_j \geq \alpha$ and $0 \leq \text{Re } z_j \leq 1$.

Note Since the theory of [10] already applies to polarized real variations, it would suffice to assume that \mathcal{H} is a variation of real Hodge structure, and that Q is a polarization of $H_{\mathbb{R}}$. Except for Sect. 3.5, all of our proofs work in this generality.

3.5 The closure of the set of integral points

Granting Theorem 3.4 for the time being, we shall now show that Condition 2.4 is true: the map $\varepsilon : T_{\mathbb{Z}} \rightarrow T(F_0\mathcal{M})$ is injective and has closed image. (Recall that $T(F_0\check{\mathcal{M}}) \simeq T(F_0\mathcal{M})$, using the polarization.) The first result is that any sequence of points in $T_{\mathbb{Z}}$ over $(\Delta^*)^n$ that converges in $T(F_0\mathcal{M})$ has to be eventually constant and invariant under monodromy. This is the only point in the proof where we use the integral structure.

Theorem 3.5 *Let $z(m) \in \mathbb{H}^n$ be a sequence of points with $\text{Im } z_j(m) \rightarrow \infty$ and $\text{Re } z_j(m) \in [0, 1]$ for $j = 1, \dots, n$. Let $h(m) \in H_{\mathbb{Z}}$ be a corresponding sequence of integral classes, such that*

$$Q(h(m), \sigma_{I,v}(z(m)))$$

is convergent for every $I \subseteq \{1, \dots, n\}$ and every $v \in F^{|I|}$. Then the sequence $h(m)$ is eventually constant, and its constant value satisfies $N_k h(m) = 0$ for $k = 1, \dots, n$.

Proof The first step is to show that $N_k h(m) = 0$ for all $k = 1, \dots, n$, and all sufficiently large m . We begin by finding a subsequence of $h(m)$ along which this is true. By assumption, the quantity $B(z(m), h(m))$ is bounded, and so the inequality in Theorem 3.4 implies that $\|h(m)\|$ is bounded. Since $h(m) \in H_{\mathbb{Z}}$, the sequence can take on only finitely many distinct values; let $h \in H_{\mathbb{Z}}$ be one of them. The inequality also implies that $\sum z_j(m) N_j h(m)$ is bounded; according to Lemma 3.6 below, $h \in W_0^{(m)} = W_{-1}$, and we can find a subsequence along which

$$\sum_{j=1}^n z_j(m) N_j h(m) \rightarrow \sum_{j=1}^n w_j N_j h,$$

for some $w \in \mathbb{H}^n$ with $\text{Im } w$ large. We then have $e^{-\sum z_j(m)N_j}h(m) \rightarrow e^{-\sum w_j N_j}h$; by taking $I = \{k\}$ and $v \in F^1$ arbitrary, it follows that

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} s_k(m) \cdot Q(h(m), \sigma_{\{k\}, v}(z(m))) = Q(h, e^{i\delta} e^{\sum w_j N_j} N_k v) \\ &= -Q(N_k h, e^{i\delta} e^{\sum w_j N_j} v). \end{aligned} \tag{3.3}$$

Now $(W, e^{i\delta} e^{\sum w_j N_j} F)$ is a mixed Hodge structure; because the vector $N_k h$ is rational and belongs to W_{-3} , we easily conclude from (3.3) that $N_k h = 0$.

The argument above actually proves that $N_k h(m) = 0$ for all sufficiently large m —otherwise, we could find a subsequence along which $N_k h(m) \neq 0$, leading to a contradiction. Consequently, $h(m) \in W_{-1}$ for $m \gg 0$; moreover, we now have $e^{-\sum z_j(m)N_j}h(m) = h(m)$, from which it follows that

$$\lim_{m \rightarrow \infty} Q(h(m), \sigma_{\emptyset, v}(z(m))) = \lim_{m \rightarrow \infty} Q(h(m), e^{i\delta} v) \tag{3.4}$$

for every $v \in F^0$. Now consider the mixed Hodge structure $(W, e^{i\delta} F)$ on $H_{\mathbb{C}}$. Since $e^{i\delta} F^0 \cap W_{-1} \cap H_{\mathbb{R}} = \{0\}$, we deduce from the existence of the limit in (3.4) that the sequence of vectors $h(m) \in H_{\mathbb{Z}}$ must be a Cauchy sequence, and hence that it must be eventually constant. \square

Lemma 3.6 *Let $h \in H_{\mathbb{R}}$, and suppose that $z_1(m)N_1 h + \dots + z_n(m)N_n h$ remains bounded for $m \rightarrow \infty$. Then $h \in W_0^{(n)}$. Moreover, for any $\alpha > 0$, there is a point $w \in \mathbb{C}^n$ with $\max_{1 \leq j \leq n} \text{Im } w_j \geq \alpha$, such that*

$$\sum_{j=1}^n z_j(m)N_j h \rightarrow \sum_{j=1}^n w_j N_j h$$

is true along a subsequence.

Proof We borrow a technique introduced by E. Cattani, P. Deligne, and A. Kaplan [8, p. 494]. Let $x_j(m) = \text{Re } z_j(m)$, and $y_j(m) = \text{Im } z_j(m)$. After passing to a subsequence, we can find constant vectors $\theta^1, \dots, \theta^r \in \mathbb{R}^n$, whose components satisfy the inequalities $0 \leq \theta_j^1 \leq \theta_j^2 \leq \dots \leq \theta_j^r$, such that

$$y_j(m) = t_1(m)\theta^1 + \dots + t_r(m)\theta^r + \eta(m),$$

where the ratios $t_1(m)/t_2(m), \dots, t_{r-1}(m)/t_r(m)$, and $t_r(m)$ are tending to infinity, and the remainder term $\eta(m)$ is convergent. We can take every $\text{Im } \eta_j(m) \geq \alpha$; moreover, we may clearly assume that the bounded sequence $x(m)$ is also convergent. Let $w(m) = x(m) + i\eta(m)$. Along the subsequence

in question, we then have

$$\sum_{j=1}^n z_j(m) N_j h = \sum_{j=1}^n w_j(m) N_j h + i \sum_{k=1}^r t_k(m) \sum_{j=1}^n \theta_j^k N_j h.$$

This expression can only be bounded if $\sum \theta_j^k N_j h = 0$ for every k ; it follows that $h \in W_0^{(n)}$, because $\sum \theta_j^r N_j \in C^{(n)}$. We now obtain the second assertion with $w = \lim_{m \rightarrow \infty} w(m)$. □

Corollary 3.7 *The map $\varepsilon : T_{\mathbb{Z}} \rightarrow T(F_0\mathcal{M})$ is injective, and $\varepsilon(T_{\mathbb{Z}})$ is a closed analytic subset; therefore Condition 2.4 is true for polarizable variations of Hodge structure of weight -1 on $(\Delta^*)^n$ with unipotent monodromy.*

Proof The map ε is injective because the induced map $T_{\mathbb{Z}} \rightarrow T(F^0\mathcal{H}_{\emptyset}^c)$ is injective. Its image is a closed analytic subset because of Theorem 3.5. □

3.6 Decompositions in \mathbb{R} -split mixed Hodge structures

The limit mixed Hodge structure of a variation of Hodge structure determines two natural decompositions of the underlying vector space $H_{\mathbb{C}}$. The first is Deligne’s decomposition $H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ of the limit mixed Hodge structure. The second is the Lefschetz decomposition of the associated representation of $\mathfrak{sl}_2(\mathbb{C})$. In this section, we relate both decompositions to the Hodge decomposition in the associated nilpotent orbit, under the assumption that the limit mixed Hodge structure is split over \mathbb{R} . This is a preparation for the proof of Theorem 3.4 in Sect. 3.7 below.

Throughout, we let (W, F) be an \mathbb{R} -split mixed Hodge structure, polarized by a nondegenerate bilinear form Q and a nilpotent operator N , such that the weight filtration of the mixed Hodge structure satisfies $W_{\bullet} = W(N)_{\bullet-m}$. Let

$$H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

be Deligne’s decomposition; since the mixed Hodge structure is split over \mathbb{R} , we have $I^{p,q} = W_{p+q} \cap F^p \cap \overline{F^q}$. The operator Y , which acts as multiplication by $p + q - m$ on $I^{p,q}$, is then a real splitting of the filtration $W(N)$; let N^+ be the real operator making (N, Y, N^+) into an $\mathfrak{sl}_2(\mathbb{C})$ -triple.

There are two natural decompositions of the vector space $H_{\mathbb{C}}$, and one purpose of this section is to relate the two. The first one is Deligne’s decomposition by the $I^{p,q}$, the second one the primitive decomposition determined by the nilpotent operator N^+ . The reason for using N^+ instead of N will become apparent below. We define the primitive subspaces for the operator N^+

as

$$I_0^{p,q} = I^{p,q} \cap \ker N.$$

Given a vector $h \in H_C$, we denote by $h^{p,q}$ its component in the space $I^{p,q}$, and then $h = \sum_{p,q} h^{p,q}$. We can also write h uniquely in the form

$$h = \sum_{p,q} \sum_{b=0}^{m-p-q} (N^+)^b h^{p,q}(b)$$

where each vector $h^{p,q}(b) \in I_0^{p,q}$ is primitive for N^+ , meaning that $Nh^{p,q}(b) = 0$.

Lemma 3.8 *There are constants $C(p, q, b, j) \in \mathbb{Q}$, depending only on the Hodge numbers of the \mathbb{R} -split mixed Hodge structure (W, F) , such that*

$$h^{p,q}(b) = \sum_{j \geq 0} C(p, q, b, j) (N^+)^j N^{b+j} h^{p+b, q+b}.$$

Proof Since N^+ is a morphism of type $(1, 1)$, a short computation shows that

$$N^a h^{p+a, q+a} = \sum_{j \geq 0} R(a, a + j, m - p - q + 2j) (N^+)^j h^{p-j, q-j}(a + j),$$

where the constants are as in Lemma 3.9 below. Since $R(a, a, m - p - q) \neq 0$ for $0 \leq a \leq m - p - q$, we can solve those equations for the $h^{p,q}(b)$ by descending induction on b to arrive at the stated formulas. \square

Lemma 3.9 *Let $v \neq 0$ be a vector satisfying $Nv = 0$ and $Yv = -\ell v$ (and therefore $\ell \geq 0$). Then $N^a (N^+)^b v = R(a, b, \ell) (N^+)^{b-a} v$, with*

$$R(a, b, \ell) = \frac{b!(\ell + a - b)!}{(\ell - b)!(b - a)!}$$

for $0 \leq a \leq b \leq \ell$, and $R(a, b, \ell) = 0$ in all other cases.

Proof This is well-known; but since the proof is short, we include it here. We have

$$\begin{aligned} N^{a+1} (N^+)^b v &= N \cdot R(a, b, \ell) (N^+)^{b-a} v \\ &= R(a, b, \ell) R(1, b - a, \ell) (N^+)^{b-a-1} v, \end{aligned}$$

from which the identity $R(a + 1, b, \ell) = R(a, b, \ell) R(1, b - a, \ell)$ follows. We also have

$$\begin{aligned} N(N^+)^{b+1}v &= (N^+N - Y) \cdot (N^+)^b v \\ &= (R(1, b, \ell) - (2b - \ell))(N^+)^b v, \end{aligned}$$

from which one sees that $R(1, b + 1, \ell) = R(1, b, \ell) + (\ell - 2b)$. Together with the evident condition that $R(1, 0, \ell) = 0$, the two equations suffice to prove the formula for $R(a, b, \ell)$ by induction. \square

The formula in Lemma 3.8 shows how the size of the primitive components depends on the two operators N^+ and N . Since we will need this fact in Sect. 3.7, we state it as a corollary.

Corollary 3.10 *Fix a norm on the vector space $H_{\mathbb{C}}$, and define $Z(N, h) = \max_{k \geq 0} \|N^k h\|$. Then there is a constant $C > 0$ and an integer $d \in \mathbb{N}$, both depending only on the Hodge numbers of (W, F) , such that*

$$\max_{p,q,b} \|h^{p,q}(b)\| \leq C \|N^+\|^d \cdot Z(N, h)$$

for every $h \in H_{\mathbb{C}}$.

We now specialize to the case $m = -1$. Then $e^{iN} F$ is a point of the period domain D by [10, Lemma 3.12], and therefore defines a polarized Hodge structure of weight -1 on $H_{\mathbb{C}}$. In particular, we have the decomposition

$$H_{\mathbb{C}} = e^{iN} F^0 \oplus e^{-iN} \overline{F^0}. \tag{3.5}$$

Any vector $h \in H_{\mathbb{C}}$ can therefore be written uniquely as $h = e^{iN} v + e^{-iN} w$, with

$$v \in F^0 = \bigoplus_{p \geq 0} I^{p,q} \quad \text{and} \quad w \in \overline{F^0} = \bigoplus_{q \geq 0} I^{p,q}.$$

The *uniqueness* of the decomposition has a useful consequence that we shall now explain. Let $w = \sum (N^+)^b w^{p,q}(b)$ be the primitive decomposition of the vector $w \in \overline{F^0}$; note that

$$w^{p,q} = \sum_{b \geq 0} (N^+)^b w^{p-b, q-b}(b),$$

which implies that $w^{p,q}(b) = 0$ unless $q + b \geq 0$. Set $g = e^{-iN} h$, and similarly write $g = \sum (N^+)^b g^{p,q}(b)$. The decomposition in (3.5) becomes $g = v + e^{-2iN} w$, and since $v \in F^0$, the vector w is uniquely defined by the condition that

$$g^{p,q} = (e^{-2iN} w)^{p,q}$$

for every $p \leq -1$ and every q . The right-hand side can be expanded as

$$\begin{aligned} & (e^{-2iN} w)^{p,q} \\ &= \sum_{k,a \geq 0} \frac{(-2i)^k}{k!} N^k (N^+)^a w^{p-a+k, q-a+k}(a) \\ &= \sum_{k,b \geq 0} \frac{(-2i)^k}{k!} R(k, k + b, 2b - 1 - p - q) (N^+)^b w^{p-b, q-b}(k + b). \end{aligned}$$

By equating primitive components, we obtain the set of equations

$$g^{p,q}(b) = \sum_{k \geq 0} \frac{(-2i)^k}{k!} R(k, k + b, -1 - p - q) w^{p,q}(k + b) \tag{3.6}$$

for $p + b \leq -1$. We point out one more time that $w^{p,q}(b) = 0$ unless $q + b \geq 0$.

We now consider (3.6) as a system of linear equations for the vectors $w^{p,q}(b) \in I_0^{p,q}$ with $q + b \geq 0$. Since the decomposition $g = v + e^{-2iN} w$ is unique, the system must have a unique solution, which means that its coefficient matrix has to be invertible. It follows that there are constants $\Gamma(p, q, b, a) \in \mathbb{Q}(i)$ such that

$$w^{p,q}(b) = \sum_{a=0}^{-1-p} \Gamma(p, q, b, a) g^{p,q}(a);$$

the upper limit for the summation stems from the condition $p + a \leq -1$. Since the proof of Theorem 3.4 in Sect. 3.7 is entirely based on the solution to the system of equations in (3.6), we summarize the result in the following proposition.

Proposition 3.11 *Consider the system of equations (for $p + b \leq -1$)*

$$y^{p,q}(b) = \sum_{k \geq 0} \frac{(-2i)^k}{k!} R(k, k + b, -1 - p - q) \cdot x^{p,q}(k + b)$$

in the unknowns $\{x^{p,q}(b)\}_{q+b \geq 0}$. Given any collection of vectors $\{y^{p,q}(b)\}_{p+b \leq -1}$, the unique solution to the system is given by the formula

$$x^{p,q}(b) = \sum_{a=0}^{-1-p} \Gamma(p, q, b, a) \cdot y^{p,q}(a),$$

where $\Gamma(p, q, a, b) \in \mathbb{Q}(i)$ are certain constants that depend only on the Hodge numbers $\dim I^{p,q}$ of the \mathbb{R} -split mixed Hodge structure (W, F) .

3.7 Proof of the main estimate

After the preliminary work in the previous section, we now come to the proof of the estimate from Theorem 3.4. Given a point $y \in \mathbb{H}^n$, we set $N = y_1 N_1 + \dots + y_n N_n$; note that the weight filtration $W(N)$ is independent of y . Together with the bilinear form Q , the nilpotent operator N polarizes the \mathbb{R} -split mixed Hodge structure (W, F) , where $W_\bullet = W(N)_{\bullet+1}$. Let Y be the real splitting of $W(N)$ determined by Deligne’s decomposition $H_{\mathbb{C}} = \bigoplus I^{p,q}$, and let (N, Y, N^+) be the corresponding $\mathfrak{sl}_2(\mathbb{C})$ -triple. An important observation is that the operator N^+ is of order $1/y_n$; this is a simple consequence of the SL_2 -Orbit Theorem of [10].

Lemma 3.12 *There are constants $C > 0$ and $\alpha > 0$ such that $\|N^+\| \leq C/y_n$ holds for all $y_1 \geq \dots \geq y_n \geq \alpha$.*

Proof Since $y_n N^+ = (N/y_n)^+$, it follows from [10, Theorem 4.20] that the operator $y_n N^+$ has a power series expansion in non-positive powers of $y_1/y_2, \dots, y_{n-1}/y_n$, convergent in a region of the form $y_2/y_1 > \beta, \dots, y_n/y_{n-1} > \beta$ for some $\beta > 0$. The assertion follows from this via dependence on parameters. More precisely, we argue as follows.

Suppose to the contrary that $y_n N^+$ was not bounded. Since $y_n N^+ = (N/y_n)^+$ depends only on the ratios $y_1/y_2, \dots, y_{n-1}/y_n$, we can then find a sequence of points $y(m)$ with $y_1(m) \geq \dots \geq y_n(m)$ and $y_n(m) \rightarrow \infty$, along which $\|y_n N^+\|$ diverges. After passage to a subsequence, we can arrange that

$$y_1(m)N_1 + \dots + y_n(m)N_n = t_1(m)M_1(m) + \dots + t_r(m)M_r(m)$$

where $t_1(m)/t_2(m), \dots, t_{r-1}(m)/t_r(m)$, and $t_r(m) = y_n(m)$ are going to infinity, and each $M_j(m)$ is a linear combination of N_1, \dots, N_n with coefficients that lie in a bounded interval $[1, K]$. By [8, Remark 4.65], the data in the SL_2 -Orbit Theorem depend real analytically on these coefficients; we can therefore use the convergence of the series as above to conclude that

$$y_n N^+ = \left(\frac{t_1(m)}{t_r(m)} M_1(m) + \dots + \frac{t_{r-1}(m)}{t_r(m)} M_{r-1}(m) + M_r(m) \right)^+$$

remains bounded as $m \rightarrow \infty$. But this clearly contradicts our original assumption, and so the lemma is proved. □

Recall that $N = y_1N_1 + \dots + y_nN_n$, and that our goal is to bound the quantity

$$Z(y, h) = \max_{k \geq 0} \|N^k h\|.$$

Also recall that $e^{i\delta} e^{iN} e^{\Gamma(s)} F = e^{-\sum x_j N_j} \tilde{\Phi}(z)$ defines a Hodge structure of weight -1 on $H_{\mathbb{C}}$. We may therefore write any vector $h \in H_{\mathbb{C}}$ uniquely in the form

$$h = e^{i\delta} e^{iN} e^{\Gamma(s)} u + e^{-i\delta} e^{-iN} e^{\overline{\Gamma(s)}} v \tag{3.7}$$

with $u \in F^0$ and $v \in \overline{F^0}$. We can use the boundedness of $y_n N^+$, together with the analysis in Sect. 3.6, to prove the following important estimate.

Proposition 3.13 *There are two constants $\alpha \geq 1$ and $C > 0$, such that we have*

$$Z(y, v) \leq C \cdot Z(y, h) \tag{3.8}$$

for every $h \in H_{\mathbb{C}}$ and every $y_1 \geq \dots \geq y_n \geq \alpha$, where $v \in \overline{F^0}$ is defined by (3.7).

Proof We let $g = e^{-\Gamma(s)} e^{-iN} e^{-i\delta} h$, where $s_j = e^{2\pi i z_j}$, and observe that $Z(y, g)$ is bounded by a constant multiple of $Z(y, h)$ because of Lemma 3.1. Let

$$g = \sum (N^+)^b g^{p,q}(b)$$

be the primitive decomposition of g determined by N^+ , with $g^{p,q}(b) \in I_0^{p,q}$ in the notation of Sect. 3.6. According to Corollary 3.10, the quantity $\max_{p,q,b} \|g^{p,q}(b)\|$ is still bounded by a fixed multiple of $Z(y, h)$.

Similarly write the primitive decomposition of the vector v as

$$v = \sum (N^+)^b v^{p,q}(b),$$

keeping in mind that $v \in \overline{F^0}$ means that $v^{p,q}(b) = 0$ unless $q + b \geq 0$. We will prove the estimate in (3.8) by showing that $\max_{p,q,b} \|v^{p,q}(b)\|$ is bounded by a constant multiple of $\max_{p,q,b} \|g^{p,q}(b)\|$, and hence by $Z(y, h)$; this clearly suffices because $\|N^+\|$ is bounded due to Lemma 3.12.

The vector v in the decomposition is uniquely determined by the condition that

$$g - e^{-\Gamma(s)} e^{-2iN} e^{-2i\delta} e^{\overline{\Gamma(s)}} v \in F^0.$$

If we set $w = e^{2iN} e^{-\Gamma(s)} e^{-2iN} \cdot e^{-2i\delta} e^{\overline{\Gamma(s)}} v$, then we can use Deligne’s decomposition $H_{\mathbb{C}} = \bigoplus I^{p,q}$ to recast that condition into the form

$$g^{p,q} = (e^{-2iN} w)^{p,q} \quad \text{for any } p \leq -1 \text{ and any } q.$$

We will show that this system of equations is a perturbation (of order $1/y_n$) of a triangular system. The following convention greatly simplifies the book-keeping:

Notation For two vectors $h_1, h_2 \in H_{\mathbb{C}}$, we shall write $h_1 \equiv h_2$ to mean that

$$h_1 - h_2 = \sum_{p,q,b} P(p, q, b)v^{p,q}(b)$$

for linear operators $P(p, q, b)$ that are allowed to depend on z (but not on v), and have to satisfy $\max\|P(p, q, b)\| \leq B/y_n$ for a constant B that is independent of z . It is easy to see that if X is a linear operator such that $\|X\|$ is bounded independently of z , then $h_1 \equiv h_2$ implies $Xh_1 \equiv Xh_2$.

We begin our analysis by observing that the operator δ is nilpotent, since it belongs to $L_{\mathbb{R}}^{-1,-1}(W, F)$. Let $\Delta = e^{-2i\delta}$; then we have

$$e^{-2i\delta} = \text{id} + \sum_{p,q \geq 1} \Delta_{-p,-q},$$

where $\Delta_{-p,-q}$ maps $I^{a,b}$ into $I^{a-p,b-q}$.

Next, we look more carefully at the relationship between w and v . To begin with, the boundedness of $y_n N^+$, proved in Lemma 3.12, implies that

$$\begin{aligned} N^b v &= \sum_{p,q,a} N^b (N^+)^a v^{p,q}(a) \\ &= \sum_{p,q,a} R(b, a, -1 - p - q) (N^+)^{a-b} v^{p,q}(a) \\ &\equiv \sum_{p,q} R(b, b, -1 - p - q) v^{p,q}(b). \end{aligned}$$

According to the formula in Lemma 3.8,

$$w^{p,q}(b) = \sum_{j \geq 0} C(p, q, b, j) (N^+)^j N^{b+j} w^{p+b,q+b};$$

to connect this with the primitive decomposition for the vector v , we compute

$$N^{b+j} w = N^{b+j} e^{2iN} e^{-\Gamma(s)} e^{-2iN} e^{-2i\delta} e^{\overline{\Gamma(s)}} v \equiv e^{-2i\delta} N^{b+j} v,$$

using Lemma 3.2 to neglect the terms that arise when commuting N^{b+j} past the two operators $e^{-\Gamma(s)}$ and $e^{\overline{\Gamma}(s)}$. Consequently,

$$N^{b+j}w \equiv e^{-2i\delta} \sum_{p,q} R(b+j, b+j, -1-p-q)v^{p,q}(b+j).$$

Again using the boundedness of $y_n N^+$, this shows that we are allowed to write

$$w^{p,q}(b) \equiv C(p, q, b, 0)N^b w^{p+b, q+b} = C(p, q, b, 0)(N^b w)^{p,q}.$$

Combining the various pieces of information, and remembering that $C(p, q, b, 0) \cdot R(b, b, -1-p-q) = 1$, we find that there are constants $D(p, q, b, j, k) \in \mathbb{Q}$ with the property that

$$w^{p,q}(b) \equiv v^{p,q}(b) + \sum_{j,k \geq 1} D(p, q, b, j, k) \cdot \Delta_{-j, -k} v^{p+j, q+k}(b). \tag{3.9}$$

Since we have $g^{p,q} = (e^{-2iN}w)^{p,q}$ for $p \leq -1$, the primitive components of g and w are related by the equations in (3.6). Using the constants $\Gamma(p, q, b, a) \in \mathbb{Q}(i)$ introduced in Proposition 3.11, we define

$$G^{p,q}(b) = \sum_{a=0}^{-1-p} \Gamma(p, q, b, a)g^{p,q}(a).$$

It follows that we can express each $w^{p,q}(b)$ with $q+b \geq 0$ as a linear combination of $G^{p,q}(b)$ and the vectors $\{w^{p,q}(a)\}_{q+a < 0}$. For $q+b \geq 0$, we therefore have

$$w^{p,q}(b) = G^{p,q}(b) + \sum_{a < -q} E(p, q, b, a)w^{p,q}(a)$$

with certain constants $E(p, q, b, a) \in \mathbb{Q}(i)$ that again depend on nothing but the Hodge numbers of (W, F) . Now we observe that for $q+a < 0$, the relation in (3.9) simplifies to

$$w^{p,q}(a) \equiv \sum_{j,k \geq 1} D(p, q, a, j, k) \cdot \Delta_{-j, -k} v^{p+j, q+k}(a),$$

due to the fact that $v^{p,q}(a) = 0$. When we combine the two formulas for $w^{p,q}(b)$ from above, we obtain for $q+b \geq 0$ an equation of the form

$$v^{p,q}(b) \equiv G^{p,q}(b) + \sum_{j,k \geq 1} \sum_{a \geq -q-k} D(p, q, a, j, k) \Delta_{-j, -k} v^{p+j, q+k}(a).$$

Recalling the definition of the symbol \equiv , this means that there are linear operators $P_{j,k}(b, c)$, mapping $I^{p,q}$ to $I^{p+j,q+k}$, and of size $\|P_{j,k}(b, c)\| \leq B/y_n$ for a suitable constant $B > 0$, such that

$$G^{p,q}(b) = v^{p,q}(b) - \sum_{j,k \geq 1} \sum_{a \geq -q-k} D(p, q, a, j, k) \Delta_{-j, -k} v^{p+j, q+k}(a) + \sum_{j,k} \sum_{c \geq -q+k} P_{j,k}(b, c) v^{p-j, q-k}(c). \tag{3.10}$$

Once again, we view this as a system of linear equations relating the primitive components $\{v^{p,q}(b)\}_{q+b \geq 0}$ to the vectors $\{G^{p,q}(b)\}_{q+b \geq 0}$.

Here comes the crucial point: Consider the system of equations (for $q + b \geq 0$)

$$G^{p,q}(b) = v^{p,q}(b) - \sum_{j,k \geq 1} \sum_{a \geq -q-k} D(p, q, a, j, k) \Delta_{-j, -k} v^{p+j, q+k}(a)$$

in the vectors $\{v^{p,q}(b)\}_{q+b \geq 0}$. It is evidently triangular; written in matrix form, the matrix of coefficients has determinant equal to 1. Since $\|P_{j,k}(b, c)\| \leq B/y_n$, we can now choose $\alpha \geq 1$ sufficiently large to guarantee that the coefficient matrix of the system in (3.10) has determinant close to 1 for $y_n \geq \alpha$. The system can then be solved for the $v^{p,q}(b)$, in such a way that $\max_{p,q,b} \|v^{p,q}(b)\|$ is bounded by a constant multiple of $\max_{p,q,b} \|G^{p,q}(b)\|$. It follows that there is a large constant $K > 0$ (depending on the Hodge numbers of (W, F) and on B) such that

$$\sum_{p,q,b} \|v^{p,q}(b)\| \leq K \cdot Z(y, h).$$

The decomposition $v = \sum (N^+)^b v^{p,q}(b)$ implies that each $N^k v$ can again be written as a combination of vectors of the form $(N^+)^b v^{p,q}(b+k)$. Since Lemma 3.12 bounds the size of N^+ , it is then easy to see that we have $Z(y, v) \leq C \cdot Z(y, h)$ for a suitable constant $C > 0$, as long as $y_n \geq \alpha$. \square

Note If we look more carefully at the calculation above, we find that each $P_{j,k}(b, c)$ is one of the Hodge components of an operator that is built up from $\delta, N, N^+, \Gamma(s)$, and $\overline{\Gamma}(s)$. What the proof actually shows is that v can be expressed by a very complicated formula in the Hodge components of those operators and the $h^{p,q}$. Similar reasoning can be used to prove that the entire Hodge decomposition of h in the Hodge structure $e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} F$ is given by formulas of this type.

Having completed the main technical step, we can now prove Theorem 3.4.

Proof Fix a real vector $h \in H_{\mathbb{R}}$, and let $z \in \mathbb{H}^n$ be any point with $x_j = \operatorname{Re} z_j \in [0, 1]$. Without loss of generality, we may assume that $y_1 \geq \dots \geq y_n \geq \alpha$, where $y_j = \operatorname{Im} z_j$. We will specify shortly how large α needs to be to obtain the asserted inequality between $Z(y, h)$ and $B(z, h)$. By definition, the various pairings

$$Q\left(h, e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} \prod_{j \in I} \frac{N_j}{s_j} v\right)$$

are bounded by $B(z, h)$ for $v \in F^{|I|}$ with $\|v\| \leq 1$. Since $0 \leq x_j \leq 1$ for each j , we may replace h by $e^{-\sum x_j N_j} h$ without affecting the statement we are trying to prove. For the same choices of I and v as above, we then have

$$\left| Q\left(h, e^{i\delta} e^{iN} e^{\Gamma(s)} \prod_{j \in I} \frac{N_j}{s_j} v\right) \right| \leq B(z, h).$$

Let us introduce the auxiliary vector $w = e^{-\Gamma(s)} e^{-iN} e^{-i\delta} h$. Since $N = \sum y_j N_j$ and $|s_j| = e^{-2\pi y_j}$, it is easy to deduce that

$$Q(N^k w, v) = (-1)^k Q(h, e^{i\delta} e^{iN} e^{\Gamma(s)} N^k v)$$

is bounded by a constant times $B(z, h)$, for any $v \in F^k$ with $\|v\| \leq 1$. The fact that the pairing is nondegenerate and compatible with the decomposition $H_{\mathbb{C}} = \bigoplus I^{p,q}$ now implies that the norm of each vector $N^k w^{p,q}$ with $p \leq -1$ is bounded by a constant multiple of $B(z, h)$. To exploit this information, we define

$$h' = e^{i\delta} e^{iN} e^{\Gamma(s)} \sum_{p \leq -1} w^{p,q} = e^{i\delta} \cdot e^{iN} e^{\Gamma(s)} e^{-iN} \cdot \sum_{p \leq -1} e^{iN} w^{p,q},$$

and observe that, as a consequence of Lemma 3.2, $Z(y, h') \leq C_1 \cdot B(z, h)$ for some constant $C_1 > 0$.

By construction, $h = h' + r$, where r belongs to $e^{i\delta} e^{iN} e^{\Gamma(s)} F^0$. Because h is real, it follows that $\overline{h'} - h' = r - \bar{r}$. This is a partial Hodge decomposition for the vector $\overline{h'} - h' \in H_{\mathbb{C}}$, relative to the Hodge structure of weight -1 defined by the point $e^{i\delta} e^{iN} e^{\Gamma(s)} F = e^{-\sum x_j N_j} \tilde{\Phi}(z) \in D$. Proposition 3.13, applied to $\overline{h'} - h'$, shows that we have $Z(y, r) \leq C_2 \cdot Z(y, h')$ for another constant $C_2 > 0$. The asserted bound on $Z(y, h)$ is now a consequence of the identity $h = h' + r$ and the inequality $Z(y, h') \leq C_1 \cdot B(z, h)$. \square

4 Admissible normal functions

4.1 Extending admissible normal functions without singularities

We now look at the problem of extending admissible normal functions with no singularities to holomorphic sections of the space $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$.

Let ν be a normal function on X for the variation \mathcal{H} , admissible relative to \bar{X} . By [34, p. 243], it corresponds to a mixed Hodge module N_ν on \bar{X} , with $W_{n-1}N_\nu = M$, and $\text{Gr}_n^W N_\nu$ the trivial Hodge module of weight n . On X , we have an extension of integral local systems

$$0 \rightarrow \mathcal{H}_{\mathbb{Z}} \rightarrow \mathcal{V}_{\mathbb{Z}} \rightarrow \mathbb{Z}_X \rightarrow 0, \tag{4.1}$$

and therefore a cohomology class $[\nu] \in H^1(X, \mathcal{H}_{\mathbb{Z}})$. Using the Leray spectral sequence for the inclusion $j: X \hookrightarrow \bar{X}$, we obtain an exact sequence

$$0 \rightarrow H^1(\bar{X}, j_*\mathcal{H}_{\mathbb{Z}}) \rightarrow H^1(X, \mathcal{H}_{\mathbb{Z}}) \rightarrow H^0(\bar{X}, R^1 j_*\mathcal{H}_{\mathbb{Z}}).$$

The following concept has been introduced by M. Green and P. Griffiths [15].

Definition 4.1 The image of $[\nu]$ in the space $H^0(\bar{X}, R^1 j_*\mathcal{H}_{\mathbb{Z}})$ is called the *singularity* of the normal function ν . When the image is zero, we shall say that $[\nu]$ is locally trivial, or that ν has no singularities.

When ν has no singularities, we evidently have $[\nu] \in H^1(\bar{X}, j_*\mathcal{H}_{\mathbb{Z}})$. The relationship of these definitions with (4.1) is the following: Taking direct images, we have a long exact sequence

$$0 \rightarrow j_*\mathcal{H}_{\mathbb{Z}} \rightarrow j_*\mathcal{V}_{\mathbb{Z}} \rightarrow \mathbb{Z}_{\bar{X}} \xrightarrow{\delta} R^1 j_*\mathcal{H}_{\mathbb{Z}} \rightarrow \dots,$$

and local triviality of $[\nu]$ is equivalent to the vanishing of the connecting homomorphism δ . Thus if the normal function has no singularities, we obtain from it an extension of sheaves of abelian groups on \bar{X} , namely

$$0 \rightarrow j_*\mathcal{H}_{\mathbb{Z}} \rightarrow j_*\mathcal{V}_{\mathbb{Z}} \rightarrow \mathbb{Z}_{\bar{X}} \rightarrow 0. \tag{4.2}$$

On the other hand, the mixed Hodge module N_ν is part of an extension

$$0 \rightarrow M \rightarrow N_\nu \rightarrow \mathbb{Q}_{\bar{X}}^H[n] \rightarrow 0,$$

with $\mathbb{Q}_{\bar{X}}^H[n]$ the trivial Hodge module of weight n on \bar{X} . Let $\mathbf{D}_{\bar{X}}(-)$ denote the duality functor on the category of mixed Hodge modules on \bar{X} . Recall that $\check{M} = \mathbf{D}_{\bar{X}}(M)(1-n)$ is isomorphic to the intermediate extension of the (normalized) dual variation $\check{\mathcal{H}} = \mathcal{H}^\vee(1)$. We also have $\mathbf{D}_{\bar{X}}(\mathbb{Q}_{\bar{X}}^H[n]) \simeq \mathbb{Q}_{\bar{X}}^Hn$.

Dualizing the extension above, and applying a Tate twist, we thus get an exact sequence

$$0 \rightarrow \mathbb{Q}_{\bar{X}}^H[n](1) \rightarrow \check{N}_v \rightarrow \check{M} \rightarrow 0,$$

with $\check{N}_v = \mathbf{D}_{\bar{X}}(N_v)(1 - n)$. Let $(\check{\mathcal{N}}_v, F)$ be the underlying filtered \mathcal{D} -module; then we have an extension of filtered \mathcal{D} -modules

$$0 \rightarrow (\mathcal{O}_{\bar{X}}, F[1]) \rightarrow (\check{\mathcal{N}}_v, F) \rightarrow (\check{\mathcal{M}}, F) \rightarrow 0. \tag{4.3}$$

Morphisms of mixed Hodge modules are strictly compatible with the Hodge filtrations on the underlying \mathcal{D} -modules; because $F_{-1}\mathcal{O}_{\bar{X}} = 0$, the exact sequence above induces an isomorphism $F_0\check{\mathcal{N}}_v \simeq F_0\check{\mathcal{M}}$.

Just as in Sect. 2.1, we can now compare the two extensions in (4.2) and (4.3) to obtain a section of $\bar{J}(\mathcal{H}) = T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$. But note that this has to be done carefully, since $F_0\check{\mathcal{M}}$ is in general not locally free near points of $\bar{X} - X$.

Proposition 4.2 *Any admissible normal function $v: J(\mathcal{H}) \rightarrow X$ without singularities can be canonically extended to a holomorphic section of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$.*

Proof Since v has no singularities, it gives rise to an extension of sheaves of abelian groups as in (4.2). Now cover the space \bar{X} by open subsets U_i , such that (4.2) is locally split. This means that we have $v_i \in H^0(U_i, j_*\mathcal{V}_{\mathbb{Z}})$, mapping to the constant section $1 \in H^0(U_i, \mathbb{Z})$; it follows that $h_{ij} = v_j - v_i \in H^0(U_i \cap U_j, j_*\mathcal{H}_{\mathbb{Z}})$. Note that v_i is well-defined up to a section of $j_*\mathcal{H}_{\mathbb{Z}}$ over U_i .

Write $j_i: U_i \hookrightarrow \bar{X}$ for the various open inclusions. Because of Lemma 2.8, each local section v_i defines a morphism of \mathcal{D} -modules $\phi_i: j_i^{-1}\check{\mathcal{N}}_v \rightarrow \mathcal{O}_{U_i}$. Restricting to the subsheaf $j_i^*(F_0\check{\mathcal{N}}_v) \simeq j_i^*(F_0\check{\mathcal{M}})$, we thus have local holomorphic sections

$$\psi_i \in H^0(U_i, (F_0\check{\mathcal{M}})^\vee) \quad \text{satisfying } \psi_j - \psi_i = h_{ij}.$$

By definition of the analytic structure on $T(F_0\check{\mathcal{M}})/T_{\mathbb{Z}}$, this means exactly that we have produced a global holomorphic section of the Néron model $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$. It is clear from the construction that this section is independent of the choices made. That we recover the original normal function on X is a straightforward consequence of Lemma 2.2. □

From now on, let us write \bar{v} for the section of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$ constructed in Proposition 4.2; we refer to it as the *extension* of the original normal function v .

Note It would be interesting to know the set of points in $\bar{J}(\mathcal{H})$ that can lie on the graph of an extended normal function.

4.2 The horizontality condition

It is clear that the extension $\bar{\nu}$ constructed in Proposition 4.2 is far from being an arbitrary section of the quotient. In fact, the proof shows that there are local liftings $\psi : F_0\check{\mathcal{M}}|_U \rightarrow \mathcal{O}_U$ that are compatible with differentiation: for any $k \geq 0$, any differential operator $D \in H^0(U, F_k\mathcal{D}_U)$, and any section $s \in H^0(U, F_{-k}\check{\mathcal{M}})$, the lifting satisfies $\psi(Ds) = D\psi(s)$. This appears to be the correct notion of horizontality for sections of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$.

Definition 4.3 A holomorphic section of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$ is said to be *horizontal* if it admits local holomorphic liftings $\psi : F_0\check{\mathcal{M}}|_U \rightarrow \mathcal{O}_U$ with the property that $\psi(\xi s) = d_\xi(\psi(s))$ for any holomorphic tangent vector field $\xi \in H^0(U, \Theta_U)$ and any section $s \in H^0(U, F_{-1}\check{\mathcal{M}})$.

It follows that $\psi(Ds) = D\psi(s)$ for $D \in H^0(U, F_k\mathcal{D}_U)$ and $s \in H^0(U, F_{-k}\check{\mathcal{M}})$ as above. Over X , the definition clearly recovers the usual definition of horizontality. We now prove the converse to Proposition 4.2.

Proposition 4.4 *Let $\mu : \bar{X} \rightarrow \bar{J}(\mathcal{H})$ be a holomorphic section that is horizontal. Then μ is the extension of an admissible normal function on X with locally trivial cohomology class.*

Proof The restriction of μ to X is a horizontal and holomorphic section of $J(\mathcal{H})$, and therefore a normal function ν . We have to prove that it is admissible, and that its cohomology class is locally trivial. To begin with the latter, consider the exact sequence of sheaves

$$0 \rightarrow j_*\mathcal{H}_{\mathbb{Z}} \rightarrow (F_0\check{\mathcal{M}})^\vee \rightarrow (F_0\check{\mathcal{M}})^\vee / j_*\mathcal{H}_{\mathbb{Z}} \rightarrow 0,$$

and recall that the quotient is the sheaf of sections of $\bar{J}(\mathcal{H})$. Via the connecting homomorphism, the section μ determines an element in $H^1(\bar{X}, j_*\mathcal{H}_{\mathbb{Z}})$, whose image in $H^1(X, \mathcal{H}_{\mathbb{Z}})$ is the class of the normal function. By construction, $[\nu]$ goes to zero in $H^0(\bar{X}, R^1 j_*\mathcal{H}_{\mathbb{Z}})$; this means that ν has no singularities.

Since admissibility is defined by a curve test [22], we let $f : \Delta \rightarrow \bar{X}$ be an arbitrary holomorphic curve with $f(\Delta^*) \subseteq X$, such that $\mathcal{H}^f = f^*\mathcal{H}$ has unipotent monodromy. By Proposition 2.22, we have a holomorphic mapping

$$\Delta \times_{\bar{X}} \bar{J}(\mathcal{H}) \rightarrow \bar{J}(\mathcal{H}^f)$$

over Δ , and so μ induces a holomorphic section of $\bar{J}(\mathcal{H}^f)$ whose restriction to Δ^* is the pullback of the normal function. Since it suffices to prove the admissibility of the latter, we have reduced the problem to the case of a disk, where we can apply the following lemma. \square

Lemma 4.5 *Let \mathcal{H} be a polarizable variation of integral Hodge structure of weight -1 on Δ^* , whose monodromy is unipotent. Then any holomorphic and horizontal section of $\bar{J}(\mathcal{H}) \rightarrow \Delta$ is the extension of an admissible normal function.*

Proof Let $\bar{X} = \Delta$ and $X = \Delta^*$. Shrinking the radius of the disk, if necessary, we may assume that the section can be lifted to a morphism $\psi : F_0\check{\mathcal{M}} \rightarrow \mathcal{O}_{\bar{X}}$ that satisfies the condition in the definition of horizontality. As before, the morphism defines a normal function with trivial cohomology class on X . Let

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{V} \rightarrow \mathbb{Z}_X(0) \rightarrow 0$$

be the corresponding extension of variations of mixed Hodge structure on X ; we need to show that \mathcal{V} is admissible. As a practical matter, it is more convenient to prove the admissibility of $\check{\mathcal{V}} = \mathcal{V}^\vee(1)$, which is part of to the dual extension

$$0 \rightarrow \mathbb{Z}_X(1) \rightarrow \check{\mathcal{V}} \rightarrow \check{\mathcal{H}} \rightarrow 0.$$

The first condition in the definition of admissibility, namely existence of the relative weight filtration, is trivially satisfied in our case, because the underlying local system $\check{\mathcal{V}}_{\mathbb{Z}} = \mathbb{Z}_X(1) \oplus \check{\mathcal{H}}_{\mathbb{Z}}$ is a direct sum.

Now let $\check{\mathcal{H}}_{\mathcal{O}}^e$ be Deligne’s canonical extension of the flat vector bundle $(\check{\mathcal{H}}_{\mathcal{O}}, \nabla)$; it is naturally a subsheaf of the minimal extension $\check{\mathcal{M}}$. Evidently, the canonical extension of $\check{\mathcal{V}}_{\mathcal{O}}$ is given by $\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{H}}_{\mathcal{O}}^e \hookrightarrow \mathcal{O}_{\bar{X}} \oplus \check{\mathcal{M}}$. It remains to verify the second condition in the definition of admissibility, namely that the Hodge bundles $F_p\check{\mathcal{V}}_{\mathcal{O}}$ extend to holomorphic subbundles of the canonical extension. In fact, we shall give a formula for these subbundles in terms of ψ .

At this point, we do not know that $\check{\mathcal{V}}$ can be extended to a mixed Hodge module on \bar{X} —in fact, this is equivalent to admissibility by [34, p. 243]. Nevertheless, we can use ψ to reconstruct the Hodge filtration on the underlying filtered \mathcal{D} -module $\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{M}}$. Guided by the exact sequence in (4.3), we define

$$F_p(\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{M}}) = \begin{cases} \text{im}((\psi, \text{id}) : F_p\check{\mathcal{M}} \rightarrow \mathcal{O}_{\bar{X}} \oplus \check{\mathcal{M}}) & \text{for } p \leq 0, \\ \mathcal{O}_{\bar{X}} \oplus F_p\check{\mathcal{M}} & \text{for } p > 0. \end{cases}$$

The horizontality condition on ψ ensures that the filtration is good, and therefore that $\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{M}}$ is a filtered \mathcal{D} -module. Now any coherent subsheaf of

$\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{H}}_{\mathcal{O}}^e$ is locally free (because \bar{X} is one-dimensional); consequently,

$$F_p(\mathcal{O}_{\bar{X}} \oplus \check{\mathcal{H}}_{\mathcal{O}}^e) = \begin{cases} \text{im}((\psi, \text{id}): F_p \check{\mathcal{H}}_{\mathcal{O}}^e \rightarrow \mathcal{O}_{\bar{X}} \oplus \check{\mathcal{H}}_{\mathcal{O}}^e) & \text{for } p \leq 0, \\ \mathcal{O}_{\bar{X}} \oplus F_p \check{\mathcal{H}}_{\mathcal{O}}^e & \text{for } p > 0, \end{cases}$$

extends the Hodge bundle $F_p \check{\mathcal{Y}}_{\mathcal{O}}$ to a holomorphic subbundle of the canonical extension. This concludes the proof that $\check{\mathcal{Y}}$, and hence ν , is admissible. \square

4.3 Graphs of admissible normal functions

In this section, we consider admissible normal functions on X with possibly nontrivial singularities. By Proposition 4.4, such a normal function cannot be extended to a section of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$. Nevertheless, the following surprising result is true.

Theorem 4.6 *Let $\nu: X \rightarrow J(\mathcal{H})$ be a normal function, admissible relative to \bar{X} . Then the topological closure of the graph $\nu(X)$ is an analytic subset of $\bar{J}(\mathcal{H})$.*

Proof This follows from the corresponding statement in the normal crossing case, contained in Corollary 5.7, by the same argument as in Sect. 2.11. \square

One consequence is an alternative proof for Conjecture 1.1. It is quite different from the existing proof by P. Brosnan and G. Pearlstein [6], but similar in spirit to the treatment of the one-variable case in M. Saito's paper [35].

Corollary 4.7 *If a normal function $\nu: X \rightarrow J(\mathcal{H})$ is admissible relative to \bar{X} , then the closure of its zero locus $Z(\nu)$ is an analytic subset of \bar{X} . In particular, when X is an algebraic variety, the zero locus $Z(\nu)$ is an algebraic subvariety.*

Proof The closure of $Z(\nu)$ is contained in the intersection of the closure of the graph of ν with the zero section of $\bar{J}(\mathcal{H})$, and is therefore analytic as well. When X is an algebraic variety, we take \bar{X} to be projective—admissibility is independent of the choice of compactification in that case—and the algebraicity of $Z(\nu)$ follows by Chow's Theorem. \square

We also note the following property of normal functions with torsion singularities, suggested by P. Brosnan. In the statement, $p: \bar{J}(\mathcal{H}) \rightarrow \bar{X}$ is the projection map, and $\nu(X) \subseteq J(\mathcal{H})$ is the graph of ν .

Proposition 4.8 *Suppose that $\nu: X \rightarrow J(\mathcal{H})$ is an admissible normal function, whose singularity at a point $x \in \bar{X} - X$ is torsion. If the topological closure of $\nu(X)$ intersects the fiber $p^{-1}(x)$, then ν extends to a holomorphic section of $\bar{J}(\mathcal{H})$ in a neighborhood of x .*

Proof This follows from Lemma 5.8, by a similar argument as in Sect. 2.10. Namely, let

$$0 \rightarrow M \rightarrow N_\nu \rightarrow \mathbb{Q}_{\bar{X}}^H[n] \rightarrow 0$$

be the extension of mixed Hodge modules on \bar{X} corresponding to the normal function ν , let $i : \{x\} \hookrightarrow \bar{X}$ be the inclusion, and set $V = H^{-n}i^*N_\nu$ and $H = H^{-n}i^*M$. Then both H and V are mixed Hodge structures, defined over \mathbb{Z} , and since the singularity of ν at x is torsion, we get $V_{\mathbb{Q}}/H_{\mathbb{Q}} \simeq \mathbb{Q}$. Also note that $H_{\mathbb{C}}/F_0H_{\mathbb{C}} \rightarrow V_{\mathbb{C}}/F_0V_{\mathbb{C}}$ is an isomorphism by strictness.

Now suppose that the closure of $\nu(X)$ meets $p^{-1}(x)$. Let $T_\nu \subseteq T(F_0\check{\mathcal{M}})$ be the preimage of the graph. Since the closure of T_ν is analytic, we can find a holomorphic curve $f : \Delta \rightarrow T(F_0\check{\mathcal{M}})$ with $f(\Delta^*) \subseteq T_\nu$, such that $f(0)$ lies over the point x . Let $f^*\nu$ denote the pullback of ν to Δ^* ; it is an admissible normal function without singularities on Δ , and the corresponding extension of mixed Hodge modules

$$0 \rightarrow M^f \rightarrow N^f \rightarrow \mathbb{Q}_{\Delta}^H[1] \rightarrow 0$$

splits over \mathbb{Z} . If we let $i_0 : \{0\} \hookrightarrow \Delta$, $H^f = H^{-1}i_0^*M^f$, and $V^f = H^{-1}i_0^*N^f$, we have $V_{\mathbb{Z}}^f \simeq H_{\mathbb{Z}}^f \oplus \mathbb{Z}$. As in Sect. 2.10, we obtain a commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathbb{Z}} & \longrightarrow & V_{\mathbb{Z}} & \longrightarrow & \mathbb{Z} \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & H_{\mathbb{Z}}^f & \longrightarrow & V_{\mathbb{Z}}^f & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

Let $v^f \in V_{\mathbb{Z}}^f$ be a lifting of $1 \in \mathbb{Z}$. It determines a point in $T(F_0\check{\mathcal{M}})_0$, and hence in the quotient $H_{\mathbb{C}}^f/F_0H_{\mathbb{C}}^f \simeq V_{\mathbb{C}}^f/F_0V_{\mathbb{C}}^f$. As before, the fact that we have $f(0) \in T(F_0\check{\mathcal{M}})_x$, together with the compatibility of the maps, implies that

$$v^f \in F_0V_{\mathbb{C}}^f + \text{im}(V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}^f).$$

Since the singularity of ν is a torsion class, the quotient V^f/V is a mixed Hodge structure of weight ≤ -1 . We again conclude that $v^f \in V_{\mathbb{Q}}$, and hence $v^f \in V_{\mathbb{Z}}$. But then $V_{\mathbb{Z}} \simeq H_{\mathbb{Z}} \oplus \mathbb{Z}$, and so ν has no singularity at the point x . \square

4.4 A Néron model for torsion singularities

The analytic space $\bar{J}(\mathcal{H})$ has all the properties that are expected for the identity component of the Néron model. In this section, we extend the construction

to produce an analytic space that graphs admissible normal functions with torsion singularities. This generalizes work by M. Saito [35] in the case where $\dim \bar{X} = 1$.

Theorem 4.9 *There is an analytic space $\bar{J}_{\text{tor}}(\mathcal{H}) \rightarrow \bar{X}$, whose horizontal and holomorphic sections are precisely the admissible normal functions with torsion singularities.*

We obtain the space $\bar{J}_{\text{tor}}(\mathcal{H})$ by a gluing construction as in [7, Sect. 2.3]; local models are given by locally defined admissible normal functions with torsion singularities. To introduce the appropriate notation, suppose that we have an open subset $U \subseteq \bar{X}$, and an admissible normal function ν on $U \cap X$ with only torsion singularities. Then ν defines a section of $p: \bar{J}(\mathcal{H}) \rightarrow \bar{X}$ over $U \cap X$. By Proposition 4.8, there is a maximal open subset $U(\nu) \subseteq U$ to which this section can be extended; the important fact is that the graph of $\nu: U(\nu) \rightarrow p^{-1}(U)$ is a closed analytic subset. For every such pair, we let $Y(U, \nu)$ be a copy of $p^{-1}(U) \subseteq \bar{J}(\mathcal{H})$, and write $p: Y(U, \nu) \rightarrow U$ for the projection map.

Let Y be the disjoint union of all the spaces $Y(U, \nu)$, and define an equivalence relation on the topological space Y by setting

$$y \sim y' \quad \text{if and only if} \quad \begin{cases} x = p(y) = p(y') \text{ lies in } U(\nu) \cap U'(\nu'), \\ \text{and } y + \nu(x) = y' + \nu'(x) \end{cases}$$

for $y \in Y(U, \nu)$ and $y' \in Y(U', \nu')$.

Lemma 4.10 *The quotient map $q: Y \rightarrow Y/\sim$ is an open map, and the topology on Y/\sim is Hausdorff.*

Proof To prove that q is an open map, it suffices to show that the image of each $Y(U, \nu)$ is an open subset of the quotient. One easily sees that the preimage of $q(Y(U, \nu))$ intersects $Y(U', \nu')$ in the open subset $p^{-1}(U(\nu) \cap U'(\nu'))$; this implies the first assertion.

For the second, it is again enough to prove that \sim defines a closed subset of $Y \times Y$. So suppose that we have two sequences of points $y_n \in Y(U, \nu)$ and $y'_n \in Y(U', \nu')$ with $y_n \sim y'_n$ for all $n \in \mathbb{N}$, such that $(y_n, y'_n) \rightarrow (y, y')$. Letting $x_n = p(y_n) = p(y'_n)$, we obtain $x_n \rightarrow x$, where $x = p(y) = p(y')$. Since the graphs of ν and ν' are closed by Proposition 4.8, we have $x \in U(\nu) \cap U'(\nu')$. The continuity of ν and ν' now implies that $y + \nu(x) = y' + \nu'(x)$, proving that $y \sim y'$. \square

Here is the proof of Theorem 4.9.

Proof In the notation from above, let $\bar{J}_{\text{tor}}(\mathcal{H}) = Y / \sim$, with the obvious projection map to \bar{X} . Evidently, no two distinct points of $Y(U, \nu)$ are identified by the equivalence relation, and so $Y(U, \nu)$ is isomorphic to its image in the quotient. Since the quotient is Hausdorff, it follows that it is an analytic space, with local analytic charts given by the $Y(U, \nu)$. It is clear from the construction that any admissible normal function ν on X with torsion singularities extends to a holomorphic section of $\bar{J}_{\text{tor}}(\mathcal{H})$: the extension is given by the zero section of $Y(\bar{X}, \nu) \rightarrow \bar{X}$, followed by the inclusion into the quotient. \square

Let $\mathcal{G} = \ker(R^1 j_* \mathcal{H}_{\mathbb{Z}} \rightarrow R^1 j_* \mathcal{H}_{\mathbb{Q}})$ be the sheaf of torsion sections in $R^1 j_* \mathcal{H}_{\mathbb{Z}}$; if the singularity of an admissible normal function on X is torsion, then it is an element of $H^0(\bar{X}, \mathcal{G})$. Note that \mathcal{G} is a constructible sheaf of finite abelian groups, with support contained in $\bar{X} - X$.

Lemma 4.11 *For a point $x \in \bar{X}$, let G_x denote the stalk of the sheaf \mathcal{G} at x . Then every element of G_x is the singularity of an admissible normal function that is defined in a neighborhood of x .*

Proof Fix an element $g \in G_x$. After replacing \bar{X} by a small open neighborhood of x , if necessary, we may assume that g belongs to $H^1(X, \mathcal{H}_{\mathbb{Z}})$ and therefore corresponds to an extension of local systems

$$0 \rightarrow \mathcal{H}_{\mathbb{Z}} \rightarrow \mathcal{V}_{\mathbb{Z}} \rightarrow \mathbb{Z}_X \rightarrow 0$$

on X . The extension splits over \mathbb{Q} because g is torsion. Since $\mathcal{V}_{\mathbb{Q}} \simeq \mathcal{H}_{\mathbb{Q}} \oplus \mathbb{Q}$, it follows that $\mathcal{V}_{\mathbb{Z}}$ underlies the variation of mixed Hodge structure $\mathcal{V} = \mathcal{H} \oplus \mathbb{Q}_X(0)$. Now \mathcal{V} is clearly admissible, and therefore corresponds to an admissible normal function, whose singularity equals the original element $g \in H^1(X, \mathcal{H}_{\mathbb{Z}})$. \square

4.5 Impossibility of a general analytic Néron model

We now describe the implications of Theorem 4.6 for the construction of the full Néron model. As mentioned in the introduction, it should have the property that its sections are the admissible normal functions.

Lemma 4.12 *Let $X \subseteq \bar{X}$ be a Zariski-open subset, and let \mathcal{H} be a polarizable variation of Hodge structure of weight -1 on X . Suppose that there is a topological space Y with the following three properties:*

- (i) *The topology on Y is Hausdorff, and there is a continuous map $Y \rightarrow \bar{X}$.*
- (ii) *There is a continuous injective map $\bar{J}(\mathcal{H}) \hookrightarrow Y$ over \bar{X} that is a homeomorphism over X .*
- (iii) *Admissible normal functions on X extend to a continuous sections of Y .*

Then the closure of the graph of an admissible normal function inside $\bar{J}(\mathcal{H})$ can meet every fiber of $p: \bar{J}(\mathcal{H}) \rightarrow \bar{X}$ in at most one point.

Proof Let $v: X \rightarrow J(\mathcal{H})$ be an admissible normal function. By assumption, it extends to a continuous section $\mu: \bar{X} \rightarrow Y$, and since Y is Hausdorff, its graph $\mu(\bar{X})$ has to be closed. It follows that the preimage of $\mu(\bar{X})$ in $\bar{J}(\mathcal{H})$ is also closed, and therefore contains the closure of $v(X)$. But this implies that $v(\bar{X})$ intersects each fiber $p^{-1}(x)$ in at most one point. \square

Now the problem is that, for a general admissible normal function with non-torsion singularities, the closure of the graph typically has fibers of positive dimension over \bar{X} . This can happen even in the simplest of examples: Sect. 6.3 exhibits a family of elliptic curves over $(\Delta^*)^2$, where the central fiber of $\bar{J}(\mathcal{H}) \rightarrow \Delta^2$ is a copy of \mathbb{C}^* . One can then easily write down an admissible normal function on $(\Delta^*)^2$ that extends holomorphically to $\Delta^2 - \{(0, 0)\}$, but such that the closure of its graph has a one-dimensional fiber over the origin.

In my eyes, examples of this kind make the existence of a Néron model that is Hausdorff as a topological space very unlikely, for the following reason: For a family of elliptic curves on $(\Delta^*)^2$ with unipotent monodromy, any reasonable candidate for the Néron model should have $\bar{J}(\mathcal{H})$ as its identity component, since the latter agrees with the classical construction [28]. By Lemma 4.12, this means that the normal function in the example cannot be a continuous section of a Néron model that is also Hausdorff. Thus it appears that one cannot do any better than Theorem 4.9 in general.

4.6 Comparison with Brosnan-Pearlstein-Saito

We now make the comparison of our construction with the Néron model defined by P. Brosnan, G. Pearlstein, and M. Saito in their preprint [7]. We denote the identity component of their model by $\bar{J}_{\text{BPS}}(\mathcal{H})$.

We begin by constructing a map on fibers. Let $i: \{x\} \hookrightarrow \bar{X}$ be the inclusion of an arbitrary point; as in Sect. 2.9, define the two integral mixed Hodge structures $H = H^{-n}i^*M$ and $\bar{H} = H^\vee(1) \simeq H^n i^! \bar{M}(n)$. Lemma 2.25 provides us with a surjection from $\bar{J}(\mathcal{H})_x$ to the generalized intermediate Jacobian $J(H)$. As explained in Sect. 2.1, $J(H) \simeq \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H)$ is exactly the fiber of $\bar{J}_{\text{BPS}}(\mathcal{H})$ over the point x . In this way, we obtain for every point $x \in \bar{X}$ a surjective map of complex Lie groups

$$\bar{J}(\mathcal{H})_x \twoheadrightarrow \bar{J}_{\text{BPS}}(\mathcal{H})_x.$$

It gives rise to a map of sets $\pi: \bar{J}(\mathcal{H}) \rightarrow \bar{J}_{\text{BPS}}(\mathcal{H})$. In the case $\dim \bar{X} = 1$, a very precise description of the map π as a composition of blowups with

specified centers has been given in [36]. Here, we shall content ourselves with showing that π is continuous.

Lemma 4.13 *The resulting map of sets $\pi : \bar{J}(\mathcal{H}) \rightarrow \bar{J}_{\text{BPS}}(\mathcal{H})$ is continuous.*

Proof Because of how the topology on $\bar{J}_{\text{BPS}}(\mathcal{H})$ is defined in [7], and because of the functoriality of our construction, it suffices to prove the statement in the case when $\bar{X} - X$ is a divisor with normal crossings and the local monodromy of $\mathcal{H}_{\mathbb{Z}}$ is unipotent. Let $\mathcal{H}_{\mathcal{O}}^e$ be Deligne’s canonical extension of $\mathcal{H}_{\mathcal{O}}$, and $\check{\mathcal{H}}_{\mathcal{O}}^e$ that of $\check{\mathcal{H}}_{\mathcal{O}}$; then $\check{\mathcal{H}}_{\mathcal{O}}^e \hookrightarrow \check{\mathcal{M}}$. The Hodge bundles extend to locally free subsheaves $F_p \check{\mathcal{H}}_{\mathcal{O}}^e = \check{\mathcal{H}}_{\mathcal{O}}^e \cap F_p \check{\mathcal{M}}$. Let $E \rightarrow \bar{X}$ be the holomorphic vector bundle corresponding to the locally free sheaf $\mathcal{H}_{\mathcal{O}}^e$, and $F_0 E \subseteq E$ the subbundle corresponding to $F_0 \mathcal{H}_{\mathcal{O}}^e$. We then have a holomorphic mapping $T(F_0 \check{\mathcal{M}}) \rightarrow T(F_0 \check{\mathcal{H}}_{\mathcal{O}}^e)$; since $(F_0 \check{\mathcal{H}}_{\mathcal{O}}^e)^\vee \simeq \mathcal{H}_{\mathcal{O}}^e / F_0 \mathcal{H}_{\mathcal{O}}^e$, this means that we get a holomorphic mapping

$$T(F_0 \check{\mathcal{M}}) \rightarrow E / F_0 E$$

from the analytic space on the left to the vector bundle on the right. Since the topology on $\bar{J}_{\text{BPS}}(\mathcal{H})$ is induced from that on $E / F_0 E$, and topology on $\bar{J}(\mathcal{H})$ from that on $T(F_0 \check{\mathcal{M}})$, the continuity of $\bar{J}(\mathcal{H}) \rightarrow \bar{J}_{\text{BPS}}(\mathcal{H})$ is immediate. \square

Note The map $\bar{J}(\mathcal{H})_x \rightarrow \bar{J}_{\text{BPS}}(\mathcal{H})_x$ constructed in Sect. 2.9 has a splitting: in fact, by Lemma 2.8, we have a map $H_{\mathbb{C}} \rightarrow T(F_0 \check{\mathcal{M}})_x$, and the composition

$$H_{\mathbb{C}} / F_0 H_{\mathbb{C}} \rightarrow T(F_0 \check{\mathcal{M}})_x \rightarrow (F_0 \check{H}_{\mathbb{C}})^\vee$$

is an isomorphism. This circumstance is useful for proving the surjectivity of π , but otherwise turns out to be something of a red herring, because the resulting mapping $\bar{J}_{\text{BPS}}(\mathcal{H}) \rightarrow \bar{J}(\mathcal{H})$ is typically neither continuous, nor compatible with normal functions (as pointed out to me by M. Saito).

Now let ν be an admissible normal function on X with locally trivial cohomology class. We can also show that its extension $\bar{\nu}$ to a holomorphic section of $\bar{J}(\mathcal{H}) \rightarrow \bar{X}$ is mapped to the extension constructed in [7].

Lemma 4.14 *Let $\bar{\nu} : \bar{X} \rightarrow \bar{J}(\mathcal{H})$ be the extension of an admissible normal function ν without singularities. Then the induced section $\pi \circ \bar{\nu}$ of $\bar{J}_{\text{BPS}}(\mathcal{H})$ agrees with the extension of ν defined in [7].*

Proof Associated to the normal function, we have an extension of variations of mixed Hodge structure

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{V} \rightarrow \mathbb{Z}_X(0) \rightarrow 0.$$

Because of admissibility, \mathcal{V} can be extended to a mixed Hodge module \mathcal{N}_v on \bar{X} with $W_{n-1}\mathcal{N}_v \simeq M$ and $\text{Gr}_n^W \mathcal{N}_v \simeq \mathbb{Q}_{\bar{X}}^H[n]$.

Fix a point $i : \{x\} \hookrightarrow \bar{X}$, and let $H = H^{-n}i^*M$ and $V = H^{-n}i^*\mathcal{N}_v$. Also define $\check{H} = H^\vee(1) \simeq H^n i^! \check{M}(n)$. Since the cohomology class of v is trivial near x , it is easy to see that we obtain an extension of mixed Hodge structures

$$0 \rightarrow H \rightarrow V \rightarrow \mathbb{Z}(0) \rightarrow 0, \tag{4.4}$$

and therefore a point in $\text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H) \simeq J(H)$; it is the value of the extended normal function in $\bar{J}_{\text{BPS}}(\mathcal{H})_x$. According to Sect. 2.1, this point is obtained by choosing a lifting $v_{\mathbb{Z}} \in V_{\mathbb{Z}}$ for $1 \in \mathbb{Z}$, and restricting it to a linear operator on $F_0\check{H}_{\mathbb{C}}$. If we take v equal to the value at x of a locally defined flat section of $\mathcal{V}_{\mathbb{Z}}$ splitting the extension of local systems, then it follows that this prescription is compatible with the definition of the extended normal function \bar{v} in Proposition 4.2. This means that $\pi(\bar{v}(x))$ gives the same point in $J(H)$, as claimed. \square

Note A shorter proof is the following: Both the extension of v constructed in [7] and $\pi \circ \bar{v}$ are continuous sections of $\bar{J}_{\text{BPS}}(\mathcal{H})$. Since they agree over X , and since X is dense in \bar{X} , it follows that they agree everywhere.

5 Local analysis of admissible normal functions

5.1 Introduction

This part of the paper is devoted to a local analysis of admissible normal functions with possibly nontrivial singularities, and to the proof of Theorem 4.6. One might expect that this would be considerably more difficult, but in fact, the methods of Sect. 3.7 extend to this case with little additional effort.

Let \mathcal{H} be a polarizable variation of Hodge structure of weight -1 on $X = (\Delta^*)^n$, and let v be a normal function, admissible relative to $\bar{X} = \Delta^n$. We represent v by an admissible variation of mixed Hodge structure \mathcal{V} , in the form of an extension

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{V} \rightarrow \mathbb{Z}_X(0) \rightarrow 0.$$

Since $\mathcal{H}_{\mathbb{Z}}$ has unipotent monodromy, the same is clearly true for $\mathcal{V}_{\mathbb{Z}}$. Let $\check{\mathcal{V}} = \mathcal{V}^\vee(1)$ denote the (normalized) dual variation of mixed Hodge structure. As in Sect. 2.1, we have the dual extension

$$0 \rightarrow \mathbb{Z}_X(1) \rightarrow \check{\mathcal{V}} \rightarrow \check{\mathcal{H}} \rightarrow 0,$$

and therefore an isomorphism $F_0\check{\mathcal{V}}_{\mathcal{O}} \simeq F_0\check{\mathcal{H}}_{\mathcal{O}}$. It gives rise to a morphism of sheaves $\mathcal{V}_{\mathbb{Z}} \rightarrow (F_0\check{\mathcal{H}}_{\mathcal{O}})^\vee$ on $(\Delta^*)^n$. Let T_v be the subset of the étalé space of

$\mathcal{V}_{\mathbb{Z}}$, consisting of those points that map to $1 \in \mathbb{Z}$. We then have a holomorphic embedding

$$\varphi: T_v \hookrightarrow T(F_0\check{\mathcal{H}}_{\mathcal{O}})$$

over X , and the goal of this section is to prove that the topological closure of $\varphi(T_v)$ inside the larger analytic space $T(F_0\check{\mathcal{M}})$ is an analytic subset.

Note As in Sect. 3.2, we choose once and for all a polarization on \mathcal{H} . It induces isomorphisms $F_0\check{\mathcal{H}}_{\mathcal{O}} \simeq F_0\mathcal{H}_{\mathcal{O}}$ and $F_0\check{\mathcal{M}} \simeq F_0\mathcal{M}$, and because it simplifies the notation, we shall use the mapping $\varphi: T_v \hookrightarrow T(F_0\mathcal{H}_{\mathcal{O}})$ instead of the original one.

5.2 A technical result

Just as in the pure case, we let $V_{\mathbb{C}}$ denote the fiber of the pullback of \mathcal{V} to \mathbb{H}^n . Let W be the resulting weight filtration on $V_{\mathbb{Z}}$, with $W_{-1} = H_{\mathbb{Z}}$ and $\text{Gr}_0^W \simeq \mathbb{Z}$. Let $N_1, \dots, N_n \in \text{End}(V_{\mathbb{Q}})$ be the logarithms of the monodromy operators; note that $\text{im } N_j \subseteq H_{\mathbb{Q}}$. Likewise, most of the notation introduced in Sect. 3.2 will now be used for $V_{\mathbb{C}}$ in place of $H_{\mathbb{C}}$.

Notation It will be convenient to let $V_{\mathbb{C},1} \subseteq V_{\mathbb{C}}$ denote the subset of elements that map to $1 \in \mathbb{C} \simeq V_{\mathbb{C}}/H_{\mathbb{C}}$. We similarly define $V_{\mathbb{R},1}$ and $V_{\mathbb{Z},1}$.

The lifting of the period map will be denoted by $\tilde{\Phi}: \mathbb{H}^n \rightarrow D$, where D is now a period domain for mixed Hodge structures. Since the original variation is admissible, we have $e^{-\sum z_j N_j} \tilde{\Phi}(z) = \Psi(s)$ with Ψ holomorphic on Δ^n . In addition, the relative monodromy weight filtration $M = M(N_1, \dots, N_n; W)$ exists and is constant on the open cone $C(N_1, \dots, N_n)$, and the pair $(M, \Psi(0))$ is a mixed Hodge structure [22, Proposition 5.2.1]. Let $\delta \in L_{\mathbb{R}}^{-1,-1}(M, \Psi(0))$ be the unique element for which (M, F) is \mathbb{R} -split, where $F = e^{-i\delta} \Psi(0)$. As in Sect. 3.2, we can put the period map for the variation of mixed Hodge structure into the standard form [30, Proof of Theorem 6.13]

$$\tilde{\Phi}(z) = e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} F,$$

where Γ is holomorphic and satisfies $\Gamma(0) = 0$. Since the period map is again horizontal, Lemma 3.1 extends to this setting. Obviously, the restriction of Γ or δ to the subspace $H_{\mathbb{C}}$ gives back the operators that were introduced in Sect. 3.2.

In the remainder of this section, we prove the following generalization of Theorem 3.5; note the similarity with the main result of E. Cattani, P. Deligne, and A. Kaplan [8, Theorem 2.16]. By a slight abuse of notation, we also let Q

denote the pairing between $V_{\mathbb{C}}$ and sections (of the pullback to \mathbb{H}^n) of $F_0\mathcal{H}_{\mathcal{O}}$, induced by the morphism $\mathcal{V}_{\mathbb{C}} \rightarrow (F_0\mathcal{H}_{\mathcal{O}})^{\vee}$ described above.

Theorem 5.1 *Let $z_j(m) \in \mathbb{H}^n$ be a sequence of points with $\text{Im } z_j(m) \rightarrow \infty$ and $\text{Re } z_j(m) \in [0, 1]$ for $j = 1, \dots, n$. Let $v(m) \in V_{\mathbb{Z},1}$ be a corresponding sequence of integral classes, such that $Q(v(m), \sigma_{I,u}(z(m)))$ converges for every $I \subseteq \{1, \dots, n\}$ and every $u \in F^{|I|} \cap H_{\mathbb{C}}$ (see Sect. 3.3). Then the following four things are true:*

- (i) *The sequence $v(m)$ is bounded, hence takes only finitely many values.*
- (ii) *Let $v \in V_{\mathbb{Z},1}$ be a point of accumulation. Then there are positive integers a_1, \dots, a_n with the property that $a_1 N_1 v + \dots + a_n N_n v = 0$.*
- (iii) *There is a vector $w \in \mathbb{C}^n$ such that*

$$e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} e^{-i\delta} v(m) \rightarrow e^{-\sum w_j N_j} e^{-i\delta} v$$

along a subsequence of the original sequence.

- (iv) *For each $k = 1, \dots, n$, we have*

$$e^{-\sum w_j N_j} e^{-i\delta} N_k v = e^{-\sum \text{Re } w_j N_j} N_k v^{0,0} = N_k v^{0,0},$$

which implies that the vector $N_k v$ is a rational Hodge class of type $(-1, -1)$ in the mixed Hodge structure $(M \cap H, e^{\sum w_j N_j} \Psi(0))$.

5.3 Proof of the technical result

The proof proceeds through a sequence of lemmas. In analogy with the notation used in Sect. 3.4, we define $N = y_1 N_1 + \dots + y_n N_n$, and observe that $M = M(N, W)$ is the relative weight filtration for N . Consequently, we have $M_{-1} \subseteq W_{-1}$ and $M_0 + W_{-1} = W_0$, and $M \cap H = W(N)[-1]$ is the shifted monodromy weight filtration for N on H .

Lemma 5.2 *There is a unique element $v_0 \in M_0 \cap F^0 \cap V_{\mathbb{R},1}$ with $Nv_0 = 0$.*

Proof The uniqueness of such an element is clear; indeed, $\ker N \cap H$ is a mixed Hodge structure of weight ≤ -1 , which implies that $F^0 \cap \ker N \cap H_{\mathbb{R}} = \{0\}$. It remains to show that a suitable element v_0 always exists. Since $M_0 + H_{\mathbb{Q}} = V_{\mathbb{Q}}$, we can certainly find an element $v \in M_0 \cap V_{\mathbb{Q}}$ that lifts $1 \in \mathbb{Q}$. Since (M, F) is \mathbb{R} -split, we can replace v by its component in the space $I^{0,0}(M, F)$ and assume that v is real and lies in $I^{0,0}(M, F)$. Then Nv belongs to $I^{-1,-1}(M \cap H, F \cap H)$ and hence to $W(N)_{-1}$, and so there is an element $h \in H_{\mathbb{R}}$ with $Nv = Nh$. Again replacing h by one of its components, we may assume that $h \in I^{0,0}(M \cap H, F \cap H)$. But now $v_0 = v - h$ satisfies all the required conditions. □

Fix a norm $\|-\|$ on the vector space $V_{\mathbb{C}}$. As in Sect. 3.7, the analysis in this section depends mostly on a single difficult statement, namely that $\|v_0\|$ remains bounded as $y_1, \dots, y_n \rightarrow \infty$. This is a special case of a more general theorem due to P. Brosnan and G. Pearlstein [6], and as in their work, relies on the SL_2 -Orbit Theorem of K. Kato, C. Nakayama, and S. Usui [23]. Observe that the pair $(W, e^{iN}F)$ defines an \mathbb{R} -split mixed Hodge structure, due to the fact that (M, F) splits over \mathbb{R} . Since $Nv_0 = 0$, it is obvious that v_0 is the unique real element in $I^{0,0}(W, e^{iN}F)$ that maps to $1 \in \text{Gr}_0^W$; said differently, v_0 is the image of $1 \in \text{Gr}_0^W$ under the canonical splitting of $(W, e^{iN}F)$ [23, Sect. 1.2].

Lemma 5.3 *There are constants $C > 0$ and $\alpha > 0$, such that $\|v_0\| \leq C$ for all $y_1, \dots, y_n \geq \alpha$.*

Proof Without loss of generality, we may suppose that $y_1 \geq \dots \geq y_n \geq \alpha$. [23, Theorem 0.5] implies that the canonical splitting of $(W, e^{iN}F)$ has a power series expansion in non-positive powers of $y_1/y_2, \dots, y_{n-1}/y_n$, and y_n ; the series converges provided that $y_1/y_2 > \beta, \dots, y_{n-1}/y_n > \beta$, and $y_n > \beta$. Arguing as in the proof of Lemma 3.12, we conclude that the canonical splitting is uniformly bounded for all $y_1, \dots, y_n \geq \alpha$, once we take α sufficiently large. The same is therefore true for the image of $1 \in \text{Gr}_0^W$ under the canonical splitting; but this image is precisely v_0 . \square

For $v \in V_{\mathbb{C}}$, define $Z(y, v) = \max_{k \geq 0} \|N^k v\|$. As before, we have to show that the norm $\|v\|$ of a real vector $v \in V_{\mathbb{R},1}$ is controlled by the size of the pairings $Q(v, \sigma_{I,u}(z))$, once y_1, \dots, y_n are sufficiently large.

Lemma 5.4 *Let $B(z, v)$ denote the supremum of $|Q(v, \sigma_{I,u}(z))|$, taken over $I \subseteq \{1, \dots, n\}$ and $u \in F^{|I|} \cap H_{\mathbb{C}}$ with $\|u\| \leq 1$. Then there are constants $C > 0$ and $\alpha > 0$, such that*

$$Z(y, v) \leq C \cdot B(z, v)$$

for every $v \in V_{\mathbb{R},1}$ and every $z \in \mathbb{H}^n$ with $y_j = \text{Im } z_j \geq \alpha$ and $0 \leq \text{Re } z_j \leq 1$.

Proof Given a vector $v \in V_{\mathbb{C}}$, we let $v^{p,q} \in I^{p,q}(M, F)$ denote its components relative to Deligne’s decomposition. As in Sect. 3.7, we may replace v by $e^{-\sum x_j N_j} v$ without affecting the statement we are trying to prove. Setting $w = e^{-\Gamma(s)} e^{-iN} e^{-i\delta} v$, we easily see that the norm of each vector $N^k w^{p,q}$ with $p \leq -1$ is bounded by a constant times $B(z, v)$. We again define

$$v' = e^{i\delta} e^{iN} e^{\Gamma(s)} e^{-iN} \cdot \sum_{p \leq -1} e^{iN} w^{p,q},$$

and observe that $Z(y, v')$ is bounded by a fixed multiple of $B(z, h)$ by a version of Lemma 3.2. A useful observation is that $v' \in H_{\mathbb{C}}$; this is because Gr_0^W is of type $(0, 0)$ at every point $z \in \mathbb{H}^n$. By construction, we have $v - v' \in e^{i\delta} e^{iN} e^{\Gamma(s)} F^0$; since $v \in V_{\mathbb{R},1}$, it is therefore possible to write

$$v = v' + e^{i\delta} e^{iN} e^{\Gamma(s)} (v_0 + h)$$

for a unique choice of $h \in F^0 \cap H_{\mathbb{C}}$.

To continue, we let $g = v' + e^{i\delta} e^{iN} e^{\Gamma(s)} v_0 - v_0$; note that this vector belongs to $H_{\mathbb{C}}$. Since $Nv_0 = 0$, and since $\|v_0\|$ is uniformly bounded due to Lemma 5.3, we still have $Z(y, g)$ bounded by a constant multiple of $B(z, v)$. We can now rewrite the equation from above as $v - v_0 = g + e^{i\delta} e^{iN} e^{\Gamma(s)} h$. Remembering that $v - v_0$ is a real vector, we obtain the relation

$$\bar{g} - g = e^{i\delta} e^{iN} e^{\Gamma(s)} h - e^{-i\delta} e^{-iN} e^{\overline{\Gamma(s)}} \bar{h}.$$

From Proposition 3.13, we deduce that $Z(y, h)$ is bounded by a constant times $Z(y, g)$, and hence by a constant multiple of $B(z, v)$, provided that $y_1, \dots, y_n \geq \alpha$. But now the formula $v = v_0 + g + e^{i\delta} e^{iN} e^{\Gamma(s)} h$, together with Lemma 5.3, shows that the same is true for $Z(y, v)$. □

Once again, this single inequality is all that one needs to prove Theorem 5.1

Proof By the inequality in Lemma 5.4, $\|v(m)\|$ remains bounded as $m \rightarrow \infty$. Since $v(m) \in V_{\mathbb{Z},1}$, the sequence can take only finitely many values, proving Theorem 5.1(i). We can then pass to a subsequence, and assume for the remainder of the argument that $v(m) = v$ for some $v \in V_{\mathbb{Z},1}$. Arguing as in the proof of Lemma 3.6, we conclude from the boundedness of $\sum y_j(m) N_j v$ that v satisfies Theorem 5.1(ii). We also see that there is a further subsequence along which $\sum z_j(m) N_j v = \sum w_j(m) N_j v$, where the sequence of $w(m) \in \mathbb{C}^n$ converges to a vector $w \in \mathbb{C}^n$. This implies Theorem 5.1(iii).

Finally, we need to establish Theorem 5.1(iv). From the convergence of $Q(v, \sigma_{\{k\},u}(z(m)))$ for $u \in F^1 \cap H_{\mathbb{C}}$, we deduce as in the proof of Theorem 3.5 that

$$N_k v \in e^{\sum w_j N_j} e^{i\delta} (F^{-1} \cap H_{\mathbb{C}}).$$

This means that the vector $e^{-\sum \text{Re } w_j N_j} N_k v$ is a real Hodge class of type $(-1, -1)$ in the mixed Hodge structure $(M \cap H, e^{i\delta+i \sum \text{Im } w_j N_j} (F \cap H))$. Lemma 5.5 implies that $N_k v$ lies in the kernel of the operator $\delta + \sum \text{Im } w_j N_j$, and that

$$e^{-\sum w_j N_j} e^{-i\delta} N_k v = e^{-\sum \text{Re } w_j N_j} N_k v = N_k v^{0,0}.$$

This gives Theorem 5.1(iv) and concludes the proof. □

Lemma 5.5 *Let (W, F) be an \mathbb{R} -mixed Hodge structure, and let $v \in W_{2p} \cap F^p$ be a real Hodge class of type (p, p) . Let $\delta \in L_{\mathbb{R}}^{-1, -1}(W, F)$ be the unique element for which $(W, e^{-i\delta}F)$ splits over \mathbb{R} . Then $\delta v = 0$, and consequently $v \in I^{p,p}(W, e^{-i\delta}F)$.*

Proof Since v defines a morphism of \mathbb{R} -mixed Hodge structures $\mathbb{R}(-p) \rightarrow (W, F)$, the functoriality of δ (see [23, Lemma 1.6] for a proof) implies that $\delta v = 0$. It follows that v is also a real Hodge class of type (p, p) in $(W, e^{-i\delta}F)$. □

5.4 Graphs of admissible normal functions

Recall that we defined T_v as the subset of the étalé space of $\mathcal{V}_{\mathbb{Z}}$, consisting of those points that map to $1 \in \mathbb{Z}$. Theorem 5.1 is strong enough to conclude that T_v has an analytic closure inside of $T(F_0\mathcal{M})$. Note that Corollary 3.7, to the effect that $\varepsilon(T_{\mathbb{Z}}) \subseteq T(F_0\mathcal{M})$ is closed analytic, can be viewed as the special case $v = 0$.

Theorem 5.6 *The topological closure of T_v inside $T(F_0\mathcal{M})$ is an analytic subset.*

Proof We shall use both the space $T(F_0\mathcal{M})$, as well as the space $T(F_0\mathcal{H}_{\mathcal{O}}^e)$ coming from the canonical extension; since $F_0\mathcal{H}_{\mathcal{O}}^e \subseteq F_0\mathcal{M}$, they are related by a holomorphic mapping $g: T(F_0\mathcal{M}) \rightarrow T(F_0\mathcal{H}_{\mathcal{O}}^e)$. We also have holomorphic mappings $\varphi: T_v \rightarrow T(F_0\mathcal{M})$ and $\psi: T_v \rightarrow T(F_0\mathcal{H}_{\mathcal{O}}^e)$ with $\psi = g \circ \varphi$. Let $T_v(v)$ denote the connected component of T_v containing a given vector $v \in V_{\mathbb{Z},1}$. It suffices to show that the image of each $T_v(v)$ under the holomorphic mapping φ has an analytic closure; this is because (i) in Theorem 5.1 assures us that only finitely many of these image closures can meet at any given point of $T(F_0\mathcal{M})$.

Fix a vector $v \in V_{\mathbb{Z},1}$. We may clearly assume for the remainder of the argument that v satisfies (ii)–(iv) in Theorem 5.1, for otherwise, the image of $T_v(v)$ is already closed in a neighborhood of $0 \in \Delta^n$ and there is nothing to prove. In particular, we have $a_1N_1v + \dots + a_nN_nv = 0$, and $e^{-\sum w_jN_j}e^{-i\delta}N_kv = N_kv^{0,0}$ for some $w \in \mathbb{C}^n$. Replacing v by $e^{-\sum \text{Re } w_jN_j}v$, we arrange that $w = 0$, at the cost of having $v \in V_{\mathbb{R},1}$.

We can use this information to show that the image of $T_v(v)$ under ψ has an analytic closure inside $T(F_0\mathcal{H}_{\mathcal{O}}^e)$. Let $D = \Delta^n - (\Delta^*)^n$, let $p: T(F_0\mathcal{H}_{\mathcal{O}}^e) \rightarrow \Delta^n$ denote the projection map, and set $E = p^{-1}(D)$. Clearly, $\psi(T_v(v))$ is a closed analytic subset of $T(F_0\mathcal{H}_{\mathcal{O}}^e) - E$, of pure dimension n . To prove that its closure remains analytic, it suffices to show that the intersection of the closure with E is contained in a countable union of images of complex manifolds of dimension at most $n - 1$. Indeed, this implies

that the intersection has $2n$ -dimensional Hausdorff measure equal to zero, and we conclude by a result of E. Bishop’s [1, Lemma 9].

By construction of the canonical extension, the mapping $\psi : T_v(v) \rightarrow T(F_0\mathcal{H}_\varphi^e)$ is given in coordinates by the formula

$$\psi : \mathbb{H}^n \rightarrow \Delta^n \times \text{Hom}(F^0H_{\mathbb{C}}, \mathbb{C}), \quad (z_1, \dots, z_n) \mapsto (e^{2\pi iz_1}, \dots, e^{2\pi iz_n}, f_z)$$

where $f_z : F^0H_{\mathbb{C}} \rightarrow \mathbb{C}$ is the linear functional

$$u \mapsto f_z(u) = Q(v, e^{i\delta} e^{\sum z_j N_j} e^{\Gamma(s)} u).$$

Now we compute that

$$\begin{aligned} & e^{-\Gamma(s)} e^{-\sum z_j N_j} e^{-i\delta} v \\ &= e^{-\Gamma(s)} e^{-i\delta} v + \sum_{k=1}^{\infty} (-1)^k e^{-\Gamma(s)} (z_1 N_1 + \dots + z_n N_n)^k v^{0,0}, \end{aligned}$$

and hence we have

$$\begin{aligned} f_z(u) &= Q(e^{-\Gamma(s)} e^{-\sum z_j N_j} e^{-i\delta} v, u) \\ &= Q(e^{-\Gamma(s)} e^{-i\delta} v, u) \\ &\quad + \sum_{k=1}^{\infty} (-1)^k Q(e^{-\Gamma(s)} (z_1 N_1 + \dots + z_n N_n)^k v^{0,0}, u) \end{aligned}$$

for every $u \in F^0H_{\mathbb{C}}$. In particular, remembering that $\Gamma(s) \in \mathfrak{q}$, we find that when u belongs to the subspace $I^{1,1}(M \cap H, F \cap H)$, then

$$f_z(u) = Q(e^{-\Gamma(s)} e^{-i\delta} v, u) - Q(z_1 N_1 v^{0,0} + \dots + z_n N_n v^{0,0}, u). \tag{5.1}$$

It is now easy to determine the points in the closure. Fix a subset $I \subseteq \{1, \dots, n\}$ of size k , and consider the stratum $D_I \subseteq D$ of points with $s_j \neq 0$ for $j \in I$ and $s_j = 0$ for $j \notin I$; note that $\dim D_I = k$. Suppose that $z(m) \in \mathbb{H}^n$ is a sequence of points for which $\psi(z(m))$ converges to a point over $s_0 \in D_I$. We have $N_j v = 0$ for $j \in I$, and since $a_1 N_1 v + \dots + a_n N_n v = 0$, the span of $N_1 v, \dots, N_n v$ has dimension at most $n - k - 1$. Using the convergence of $f_{z(m)}(u)$ in (5.1) and arguing as in the proof of Lemma 3.6, we see that $\sum z_j(m) N_j v^{0,0}$ converges to $\sum w_j N_j v^{0,0}$ for some vector $w \in \mathbb{C}^n$. Thus every limit point over $s_0 \in D_I$ is of the form (s_0, f) , where $f : F^0H_{\mathbb{C}} \rightarrow \mathbb{C}$ is given by the formula

$$f(u) = Q(e^{-\Gamma(s_0)}e^{-i\delta}v, u) + \sum_{k=1}^{\infty} (-1)^k Q(e^{-\Gamma(s_0)}(w_1N_1 + \dots + w_nN_n)^k v^{0,0}, u)$$

for some choice of $w \in \mathbb{C}^n$. Evidently, such points are parametrized by a linear space of dimension at most $n - k + 1$. It follows that the intersection of $p^{-1}(D_I)$ with the closure of $T_v(v)$ is contained in a complex-analytic subset of dimension at most $\dim D_I + (n - 1 + k) = n - 1$; as explained above, this suffices to conclude that $\psi(T_v(v))$ has an analytic closure inside $T(F_0\mathcal{H}_\theta^e)$.

To finish the proof, we observe that the preimage of $\overline{\psi(T_v(v))}$ under g is an analytic subset of $T(F_0\mathcal{M})$ whose intersection with $T(F_0\mathcal{H}_\theta^e)$ equals $\varphi(T_v(v))$. By a well-known result in complex analysis, this implies that the closure of $\varphi(T_v(v))$ inside $T(F_0\mathcal{M})$ is itself analytic, and thereby concludes the proof. \square

Similarly, the graph of the normal function $v: X \rightarrow J(\mathcal{H})$ has an analytic closure inside of $\bar{J}(\mathcal{H})$.

Corollary 5.7 *Let $X = (\Delta^*)^n$ and $\bar{X} = \Delta^n$, and let $v: X \rightarrow J(\mathcal{H})$ be a normal function that is admissible relative to \bar{X} . Then the topological closure of the graph $v(X)$ inside of $\bar{J}(\mathcal{H})$ is an analytic subvariety.*

Proof Since the quotient map $T(F_0\mathcal{M}) \rightarrow \bar{J}(\mathcal{H})$ is open by Lemma 2.11, this follows immediately from Theorem 5.6. \square

When the singularity of v is a nonzero torsion class, then the graph of v is already closed (this observation is due to P. Brosnan).

Lemma 5.8 *Suppose that the singularity of v at $0 \in \Delta^n$ is a nonzero torsion class. Then the closure of the graph of v in $\bar{J}(\mathcal{H})$ does not meet $p^{-1}(0)$.*

Proof It suffices to show that if the singularity of v is torsion, but the closure of T_v in $T(F_0\mathcal{M})$ contains a point over $0 \in \Delta^n$, then v actually has no singularity. If there is such a point in the closure, then by Theorem 5.1, there exists a class $v \in V_{\mathbb{Z},1}$ that satisfies the conditions listed there; in particular, $a_1N_1v + \dots + a_nN_nv = 0$ for positive integers a_1, \dots, a_n . Consider now the complex

$$\left[H_{\mathbb{Q}} \rightarrow \bigoplus_j N_j(H_{\mathbb{Q}}) \rightarrow \bigoplus_{j < k} N_jN_k(H_{\mathbb{Q}}) \rightarrow \dots \rightarrow N_1 \dots N_n(H_{\mathbb{Q}}) \right]$$

that computes the intersection cohomology of the local system $\mathcal{H}_{\mathbb{Q}}$ on the polydisk Δ^n [11, p. 219]. With rational coefficients, the singularity of v is

represented by the class of (N_1v, \dots, N_nv) in the first cohomology group of the complex. Since the singularity is a torsion class, there is a vector $h \in H_{\mathbb{Q}}$ such that $N_jv = N_jh$ for all j . Now $a_1N_1h + \dots + a_nN_nh = 0$, and so we have $h \in M_{-1} \cap H_{\mathbb{Q}}$, which implies that $N_jv \in M_{-3} \cap H_{\mathbb{Q}}$. Assertion (iv) in Theorem 5.1 then forces $N_jv = 0$ for all $j = 1, \dots, n$, and so v has no singularities on Δ^n . \square

6 Examples

This is a collection of three examples that illustrates various properties of our Néron model. Note that the variations of Hodge structure considered here are naturally polarized, and so we have $F_0\check{\mathcal{M}} \simeq F_0\mathcal{M}$ and $T(F_0\check{\mathcal{M}}) \simeq T(F_0\mathcal{M})$.

6.1 Non-unipotent monodromy

In this section, we describe a simple one-parameter family of elliptic curves in which the local monodromy is not unipotent. This illustrates the difference between $\bar{J}(\mathcal{H})$ and the identity component of the Néron model constructed in [7]. Let $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be the elliptic curve with an automorphism of order six; here $\tau = e^{i\pi/3}$ satisfies $\tau^2 = \tau - 1$, and the automorphism is given by multiplication by τ . We consider the trivial family $E \times \Delta^*$, as well as its quotient by $\mathbb{Z}/6\mathbb{Z}$; a generator acts on Δ as multiplication by τ , and on E by the automorphism. We denote the local system corresponding to the quotient by \mathcal{H} ; our aim is to describe the structure of $\bar{J}(\mathcal{H}) \rightarrow \Delta$.

We first work out the monodromy. Let α and β be the standard basis for $H_1(E, \mathbb{Z})$; in the usual fundamental domain inside \mathbb{C} , the cycle α is the image of the line segment from 0 to 1, and the cycle β that of the segment from 0 to τ . Drawing a picture, it is clear that the automorphism acts by

$$\alpha \mapsto \beta, \quad \beta \mapsto \beta - \alpha.$$

Letting α^* and β^* denote the dual basis for $H^1(E, \mathbb{Z})$, we also have

$$\alpha^* \mapsto -\beta^*, \quad \beta^* \mapsto \alpha^* + \beta^*.$$

Thus the monodromy operator T is given by

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

and one easily checks that it has eigenvalues τ and $\bar{\tau} = -\tau^2$. Also, $\det(T - \text{id}) = 1$, and so the local system (over \mathbb{Z}) has vanishing H^0 and H^1 . It is clear

from the construction that $\alpha^* + \tau\beta^*$ is an eigenvector for τ (indeed, it restricts to a holomorphic 1-form on each fiber).

Let s be the holomorphic coordinate on Δ . For our construction, we need the minimal extension of the flat vector bundle with monodromy T ; according to [31], this is given by the Deligne lattice on which the residues of the connection lie in $(-1, 0]$. Thus the correct extension is given by $\mathcal{O}e_1 \oplus \mathcal{O}e_2$, with connection

$$\nabla e_1 = -e_1 \otimes \frac{ds}{6s}, \quad \nabla e_2 = -e_2 \otimes \frac{5ds}{6s}.$$

Let $\mathbb{H} \rightarrow \Delta^*$, with $s = e^{2\pi iz}$, be the universal covering space; on \mathbb{H} , a flat section $\sigma(z)$ with $\sigma(z + 1) = \tau\sigma(z)$ is then found by solving $f'(z) - \pi if(z)/3 = 0$, and so

$$\sigma(z) = e^{\pi iz/3} e_1.$$

Neglecting constants, we have $\sigma(z) = \alpha^* + \tau\beta^*$; thus $\omega = e_1$ is a section of F^0 of the canonical extension (since it gives a holomorphic 1-form on each fiber), and

$$\alpha^* + \tau\beta^* = e^{\pi iz/3} \omega.$$

Thus we see that

$$\int_{m\alpha+n\beta} \omega = (m + \tau n) \cdot e^{-\pi iz/3} = (m + \tau n) \cdot e^{\pi y/3} \cdot e^{-\pi ix/3},$$

which goes to infinity with y unless $m = n = 0$. It follows that the closure of the family of integral lattices inside the line bundle (whose dual is spanned by ω) only adds one point; thus the fiber of the Néron model $\bar{J}(\mathcal{H})$ over $0 \in \Delta$ is a copy of \mathbb{C} . This is what it should be, given that we started from a family of elliptic curves.

Next, we look at admissible normal functions and their extensions. By definition, admissibility can be tested by pulling back along a branched cover ($s = t^6$ in our case) to make the monodromy unipotent [34]. Thus we only need to consider the family $E \times \Delta^*$. Admissibility implies that the normal function extends to a holomorphic mapping $\Delta \rightarrow E$. Lifting this to $g: \Delta \rightarrow \mathbb{C}$, we have

$$g(\tau t) - \tau g(t) \in \mathbb{Z} + \mathbb{Z}\tau,$$

because the normal function is pulled back from the original family. It is easy to see that we can choose g so that, in fact, $g(\tau t) = \tau g(t)$. This choice of g represents the pullback of the extended normal function; its value over the origin is $g(0) = 0$, and so the pullback of any admissible normal function to

the family $E \times \Delta$ has to go through the origin in \mathbb{C} . This is consistent with the Néron model constructed by P. Brosnan, G. Pearlstein, and M. Saito in [7]: its fiber over the origin is a single point, because the local system \mathcal{H} has no nontrivial sections on Δ^* .

It should be noted, however, that there are no constraints on the graphs of normal functions in our Néron model $\bar{J}(\mathcal{H}) \rightarrow \Delta$. In fact, as shown in Proposition 4.4, any holomorphic section of $\bar{J}(\mathcal{H}) \rightarrow \Delta$ is an admissible normal function; the reason why the pullback of such a section to $E \times \Delta$ has to pass through the origin is that the image of $\bar{J}(\mathcal{H})_0 \rightarrow E$ is a point.

6.2 A singular Néron model

The example in this section was suggested by M. Saito; it shows that the analytic space $\bar{J}(\mathcal{H})$ may be singular if $\dim X \geq 2$. We let $H_{\mathbb{Z}} = \mathbb{Z}^4$, with polarization given by the matrix

$$Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The monodromy action is given by the two nilpotent operators

$$N_1 = N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\omega \in \mathbb{C}$ be such that $\text{Im } \omega > 0$. Then $H_{\mathbb{Z}}$ carries an \mathbb{R} -split mixed Hodge structure $H_{\mathbb{C}} = I^{1,-1} \oplus I^{-1,1} \oplus I^{0,-2} \oplus I^{-2,0}$, where we set

$$I^{1,-1} = \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \omega \end{pmatrix}, \quad I^{-1,1} = \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \bar{\omega} \end{pmatrix},$$

$$I^{0,-2} = \mathbb{C} \begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}, \quad I^{-2,0} = \mathbb{C} \begin{pmatrix} 1 \\ \bar{\omega} \\ 0 \\ 0 \end{pmatrix}.$$

These data define an \mathbb{R} -split nilpotent orbit on $(\Delta^*)^2$, by the rule $(z_1, z_2) \mapsto e^{z_1 N_1 + z_2 N_2} F$, where F is given by the $I^{p,q}$. Evidently, it is the pullback of a nilpotent orbit from Δ^* , by the map $(z_1, z_2) \mapsto z_1 z_2$.

We now describe the sheaf $F_0\mathcal{M}$ and the analytic space $T(F_0\mathcal{M})$ over Δ^2 . Let the coordinates on Δ^2 be (s_1, s_2) . The Deligne extension is a trivial vector bundle of rank 4, with Hodge filtration given by the $I^{p,q}$. Thus $F_0\mathcal{M}$ is spanned by three sections,

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \omega \end{pmatrix}, \quad e_1 = \frac{1}{s_1} \begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \frac{1}{s_2} \begin{pmatrix} 1 \\ \omega \\ 0 \\ 0 \end{pmatrix}.$$

This gives a presentation for $F_0\mathcal{M}$ in the form

$$\mathcal{O} \xrightarrow{\begin{pmatrix} 0 \\ -s_1 \\ s_2 \end{pmatrix}} \mathcal{O}^3 \rightarrow F_0\mathcal{M} \rightarrow 0,$$

and so $T(F_0\mathcal{M})$ is the subset of $\Delta^2 \times \mathbb{C}^3$ given by the equation $s_1 v_1 = s_2 v_2$, using coordinates $(s_1, s_2, v_0, v_1, v_2)$. Thus $T(F_0\mathcal{M})$ is a vector bundle of rank 2 outside the origin, while the fiber over the origin is \mathbb{C}^3 . Moreover, the analytic space $T(F_0\mathcal{M})$ is clearly singular along the entire line $\mathbb{C}(0, 0, v_0, 0, 0)$.

Next, we look at the embedding of the set of integral points $T_{\mathbb{Z}}$. Let $h \in \mathbb{Z}^4$ be any integral vector. We compute that

$$Q(e_0, e^{-(z_1 N_1 + z_2 N_2)} h) = (z_1 + z_2)(h_3 + h_4 \omega) - (h_1 + h_2 \omega),$$

$$Q(e_j, e^{-(z_1 N_1 + z_2 N_2)} h) = -\frac{h_3 + h_4 \omega}{s_j} \quad (\text{for } j = 1, 2).$$

This means that $\varepsilon(T_{\mathbb{Z}}) \subseteq T(F_0\mathcal{M})$ is the closure of the image of the holomorphic mapping $\mathbb{H}^2 \times \mathbb{Z}^4 \rightarrow \Delta^2 \times \mathbb{C}^3$, which sends the point (z, h) to

$$\left(e^{2\pi i z_1}, e^{2\pi i z_2}, (z_1 + z_2)(h_3 + h_4 \omega) - (h_1 + h_2 \omega), \right. \\ \left. -\frac{h_3 + h_4 \omega}{e^{2\pi i z_1}}, -\frac{h_3 + h_4 \omega}{e^{2\pi i z_2}} \right).$$

Over $s_1 s_2 = 0$, the points in the closure are of the form $(s_1, s_2, -(h_1 + h_2 \omega), 0, 0)$. Let $J_0 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$ be the torus corresponding to the monodromy-invariant part of the mixed Hodge structure. The quotient $T(F_0\mathcal{M})/T_{\mathbb{Z}}$ has the following structure: over $(\Delta^*)^2$, the fibers are the two-dimensional intermediate Jacobians; over $(0, 0)$, the fiber is $J_0 \times \mathbb{C}^2$; over the remaining points with $s_1 s_2 = 0$, the fiber is $J_0 \times \mathbb{C}$. Moreover, $T(F_0\mathcal{M})/T_{\mathbb{Z}}$ is singular along the torus $J_0 \times \{(0, 0)\}$ over the origin.

Note In this case, the Zucker extension is not Hausdorff. In fact, the integral points are embedded into the ambient space $\Delta^2 \times \mathbb{C}^2$ via the holomorphic mapping $\mathbb{H}^2 \times \mathbb{Z}^4 \rightarrow \Delta^2 \times \mathbb{C}^2$, which takes the point (z, h) to

$$(e^{2\pi iz_1}, e^{2\pi iz_2}, (z_1 + z_2)(h_3 + h_4\omega) - (h_1 + h_2\omega), -(h_3 + h_4\omega)).$$

The closure of the image is much bigger than just the set of monodromy-invariant classes in $H_{\mathbb{Z}}$; to obtain the Zucker extension, therefore, one is taking a quotient by a non-closed equivalence relation, which can never produce a Hausdorff space.

6.3 A normal function with non-torsion singularity

In this section, we shall look at a simple example of a normal function on $(\Delta^*)^2$ with a non-torsion singularity at the origin in Δ^2 . The interesting point here is that the closure of its graph has a one-dimensional fiber over the origin.

The example is a family of elliptic curves; the corresponding variation of Hodge structure of weight -1 is a nilpotent orbit, which we describe by giving its limit mixed Hodge structure. So let $H_{\mathbb{Z}} = \mathbb{Z}^2$, with nilpotent operators

$$N_1 = N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and define the limit mixed Hodge structure by letting $I^{0,0} = \mathbb{C}(0, 1)$ and $I^{-1,-1} = \mathbb{C}(1, 0)$. The period mapping of the associated variation of Hodge structure is then given by $\tilde{\Phi}(z) = e^{z_1 N_1 + z_2 N_2} F$, and so the vector $(z_1 + z_2, 1)$ spans $\tilde{\Phi}(z)^0$.

We now introduce an admissible normal function through its variation of mixed Hodge structure. Let $V_{\mathbb{Z}} = H_{\mathbb{Z}} \oplus \mathbb{Z}$, and extend the operators above to

$$N_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

Thus the vector $v = (0, 0, 1)$ belongs to $V_{\mathbb{Z},1}$, and satisfies $N_1 v + N_2 v = 0$. Let $W_{-1} = H_{\mathbb{Z}}$ and $W_0 = V_{\mathbb{Z}}$. The \mathbb{R} -split mixed Hodge structure (M, F) with $I^{0,0}(M, F) = \mathbb{C}(0, 1, 0) \oplus \mathbb{C}(\lambda, 0, 1)$ and $I^{-1,-1}(M, F) = \mathbb{C}(1, 0, 0)$ defines a mixed nilpotent orbit $(W, e^{z_1 N_1 + z_2 N_2} F)$, and one can easily check that it is admissible. Let ν denote the corresponding admissible normal function on $(\Delta^*)^2$.

We will now determine the closure of T_ν inside $T(F_0\mathcal{M})$. In this situation, $F_0\mathcal{M}$ is a trivial line bundle on Δ^2 , whose pullback to \mathbb{H}^2 is spanned by the section $(z_1 + z_2, 1)$. The embedding $T_\nu \hookrightarrow T(F_0\mathcal{M})$ now takes the form

$$\mathbb{H}^2 \times V_{\mathbb{Z},1} \rightarrow \Delta^2 \times \mathbb{C},$$

and is given by the formula

$$(z_1, z_2, a, b, 1) \mapsto (e^{2\pi iz_1}, e^{2\pi iz_2}, a - b(z_1 + z_2) + (z_2 - z_1)).$$

From this, it is easy to determine the closure of the graph. Over a point $(s_1, 0)$ with $s_1 \neq 0$, we only get points in the closure when $b = 1$, and so the fiber consists of all points $a - 2z_1$ with $e^{2\pi iz_1} = s_1$. Similarly, the fiber over $(0, s_2)$ with $s_2 \neq 0$ is the discrete set of points $a + 2z_2$ with $e^{2\pi iz_2} = s_2$. More interesting is the fiber over $(0, 0) \in \Delta^2$. By taking $a = b = 0$ and $z_2 = z_1 + w$ with $w \in \mathbb{C}$ arbitrary and $\text{Im } z_1 \rightarrow \infty$, we see that the fiber consists of all of \mathbb{C} .

The quotient $\bar{J}(\mathcal{H}) = T(F_0\mathcal{M})/T_{\mathbb{Z}}$ is a family of elliptic curves over $(\Delta^*)^2$, with fibers over $s_1s_2 = 0$ copies of \mathbb{C}^* . The discussion above shows that ν extends to an admissible normal function over $\Delta^2 - \{(0, 0)\}$, but that the closure of the graph of ν inside $\bar{J}(\mathcal{H})$ contains the entire fiber \mathbb{C}^* over $(0, 0)$. As mentioned in Sect. 4.3, this is evidence that there can probably not exist a Néron model (in the sense originally intended by P. Griffiths) for this family that is Hausdorff as a topological space.

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