A LOG RESOLUTION FOR THE THETA DIVISOR OF A HYPERELLIPTIC CURVE

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Abstract. In this paper, we prove that the theta divisor of a smooth hyperelliptic curve has a natural and explicit embedded resolution of singularities using iterated blowups of Brill-Noether subvarieties. We also show that the Brill-Noether stratification of the hyperelliptic Jacobian is a Whitney stratification.

Introduction

Let C be a smooth projective curve of genus $g \geq 1$. Let $\operatorname{Jac}(C)$ be the Jacobian of C, and let $\Theta \subseteq \operatorname{Jac}(C)$ be the theta divisor. The purpose of this paper is to give a natural and explicit log resolution of the pair $(\operatorname{Jac}(C), \Theta)$ when C is a hyperelliptic curve.

Recall that the Brill-Noether variety $W^r_{g-1}(C)$ parametrizes line bundles $L \in \operatorname{Pic}^{g-1}(C)$ of degree $g-1$ with $h^0(L) \geq r+1$. According to a theorem by Riemann, we can choose an isomorphism $\operatorname{Jac}(C) \cong \operatorname{Pic}^{g-1}(C)$ so that the theta divisor Θ becomes identified with $W_{g-1}(C) := W^0_{g-1}(C)$. The Abel-Jacobi map from the symmetric product $C_{g-1} := \operatorname{Sym}^{g-1}(C)$ gives a resolution of singularities of Θ, which is useful for answering many geometric questions about Jacobian varieties. However, if one wants to investigate the geometry of the embedding $\Theta \subseteq \operatorname{Jac}(C)$, one needs instead a log resolution of the pair $(\operatorname{Jac}(C), \Theta)$. Inspired by a global study of the vanishing cycle functor for divisors, we are lead to the question of finding an explicit log resolution in the case of hyperelliptic theta divisors. Since the log resolution is of a purely geometric nature, we leave the actual computation of vanishing cycles to another paper.

When C is a hyperelliptic curve of genus $g \geq 1$, we have a lot of very precise information about the chain of subvarieties

$$\Theta = W_{g-1}(C) \supseteq W^1_{g-1}(C) \supseteq \cdots \supseteq W^n_{g-1}(C), \tag{1}$$

where $n = \left\lfloor \frac{g-1}{2} \right\rfloor$ is the maximal integer such that $W^n_{g-1}(C) \neq \varnothing$. First, the dimension of $W^r_{g-1}(C)$ is equal to $g-1-2r$ and $W^r_{g-1}(C)$ is reduced (see Proposition A.1). Second, the singular locus of $W^r_{g-1}(C)$ is exactly $W^{r+1}_{g-1}(C)$. Third, the multiplicity of the theta divisor at a point $L \in \operatorname{Pic}^{g-1}(C)$ is equal to $h^0(L)$ by the Riemann singularity theorem, and so $W^r_{g-1}(C)$ is exactly the set of points of multiplicity $\geq r+1$ on Θ. (See [1, Chapter IV, §4] for details.)

These facts immediately suggest that one might be able to get a log resolution of the pair $(\operatorname{Jac}(C), \Theta)$ by successively blowing up the Brill-Noether subvarieties $W^r_{g-1}(C)$ in the order from smallest to largest. This guess turns out to be correct, but it requires quite a bit of work to prove rigorously that it works.

More precisely, we use the following iterative procedure, consisting of n steps. In the first step, we blow up $\operatorname{Jac}(C)$ along the smallest subvariety $W^n_{g-1}(C)$, and denote the blowup by $\pi_1\colon \operatorname{bl}_1(\operatorname{Jac}(C)) \to \operatorname{Jac}(C)$. In the second step, we blow up $\operatorname{bl}_1(\operatorname{Jac}(C))$ along the strict transform of $W^{n-1}_{g-1}(C)$, and denote the new blowup by $\pi_2\colon \operatorname{bl}_2(\operatorname{Jac}(C)) \to \operatorname{Jac}(C)$. In the i-th step, we blow up $\operatorname{bl}_{i-1}(\operatorname{Jac}(C))$ along the strict transform of $W^{n+1-i}_{g-1}(C)$, and denote the new blowup by $\pi_i\colon \operatorname{bl}_i(\operatorname{Jac}(C)) \to \operatorname{Jac}(C)$. This process stops after the

n-th step. The strict transforms of the exceptional divisor give us a sequence of divisors $Z_0, Z_1, \ldots, Z_{n-1}$, with Z_i sitting over the locus $W^{n-i}_{g-1}(C)$ of points of multiplicity $\geq n+1-i$. Let $\tilde{\Theta}$ denote the strict transform of the theta divisor. With this notation, our main result is the following.

Theorem A. *If C is a smooth hyperelliptic curve, then in the sequence of blowups described above, $\pi_n : \mathrm{bl}_n(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$ is a log resolution of $(\mathrm{Jac}(C), \Theta)$, where*

$$\pi_n^*(\Theta) = \tilde{\Theta} + \sum_{i=0}^{n-1} (n+1-i) Z_i$$

is a divisor with simple normal crossing support. Moreover, at the i-th stage of the construction, the strict transform of $W^{n-i}_{g-1}(C)$ becomes smooth, and so each blowup in the sequence is a blowup along a smooth center.

We can also describe the generic structure of the exceptional divisors.

Corollary B. *For $r = 1, \ldots, n$, every fiber of the projection*

$$Z_{n-r} \setminus (Z_0 \cup \cdots \cup Z_{n-r-1} \cup Z_{n-r+1} \cup \cdots \cup Z_{n-1}) \to W^r_{g-1}(C) \setminus W^{r+1}_{g-1}(C)$$

is isomorphic to the complement of a hypersurface of degree $r+1$ in $\mathbf{P}^{2r}$; the hypersurface is the $(r-1)^{\text{th}}$ secant variety of a rational normal curve of degree $2r$ in $\mathbf{P}^{2r}$.

There are a few other examples in the literature where this simple-minded procedure of successive blowups along singular loci produces a log resolution:

(1) Let X be the affine space of n-by-n matrices and let D be the hypersurface defined by the vanishing of the determinant. Let $D_i \subseteq D$ be the set of matrices of rank $\leq n-i$. According to [1, p. 69], one has $(D_i)_{\text{sing}} = D_{i+1}$, and D_i is exactly the set of points of multiplicity $\geq i$ on D. It is proved in [8, Chapter 4] and in [10] (using complete collineations) that one can construct a log resolution of the pair (X, D) by successively blowing up $D_n, D_{n-1}, \ldots, D_2$.

(2) Let $X = \mathbf{P}H^0(C, M)$ and let $D = \mathrm{Sec}^n(C)$ be the n-th secant variety of a smooth projective curve C, embedded by a line bundle M with $h^0(M) = 2n+3$ that separates $2n+2$ points. Setting $D_i = \mathrm{Sec}^{n-i+1}(C)$, Bertram [2, Page 440] proved that $(D_i)_{\text{sing}} = D_{i+1}$ and that D_i is again the set of points of multiplicity $\geq i$ on D. He also showed [2, Corollary 2.4] that successively blowing up $D_n, D_{n-1}, \ldots, D_2$ produces a log resolution of the pair (X, D).

Another common feature of these examples is that the divisor in question is a determinantal variety: this is easy to see for the space of matrices or for the secant variety of a rational normal curve, and is also true for the theta divisor of a hyperelliptic curve (which is the determinantal variety associated to a morphism of vector bundles over $\mathrm{Pic}^{g-1}(C)$).

The subvarieties in (1) induces a stratification

$$\mathrm{Jac}(C) = (\mathrm{Jac}(C) - \Theta) \sqcup \bigsqcup_{0 \leq r \leq n} (W^r_{g-1}(C) - W^{r+1}_{g-1}(C)),$$

which is called the Brill-Noether stratification.

Proposition C. *If C is a smooth hyperelliptic curve, then the Brill-Noether stratification of $\mathrm{Jac}(C)$ defined above is a Whitney stratification.*

Ideas of the proof. Let us discuss the method we use to prove Theorem A. Our main tool is Bertram's blowup construction for a chain of maps [2] (for a detailed and complete review of this construction, see §1).

One inconvenient point in the process above is that the Brill-Noether varieties $W^r_{g-1}(C)$ are not smooth, which makes it hard to keep track of conormal bundles and exceptional divisors in the various blowups. Fortunately, on a hyperelliptic curve, each $W^r_{g-1}(C)$ has a natural resolution of singularities by C_{g-1-2r}, the $(g-1-2r)$-th symmetric product of the curve, viewed as the space of effective divisors of degree $g-1-2r$ on C. If we let g^1_2 be the line bundle corresponding to the hyperelliptic map $h\colon C \to \mathbf{P}^1$, then the resolution of singularities is the Abel-Jacobi mapping

$$\delta_{n-r} : C_{g-1-2r} \to W^r_{g-1}(C), \quad D \mapsto rg^1_2 \otimes \mathcal{O}_C(D).$$

Since it is easier to blow up smooth varieties, we therefore modify the construction from above, and instead of the subvarieties $W^r_{g-1}(C)$, we work with the chain of maps $\{\delta_i\}_{i=0}^n$. The advantage is that we do not need to analyze the singularities of the proper transforms of $W^r_{g-1}(C)$ and how they intersect with exceptional divisors; instead, we transform the problem into checking that certain maps are embeddings (see Lemma 1.19), which eventually reduces to the calculation of certain conormal bundles. The projectivized conormal bundles that show up as exceptional divisors are closely related to secant bundles over symmetric products of $\mathbf{P}^1$; for that reason, Bertram's results about these secant bundles are another important tool that we use.

Outline of the paper. In §1, we recall Bertram's blowup construction in details. In §2, we set up notations for Abel-Jacobi maps and reduce the proof of Theorem A to two propositions (Proposition 2.2 and Proposition 2.1), which deal with the properties of two specific chains of maps between symmetric products and Jacobians. In §3, we review the construction of secant bundles and describe Bertram's results. In §4, we study some basic properties of Abel-Jacobi maps and the addition maps among symmetric products. In §5-§6, we prove Proposition 2.2 and Proposition 2.1, and thereby complete the proof of Theorem A for hyperelliptic curves of odd genus. The proof of Corollary B can be found at the end of §5. Finally, in §7, we outline a proof for hyperelliptic curves of even genus, which goes along the same lines but requires a few changes in the notation. In §8, we prove Proposition C, which is a technical result needed in a later paper. In §9, we propose some questions in the direction of this paper.

Notation.

(1) If V is a vector space, $\mathbf{P}(V)$ stands for the projective space of one-dimensional quotients of V. We use the same notation for vector bundles.

(2) Let $f : X \to Y$ be a morphism between smooth projective varieties. Let $Y_1 \subseteq Y$ be a subvariety. We use the notation

$$f^{-1}(Y_1) := Y_1 \times_Y X,$$

exclusively for the scheme-theoretic preimage, which is the fiber product of the two morphisms $X \to Y$ and $Y_1 \to Y$.

(3) Let $f : X \to Y$ be a morphism between smooth varieties. We denote by

$$df : f^*T^*_Y \to T^*_X$$

the induced morphism between cotangent bundles. If df is surjective, then

$$N^*_f = \operatorname{Ker}\bigl(df : f^*T^*_Y \to T^*_X\bigr)$$

denotes the conormal bundle of the morphism. In the case of a smooth subvariety $X \subseteq Y$, we also use the notation $N^*_{X|Y}$.

Acknowledgement. We thank Rob Lazarsfeld for suggesting the statement of the main theorem and we thank Gavril Farkas, Zhuang He, Carl Lian, András Lőrincz, Mirko Mauri and Botong Wang for helpful discussions. We thank Nero Budur for the discussion of the reducedness of $W^r_d(C)$. We also thank the Max-Planck-Institute for Mathematics for its hospitality and for providing us with excellent working conditions. Ch.S. was partially supported by NSF grant DMS- 1551677 and by a Simons Fellowship.

1. Bertram's blowup construction

In this section, we review Bertram's construction from [2, §2]. We consider sequences of blowups whose centers are determined by a chain of morphisms; the main result of this section is an inductive criterion for checking that such a sequence of blowups is a log resolution (see Lemma 1.19). Let X be a projective variety, not necessarily smooth.

Definition 1.1. A *proper chain* is a sequence of morphisms $\{f_i : X_i \to X\}_{i=0}^n$ from projective varieties X_i with the property that for each $0 \leq i < j \leq n$, there exists a commutative diagram

$$\begin{tikzcd} X_{i,j} \arrow[r, two heads, "g_{i,j}"] \arrow[d, "h_{i,j}"] & X_i \arrow[d, "f_i"] \\ X_j \arrow[r, "f_j"] & X \end{tikzcd}$$

so that $g_{i,j}$ is surjective and there is a proper inclusion $f_i(X_i) \subsetneq f_j(X_j)$. Note that $X_{i,j}$ is not necessarily the fiber product of f_i and f_j.

Let $\{f_j : X_j \to X\}$ be a proper chain. We now define the associated sequence of blowups, which exists under the assumption that certain morphisms are closed embeddings.

Definition 1.2. If f_0 is an *embedding*, we identify X_0 with its image and define:

$$\begin{aligned} \mathrm{bl}_1(X) &:= \text{the blowup of } X \text{ along } X_0. \\ \mathrm{bl}_1(X_j) &:= \text{the blowup of } X_j \text{ along } f_j^{-1}(X_0). \\ \mathrm{bl}_1(f_j) &:= \text{the unique lift of } f_j \text{ to a map } \mathrm{bl}_1(X_j) \to \mathrm{bl}_1(X). \end{aligned}$$

Note that $\mathrm{bl}_1(f_j)$ exists by the universal property of blowing up. Assuming that $\mathrm{bl}_i(X)$, $\mathrm{bl}_i(X_j)$ and $\mathrm{bl}_i(f_j)$ are already defined for some $1 \leq i \leq n-1$ and for all $j \geq i$, and that $\mathrm{bl}_i(f_i)$ is an *embedding*, we identify $\mathrm{bl}_i(X_i)$ with its image in $\mathrm{bl}_i(X)$, and further define:

$$\begin{aligned} \mathrm{bl}_{i+1}(X) &:= \text{the blowup of } \mathrm{bl}_i(X) \text{ along } \mathrm{bl}_i(X_i). \\ \mathrm{bl}_{i+1}(X_j) &:= \text{the blowup of } \mathrm{bl}_i(X_j) \text{ along } \mathrm{bl}_i(f_j)^{-1}(\mathrm{bl}_i(X_i)). \\ \mathrm{bl}_{i+1}(f_j) &:= \text{the unique lift of } \mathrm{bl}_i(f_j) \text{ to a map } \mathrm{bl}_{i+1}(X_j) \to \mathrm{bl}_{i+1}(X). \end{aligned}$$

Notation 1.3. To have a uniform notation, we set

$$\mathrm{bl}_0(f_i) := f_i, \quad \mathrm{bl}_0(X_i) := X_i, \quad \mathrm{bl}_0(X) := X.$$

This notation is going to be useful in the inductive proofs below.

Definition 1.4. Let $\{f_i : X_i \to X\}_{i=0}^n$ be a proper chain. If $\mathrm{bl}_{n+1}(X)$ is defined, we say that $\{f_i\}$ is a *chain of centers*. Concretely, this amounts to the (recursive) condition that the $n+1$ morphisms f_0, $\mathrm{bl}_1(f_1)$, $\dots$, $\mathrm{bl}_n(f_n)$ should all be closed embeddings. If X as well as the $n+1$ varieties X_0, $\mathrm{bl}_1(X_1)$, $\dots$, $\mathrm{bl}_n(X_n)$ are all smooth, then we say that $\{f_i\}$ is a *chain of smooth centers*. In this case, all the blowups in the sequence are blowups along smooth centers, and therefore the varieties $\mathrm{bl}_i(X)$ are smooth for $i = 1, \dots, n+1$.

We formulate an additional definition which ensures that the exceptional divisors in the final blowup $\mathrm{bl}_{n+1}(X)$ form a simple normal crossing divisor. We are going to refer to these conditions as the *NCD conditions.*

Definition 1.5. Suppose $\{f_i : X_i \to X\}_{i=0}^n$ is a chain of smooth centers. For each $i \le n$, let $E_{i,i+1} \subseteq \mathrm{bl}_{i+1}(X)$ be the exceptional divisor for the blowing-up of $\mathrm{bl}_i(X)$ along $\mathrm{bl}_i(X_i)$. For each $j \ge i+1$, let $E_{i,j} \subseteq \mathrm{bl}_j(X)$ be the scheme-theoretic inverse image of $E_{i,i+1}$ under the later blowups, as in the following diagram (with Cartesian squares):

$$\begin{tikzcd}
\mathrm{bl}_j(X) \arrow[r] & \mathrm{bl}_{i+1}(X) \arrow[r] & \mathrm{bl}_i(X) \\
E_{i,j} \arrow[r] \arrow[u, hook] & E_{i,i+1} \arrow[r] \arrow[u, hook] & \mathrm{bl}_i(X_i) \arrow[u, hook]
\end{tikzcd}$$

We say that $\{f_i\}_{i=0}^n$ is an *NCD chain* if, for each $0 \le i < j \le n$, the intersection

$$\mathrm{bl}_j(X_j) \cap E_{i,j} \subseteq \mathrm{bl}_j(X)$$

is transverse and the divisor $E_{0,j} + \cdots + E_{j-1,j} \subseteq \mathrm{bl}_j(X)$ has simple normal crossings. If this is the case, we use the notation

$$E_i := E_{i,n+1}$$

for the divisors in the final blowup, and we say that $\{E_i\}_{i=0}^n$ is *the set of exceptional divisors* of the chain $\{f_i\}_{i=0}^n$. It follows that in any NCD chain, the divisor

$$E_0 + E_1 + \cdots + E_n \subseteq \mathrm{bl}_{n+1}(X)$$

is a simple normal crossing divisor.

Remark 1.6. Using the above notation, suppose $\{f_i : X_i \to X\}_{i=0}^n$ is a chain of smooth centers. Then there is a natural embedding of $X - f_n(X_n)$ into $\mathrm{bl}_{n+1}(X)$ such that

$$\mathrm{bl}_{n+1}(X) - \bigcup_{0 \le i \le n} E_i = X - f_n(X_n).$$

This follows from the construction of the iterated blowups, because the set on the right is exactly the complement of all the centers in blowups.

Notation 1.7. Let S be a smooth variety and let $\{f_i : X_i \to X\}_{i=0}^n$ be a proper chain. It induces a new chain $\{f_i \times \mathrm{id} : X_i \times S \to X \times S\}_{i=0}^n$, i.e. the collection of maps that are f_i on the first factor and the identity on S.

Lemma 1.8. *Let $\{f_i : X_i \to X\}_{i=0}^n$ be an NCD chain. For each $0 \le i < j \le n+1$, let $E_{i,j} \subseteq \mathrm{bl}_j(X)$ and $F_{i,j} \subseteq \mathrm{bl}_j(X \times S)$ be the exceptional divisors associated to the chains $\{f_i\}_{i=0}^n$ and $\{f_i \times \mathrm{id}\}_{i=0}^n$ as in Definition 1.5. Then for $0 \le i < j \le n+1$, we have*

$$\mathrm{bl}_i(X \times S) = \mathrm{bl}_i(X) \times S, \quad \mathrm{bl}_i(X_j \times S) = \mathrm{bl}_i(X_j) \times S,$$

$$\mathrm{bl}_i(f_j \times \mathrm{id}) = \mathrm{bl}_i(f_j) \times \mathrm{id}, \quad F_{i,j} = E_{i,j} \times S.$$

Moreover, $\{f_i \times \mathrm{id}\}_{i=0}^n$ is again an NCD chain.

Proof. We prove these statements by induction on $i < j$. A general fact we use is that if $A \subseteq B$ is an embedding of smooth varieties, C is another smooth variety, then

$$\mathrm{bl}_{A \times C}(B \times C) = \mathrm{bl}_A(B) \times C.$$

For $i = 0, 1$, the statement follows from definition and the general fact above. Assume the statement holds for a given value of i. To prove it for $i+1$, we blow up along $\mathrm{bl}_i(X_i \times S)$ and its preimages. By the induction hypothesis, we have

$$\mathrm{bl}_{i+1}(X \times S) = \mathrm{bl}_{\mathrm{bl}_i(X_i \times S)}\big(\mathrm{bl}_i(X \times S)\big) = \mathrm{bl}_{\mathrm{bl}_i(X_i) \times S}\big(\mathrm{bl}_i(X) \times S\big) = \mathrm{bl}_{i+1}(X) \times S.$$

For any $j \geq i+1$, we likewise have

$$\mathrm{bl}_i(f_j \times \mathrm{id})^{-1}\big(\mathrm{bl}_i(X_i \times S)\big) = \mathrm{bl}_i(f_j)^{-1}\big(\mathrm{bl}_i(X_i)\big) \times S,$$

consequently, $\mathrm{bl}_{i+1}(X_j \times S)$ is the blow up of $\mathrm{bl}_i(X_j \times S)$ along $\mathrm{bl}_i(f_j \times \mathrm{id})^{-1}\big(\mathrm{bl}_i(X_i \times S)\big)$, which is $\mathrm{bl}_{i+1}(X_j) \times S$. Therefore the induced map

$$\mathrm{bl}_{i+1}(f_j \times \mathrm{id}) : \mathrm{bl}_{i+1}(X_j \times S) \to \mathrm{bl}_{i+1}(X \times S)$$

can be identified with $\mathrm{bl}_{i+1}(f_j) \times \mathrm{id}$. The NCD conditions can be proved in a similar fashion because the exceptional divisors and their intersections respect the product with S for the same reason. □

To prove that a proper chain is a chain of smooth centers and satisfies the NCD conditions, it is useful to have the following definition.

Definition 1.9. Suppose that $\{f_i : X_i \to X\}_{i=0}^n$ and $\{g_i : Y_i \to Y\}_{i=0}^n$ are two chains of centers. We say that a map $\phi : X \to Y$ is a *map of chains of centers* if, recursively,

$$\mathrm{bl}_i(\phi)^{-1}\big(\mathrm{bl}_i(Y_i)\big) = \mathrm{bl}_i(X_i), \quad 0 \leq i \leq n,$$

where $\mathrm{bl}_i(\phi)$ is defined at each stage by the universal property of blowing up. We say that the map ϕ is an *injective* map of chains of centers if in addition, $\mathrm{bl}_{n+1}(\phi)$ is injective.

Remark 1.10. Definition 1.9 can also be expressed in the following manner. A map $\phi : X \to Y$ is a map of chains of centers if, at each stage, the diagram

$$\begin{tikzcd} \mathrm{bl}_i(Y_i) \arrow[r] \arrow[d, hook, "\mathrm{bl}_i(g_i)"] & \mathrm{bl}_i(X_i) \arrow[d, hook, "\mathrm{bl}_i(f_i)"] \\ \mathrm{bl}_i(Y) \arrow[r, "\mathrm{bl}_i(\phi)"] & \mathrm{bl}_i(X) \end{tikzcd}$$

is Cartesian for every $0 \leq i \leq n$. Here the two vertical arrows are closed embeddings because $\{f_i\}$ and $\{g_i\}$ are chains of centers. Note that this is again a recursive definition, because we need the i-th diagram to be Cartesian in order to define the next map $\mathrm{bl}_{i+1}(\phi)$ by the universal property of blowing up.

Lemma 1.11. *Suppose that X, X_i, $X_{i,j}$ are smooth projective varieties for $0 \leq i < j \leq n$. Let $\{\phi_j : X_j \to X\}_{j=0}^n$ be a chain of smooth centers. Suppose that for each j, there is a chain of smooth centers $\{f_{i,j} : X_{i,j} \to X_j\}_{i=0}^{j-1}$ and that ϕ_j is a map of chains of centers. If each auxiliary chain $\{f_{i,j}\}_{i=0}^{j-1}$ satisfies the NCD conditions, then the original chain $\{\phi_j\}_{j=0}^n$ also satisfies the NCD conditions.*

Proof. The two assumptions – that ϕ_j is a map of chains of centers and that $\{\phi_j\}_{j=0}^n$ is a chain of smooth centers – imply that ϕ_j is an injective map of chains of centers. By Remark 1.6, the NCD conditions on $\{f_{i,j}\}_{i=0}^{j-1}$ guarantee that the complement of $X_j - f_{j-1,j}(X_{j-1,j})$ in $\mathrm{bl}_j(X_j)$ is a simple normal crossing divisor with j components. Therefore we can apply [2, Lemma 2.1]. Moreover, it follows from the proof of [2, Lemma 2.1] that for each $j \leq n$, $\mathrm{bl}_j(X_j)$ intersects each exceptional divisor in $\mathrm{bl}_j(X)$ transversely, and that the new exceptional divisors in $\mathrm{bl}_{j+1}(X)$ also intersect transversely. □

In the rest of this section, we discuss how to impose conditions in order to show that a proper chain $\{\phi_j\}$ has smooth centers, in the same spirit as in Lemma 1.11. These conditions are embedded in the proof of [2, Proposition 2.2]. By formulating them abstractly, we hope it will make our proof of Theorem A more transparent.

Lemma 1.12. *Let $f : X \to Y$ be a morphism between two smooth projective varieties. Let $F \subseteq Y$ be a smooth divisor. If $E := f^{-1}(F)$ is also a smooth divisor, then*

$$f^* N^*_{F|Y} = N^*_{E|X}.$$

Proof. Since F is a smooth divisor, we have $N^*_{F|Y} = \mathcal{O}_F(-F)$ by the conormal sequence. The same holds for the scheme-theoretic preimage $E = f^{-1}(F)$. Therefore

$$f^* N^*_{F|Y} = f^* \mathcal{O}_F(-F) = \mathcal{O}_E(-E) = N^*_{E|X}. \qquad \square$$

Lemma 1.13. *Let $f : X \to Y$ be a morphism between two smooth projective varieties. Let $Z \subseteq Y$ be a smooth subvariety such that $W := f^{-1}(Z)$ is smooth and properly contained in X, and denote by $\tilde{f} : \tilde{X} \to \tilde{Y}$ the induced morphism between the two blowups $\tilde{Y} = \mathrm{bl}_Z Y$ and $\tilde{X} = \mathrm{bl}_W X$, as in the following diagram.*

$$\begin{tikzcd} E \arrow[r, hook] & \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{Y} \arrow[d] & F \arrow[l, hook'] \\ & X \arrow[r, "f"] & Y & \end{tikzcd}$$

Then $\tilde{f}^{-1}(F) = E$, where F and E are the exceptional divisors, and

$$\tilde{f}^* N^*_{F|\tilde{Y}} = N^*_{E|\tilde{X}}. \tag{2}$$

Proof. Only (2) is not proved in [2, Page 442, Fact A]. By assumption, Z and W are smooth, therefore E and F are smooth divisors and we can apply Lemma 1.12 to get (2). $\square$

Lemma 1.14. *Keeping the assumptions from Lemma 1.13, further suppose that*

(a) *the map $\tilde{f}\big|_E : E \to F$ is an embedding,*
(b) *the map $f : X - W \to Y - Z$ is an embedding.*

Then $\tilde{f}$ is an embedding.

Proof. By condition (b), $\tilde{f}$ is an embedding away from E. Condition (a) implies that $\tilde{f}$ is set-theoretically injective over E. Therefore, it suffices to show that

$$d\tilde{f} : \tilde{f}^* T^*_{\tilde{Y}} \to T^*_{\tilde{X}}$$

is surjective over E. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{f}^* N^*_{F|\tilde{Y}} \arrow[r] \arrow[d, "\cong"] & \tilde{f}^* T^*_{\tilde{Y}}\big|_E \arrow[r] \arrow[d, "d\tilde{f}"] & \tilde{f}^* T^*_F \arrow[r] \arrow[d, "d(\tilde{f}|_E)"] & 0 \\ 0 \arrow[r] & N^*_{E|\tilde{X}} \arrow[r] & T^*_{\tilde{X}}\big|_E \arrow[r] & T^*_E \arrow[r] & 0 \end{tikzcd}$$

By (2), the arrow on the left is an isomorphism. Since $\tilde{f}\big|_E$ is an embedding, the arrow on the right is surjective. By the snake lemma, we conclude that the arrow in the middle is also surjective, and conclude that $\tilde{f}$ is an embedding. $\square$

Lemma 1.14 generalizes to maps of chains of centers. We state the result in a way that is suitable for proofs by induction. Let $\{f_i : X_i \to X\}_{i=0}^{j-1}$ and $\{g_i : Y_i \to Y\}_{i=0}^{j-1}$ be two chains of centers, and let $\phi : X \to Y$ be a map of chains of centers, meaning that

$$\mathrm{bl}_i(\phi)^{-1}\big(\mathrm{bl}_i(X_i)\big) = \mathrm{bl}_i(Y_i), \quad \forall 0 \le i \le j-1. \tag{3}$$

For each i, consider the following diagram:

$$\begin{tikzcd} E_i \arrow[r, hook] \arrow[d] & \mathrm{bl}_j(X) \arrow[r, "\mathrm{bl}_j(\phi)"] \arrow[d] & \mathrm{bl}_j(Y) \arrow[d] & F_i \arrow[l, hook'] \arrow[d] \\ \mathrm{bl}_i(X_i) \arrow[r, hook] & \mathrm{bl}_i(X) \arrow[r, "\mathrm{bl}_i(\phi)"] & \mathrm{bl}_i(Y) & \mathrm{bl}_i(Y_i) \arrow[l, hook'] \end{tikzcd} \tag{4}$$

Here $\{E_i\}, \{F_i\}$ are the sets of exceptional divisors of the chains $\{f_i\}_{i=0}^{j-1}$, $\{g_i\}_{i=0}^{j-1}$, as in Definition 1.5. Because of (3), we have

$$\mathrm{bl}_j(\phi)^{-1}(F_i) = E_i, \quad \forall 0 \leq i \leq j-1. \tag{5}$$

Notation 1.15. For a sequence of divisors $\{E_i\}_{i=0}^{n}$, we define

$$E_i^\circ := E_i - (E_0 \cup E_1 \cup \cdots \cup E_{i-1})$$

Note that we are removing *only* the intersections with the previous divisors.

Lemma 1.16. *Using the notation above, assume in addition that X, Y are smooth projective varieties and that $\{f_i : X_i \to X\}_{i=0}^{j-1}$, $\{g_i : Y_i \to Y\}_{i=0}^{j-1}$ are chains of smooth centers. Further assume that*

(a) *the two chains $\{f_i\}_{i=0}^{j-1}, \{g_i\}_{i=0}^{j-1}$ satisfy the NCD conditions,*
(b) *for every $0 \leq i \leq j-1$, the induced map*

$$\mathrm{bl}_j(\phi) : E_i^\circ \to F_i^\circ$$

is an embedding,
(c) *the map $\phi : X - f_{j-1}(X_{j-1}) \to Y$ is an embedding.*

Then $\mathrm{bl}_j(\phi)$ is an embedding.

Proof. The transversality condition (a) guarantees that F_i and E_i are smooth divisors in smooth projective varieties. Therefore, by (5) and Lemma 1.12, we have

$$\mathrm{bl}_j(\phi)^* N^*_{F_i | \mathrm{bl}_j(Y)} = N^*_{E_i | \mathrm{bl}_j(X)}.$$

By restriction to E_i° and F_i°, we get

$$\mathrm{bl}_j(\phi)^* N^*_{F_i^\circ | \mathrm{bl}_j(Y)} = N^*_{E_i^\circ | \mathrm{bl}_j(X)}. \tag{6}$$

Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{bl}_j(\phi)^* N^*_{F_i^\circ | \mathrm{bl}_j(Y)} \arrow[r] \arrow[d, "\cong"] & \mathrm{bl}_j(\phi)^* T^*_{\mathrm{bl}_j(Y)}\big|_{E_i^\circ} \arrow[r] \arrow[d, "d\,\mathrm{bl}_j(\phi)"] & \mathrm{bl}_j(\phi)^* T^*_{F_i^\circ} \arrow[r] \arrow[d, "d(\mathrm{bl}_j(\phi)|_{E_i^\circ})"] & 0 \\ 0 \arrow[r] & N^*_{E_i^\circ | \mathrm{bl}_j(X)} \arrow[r] & T^*_{\mathrm{bl}_j(X)}\big|_{E_i^\circ} \arrow[r] & T^*_{E_i^\circ} \arrow[r] & 0 \end{tikzcd}$$

Using (6), the condition (b) and the snake lemma, we see that

$$d\,\mathrm{bl}_j(\phi) : \mathrm{bl}_j(\phi)^* T^*_{\mathrm{bl}_j(Y)} \to T^*_{\mathrm{bl}_j(X)}$$

is surjective over each E_i°. By Remark 1.6, the set $X - f_n(X_n)$ naturally embeds into $\mathrm{bl}_j(X)$ with complement $\cup_i E_i$, and condition (c) says that $d\,\mathrm{bl}_j(\phi)$ is surjective away from $\cup_i E_i$. Since $\bigcup_i E_i = \bigcup_i E_i^\circ$, we conclude that $\mathrm{bl}_j(\phi)$ is an embedding. $\square$

Remark 1.17. In condition (b), we do not ask for $\mathrm{bl}_j(\phi) : E_i \to F_i$ to be an embedding; the reason is that it is much easier to check this condition on open subsets of exceptional divisors in the proof of Theorem A.

To apply Lemma 1.16, one needs to prove that a map is a map of chains, which amounts to verifying the conditions in (3). The following lemma reduces this to exceptional divisors and their complements.

Lemma 1.18. *Let $\phi : X \to Y$ be a morphism between smooth projective varieties. Let $\{E_i\}_{i=0}^{j-1}$ and $\{F_i\}_{i=0}^{j-1}$ be two sequences of smooth divisors in X and Y such that*

$$\phi^{-1}(F_i) = E_i, \quad \forall 0 \le i \le j-1.$$

Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be two smooth subvarieties. Assume that for each $0 \le i \le j-1$,

(a) *the intersections $X_1 \cap E_i$ and $Y_1 \cap F_i$ are transverse,*
(b) *$\phi^{-1}(Y_1 \cap F_i^\circ) = X_1 \cap E_i^\circ$, where we use Notation 1.15,*
(c) *$\phi^{-1}(Y_1 - \cup_i F_i) = X_1 - \cup_i E_i$.*

Then $\phi^{-1}(Y_1) = X_1$ scheme-theoretically.

Proof. From the assumption, we know that the set-theoretic preimage of Y_1 under ϕ is X_1. In order to show that this also holds scheme-theoretically, we need to know that $\phi^*\mathcal{I}_{Y_1} \to \mathcal{I}_{X_1}$ is surjective. Since $N^*_{X_1|X} = \mathcal{I}_{X_1}/\mathcal{I}^2_{X_1}$ and $N^*_{Y_1|Y} = \mathcal{I}_{Y_1}/\mathcal{I}^2_{Y_1}$, Nakayama's lemma shows that this is equivalent to the surjectivity of

$$d\phi : \phi^* N^*_{Y_1|Y} \to N^*_{X_1|X}.$$

This can be checked over $X_1 - \cup_i E_i = X_1 - \cup_i E_i^\circ$ and $X_1 \cap E_i^\circ$ separately. Condition (c) implies that $d\phi$ is surjective over $X_1 - \cup_i E_i$. On the other hand, using condition (a), we have the following commutative diagram

$$\begin{tikzcd}
\phi^* N^*_{Y_1|Y}\big|_{Y_1\cap F_i^\circ} \arrow[r, "d\phi|_{X_1\cap E_i^\circ}"] \arrow[d, "\cong"] & N^*_{X_1|X}\big|_{X_1\cap E_i^\circ} \arrow[d, "\cong"] \\
\phi^* N^*_{Y_1\cap F_i^\circ|F_i^\circ} \arrow[r, "d(\phi|_{E_i^\circ})"] & N^*_{X_1\cap E_i^\circ|E_i^\circ}.
\end{tikzcd}$$

The bottom map is induced by $\phi|_{E_i^\circ} : E_i^\circ \to F_i^\circ$. The vertical maps are isomorphisms because of the transversality condition (which implies that the intersections $X_1 \cap E_i^\circ$ and $Y_1 \cap F_i^\circ$ are transverse). Therefore condition (b) implies that $d\phi$ is also surjective over $X_1 \cap E_i^\circ$ for each i. $\square$

Putting everything together, we have the following inductive criterion for a proper chain to have smooth centers and satisfy the NCD conditions.

Lemma 1.19. *Let $\{f_j : X_j \to X\}_{j=0}^n$ be an NCD chain. Let $f : Y \to X$ be a morphism from a smooth projective variety, such that $f_n(X_n) \subsetneq f(Y)$. Suppose that the the following three conditions are satisfied:*

(a) *There is an NCD chain $\{g_j : Y_j \to Y\}_{j=0}^n$ such that $f : Y \to X$ becomes a map of chains of centers: concretely, this means that for $0 \le j \le n$, the diagram*

$$\begin{tikzcd}
\mathrm{bl}_j(Y_j) \arrow[r] \arrow[d, hook, "\mathrm{bl}_j(g_j)"] & \mathrm{bl}_j(X_j) \arrow[d, hook, "\mathrm{bl}_j(f_j)"] \\
\mathrm{bl}_j(Y) \arrow[r, "\mathrm{bl}_j(f)"] & \mathrm{bl}_j(X)
\end{tikzcd}$$

is Cartesian (and the vertical arrows are closed embeddings).

(b) *For $0 \le j \le n$, let $F_j \subseteq \mathrm{bl}_{n+1}(Y)$ and $E_j \subseteq \mathrm{bl}_{n+1}(X)$ be the exceptional divisors of the chains $\{g_j\}_{j=0}^n$ and $\{f_j\}_{j=0}^n$, as in Definition 1.5. Then each map*

$$\mathrm{bl}_{n+1}(f) : F_j^\circ \to E_j^\circ$$

is a closed embedding,

(c) *The map $f : Y - g_n(Y_n) \to X$ is an embedding.*

Then the augmented chain $\{f_j : X_j \to X\}_{j=0}^n \cup \{f : Y \to X\}$ is also an NCD chain.

Proof. The condition $f_n(X_n) \subsetneq f(Y)$ guarantees that $\{f_j : X_j \to X\}_{j=0}^n \cup \{f : Y \to X\}$ is still a proper chain. According to (a), we have $\mathrm{bl}_j(f)^{-1}\big(\mathrm{bl}_j(X_j)\big) = \mathrm{bl}_j(Y_j)$ for $0 \leq j \leq n$. This means that the notation is consistent, and that $\mathrm{bl}_j(f)\colon \mathrm{bl}_j(Y) \to \mathrm{bl}_j(X)$ for $1 \leq j \leq n+1$ are also the first $n+1$ morphisms in the augmented chain. Therefore we only need to prove the following three statements:

(1) $\mathrm{bl}_{n+1}(Y)$ is smooth.
(2) $\mathrm{bl}_{n+1}(f) : \mathrm{bl}_{n+1}(Y) \to \mathrm{bl}_{n+1}(X)$ is a closed embedding.
(3) The augmented chain $\{f_j\}_{j=0}^n \cup \{f\}$ is NCD.

Since (a) says that $\{g_j : Y_j \to Y\}_{j=0}^n$ is a chain of smooth centers, $\mathrm{bl}_{n+1}(Y)$ is smooth, and (1) is proved. Using (a), (b) and (c), we can apply Lemma 1.16 to $f : Y \to X$, viewed as a map of chains of centers, to conclude that

$$\mathrm{bl}_{n+1}(f) : \mathrm{bl}_{n+1}(Y) \to \mathrm{bl}_{n+1}(X)$$

is an embedding, proving (2). Finally, we know from (1) and (2) that the augmented chain $\{f_j\}_{j=0}^n \cup \{f\}$ is a chain of smooth centers. Because of (a), we can apply Lemma 1.11 to the augmented chain and conclude that it satisfies the NCD conditions. □

Remark 1.20. According to this lemma, to prove that a chain $\{f_j : X_j \to X\}_{j=0}^n$ is NCD, it suffices to construct auxiliary chains of maps to each X_j that satisfy various conditions on pull-backs and exceptional divisors. By Lemma 1.18, the condition in (a) can also be checked by intersection with exceptional divisors. In essence, in the proof of Theorem A, all the required conditions boil down to calculations of conormal bundles.

Since the exceptional divisors in smooth blowups are projective bundles, to verify the conditions (a) and (b) in Lemma 1.19 in practice, we need a relative version of some of the lemmas above.

Lemma 1.21. *Let $\phi : X \to Y$ be a B-morphism of smooth algebraic varieties over a smooth variety B, such that f and g are smooth morphisms.*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Denote the induced map over a closed point $t \in B$ by the symbol $\phi_t : X_t \to Y_t$. Then

(1) *If ϕ_t is an embedding for each t, then ϕ is an embedding.*
(2) *Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be smooth subvarieties. If $Y_t \cap Y_1, X_t \cap X_1$ are smooth and $\phi_t^{-1}(Y_t \cap Y_1) = X_t \cap X_1$ for each t, then $\phi^{-1}(Y_1) = X_1$.*
(3) *Suppose ϕ is an embedding, then $\mathrm{bl}_X(Y)$ is a B-variety and the fiber over $t \in B$ is $\mathrm{bl}_{X_t}(Y_t)$.*

Proof. For (1), it suffices to check that the differential $d\phi : \phi^* T_Y^* \to T_X^*$ is surjective, which can be checked by restriction to each X_t; the proof is similar to that of Lemma 1.14 by using the isomorphism

$$\phi^* N^*_{Y_t|Y} \cong N^*_{X_t|X}.$$

For (2), it suffices to check the surjectivity of the induced map

$$d\phi : \phi^* N^*_{Y_1|Y} \to N^*_{X_1|X}.$$

The proof is similar to that of Lemma 1.18 and uses the isomorphism

$$N^*_{Y_1|Y}\big|_{Y_1\cap Y_t} \cong N^*_{Y_1\cap Y_t|Y_t}$$

because $Y_1 \cap Y_t$ is smooth; likewise for $N^*_{X_1|X}$. (3) follows from local computations. □

2. Abel-Jacobi maps and addition maps on symmetric products

In this section, we reduce the proof of Theorem A to two somewhat technical propositions about certain chains of maps being NCD chains, using the general framework in §1. Let C be a hyperelliptic curve of odd genus $g = 2n+1$. The even genus case is very similar, and will be treated separately in §7. Let g_2^1 be the line bundle corresponding to the hyperelliptic map $h : C \to \mathbf{P}^1$ and denote by $C_j := \mathrm{Sym}^j(C)$ the j-th symmetric product of C; we view the closed points of C_j as effective divisors of degree j on the curve C, and let C_0 be the one-point set consisting of the trivial divisor. Consider the following chain of maps $\{\delta_j\}_{j=0}^n$ to $\mathrm{Jac}(C)$, where

$$\begin{aligned} \delta_j : C_{2j} &\to \mathrm{Jac}(C) = \mathrm{Pic}^{g-1}(C), \quad 0 \le j \le n, \\ D &\mapsto (n-j)g_2^1 \otimes \mathcal{O}_C(D). \end{aligned} \tag{7}$$

By the Abel-Jacobi theorem, we have $\delta_j(C_{2j}) = W^{n-j}_{g-1}$ and there is a natural embedding $\mathbf{P}^j = \delta_j^{-1}(ng_2^1) \hookrightarrow C_{2j}$.

It is easy to see that $\{\delta_j\}_{j=0}^n$ is a proper chain in the sense of Definition 1.1. Indeed, for $i < j$, we have a commutative diagram

$$\begin{tikzcd} C_{2i} \times \mathbf{P}^{j-i} \arrow[r, two heads, "p_1"] \arrow[d, "\gamma_{i,j}"] & C_{2i} \arrow[d, "\delta_i"] \\ C_{2j} \arrow[r, "\delta_j"] & \mathrm{Jac}(C), \end{tikzcd} \tag{8}$$

where p_1 denotes the projection to the first factor. Since p_1 is surjective and $\delta_i(C_{2i}) = W^{n-i}_{g-1} \subsetneq W^{n-j}_{g-1} = \delta_j(C_{2j})$, the chain $\{\delta_j\}_{j=0}^n$ is a proper chain, as claimed.

The proof of Theorem A for hyperelliptic curves of odd genus can be reduced to the following proposition, whose proof we postpone until §5. It describes the properties of the chain $\{\delta_j\}_{j=0}^n$, using the language introduced in the previous section.

Proposition 2.1. *The chain $\{\delta_j : C_{2j} \to \mathrm{Jac}(C)\}_{j=0}^n$ is an NCD chain. Concretely, this means the following things:*

(a) *Each map $\mathrm{bl}_j(\delta_j) : \mathrm{bl}_j(C_{2j}) \to \mathrm{bl}_j(\mathrm{Jac}(C))$ is an embedding of smooth projective varieties, whose image intersects the union of all the exceptional divisors in $\mathrm{bl}_j(\mathrm{Jac}(C))$ transversely.*

(b) *There is a natural embedding $\mathrm{Jac}(C) - \delta_n(C_{2n}) \hookrightarrow \mathrm{bl}_{n+1}(\mathrm{Jac}(C))$, whose complement has $n+1$ smooth components with normal crossings.*

Assuming this proposition, we can easily prove the main theorem (for hyperelliptic curves of odd genus $g = 2n+1$).

Proof of Theorem A. For the sake of clarity, let us denote by $\{\mathrm{bl}'_i(\mathrm{Jac}(C))\}_{i=1}^n$ the sequence of blowups described in the introduction, where at the i-th stage to get $\mathrm{bl}'_i(\mathrm{Jac}(C))$, we blow up the strict transform of the Brill-Noether variety $W^{n+1-i}_{g-1}(C)$. We are going to argue that, in fact, $\mathrm{bl}'_i(\mathrm{Jac}(C)) = \mathrm{bl}_i(\mathrm{Jac}(C))$. First, since $g = 2n+1$, the image of $\mathrm{bl}_n(C_{2n})$ in $\mathrm{bl}_n(\mathrm{Jac}(C))$ is a divisor, and so $\mathrm{bl}_{n+1}(\mathrm{Jac}(C)) = \mathrm{bl}_n(\mathrm{Jac}(C))$; therefore both sequences really have only n steps. Note that $\delta_i(C_{2i})$ is equal to the subset $W^{n-i}_{g-1}(C)$ of $\mathrm{Jac}(C) = \mathrm{Pic}^{g-1}(C)$. By Proposition 2.1, the map $\mathrm{bl}_i(C_{2i}) \to \mathrm{bl}_i(\mathrm{Jac}(C))$

is an embedding. By induction on $1 \leq i \leq n$, it then follows easily that $\mathrm{bl}'_i(\mathrm{Jac}(C)) = \mathrm{bl}_i(\mathrm{Jac}(C))$, and that the proper transform of $W^{n-i}_{g-1}(C)$ under the birational morphism $\pi_i : \mathrm{bl}_i(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$ is equal to the image of $\mathrm{bl}_i(C_{2i}) \hookrightarrow \mathrm{bl}_i(\mathrm{Jac}(C))$, hence smooth.

The conclusion is that $\mathrm{bl}'_n(\mathrm{Jac}(C)) = \mathrm{bl}_n(\mathrm{Jac}(C))$ is smooth, and that the strict transform $\tilde{\Theta}$ is the image of $\mathrm{bl}_n(C_{2n}) \hookrightarrow \mathrm{bl}_n(\mathrm{Jac}(C))$, hence also smooth. Since $\{\delta_i\}_{i=0}^n$ is an NCD chain, the pullback $\pi_n^*\Theta$ is a divisor with simple normal crossings. The multiplicity of the exceptional divisor Z_i equals the multiplicity of Θ at a point in $W^{n-i}_{g-1} - W^{n+1-i}_{g-1}$, which is $n+1-i$ by the Riemann Singularity Theorem. $\square$

The proof of Proposition 2.1 is by a rather tricky inductive argument. Along the way, we need several auxiliary chains that we now describe. The first such chain lives over C_{2j}, and its shape is suggested by the commutative diagram in (8).

For each $j \geq 1$, we define the following chain of maps $\{\gamma_{i,j}\}_{i=0}^{j-1}$ to C_{2j}, where

$$\gamma_{i,j} : C_{2i} \times \mathbf{P}^{j-i} \hookrightarrow C_{2i} \times C_{2j-2i} \to C_{2j}, \quad 0 \leq i < j. \tag{9}$$

The first map is induced by the embedding $\mathbf{P}^{j-i} \hookrightarrow C_{2j-2i}$ and the second map is the addition map on symmetric products. Again, it is not hard to check that this is a proper chain. Indeed, for $i < j < k$, we have the following commutative diagram:

$$\begin{tikzcd} C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j} \arrow[r, two heads, "\mathrm{Id}\times r"] \arrow[d, "\gamma_{i,j}\times\mathrm{id}"] & C_{2i} \times \mathbf{P}^{k-i} \arrow[d, "\gamma_{i,k}"] \\ C_{2j} \times \mathbf{P}^{k-j} \arrow[r, "\gamma_{j,k}"] & C_{2k} \end{tikzcd} \tag{10}$$

Here r is the restriction of the addition map $C_{2(j-i)} \times C_{2(k-j)} \to C_{2(k-i)}$, which can also be viewed as the addition map for symmetric products of $\mathbf{P}^1$ if we think of $\mathbf{P}^\ell$ as $\mathrm{Sym}^\ell \mathbf{P}^1$. In Lemma 4.2, we will show that $\gamma_{i,j}(C_{2i} \times \mathbf{P}^{j-i})$ parametrizes effective divisors D of degree $2j$ such that $h^0(\mathcal{O}_C(D)) \geq j - i + 1$. Therefore

$$\gamma_{i,j}(C_{2i} \times \mathbf{P}^{j-i}) \subsetneq \gamma_{j,k}(C_{2j} \times \mathbf{P}^{k-j}),$$

and so $\{\gamma_{i,j}\}_{i=0}^{j-1}$ is also a proper chain, as claimed.

The second auxiliary chain lives over $C_{2j} \times \mathbf{P}^{k-j}$, and its shape is again suggested by (10). Using Notation 1.7, for any $j < k$, consider the chain

$$\left\{\gamma_{i,j} \times \mathrm{id} : (C_{2i} \times \mathbf{P}^{j-i}) \times \mathbf{P}^{k-j} \to C_{2j} \times \mathbf{P}^{k-j}\right\}_{i=0}^{j-1}$$

which is induced by taking the product of the chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$ with $\mathbf{P}^{k-j}$. (Fortunately, because of the product structure of this chain, the process stops here and no further chains are needed.)

The key step in the proof of Proposition 2.1 is the following result about the properties of the two chains $\{\gamma_{i,j}\}_{i=0}^{j-1}$ and $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$.

Proposition 2.2. *For each $k \geq 1$, the chain*

$$\{\gamma_{i,k} : C_{2i} \times \mathbf{P}^{k-i} \to C_{2k}\}_{i=0}^{k-1}$$

is an NCD chain, and for each $1 \leq j < k$, the map

$$\gamma_{j,k} : C_{2j} \times \mathbf{P}^{k-j} \to C_{2k}$$

is a map of chains of centers from $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ to $\{\gamma_{i,k}\}_{i=0}^{j-1}$. Concretely, this means the following things:

(a) *For $0 \leq i < k$, the map $\mathrm{bl}_i(\gamma_{i,k}) : \mathrm{bl}_i(C_{2i} \times \mathbf{P}^{k-i}) \to \mathrm{bl}_i(C_{2k})$ is a closed embedding between smooth projective varieties, whose image intersects the union of all the exceptional divisors in $\mathrm{bl}_i(C_{2k})$ transversely.*

(b) *There is a natural embedding* $C_{2k} - \gamma_{k-1,k}(C_{2k-2} \times \mathbf{P}^1) \hookrightarrow \mathrm{bl}_k(C_{2k})$*, whose complement has* k *smooth components with normal crossings.*
(c) *For each* $i < j < k$*, one has a Cartesian diagram*

$$\begin{tikzcd} \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i}\big) \times \mathbf{P}^{k-j} \arrow[r] \arrow[d, hook, "\mathrm{bl}_i(\gamma_{i,j}) \times \mathrm{id}"] & \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i}\big) \arrow[d, hook, "\mathrm{bl}_i(\gamma_{i,k})"] \\ \mathrm{bl}_i(C_{2j}) \times \mathbf{P}^{k-j} \arrow[r, "\mathrm{bl}_i(\gamma_{j,k})"] & \mathrm{bl}_i(C_{2k}) \end{tikzcd}$$

3. Secant bundles and maps between them

The proofs of Proposition 2.2 and Proposition 2.1 rely on certain results about secant bundles over symmetric products of curves. In this section, we review the necessary definitions and results, following the notation in [2].

Let C be a smooth projective curve of genus $g \geq 0$, let M be a line bundle on C, and let $j \geq 0$ be an integer. We denote by $C_j = \mathrm{Sym}^j C$ the j-th symmetric product of the curve. Consider the following diagram:

$$\begin{tikzcd} \mathscr{D}_{j+1} \arrow[r, hook] & C \times C_{j+1} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & C_{j+1} \end{tikzcd}$$

Here $\mathscr{D}_{j+1} = C \times C_j$ is the universal divisor over C_{j+1}, embedded via $(p, D) \mapsto (p, p + D)$. We say that M separates d points if

$$h^0(C, M) = h^0(C, M(-D)) + d, \quad \forall D \in C_d.$$

If M separates $j + 1$ points, then the sequence of sheaves

$$0 \to p_1^* M \otimes \mathcal{O}(-\mathscr{D}_{j+1}) \to p_1^* M \to p_1^* M \otimes \mathcal{O}_{\mathscr{D}_{j+1}} \to 0 \tag{11}$$

on $C \times C_{j+1}$ remains exact when pushed down to C_{j+1}.

Definition 3.1. The secant bundle (with respect to M) of j-planes over C_{j+1} is

$$B^j(M) := \mathbf{P}(p_2)_*(p_1^* M \otimes \mathcal{O}_{\mathscr{D}_{j+1}}).$$

This is a $\mathbf{P}^j$-bundle over the symmetric product C_{j+1}; for $j = 0$, we have $B^0(M) = C$. If M separates $j + 1$ points, the natural map to $\mathbf{P}H^0(C, M)$ is

$$\beta_j : B^j(M) \to \mathbf{P}(p_2)_*(p_1^* M) = \mathbf{P}H^0(C, M) \times C_{j+1} \to \mathbf{P}H^0(C, M), \tag{12}$$

where the last map is the projection to $\mathbf{P}H^0(C, M)$.

Assuming that M separates $k + 1$ points, we get a proper chain

$$\big\{\beta_j : B^j(M) \to \mathbf{P}H^0(C, M)\big\}_{j=0}^{k}. \tag{13}$$

In order to study this chain, we need certain auxiliary chains, just as in the previous section. For $i < j$, the addition map $r : C_{i+1} \times C_{j-i} \to C_{j+1}$ induces a map

$$\alpha_{i,j} : B^i(M) \times C_{j-i} \to B^j(M). \tag{14}$$

For each $j \geq 1$, these maps gives us another proper chain

$$\big\{\alpha_{i,j} : B^i(M) \times C_{j-i} \to B^j(M)\big\}_{i=0}^{j-1}.$$

Lastly, using the construction in Notation 1.7, we have for each $j < k$ a proper chain

$$\big\{\alpha_{i,j} \times \mathrm{id} : (B^i(M) \times C_{j-i}) \times C_{k-j} \to B^j(M) \times C_{k-j}\big\}_{i=0}^{j-1}.$$

In [2, Proposition 2.2, Proposition 2.3], Bertram proved the following result.

Proposition 3.2 (Bertram). *Let M be a line bundle on C and $0 \leq j < k$.*

(a) *Both $\{\alpha_{i,k}\}_{i=0}^{k-1}$ and $\{\alpha_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ are chains of smooth centers, and the map*

$$\alpha_{j,k} : B^j(M) \times C_{k-j} \to B^k(M)$$

is an injective map of chains from $\{\alpha_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ to the truncation $\{\alpha_{i,k}\}_{i=0}^{j-1}$.

(b) *If M separates $2k+2$ points, then $\{\beta_j\}_{j=0}^{k}$ is a chain of smooth centers, and*

$$\beta_k : B^k(M) \to \mathbf{P}H^0(C, M)$$

is an injective map of chains from $\{\alpha_{j,k}\}_{j=0}^{k-1}$ to the truncation $\{\beta_j\}_{j=0}^{k-1}$.

Remark 3.3. Using Definition 1.5, Bertram's proof actually shows that $\{\alpha_{i,k}\}_{i=0}^{k-1}$ and $\{\beta_j\}_{j=0}^{k}$ are NCD chains. But these facts will not be used later.

Let us spell out in detail what Bertram's theorem says in the case of $\mathbf{P}^1$, where the images of the secant bundles for $\mathscr{O}_{\mathbf{P}^1}(d)$ are the secant varieties to the rational normal curve of degree d in $\mathbf{P}^d$.

Corollary 3.4. *Let $d \geq 2k+1$ and consider the line bundle $M = \mathscr{O}_{\mathbf{P}^1}(d)$ on $\mathbf{P}^1$.*

(a) *For $0 \leq i < j < k$, the diagram*

$$\begin{tikzcd}
\mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{j-i}\big) \times \mathbf{P}^{k-j} \arrow[r] \arrow[d, hook, "\mathrm{bl}_i(\alpha_{i,j}) \times \mathrm{id}"] & \mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{k-i}\big) \arrow[d, hook, "\mathrm{bl}_i(\alpha_{i,k})"] \\
\mathrm{bl}_i B^j(M) \times \mathbf{P}^{k-j} \arrow[r, "\mathrm{bl}_i(\alpha_{j,k})"] & \mathrm{bl}_i B^k(M)
\end{tikzcd}$$

is Cartesian and the two vertical arrows are injective.

(b) *For $0 \leq i < j$, the diagram*

$$\begin{tikzcd}
\mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{j-i}\big) \arrow[r] \arrow[d, hook, "\mathrm{bl}_i(\alpha_{i,j})"] & \mathrm{bl}_i B^i(M) \arrow[d, hook, "\mathrm{bl}_i(\beta_i)"] \\
\mathrm{bl}_i B^j(M) \arrow[r, "\mathrm{bl}_i(\beta_j)"] & \mathrm{bl}_i \mathbf{P}^d
\end{tikzcd}$$

is Cartesian and the two vertical arrows are injective.

Proof. Since $M = \mathscr{O}_{\mathbf{P}^1}(d)$ separates $d+1$ points on $\mathbf{P}^1$, we can apply Proposition 3.2 and use the isomorphisms $\mathbf{P}^k \cong \mathrm{Sym}^k \mathbf{P}^1$ and $\mathbf{P}^d \cong \mathbf{P}H^0(\mathbf{P}^1, M)$. □

4. Properties of Abel-Jacobi maps and addition maps

In this section, as a preliminary for the proof of Proposition 2.2 and Proposition 2.1, we establish some basic properties of the map $\gamma_{i,j} : C_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$ and of the Abel-Jacobi map $\delta_j : C_{2j} \to \mathrm{Jac}(C)$. In particular, their conormal bundles are calculated in terms of the secant bundles over symmetric products of $\mathbf{P}^1$. In fact, it is known that the conormal bundle of Abel-Jacobi maps can be described in terms of Steiner bundles (see [4, Theorem 1.1]). For our purpose, it is more natural to use secant bundles.

Notation 4.1. For each j, we define $U_{2j} := C_{2j} - \gamma_{j-1,j}(C_{2j-2} \times \mathbf{P}^1)$. By Remark 1.6, this is exactly the open subset of C_{2j} which is the complement of exceptional divisors in $\mathrm{bl}_j(C_{2j})$ associated to the chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$.

Lemma 4.2. *Let C be a hyperelliptic curve of odd genus $g = 2n+1$.*

(a) *The map* $\delta_j : U_{2j} \to \mathrm{Jac}(C)$ *is an embedding for* $0 \leq j \leq n$, *and for* $0 \leq i < j \leq n$, *the restriction of the diagram* (8) *is Cartesian:*

$$\begin{tikzcd} U_{2i} \times \mathbf{P}^{j-i} \arrow[r, "p_1"] \arrow[d, hook, "\gamma_{i,j}"] & U_{2i} \arrow[d, hook, "\delta_i"] \\ C_{2j} \arrow[r, "\delta_j"] & \mathrm{Jac}(C) \end{tikzcd}$$

(b) *The map* $\gamma_{i,j} : U_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$ *is an embedding for* $0 \leq i < j \leq n$, *and for* $0 \leq i < j < k \leq n$, *the restriction of the diagram* (10) *is Cartesian:*

$$\begin{tikzcd} (U_{2i} \times \mathbf{P}^{j-i}) \times \mathbf{P}^{k-j} \arrow[r, "\mathrm{id} \times r"] \arrow[d, hook, "\gamma_{i,j} \times \mathrm{id}"] & U_{2i} \times \mathbf{P}^{k-i} \arrow[d, hook, "\gamma_{i,k}"] \\ C_{2j} \times \mathbf{P}^{k-j} \arrow[r, "\gamma_{j,k}"] & C_{2k} \end{tikzcd}$$

(c) *In particular, for* $\mathbf{P}^k \subseteq C_{2k}$, *we have* $\gamma_{j,k}^{-1}(\mathbf{P}^k) = \mathbf{P}^j \times \mathbf{P}^{k-j}$.

Proof. Let $D \in C_{2j}$ be a degree $2j$ divisor such that $h^0(\mathcal{O}_C(D)) = r + 1$. Since C is a hyperelliptic curve, there is a unique decomposition

$$D = E + \sum_{\ell=1}^{r} (p_\ell + q_\ell)$$

such that $p_\ell + q_\ell$ are hyperelliptic pairs and E is a degree $2j - 2r$ divisor with $h^0(\mathcal{O}_C(E)) = 1$. In particular, the map

$$\gamma_{i,j} : U_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$$

is injective, and its image consists of divisors D of degree $2j$ such that $h^0(\mathcal{O}_C(D)) = j - i + 1$. Similarly, using the Abel-Jacobi theorem, we know that for any $L \in \mathrm{Jac}(C)$ with $h^0(L) = r + 1$, there is a unique decomposition

$$L = rg_2^1 \otimes L' \tag{15}$$

such that $h^0(L') = 1$.

For (a), since $\delta_j(D) = (k - j)g_2^1 \otimes \mathcal{O}_C(D)$, it follows from the uniqueness of the decomposition (15) that $\delta_j : U_{2j} \to \mathrm{Jac}(C)$ is injective, and that its image consists of line bundles $L \in \mathrm{Jac}(C)$ such that $h^0(L) = k - j + 1$. Suppose that $D \in C_{2j}$ is such that $\delta_j(D) \in \delta_i(U_{2i})$. Then

$$h^0\big((k - j)g_2^1 \otimes \mathcal{O}_C(D)\big) = h^0(\delta_j(D)) = k - i + 1.$$

Using (15), we have $h^0(\mathcal{O}_C(D)) = j - i + 1$ and conclude that $D \in \gamma_{i,j}(U_{2i} \times \mathbf{P}^{j-i})$, by its characterization in C_{2j}. The argument for the set-theoretic part of (b),(c) is similar.

To show δ_j and $\gamma_{i,j}$ are embeddings, one needs the surjectivity of $d\delta_j$ and $d\gamma_{i,j}$, which follows from Lemma 4.4(a) and Lemma 4.7(a) below. To show the diagram in (a) is Cartesian, we need to show $\delta_j^{-1}(U_{2i}) = U_{2i} \times \mathbf{P}^{j-i}$; this amounts to the surjectivity of

$$d\delta_j : \delta_j^* N^*_{U_{2i}|\,\mathrm{Jac}(C)} \to N^*_{U_{2i} \times \mathbf{P}^{j-i}|C_{2j}},$$

which follows from Corollary 4.8. Similarly, the statement that the diagram in (b) is Cartesian follows from Corollary 4.6. □

Notation 4.3. By the proof of Lemma 4.2, the open subset $U_{2j} \subseteq C_{2j}$ consists of divisors D of degree $2j$ such that there is no hyperelliptic pair contained as an effective subdivisor of D. In particular, any $D \in U_{2j}$ gives a degree $2j$ divisor on $\mathbf{P}^1$ via the hyperelliptic map $h : C \to \mathbf{P}^1$. We denote this divisor on $\mathbf{P}^1$ by the symbol h_*D and define

$$\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) := \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathbf{P}^1}(-h_*D),$$

which is a line bundle of degree $g-1-2j$ on $\mathbf{P}^1$. Note that we have $h_*\mathcal{O}_D \cong \mathcal{O}_{h_*D}$.

The divisor h_*D shows up in the following way. Recall that since the curve C is hyperelliptic, we have $\omega_C \cong h^*\mathcal{O}_{\mathbf{P}^1}(g-1)$, and therefore

$$h_*\omega_C \cong \omega_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(g-1).$$

If we twist the canonical bundle by an effective divisor $D \in U_{2j}$, we instead get

$$h_*\omega_C(-D) \cong \omega_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D). \tag{16}$$

To see this, push the short exact sequence $0 \to \omega_C(-D) \to \omega_C \to \omega_C \otimes \mathcal{O}_D \to 0$ forward along $h\colon C \to \mathbf{P}^1$, and use the fact that $h_*(\omega_C \otimes \mathcal{O}_D) \cong \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{h_*D}$, because of the projection formula and $h_*\mathcal{O}_D \cong \mathcal{O}_{h_*D}$.

Lemma 4.4. *For $0 \le i < j$, consider $\gamma_{i,j} : C_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$. Then*

(a) *$d\gamma_{i,j} : \gamma_{i,j}^* T^*_{C_{2j}} \to T^*_{C_{2i}\times\mathbf{P}^{j-i}}$ is surjective when restricted to $U_{2i} \times \mathbf{P}^{j-i}$.*

(b) *For $i = 0$, we have an isomorphism*

$$\mathbf{P}N^*_{\mathbf{P}^j|C_{2j}} \cong B^{j-1}(\mathcal{O}_{\mathbf{P}^1}(g-1)),$$

the latter is the secant bundle over $\mathrm{Sym}^j\mathbf{P}^1 = \mathbf{P}^j$ with respect to $\mathcal{O}_{\mathbf{P}^1}(g-1)$.

(c) *For $i \ge 1$, the space $\mathbf{P}N^*_{\gamma_{i,j}}\big|_{U_{2i}\times\mathbf{P}^{j-i}}$ is a smooth variety over U_{2i}, such that over $D \in U_{2i}$ we have an isomorphism:*

$$\mathbf{P}N^*_{\gamma_{i,j}}\big|_{\{D\}\times\mathbf{P}^{j-i}} \cong B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)),$$

the latter is the secant bundle over $\mathbf{P}^{j-i}$ with respect to $\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$.

Proof. As a warm-up, let us calculate the conormal bundle of $\mathbf{P}^j$ inside C_{2j}. Recall that for any divisor $D \in C_{2j}$, there is a canonical identification

$$T^*_{C_{2j}}\big|_D \cong H^0(C, \omega_C \otimes \mathcal{O}_D).$$

Using the isomorphism $\mathbf{P}^j \cong \mathrm{Sym}^j\mathbf{P}^1$, the morphism $\mathbf{P}^j \to C_{2j}$ associates to an effective divisor E of degree j on $\mathbf{P}^1$ the effective divisor h^*E of degree $2j$ on C. We have

$$T^*_{\mathbf{P}^j}\big|_E \cong H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E)$$

for the cotangent space of $\mathbf{P}^j$ at the point E, and

$$\begin{aligned} T^*_{C_{2j}}\big|_{h^*E} &\cong H^0(C, \omega_C \otimes h^*\mathcal{O}_E) \cong H^0(\mathbf{P}^1, h_*\omega_C \otimes \mathcal{O}_E) \\ &\cong H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E) \oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_E) \end{aligned}$$

for the contangent space of C_{2j} at the point h^*E. Moreover, the morphism between the two contangent spaces is the projection to the first summand. It follows that

$$T^*_{C_{2j}}\big|_{h^*E} \to T^*_{\mathbf{P}^j}\big|_E$$

is surjective (which means that $\mathbf{P}^j \to C_{2j}$ is a closed embedding), with kernel

$$N^*_{\mathbf{P}^j|C_{2j}}\big|_E \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_E).$$

This isomorphism is natural in E, and therefore

$$N^*_{\mathbf{P}^j|C_{2j}} \cong (p_2)_*\big(p_1^*\mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathscr{E}_j}\big).$$

where $\mathscr{E}_j$ denotes the universal divisor over $\mathbf{P}^j \cong \mathrm{Sym}^j\mathbf{P}^1$, and the notation is as in the following diagram:

$$\begin{tikzcd} \mathscr{E}_j \arrow[r, hook] & \mathbf{P}^1 \times \mathrm{Sym}^j\mathbf{P}^1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{P}^1 & & \mathrm{Sym}^j\mathbf{P}^1 \end{tikzcd} \tag{17}$$

In particular, the projectivized conormal bundle is the secant bundle $B^{j-1}\big(\mathcal{O}_{\mathbf{P}^1}(g-1)\big)$, proving (b).

For (a) and (c), consider $D \in U_{2i}$ and $E \in \mathbf{P}^{j-i}$. The morphism $\gamma_{i,j} : U_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$ takes the pair (D, E) to the divisor $D + h^*E$ of degree $2j$ on C. As before, the cotangent spaces of the three varieties are canonically identified with

$$\begin{aligned} T^*_{U_{2i}}\big|_D &\cong H^0(C, \omega_C \otimes \mathcal{O}_D), \\ T^*_{\mathbf{P}^{j-i}}\big|_E &\cong H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E), \\ T^*_{C_{2j}}\big|_{D+h^*E} &\cong H^0(C, \omega_C \otimes \mathcal{O}_{D+h^*E}). \end{aligned}$$

Because $D \in U_{2i}$, we have $H^0(C, \mathcal{O}_C(D + h^*E)) = H^0(C, \mathcal{O}_C(h^*E))$. After a little bit of diagram chasing, this gives us a short exact sequence

$$0 \to H^0(C, \omega_C(-D) \otimes \mathcal{O}_{h^*E}) \to H^0(C, \omega_C \otimes \mathcal{O}_{D+h^*E}) \to H^0(C, \omega_C \otimes \mathcal{O}_D) \to 0.$$

The morphism between the cotangent spaces of C_{2j} and U_{2i} is the morphism in this short exact sequence; consequently,

$$\mathrm{Ker}\Big(T^*_{C_{2j}}\big|_{D+h^*E} \to T^*_{U_{2i}}\big|_D\Big) \cong H^0(C, \omega_C(-D) \otimes h^*\mathcal{O}_E) \cong H^0(\mathbf{P}^1, h_*\omega_C(-D) \otimes \mathcal{O}_E).$$

Using (16), we have

$$H^0(\mathbf{P}^1, h_*\omega_C(-D) \otimes \mathcal{O}_E) \cong H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E) \oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_E).$$

Since the morphism to the cotangent space of $\mathbf{P}^{j-i}$ is the projection to the first summand in this decomposition, we deduce that

$$d\gamma_{i,j} : T^*_{C_{2j}}\big|_{D+h^*E} \to T^*_{U_{2i}}\big|_D \oplus T^*_{\mathbf{P}^{j-i}}\big|_E$$

is surjective, proving (a); and that its kernel is isomorphic to

$$N^*_{\gamma_{i,j}}\big|_{(D,E)} \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_E).$$

This isomorphism is again natural in E, and therefore

$$N^*_{\gamma_{i,j}}\big|_{\{D\}\times\mathbf{P}^{j-i}} \cong (p_2)_*\big(p_1^*\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_{\mathscr{E}_{j-i}}\big).$$

This is a vector bundle on $\mathbf{P}^{j-i}$ because the line bundle $\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$ separates $j-i$ points (on account of the inequality $g-1-2i > j-i$). The projectivized conormal bundle is therefore a projective bundle over U_{2i}, hence smooth, and its fiber over $D \in U_{2i}$ is the secant bundle $B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$, proving (c). □

Remark 4.5. This lemma is parallel to [2, Lemma 1.3], with the difference that the relevant divisor is h_*D (and not $2h_*D$ as in Bertram's case).

From the proof, we can deduce one additional useful fact. For $0 \leq i < j < k \leq n$, consider the commutative diagram from (10), which looks like this:

$$\begin{tikzcd}
(U_{2i} \times \mathbf{P}^{j-i}) \times \mathbf{P}^{k-j} \arrow[r, "\mathrm{id} \times r"] \arrow[d, "\gamma_{i,j} \times \mathrm{id}"] & U_{2i} \times \mathbf{P}^{k-i} \arrow[d, "\gamma_{i,k}"] \\
C_{2j} \times \mathbf{P}^{k-j} \arrow[r, "\gamma_{j,k}"] & C_{2j}
\end{tikzcd}$$

Corollary 4.6. *For $D \in U_{2i}$, the induced map of conormal bundles*

$$\epsilon : (\mathrm{id} \times r)^* N^*_{\gamma_{i,k}}\big|_{\{D\} \times \mathbf{P}^{k-i}} \to N^*_{\gamma_{i,j} \times \mathrm{id}}\big|_{\{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}}$$

on $\{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}$ is surjective, and the diagram

$$\begin{tikzcd}
\mathbf{P} N^*_{\gamma_{i,j} \times \mathrm{id}}\big|_{\{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}} \arrow[r, "\alpha"] \arrow[d, "\cong"] & \mathbf{P} N^*_{\gamma_{i,k}}\big|_{\{D\} \times \mathbf{P}^{k-i}} \arrow[d, "\cong"] \\
B^{j-i-1}(M) \times \mathbf{P}^{k-j} \arrow[r, "\alpha_{j-i-1,k-i-1}"] & B^{k-i-1}(M)
\end{tikzcd}$$

commutes. Here $\alpha_{j-i-1,k-i-1}$ is the map in (14) for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D)$; and α is induced by ϵ and the projection to $\mathbf{P} N^*_{\gamma_{i,k}}\big|_{\{D\} \times \mathbf{P}^{k-i}}$.

Proof. To simplify the notation, fix a point $D \in U_{2i}$ and define $M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D)$. According to the proof of Lemma 4.4, for $E_1 \in \mathbf{P}^{j-i}$ and $E_2 \in \mathbf{P}^{k-j}$, the map

$$\epsilon\big|_{(D,E_1,E_2)} : N^*_{\gamma_{i,k}}\big|_{(D,E_1+E_2)} \to N^*_{\gamma_{i,j}}\big|_{(D,E_1)}$$

between the fibers of the two conormal bundles is identified with the map

$$H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1+E_2}\big) \to H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1}\big),$$

which is obviously surjective. The remaining assertion is clear from (14). $\square$

Next, we prove the analogous results for the Abel-Jacobi map.

Lemma 4.7. *For $0 \leq j \leq n$, consider $\delta_j : C_{2j} \to \mathrm{Jac}(C)$.*

(a) *$d\delta_j : \delta_j^* T^*_{\mathrm{Jac}(C)} \to T^*_{C_{2j}}$ is surjective when restricted to U_{2j}.*
(b) *The fiber of $N^*_{\delta_j}$ over $D \in U_{2j}$ is $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D))$.*

Proof. The proof is similar to that of Lemma 4.4. We have

$$T^*_{\mathrm{Jac}(C)} \cong H^0(C, \omega_C) \otimes \mathcal{O}_{\mathrm{Jac}(C)},$$

and the morphism between the two contangent spaces is identified with

$$H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \mathcal{O}_D).$$

Since $D \in U_{2j}$, this morphism is surjective, and its kernel equals

$$N^*_{\delta_j}\big|_D \cong H^0\big(C, \omega_C(-D)\big) \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D)\big),$$

using the isomorphism in (16). $\square$

Again, we record one additional useful fact. For $0 \leq i < j \leq n$, consider the commutative diagram in (8), which looks like this:

$$\begin{tikzcd}
U_{2i} \times \mathbf{P}^{j-i} \arrow[r, "p_1"] \arrow[d, "\gamma_{i,j}"] & U_{2i} \arrow[d, "\delta_i"] \\
C_{2j} \arrow[r, "\delta_j"] & \mathrm{Jac}(C)
\end{tikzcd}$$

Corollary 4.8. *For $D \in U_{2i}$, the induced map of conormal bundles*

$$\epsilon : p_1^* N^*_{\delta_i}\big|_D \to N^*_{\gamma_{i,j}}\big|_{\{D\}\times \mathbf{P}^{j-i}}$$

over $\{D\} \times \mathbf{P}^{j-i}$ is surjective, and the diagram

$$\begin{array}{ccc} \mathbf{P}N^*_{\gamma_{i,j}}\big|_{\{D\}\times\mathbf{P}^{j-i}} & \xrightarrow{\beta} & \mathbf{P}N^*_{\delta_i}\big|_D \\ \big\downarrow \cong & & \big\downarrow \cong \\ B^{j-i-1}(M) & \xrightarrow{\beta_{j-i-1}} & \mathbf{P}H^0(\mathbf{P}^1, M) \end{array}$$

commutes. Here the first vertical isomorphism comes from Lemma 4.4(c); β_{j-i-1} is the map (12) *for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$; and β is induced by ϵ and the projection to $\mathbf{P}N^*_{\delta_i}\big|_D$.*

Proof. To simplify the notation, fix a point $D \in U_{2i}$ and define $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. According to the proof of Lemma 4.7, for $E \in \mathbf{P}^{j-i}$, the map

$$\epsilon\big|_{(D,E)} : N^*_{\delta_i}\big|_D \to N^*_{\gamma_{i,j}}\big|_{(D,E)}$$

between the fibers of the two conormal bundles is identified with the map

$$H^0(\mathbf{P}^1, M) \to H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_E\big),$$

which is surjective for degree reasons. The remaining assertion is clear from (12). $\square$

5. The proof of Proposition 2.1

The proofs of Proposition 2.1 and Proposition 2.2 follow the same lines. Since the notation for Proposition 2.2 is more complicated, we are going to postpone its proof to §6. In this section, we assume Proposition 2.2 and prove Proposition 2.1. We also assume Claim 6.2, which is proved along with Proposition 2.2.

Recall that $g = 2n+1$. For each $1 \le k \le n$, we study the map

$$\delta_k : C_{2k} \to \mathrm{Jac}(C)$$

as a map between the two chains $\{\gamma_{i,k}\}_{i=0}^{k-1}$ and $\{\delta_i\}_{i=0}^{k-1}$. To keep track of the exceptional divisors in different blowup spaces, for $0 \le i < j \le k$, we denote by

$$E_{i,j} \subseteq \mathrm{bl}_j(C_{2k}), \quad F_{i,j} \subseteq \mathrm{bl}_j(\mathrm{Jac}(C))$$

the exceptional divisors of the two chains $\{\gamma_{i,k}\}_{i=0}^{k-1}$ and $\{\delta_i\}_{i=0}^{k-1}$ as in Definition 1.5. In particular, we have the following diagram

$$\begin{array}{ccccccc} E_{i,j} & \hookrightarrow & \mathrm{bl}_j(C_{2k}) & \xrightarrow{\mathrm{bl}_j(\delta_k)} & \mathrm{bl}_j(\mathrm{Jac}(C)) & \hookleftarrow & F_{i,j} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_{i,i+1} & \hookrightarrow & \mathrm{bl}_{i+1}(C_{2k}) & \xrightarrow{\mathrm{bl}_{i+1}(\delta_k)} & \mathrm{bl}_{i+1}(\mathrm{Jac}(C)) & \hookleftarrow & F_{i,i+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i} & \hookrightarrow & \mathrm{bl}_i(C_{2k}) & \xrightarrow{\mathrm{bl}_i(\delta_k)} & \mathrm{bl}_i(\mathrm{Jac}(C)) & \hookleftarrow & \mathrm{bl}_i(C_{2i}) \end{array}$$

For the sets of divisors $\{E_{i,j}\}_{i=0}^{j-1}$ and $\{F_{i,j}\}_{i=0}^{j-1}$, we use Notation 1.15. The key observation is the following. We use U_{2i} and h_*D from Notation 4.1 and Notation 4.3.

Claim 5.1. *For $0 \leq i < j \leq k \leq n$, we have $\mathrm{bl}_j(\delta_k)(E^\circ_{i,j}) \subseteq F^\circ_{i,j}$ and the induced map*

$$\mathrm{bl}_j(\delta_k)\colon E^\circ_{i,j} \to F^\circ_{i,j}$$

is a morphism of U_{2i}-varieties which is $\mathrm{bl}_{j-i-1}(\beta_{k-i-1})$ on fibers: over a point $D \in U_{2i}$, the corresponding map is

$$\beta_{k-i-1} : B^{k-i-1}\big(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big) \to \mathbf{P}H^0\big(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big)$$

from (12) *and $\mathrm{bl}_{j-i-1}(\beta_{k-i-1})$ is the map associated to the chain $\{\beta_j\}_{j=0}$ in* (13). *In particular, by Corollary 3.4(b) and Lemma 1.21, the map*

$$\mathrm{bl}_j(\delta_j) : E^\circ_{i,j} \to F^\circ_{i,j}$$

is an embedding for $0 \leq i < j \leq n$.

Granting Claim 5.1 for the time being, let us prove Proposition 2.1.

Proof of Proposition 2.1. We prove by induction on $0 \leq k \leq n$ that the truncated chain $\{\delta_i\}_{i=0}^k$ is NCD. The base case $k = 0$ follows from the smoothness of C_0 and $\mathrm{Jac}(C)$. Fix $k \geq 1$, and suppose that the truncated chain $\{\delta_i\}_{i=0}^{k-1}$ is NCD. We would like to apply Lemma 1.19 to prove that the same is true for $\{\delta_i\}_{i=0}^k$. Proposition 2.2 implies that the chain $\{\gamma_{j,k} : C_{2j} \times \mathbf{P}^{k-j} \to C_{2k}\}_{j=0}^{k-1}$ is an NCD chain. According to Lemma 1.19, it therefore suffices to verify the following three conditions:

(a) The map $\delta_k : C_{2k} \to \mathrm{Jac}(C)$ is a map of chains of centers, i.e.

$$\mathrm{bl}_j(\delta_k)^{-1}(\mathrm{bl}_j(C_{2j})) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}, \quad \forall 0 \leq j < k.$$

(b) The map $\mathrm{bl}_j(\delta_j) : E^\circ_{i,j} \to F^\circ_{i,j}$ is an embedding for each $i < j$,

(c) The map $\delta_k : C_{2k} - \gamma_{k-1,k}(C_{2k-2} \times \mathbf{P}^1) \to \mathrm{Jac}(C)$ is an embedding.

The condition in (b) follows from Claim 5.1, and the condition in (c) follows from Lemma 4.2(a). It remains to check the condition in (a) for all pairs (j, k) such that $0 \leq j < k$. By induction on k, we can assume that it holds for every pair (j', k') such that $k' \leq k-1$.

Now we argue by a further induction on $0 \leq j < k$. The base case $(0, k)$ follows from the fact that $\delta_k^{-1}(C_0) = \mathbf{P}^k$. Assume that we have

$$\mathrm{bl}_i(\delta_k)^{-1}(\mathrm{bl}_i(C_{2i})) = \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i}, \quad \forall 0 \leq i < j, \tag{18}$$

we want to show

$$\mathrm{bl}_j(\delta_k)^{-1}(\mathrm{bl}_j(C_{2j})) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}. \tag{19}$$

To this end, we would like to apply Lemma 1.18 to the map

$$\mathrm{bl}_j(\delta_k) : \mathrm{bl}_j(C_{2k}) \to \mathrm{bl}_j(\mathrm{Jac}(C)).$$

The key point is to understand the intersection of $\mathrm{bl}_j(C_{2j})$ with the exceptional divisors in $\mathrm{bl}_j(\mathrm{Jac}(C))$ via the map $\delta_j : C_{2j} \to \mathrm{Jac}(C)$. By induction hypothesis (for k) and Proposition 2.2, the chains $\{\delta_i\}_{i=0}^{k-1}, \{\gamma_{i,j}\}_{i=0}^{k-1}$ are NCD chains. Since $j \leq k-1$, we know that $\sum_{i<j} E_{i,j} \subseteq \mathrm{bl}_j(C_{2k})$, $\sum_{i<j} F_{i,j} \subseteq \mathrm{bl}_j(\mathrm{Jac}(C))$ are simple normal crossing divisors in smooth projective varieties, and that the intersections

$$\big(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}\big) \cap E_{i,j}, \quad \mathrm{bl}_j(C_{2j}) \cap F_{i,j}, \quad \forall 0 \leq i < j,$$

are transverse. By (18) and Lemma 1.13, we have

$$\mathrm{bl}_j(\delta_k)^{-1}(F_{i,j}) = E_{i,j}, \quad \forall 0 \leq i < j.$$

Hence by Lemma 1.18, to prove (19), it suffices to show that

$$\mathrm{bl}_j(\delta_k)^{-1}(\mathrm{bl}_j(C_{2j}) \cap F^\circ_{i,j}) = (\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}) \cap E^\circ_{i,j}, \quad \forall 0 \leq i < j, \tag{20}$$

$$\mathrm{bl}_j(\delta_k)^{-1}\left(\mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j}\right) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} - \bigcup_{i<j} E_{i,j}. \tag{21}$$

For (21), by induction hypothesis (for k), the map $\delta_j : C_{2j} \to \mathrm{Jac}(C)$ is a map of chains of centers. Therefore by Lemma 1.13, $\mathrm{bl}_j(C_{2j}) \cap F_{i,j}$ are the exceptional divisors associated to the chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$ in Definition 1.5. Then by Remark 1.6, we have

$$\mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j} = C_{2j} - \gamma_{j-1,j}(C_{2j-2} \times \mathbf{P}^1) = U_{2j}. \tag{22}$$

Using Lemma 1.8, we also have

$$\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} - \bigcup_{i<j} E_{i,j} = U_{2j} \times \mathbf{P}^{k-j}. \tag{23}$$

Finally, by Lemma 4.2(a), we conclude that

$$\mathrm{bl}_j(\delta_k)^{-1}(U_{2j}) = \delta_k^{-1}(U_{2j}) = U_{2j} \times \mathbf{P}^{k-j}.$$

Now we turn to the proof of (20). By Proposition 2.2, the maps

$$\delta_j : C_{2j} \to \mathrm{Jac}(C), \quad \gamma_{j,k} : C_{2j} \times \mathbf{P}^{k-j} \to C_{2k}$$

are maps of chains of centers. Using Lemma 1.13, Claim 5.1 for $\mathrm{bl}_j(\delta_j)$, and Claim 6.2, the diagram

$$\begin{tikzcd}
(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}) \cap E^\circ_{i,j} \arrow[r] \arrow[d, hook] & \mathrm{bl}_j(C_{2j}) \cap F^\circ_{i,j} \arrow[d, hook] \\
E^\circ_{i,j} \arrow[r, "\mathrm{bl}_j(\delta_k)"] & F^\circ_{i,j}
\end{tikzcd}$$

is a diagram of U_{2i}-varieties. Over a point $D \in U_{2i}$, the corresponding diagram is

$$\begin{tikzcd}
\mathrm{bl}_{j-i-1} B^{m-i-1}(M) \times \mathbf{P}^{k-j} \arrow[r] \arrow[d, hook] & \mathrm{bl}_{j-i-1} B^{j-i-1}(M) \arrow[d, hook] \\
\mathrm{bl}_{j-i-1} B^{k-i-1}(M) \arrow[r, "\mathrm{bl}_{j-i-1}(\beta_{k-i-1})"] & \mathrm{bl}_{j-i-1} \mathbf{P}H^0(M)
\end{tikzcd}$$

Here the vertical maps are $\mathrm{bl}_{j-i-1}(\alpha_{j-i-1,k-i-1})$ and $\mathrm{bl}_{j-i-1}(\beta_{j-i-1})$, and M denotes the line bundle $\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$, which has degree

$$g - 1 - 2i = 2n - 2i \geq 2(k-i-1) + 1,$$

as $k \leq n$. Then we can apply Corollary 3.4(b) to get

$$\mathrm{bl}_{j-i-1}(\beta_{k-i-1})^{-1}(\mathrm{bl}_{j-i-1} B^{j-i-1}(M)) = \mathrm{bl}_{j-i-1} B^{j-i-1}(M) \times \mathbf{P}^{k-j}.$$

Apply Lemma 1.21 to $\mathrm{bl}_j(\delta_k) : E^\circ_{i,j} \to F^\circ_{i,j}$ as a map of U_{2i}-varieties, this proves (20) and thus finishes the inductive proof for the condition in (a). Then we finish the proof of Proposition 2.1. □

In the rest of this section, let us prove Claim 5.1. To illustrate the geometric picture, let us do the first few cases by hand, before dealing with the general case. We identify C_0 with its image $\{ng^1_2\}$ in $\mathrm{Jac}(C)$ and we have $\delta_k^{-1}(C_0) = \mathbf{P}^k$.

Case $k = 0$. There is no exceptional divisor yet, so the statement is vacuous.

Case $k = 1$. We blow up $\mathrm{Jac}(C)$ and C_2 along C_0 and $\delta_1^{-1}(C_0) = \mathbf{P}^1$ respectively. By Corollary 4.8, we have a commutative diagram

$$\begin{tikzcd} E^\circ_{0,1} = E_{0,1} \arrow[r, "\mathrm{bl}_1(\delta_1)"] \arrow[d, equal] & F^\circ_{0,1} = F_{0,1} \arrow[d, equal] \\ B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \arrow[r, "\beta_0"] & \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1)), \end{tikzcd}$$

which proves Claim 5.1 for $k = 1$.

Case $k = 2$. We have the following commutative diagram; the spaces to the right and left are the blowup centers and their corresponding divisors:

$$\begin{tikzcd} & \mathbf{P}^2 & C_4 \arrow[r, "\delta_2"] & \mathrm{Jac}(C) & C_0 & \\ \mathrm{bl}_1(C_2) \times \mathbf{P}^1 & E_{0,1} \arrow[u] & \mathrm{bl}_1(C_4) \arrow[u] \arrow[r, "\mathrm{bl}_1(\delta_2)"] & \mathrm{bl}_1(\mathrm{Jac}(C)) \arrow[u] & F_{0,1} \arrow[u] & \mathrm{bl}_1(C_2) \\ E_{1,2} \arrow[u] & E_{0,2} \arrow[u] & \mathrm{bl}_2(C_4) \arrow[u] \arrow[r, "\mathrm{bl}_2(\delta_2)"] & \mathrm{bl}_2(\mathrm{Jac}(C)) \arrow[u] & F_{0,2} \arrow[u] & F_{1,2} \arrow[u] \end{tikzcd}$$

First we blow up $\mathrm{Jac}(C)$ along C_0, and C_4 along $\delta_2^{-1}(C_0) = \mathbf{P}^2$, to get the two divisors $E_{0,1}$ and $F_{0,1}$. By Corollary 4.8, we have a commutative diagram ($j = 1$ and $i = 0$):

$$\begin{tikzcd} E_{0,1} \arrow[r, "\mathrm{bl}_1(\delta_2)"] \arrow[d, equal] & F_{0,1} \arrow[d, equal] \\ B^1(\mathcal{O}_{\mathbf{P}^1}(g-1)) \arrow[r, "\beta_1"] & \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \end{tikzcd}$$

Then we blow up $\mathrm{bl}_1(\mathrm{Jac}(C))$ along $\mathrm{bl}_1(C_2)$, and $\mathrm{bl}_1(C_4)$ along $\mathrm{bl}_1(C_2) \times \mathbf{P}^1$, to get the new exceptional divisors $E_{1,2}$ and $F_{1,2}$. The divisors $E_{0,2}, F_{0,2}$ are the strict transforms of $E_{0,1}, F_{0,1}$ under the new blowup. By the case $k = 1$, we know that

$$\begin{aligned} &\mathrm{bl}_1(C_2) \cap F_{0,1} = B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)), \\ &\big(\mathrm{bl}_1(C_2) \times \mathbf{P}^1\big) \cap E_{0,1} = B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \times \mathbf{P}^1. \end{aligned} \tag{24}$$

Since the intersections are transverse, $E_{0,2}$ and $F_{0,2}$ are the blowups of $E_{0,1}$ and $F_{0,1}$ along these intersections, which identifies $\mathrm{bl}_2(\delta_2)$ with $\mathrm{bl}_1(\beta_1)$:

$$\begin{tikzcd} E_{0,2} \arrow[r, "\mathrm{bl}_2(\delta_2)"] \arrow[d, equal] & F_{0,2} \arrow[d, equal] \\ \mathrm{bl}_1 B^1(\mathcal{O}_{\mathbf{P}^1}(g-1)) \arrow[r, "\mathrm{bl}_1(\beta_1)"] & \mathrm{bl}_1 \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \end{tikzcd}$$

The bottom map is induced by $\beta_1 : B^1(\mathcal{O}_{\mathbf{P}^1}(g-1)) \to \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1))$ by the blowup along $B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \times \mathbf{P}^1$ and $B^0(\mathcal{O}_{\mathbf{P}^1}(g-1))$ associated to the chains $\{\alpha_{i,1}\}_{i=0}^0$ and $\{\beta_i\}_{i=0}^0$ in §3.

The remaining case is $j = 2$, $i = 1$. By (24), $F_{1,2} - F_{0,2}$ is the exceptional divisor for the blowup of $\mathrm{bl}_1(\mathrm{Jac}(C))$ along

$$\mathrm{bl}_1(C_2) - B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) = C_2 - \mathbf{P}^1 = U_2.$$

Since U_2 already embeds into $\mathrm{Jac}(C)$, by Lemma 4.7(b), we know that $F^\circ_{1,2} = F_{1,2} - F_{0,2}$ is a U_2-variety with fiber over $D \in U_2$ being $\mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$, where h_*D is the degree 2 divisor on $\mathbf{P}^1$ associated to D. Similarly, $E^\circ_{1,2} = E_{1,2} - E_{0,2}$ is the exceptional divisor for the blowup of $\mathrm{bl}_1(C_4)$ along

$$\mathrm{bl}_1(C_2) \times \mathbf{P}^1 - B^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \times \mathbf{P}^1 = U_2 \times \mathbf{P}^1.$$

Moreever, the restriction map $\mathrm{bl}_2(\delta_2) : E^\circ_{1,2} \to F^\circ_{1,2}$ is a map of U_2-varieties with fibers over $D \in U_2$ being

$$\beta_0 : \mathbf{P}^1 = B^0(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)) \to \mathbf{P}H^0(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)).$$

This proves Claim 5.1 when $k = 2$.

Proof of Claim 5.1. We prove the claim by induction on $1 \leq k \leq n+1$. The base case $k = 1$ is covered by the previous discussion. Fix $k \geq 2$ and assume the claim holds for all smaller values of k. From the proof of Proposition 2.1, we see that it implies the truncated chain $\{\delta_i\}_{i=0}^{k-1}$ is NCD and that the map $\delta_j : C_{2j} \to \mathrm{Jac}(C)$ is map of chains of centers for all $j \leq k-1$.

To prove the claim for k, we fix i $(< k)$ and do an extra induction on j such that $i < j < k$. For a divisor $D \in U_{2i}$, we use the notation

$$M := \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D).$$

The base case is $j = i+1$ and we want to show that under the map

$$\mathrm{bl}_{i+1}(\delta_k) : \mathrm{bl}_{i+1}(C_{2k}) \to \mathrm{bl}_{i+1}(\mathrm{Jac}(C)),$$

$E^\circ_{i,i+1}$ is mapped into $F^\circ_{i,i+1}$. The divisor $F_{i,i+1} \subseteq \mathrm{bl}_{i+1}(\mathrm{Jac}(C))$ is the exceptional divisor for the blowup of $\mathrm{bl}_i(\mathrm{Jac}(C))$ along $\mathrm{bl}_i(C_{2i})$. Since the chain $\{\delta_j\}_{j=0}^{k-1}$ is NCD and $i \leq k-1$, we have transverse intersections

$$\mathrm{bl}_i(C_{2i}) \cap F_{\ell,i}, \quad \forall \ell < i.$$

Therefore by (22), $F^\circ_{i,i+1} = F_{i,i+1} - \cup_{\ell<i} F_{\ell,i+1}$ is the exceptional divisor over

$$\mathrm{bl}_i(C_{2i}) - \bigcup_{\ell<i} F_{\ell,i} = U_{2i}.$$

Since U_{2i} embeds into $\mathrm{Jac}(C)$, the exceptional divisor $F^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over $D \in U_{2i}$ being $\mathbf{P}H^0(M)$, by Lemma 4.7(b). Similarly by Proposition 2.2, for each $\ell < i$, the intersections $(\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{j-i}) \cap E_{\ell,i}$ is transverse. Therefore by (23), $E^\circ_{i,i+1}$ is the exceptional divisor for the blow up of $\mathrm{bl}_i(C_{2j})$ along

$$\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i} - \bigcup_{\ell<i} E_{\ell,i} = U_{2i} \times \mathbf{P}^{k-i}.$$

which is a U_{2i}-variety with fiber over $D \in U_{2i}$ being $B^{k-i-1}(M)$, and by Corollary 4.8, the restriction of $\mathrm{bl}_{i+1}(\delta_k)$ to the fiber is

$$\beta_{k-i-1} : B^{k-i-1}(M) \to \mathbf{P}H^0(M).$$

We conclude the base case.

Assume the claim is true for $j-1$. We are interested in the diagram

$$\begin{tikzcd}
E^\circ_{i,j} \arrow[r, hook] \arrow[d] & \mathrm{bl}_j(C_{2k}) \arrow[r, "\mathrm{bl}_j(\delta_k)"] \arrow[d] & \mathrm{bl}_j(\mathrm{Jac}(C)) \arrow[d] & F^\circ_{i,j} \arrow[l, hook'] \arrow[d] \\
E^\circ_{i,j-1} \arrow[r, hook] & \mathrm{bl}_{j-1}(C_{2k}) \arrow[r, "\mathrm{bl}_{j-1}(\delta_k)"] & \mathrm{bl}_{j-1}(\mathrm{Jac}(C)) & F^\circ_{i,j-1} \arrow[l, hook'] \\
 & \mathrm{bl}_{j-1}(C_{2j-2}) \times \mathbf{P}^{k-j+1} \arrow[u, hook] & \mathrm{bl}_{j-1}(C_{2j-2}) \arrow[u, hook] &
\end{tikzcd}$$

Since the truncated chain $\{\delta_i\}_{i=0}^{k-1}$ is NCD and $j-1 \leq k-1$, the intersection

$$F^\circ_{i,j-1} \cap \mathrm{bl}_{j-1}(C_{2j-2})$$

is transverse. Then by induction hypothesis on k, it is a U_{2i}-variety with fiber over $D \in U_{2i}$ being $\mathrm{bl}_{j-i-2} B^{j-i-2}(M)$. Hence $F^\circ_{i,j}$ is the blowup of $F^\circ_{i,j-1}$ along $F^\circ_{i,j-1} \cap \mathrm{bl}_{j-1}(C_{2j-2})$, and by Lemma 1.21, it is a U_{2i}-variety such that the fiber over D is the blow up of $\mathrm{bl}_{j-i-2} \mathbf{P}H^0(M)$ along $\mathrm{bl}_{j-i-2} B^{j-i-2}(M)$, which by definition is $\mathrm{bl}_{j-i-1} \mathbf{P}H^0(M)$. The calculation of $E^\circ_{i,j}$ is similar (using Proposition 2.2); this time, the induced map of $\mathrm{bl}_j(\delta_k)$ on fibers is

$$\mathrm{bl}_{j-i-1}(\beta_{k-i-1}) : \mathrm{bl}_{j-i-1} B^{k-i-1}(M) \to \mathrm{bl}_{j-i-1} \mathbf{P}H^0(M),$$

because β_{k-i-1} is a map of chains of centers by Corollary 3.4(b). This finishes the inductive proof on j and k for Claim 5.1. □

Proof of Corollary B. For $0 \leq i \leq n-1$, the exceptional divisor Z_i from the Introduction is the exceptional divisor $F_{i,n}$ in the notation of this section. Over any point $D \in U_{2i}$ and write $M := \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$, Claim 5.1 implies that the fiber of the projection

$$F^\circ_{i,i+1} = F_{i,i+1} \setminus F_{0,i+1} \cup \cdots \cup F_{i-1,i+1} \to W^{n-i}_{g-1} \setminus W^{n-i+1}_{g-1} \cong U_{2i}$$

is $\mathbf{P}H^0(M) = \mathbf{P}^{2(n-i)}$. Since $\{\delta_i\}$ is a NCD chain, using Claim 5.1, we know that the fiber of $F_{i,n} \setminus F_{0,n} \cup \cdots \cup F_{i-1,n}$ over D is obtained by iterated blow-ups of $\mathbf{P}H^0(\mathbf{P}^1, M)$ associated to the chain $\{\beta_{k-i-1} : B^{k-i-1}(M) \to \mathbf{P}H^0(M)\}_{k=i+1}^n$. To calculate the fiber of $F_{i,n} \setminus F_{0,n} \cup \cdots \cup F_{i-1,n} \cup F_{i+1,n} \cup \cdots \cup F_{n-1,n}$ over D, we can apply Remark 1.6 because $\{\beta_j\}$ is a smooth chain. It follows that the complement of the fiber inside $\mathbf{P}^{2(n-i)}$ is the image of β_{n-i-1}, which is the $(n-i-1)^{\text{th}}$ secant variety of the rational normal curve $\beta_0(B^0(M))$. This secant variety has degree $(n-i)+1$ and the rational curve has degree $2(n-i)$. This finishes the proof of Corollary B by applying $i = n-r$. □

6. The proof of Proposition 2.2

In this section, we prove Proposition 2.2, whose proof follows the exact same lines as Proposition 2.1. Therefore, some details will be omitted. The essential difference is that we replace Lemma 4.7 and Corollary 3.4(b) with Lemma 4.4 and Corollary 3.4(a). Besides the already existing induction, we prove by an extra induction on t such that the inductive hypothesis on t plays the role of Proposition 2.2 in the proof of Proposition 2.1. In particular, this says that we do not need to construct another map of chains to $C_{2j} \times \mathbf{P}^{t-j}$ and show it is a NCD chain, which is guaranteed by the induction hypothesis and Lemma 1.8.

For each $1 \leq k < t$, we study the map

$$\gamma_{k,t} : C_{2k} \times \mathbf{P}^{t-k} \to C_{2t}$$

as a map between chains $\{\gamma_{i,k} \times \mathrm{id}\}_{i=0}^{k-1}$ and $\{\gamma_{i,t}\}_{i=0}^{k-1}$. By Lemma 1.8, we have identifications

$$\mathrm{bl}_j(C_{2k} \times \mathbf{P}^{t-k}) = \mathrm{bl}_j(C_{2k}) \times \mathbf{P}^{t-k}.$$

which will be used throughout this section. For $0 \leq i < j \leq k$, let us denote

$$G_{i,j} \subseteq \mathrm{bl}_j(C_{2k}) \times \mathbf{P}^{t-k}, \quad E_{i,j} \subseteq \mathrm{bl}_j(C_{2t})$$

to be the exceptional divisors associated to the chain

$$\{\gamma_{i,k} \times \mathrm{id} : (C_{2i} \times \mathbf{P}^{k-i}) \times \mathbf{P}^{t-k} \to C_{2k} \times \mathbf{P}^{t-k}\}_{i=0}^{k-1}$$

and $\{\gamma_{i,t}\}_{i=0}^{k-1}$ as in Definition 1.5. In particular, we have the following diagram

$$\begin{tikzcd}
G_{i,j} \arrow[r, hook] \arrow[d] & \mathrm{bl}_j(C_{2k}) \times \mathbf{P}^{t-k} \arrow[r, "\mathrm{bl}_j(\gamma_{k,t})"] \arrow[d] & \mathrm{bl}_j(C_{2t}) \arrow[d] & E_{i,j} \arrow[l, hook'] \arrow[d] \\
\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k} \arrow[r, hook] & \mathrm{bl}_i(C_{2k}) \times \mathbf{P}^{t-k} \arrow[r, "\mathrm{bl}_i(\gamma_{k,t})"] & \mathrm{bl}_i(C_{2t}) & \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{t-i} \arrow[l, hook']
\end{tikzcd}$$

For the sets of divisors $\{G_{i,j}\}_{i=0}^{j-1}$ and $\{E_{i,j}\}_{i=0}^{j-1}$, we use Notation 1.15.

Remark 6.1. By Lemma 1.8, the exceptional divisors $G_{i,j}$ is equal to $H \times \mathbf{P}^{t-k}$, where $H \subseteq \mathrm{bl}_j(C_{2k})$ are the preimages of the exceptional divisors associated to the blow up along $\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i}$. Since H is the "$E_{i,j}$" divisor inside $\mathrm{bl}_j(C_{2k})$, and our notation $E_{i,j}$ cannot distinguish whether or not it lies in $\mathrm{bl}_j(C_{2k})$ or $\mathrm{bl}_j(C_{2t})$, we decide to use the new notation $G_{i,j}$.

The proof of Proposition 2.2 relies on the following

Claim 6.2. *For $0 \le i < j \le k$, we have*

$$\mathrm{bl}_j(\gamma_{k,t})(G^\circ_{i,j}) \subseteq E^\circ_{i,j}$$

and the induced map is a morphism of U_{2i}-varieties which is $\mathrm{bl}_{j-i-1}(\alpha_{k-i-1,t-i-1})$ on fibers, where $U_{2i} = C_{2i} - \gamma_{i-1,i}(C_{2i-2} \times \mathbf{P}^1)$. Over a point $D \in U_{2i}$, the corresponding map is

$$\alpha_{k-i-1,t-i-1} : B^{k-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)) \times \mathbf{P}^{t-k} \to B^{t-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)),$$

where h_*D is the divisor in Notation 4.3. In particular, by Corollary 3.4(a) and Lemma 1.21, the map

$$\mathrm{bl}_j(\gamma_{j,t}) : G^\circ_{i,j} \to E^\circ_{i,j}$$

is an embedding for $0 \le i < j$.

Proof of Proposition 2.2. We prove by induction on t that the chain

$$\{\gamma_{i,t} : C_{2i} \times \mathbf{P}^{t-i} \to C_{2t}\}_{i=0}^{t-1}$$

is NCD and for each $1 \le k < t$, the map $\gamma_{k,t}$ is a map of chains of centers between $\{\gamma_{i,k} \times \mathrm{id}\}_{i=0}^{k-1}$ and $\{\gamma_{i,t}\}_{i=0}^{k-1}$. The base case $t = 1$ follows from $\gamma_{0,1} : \mathbf{P}^1 \hookrightarrow C_2$ is an embedding of smooth varieties. Assume these are true for all $t' \le t-1$. To verify it for t, we prove by induction on $0 \le k \le t-1$ that the truncated chain $\{\gamma_{i,t}\}_{i=0}^{k}$ is a NCD chain. The base case $k = 0$ follows from $\gamma_{0,t} : \mathbf{P}^t \hookrightarrow C_{2t}$ is an embedding of smooth varieties.

Assume that $k \ge 1$ and the truncated chain $\{\gamma_{i,t}\}_{i=0}^{k-1}$ is NCD, we would like to apply Lemma 1.19 to prove that the same is true for $\{\gamma_{i,t}\}_{i=0}^{k}$. By the induction hypothesis (for t), the chain $\{\gamma_{i,k}\}_{i=0}^{k-1}$ is NCD. Hence Lemma 1.8 implies that there is a NCD chain

$$\{\gamma_{i,k} \times \mathrm{id} : (C_{2i} \times \mathbf{P}^{k-i}) \times \mathbf{P}^{t-k} \to C_{2k} \times \mathbf{P}^{t-k}\}_{i=0}^{k-1}.$$

According to Lemma 1.19, it therefore suffices to verify the following three conditions:

(a) the map $\gamma_{k,t}$ is a map of chains of centers, i.e.

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{t-j}) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}, \quad \forall 0 \le j < k.$$

(b) the map $\mathrm{bl}_j(\gamma_{j,t}) : G^\circ_{i,j} \to E^\circ_{i,j}$ is an embedding for each $i < j$,

(c) the map $\gamma_{k,t} : C_{2k} \times \mathbf{P}^{t-k} - \gamma_{k-1,k}(C_{2k-2} \times \mathbf{P}^1) \times \mathbf{P}^{t-k} \to C_{2t}$ is an embedding.

The condition in (c) follows from (22) and Lemma 4.2(b). The condition in (b) is a consequence of Claim 6.2. It remains to check the condition in (a) and Claim 6.2 for all pairs (j, k) such that $0 \leq j < k < t$. The base case $(0, k)$ follows from Lemma 4.2(c) and Lemma 4.4(b). By induction on k, we can assume that it holds for every pair (j', k') such that $k' \leq k - 1$.

We first prove Claim 6.2 for k. For fixed i and k, we induct on j, the number of blowups. For $D \in U_{2i}$, we denote $M := \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D)$. The base case is $j = i + 1$. As in the proof of Claim 5.1, since the truncated chain $\{\gamma_{\ell,t}\}_{\ell=0}^{k-1}$ is NCD and $i \leq k - 1$, therefore $E^\circ_{i,i+1} = E_{i,i+1} - \cup_{\ell<i} E_{\ell,i+1}$ is the exceptional divisor over

$$\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{t-i} - \bigcup_{\ell<i} E_{\ell,i} = U_{2i} \times \mathbf{P}^{t-i} \subseteq \mathrm{bl}_i(C_{2t}).$$

By Lemma 4.4(c), we conclude that $E^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over $D \in U_{2i}$ being $B^{t-i-1}(M)$. Similarly, $G^\circ_{i,i+1}$ is the exceptional divisor for the blow up of $\mathrm{bl}_i(C_{2k}) \times \mathbf{P}^{t-k}$ along $(U_{2i} \times \mathbf{P}^{k-i}) \times \mathbf{P}^{t-k}$, which is a U_{2i}-variety with fiber over $D \in U_{2i}$ being $B^{k-i-1}(M) \times \mathbf{P}^{t-k}$. Hence we have

$$\mathrm{bl}_{i+1}(\gamma_{k,t})(G^\circ_{i,i+1}) \subseteq E^\circ_{i,i+1},$$

and its restriction to the fiber over D is

$$\alpha_{k-i-1,t-i-1} : B^{k-i-1}(M) \times \mathbf{P}^{t-k} \to B^{t-i-1}(M).$$

This concludes the base case on j. Assume the Claim 6.2 is true for $j - 1$. Using again the induction hypothesis that the truncated chain $\{\gamma_{\ell,t}\}_{\ell=0}^{k-1}$ is NCD, we see that $E^\circ_{i,j}$ is the blow up of $E^\circ_{i,j-1}$ along $E^\circ_{i,j-1} \cap (\mathrm{bl}_{j-1}(C_{2j-2}) \times \mathbf{P}^{t-j+1})$, and by Lemma 1.21, the fiber of $E^\circ_{i,j}$ over D is the blow up of $\mathrm{bl}_{j-i-2} B^{t-i-1}(M)$ along $\mathrm{bl}_{j-i-2} B^{j-i-2}(M) \times \mathbf{P}^{t-j+1}$, which is $\mathrm{bl}_{j-i-1} B^{t-i-1}(M)$. Similar calculation gives $G^\circ_{i,j}$ as a U_{2i}-variety with fiber $\mathrm{bl}_{j-i-1} B^{k-i-1}(M)$ and the induced map over D is

$$\mathrm{bl}_{j-i-1}(\alpha_{k-i-1,t-i-1}) : \mathrm{bl}_{j-i-1} B^{k-i-1}(M) \times \mathbf{P}^{t-k} \to \mathrm{bl}_{j-i-1} B^{t-i-1}(M).$$

This finishes the inductive proof on j for Claim 6.2.

To prove the condition in (a) for k, we fix k and argue by a further induction on j such that $0 \leq j < k$. The base case $j = 0$ follows from $\gamma^{-1}_{k,t}(\mathbf{P}^t) = \mathbf{P}^k \times \mathbf{P}^{t-k}$, by Lemma 4.2(c). Assume that we have

$$\mathrm{bl}_i(\gamma_{k,t})^{-1}(\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{t-i}) = \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}, \quad \forall 0 \leq i < j, \tag{25}$$

we want to show

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{t-j}) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}.$$

To this end, we apply Lemma 1.18 to the map

$$\mathrm{bl}_j(\gamma_{k,t}) : \mathrm{bl}_j(C_{2k}) \times \mathbf{P}^{t-k} \to \mathrm{bl}_j(C_{2t}).$$

As in the proof of Proposition 2.1, using the induction hypothesis that $\{\gamma_{i,t}\}_{i=0}^{k-1}, \{\gamma_{i,j}\}_{i=0}^{k-1}$ are NCD chains (which implies that $\{\gamma_{i,k} \times \mathrm{id}_{\mathbf{P}^{t-k}}\}_{i=0}^{k-1}$ is also a NCD chain) and (25), it suffices to verify that for $0 \leq i < j$, we have

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}((\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{t-j}) \cap E^\circ_{i,j}) = (\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}) \cap G^\circ_{i,j},$$

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{t-j} - \bigcup_{\ell<j} E_{\ell,j}) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k} - \bigcup_{\ell<j} G_{\ell,j}.$$

For the second equality, by similar calculations in (23), we have

$$\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{t-j} - \bigcup_{\ell<j} E_{\ell,j} = U_{2j} \times \mathbf{P}^{t-j},$$
$$\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k} - \bigcup_{\ell<j} G_{\ell,j} = U_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}.$$

By Lemma 4.2(b), we have

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}(U_{2j} \times \mathbf{P}^{t-j}) = \gamma_{k,t}^{-1}(U_{2j} \times \mathbf{P}^{t-j}) = U_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}.$$

The first equality uses the induction hypothesis $\{\gamma_{i,k}\}_{i=0}^{k-1}, \{\gamma_{i,t}\}_{i=0}^{k-1}$ are NCD chains, $\gamma_{j,t}$ and $\gamma_{j,k} \times \mathrm{id}_{\mathbf{P}^{t-k}}$ are maps of chains of centers, the Claim 6.2 for k and Corollary 3.4(a). This finishes the inductive proof for the condition in (a) on k. Therefore we finish the proof of Proposition 2.2. $\square$

7. Even genus case

In this section, let C be a smooth hyperelliptic curve of even genus $g = 2n + 2$. We sketch a proof of Theorem A for C. The ideas are essentially the same by reducing to the calculation of conormal bundles, but for parity reasons, the corresponding maps need some modification. First, we have a chain of maps $\{\delta_j\}_{j=0}^{n}$ to $\mathrm{Jac}(C)$, where

$$\delta_j : C_{2j+1} \to \mathrm{Pic}^{g-1}(C) = \mathrm{Jac}(C), \quad 0 \le j \le n$$
$$D \mapsto (n-j)g_2^1 \otimes \mathcal{O}_C(D).$$

The image $\delta_j(C_{2j+1})$ is W_{g-1}^{n-j}, hence this is a proper chain. By Abel-Jacobi theorem, for each $\ell \ge 1$, we have $\mathbf{P}^\ell \subseteq C_{2\ell}$, then for each $j \ge 1$, there is a proper chain of maps $\{\gamma_{i,j}\}_{i=0}^{j-1}$ induced by the addition maps:

$$\gamma_{i,j} : C_{2i+1} \times \mathbf{P}^{j-i} \hookrightarrow C_{2i+1} \times C_{2j-2i} \to C_{2j+1}, \quad 0 \le i < j.$$

The even genus case of Theorem A is reduced to the following analogue of Proposition 2.2 and Proposition 2.1.

Proposition 7.1.

(1) *For each $j \ge 1$, the chain*
$$\{\gamma_{i,j} : C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}\}_{i=0}^{j-1}$$
is a NCD chain and for each $1 \le i < j$, the map $\gamma_{i,j}$ is a map of chains of i smooth centers.

(2) *The chain $\{\delta_j : C_{2j+1} \to \mathrm{Jac}(C)\}_{j=0}^{n}$ is a NCD chain and for each $1 \le j \le n$, the map $\delta_j : C_{2j+1} \to \mathrm{Jac}(C)$ is a map of chains of j smooth centers.*

As in the proof of Proposition 2.2 and Proposition 2.1, the proof of Proposition 7.1 relies on a parallel statement for Abel-Jacobi maps and conormal bundles as in Lemma 4.2 and Lemma 4.4. The proofs are left to the reader.

For each $j \ge 0$, denote

$$U_{2j+1} := C_{2j+1} - \gamma_{j-1,j}(C_{2j-1} \times \mathbf{P}^1).$$

Note that for $j = 0$, we have $U_1 = C$. Any divisor $D \in U_{2j+1}$ gives a degree $2j+1$ divisor on $\mathbf{P}^1$ via the hyperelliptic map $C \to \mathbf{P}^1$, which is denoted to be h_*D and the associated line bundle is

$$\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) := \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathbf{P}^1}(-h_*D).$$

Lemma 7.2. *Let C be a hyperelliptic curve of genus $g = 2n + 2$. Then:*

(a) *for any* $0 \le i < j$, *the map* $\gamma_{i,j} : C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}$ *is an embedding over* $U_{2i+1} \times \mathbf{P}^{j-i}$ *and for* $0 \le \ell < i < j$ *we have*

$$\gamma_{i,j}^{-1}(U_{2\ell+1} \times \mathbf{P}^{j-\ell}) = U_{2\ell+1} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}.$$

(b) *for* $0 \le j \le n$, *the map* $\delta_j : C_{2j+1} \to \mathrm{Jac}(C)$ *is an embedding over* U_{2j+1}, *and for* $0 \le i < j$, *we have*

$$\delta_j^{-1}(U_{2i+1}) = U_{2i+1} \times \mathbf{P}^{j-i}.$$

In particular, since $U_1 = C$ *we have*

$$\delta_j^{-1}(C) = C \times \mathbf{P}^j \subseteq C_{2j+1}.$$

Lemma 7.3. *For* $0 \le i < j$, *consider the map* $\gamma_{i,j} : C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}$ *and let* $D \in U_{2i+1}$. *Then*

(a) $d\gamma_{i,j} : \gamma_{i,j}^* T^*_{C_{2j+1}} \to T^*_{C_{2i+1} \times \mathbf{P}^{j-i}}$ *is surjective when restricted to* $U_{2i+1} \times \mathbf{P}^{j-i}$.

(b) $\mathbf{P}N^*_{\gamma_{i,j}}|_{U_{2i+1} \times \mathbf{P}^{j-i}}$ *is a smooth variety over* U_{2i+1} *such that over* $D \in U_{2i+1}$ *we have an isomorphism*

$$\mathbf{P}N^*_{\gamma_{i,j}}|_{\{D\} \times \mathbf{P}^{j-i}} \cong B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)).$$

Furthermore, for $\ell < i$, *consider the commutative diagram*

$$\begin{tikzcd}
(C_{2\ell+1} \times \mathbf{P}^{i-\ell}) \times \mathbf{P}^{j-i} \arrow[r, "\mathrm{id} \times r"] \arrow[d, "\gamma_{\ell,i} \times \mathrm{id}"] & C_{2\ell+1} \times \mathbf{P}^{j-\ell} \arrow[d, "\gamma_{\ell,j}"] \\
C_{2i+1} \times \mathbf{P}^{j-i} \arrow[r, "\gamma_{i,j}"] & C_{2j+1}
\end{tikzcd}$$

Here r *is the addition map. For any* $D \in U_{2\ell+1}$, *there is an induced map of conormal bundles on* $\{D\} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}$:

$$\epsilon : (\mathrm{id} \times r)^* N^*_{\gamma_{\ell,j}}|_{\{D\} \times \mathbf{P}^{j-\ell}} \to N^*_{(\gamma_{\ell,i} \times \mathrm{id})}|_{\{D\} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}}.$$

Then:

(c) ϵ *is surjective.*

(d) *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{P}N^*_{(\gamma_{\ell,i} \times \mathrm{id})}|_{\{D\} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}} \arrow[r, "\alpha"] \arrow[d, "\cong"] & \mathbf{P}N^*_{\gamma_{\ell,i}}|_{\{D\} \times \mathbf{P}^{j-\ell}} \arrow[d, "\cong"] \\
B^{i-\ell-1}(M) \times \mathbf{P}^{j-i} \arrow[r, "\alpha_{i-\ell-1,j-\ell-1}"] & B^{j-\ell-1}(M)
\end{tikzcd}$$

Here $\alpha_{i-\ell-1,j-\ell-1}$ *is the map (14) for the curve* $\mathbf{P}^1$ *and the line bundle* $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$; *and* α *is the map induced by* ϵ *composed with a projection to* $\mathbf{P}N^*_{\gamma_{\ell,j}}(D \times \mathbf{P}^{j-\ell})$.

Lemma 7.4. *With the notation in Lemma 7.3. For* $0 \le j \le n$, *consider the map* $\delta_j : C_{2j+1} \to \mathrm{Jac}(C)$. *Then:*

(a) $d\delta_j : \delta_j^* T^*_{\mathrm{Jac}(C)} \to T^*_{C_{2j+1}}$ *is surjective when restricted to* U_{2j+1}.

(b) *the fiber of* $N^*_{\delta_j}$ *over* $D \in U_{2j+1}$ *is* $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$.

Furthermore, consider the diagram for $i < j$,

$$\begin{tikzcd}
C_{2i+1} \times \mathbf{P}^{j-i} \arrow[r, "\pi_{C_{2i+1}}"] \arrow[d, "\gamma_{i,j}"] & C_{2i+1} \arrow[d, "\delta_i"] \\
C_{2j+1} \arrow[r, "\delta_j"] & \mathrm{Jac}(C)
\end{tikzcd}$$

Then for $D \in C_{2i+1}$, we get the induced map of conormal bundles over $\{D\} \times \mathbf{P}^{j-i}$:

$$\epsilon : \pi^*_{C_{2i+1}} N^*_{\delta_i}|_D \to N^*_{\gamma_{i,j}}|_{\{D\}\times\mathbf{P}^{j-i}},$$

and we have

(c) *ϵ is surjective.*

(d) *The following diagram commutes:*

$$\begin{array}{ccc} \mathbf{P}N^*_{\gamma_{i,j}}|_{\{D\}\times\mathbf{P}^{j-i}} & \xrightarrow{\beta} & \mathbf{P}N^*_{\delta_i}|_D \\ \downarrow\cong & & \downarrow\cong \\ B^{j-i-1}(M) & \xrightarrow{\beta_{j-i-1}} & \mathbf{P}H^0(\mathbf{P}^1, M) \end{array}$$

The first vertical isomorphism comes from (b) of Lemma 7.3 and β_{j-i-1} is the map (12) for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. The map β is induced from ϵ composed with a projection to $\mathbf{P}N^*_{\delta_i}|_D$.

8. Brill-Noether Stratifications are Whitney

In this section, let C be a smooth projective hyperelliptic curve of genus g. We show that the Brill-Noether stratification of $\mathrm{Jac}(C)$ determined by

$$\mathrm{Jac}(C) \supseteq \Theta = W_{g-1}(C) \supseteq W^1_{g-1}(C) \supseteq \ldots \supseteq W^n_{g-1}(C),$$

is a Whintey stratification, where $n = \lfloor \frac{g-1}{2} \rfloor$. We will assume $g = 2n+1$; the even genus case is similar.

8.1. Whitney stratifications.

Let Z be a smooth real manifold and let $X, Y \subseteq Z$ be two embedded smooth real sub-manifolds. Suppose $Y \subseteq \overline{X}$, where the closure is taken inside Z with respect to the Euclidean topology.

Definition 8.1. We say that the pair (X, Y) satisfies the *Whitney conditions* if for any point $y \in Y$ the following two conditions hold:

(A) If $\{x_i\} \subseteq X$ is a sequence of points converging to y, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear space T of the same dimension, then $T_yY \subseteq T$.

(B) If $\{x_i\} \subseteq X$ and $\{y_i\} \subseteq Y$ are two sequences of points that both converge to y, if the sequence of real secant lines between x_i and y_i converges to a real line L, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear subspace T of the same dimension, then $L \subseteq T$.

The Whitney condition (B) involves real secant lines (in local coordinates), and is therefore not so easy to verify in practice. Instead, in the case of complex algebraic varieties, there is a condition (W) introduced by Kuo [9] and Verdier [11], which implies the Whitney conditions and is easier to work with in our situation. It is proved by Teissier that, for complex analytic stratifications, the Whitney conditions are equivalent to condition (W), but we will not need this fact.

Definition 8.2 (Distance)**.** Let V be a complex vector space and let $A, B \subseteq V$ be two linear subspaces. Fix an inner product $(-,-)$ on V. The *distance between A and B* is defined to be

$$d(A,B) := \sup_{\substack{a\in A,\\ a\neq 0}} \inf_{\substack{b\in B,\\ b\neq 0}} \sin\theta(a,b).$$

Here $\theta(a,b)$ is the angle between two vectors a, b determined by the inner product $(-,-)$.

Here are some basic properties of $d(A,B)$. Note that it is not symmetric in A and B.

Fact 8.3. -

(1) $d(A, B) = 0$ if and only if $A \subseteq B$.
(2) Let $A \subseteq A'$ be two subspaces, then $d(A, B) \leq d(A', B)$.
(3) Identify V with the conjugate dual space V^* via the inner product $(-,-)$ so that $\ker(V^* \to B^*)$ is identified with the orthogonal complement $B^\perp$. Then

$$d\left(\ker(V^* \to B^*), \ker(V^* \to A^*)\right) = d(B^\perp, A^\perp) = d(A, B).$$

After choosing an orthonormal basis, this comes down to the fact that a linear operator and its adjoint (between two finite dimensional Hilbert spaces) have the same operator norm.

From now on, let Z be a complex manifold. Let X, Y be two embedded smooth complex submanifolds of Z such that $Y \subseteq \overline{X}$.

Definition 8.4. We say that the pair (X, Y) satisfies *Condition* (W) if for any point $y \in Y$, and for any sequence of points $\{x_i\} \subseteq X$ converging to y, there exists a constant $C > 0$ such that for $i \gg 0$, we have

$$d(T_yY, T_{x_i}X) \leq C \cdot d(y, x_i)$$

where we view $T_{x_i}X$ as a subspace of T_yZ using a local trivialization of the tangent bundle T_Z, and $d(y, x_i)$ is the Euclidean distance between y and x_i in a local coordinate chart.

Kuo [9] (see also Verdier [11, Théorème 1.5]) proved the following.

Theorem 8.5. *Let Z be a complex manifold. Let X, Y be two embedded smooth complex submanifolds of Z such that Y is contained in the closure of X. If the pair (X, Y) satisfies Condition (W), then the pair (X, Y) satisfies the Whitney conditions $(A), (B)$.*

We are going to use this result in the following form.

Lemma 8.6. *Same assumptions as above. Assume the pair (X, Y) satisfies the Whitney condition (A). Then the pair (X, Y) satisfies the Whitney condition (B) if the following condition holds: Let $y \in Y$ be any point, and let $\{x_i\} \subseteq X$ be a sequence of points converging to y such that $T = \lim_{i\to\infty} T_{x_i}X$ exists. Then there is a constant C such that*

$$d(T, T_{x_i}X) \leq C \cdot d(y, x_i). \tag{26}$$

Equivalently, there exists a constant C such that

$$d((N^*_{X|Z})_{x_i}, \lim_{i\to\infty}(N^*_{X|Z})_{x_i}) \leq C \cdot d(y, x_i). \tag{27}$$

Proof. By Whitney condition (A), we know that $T_yY \subseteq T$. By the property (2) of the distance function in Fact 8.3, we conclude that

$$d(T_yY, T_{x_i}X) \leq d(T, T_{x_i}X) \leq C \cdot d(y, x_i).$$

This verifies the Condition (W) and thus gives the Whitney condition (B) by Theorem 8.5. The last statement uses the property (3) of the distance function in Fact 8.3. □

Definition 8.7. Let X be a complex algebraic variety and suppose there is a finite algebraic stratification

$$X = \bigsqcup S_i$$

by connected algebraic varieties whose irreducible components are smooth. We say this is a Whitney stratification if for any $S_j \subseteq \overline{S_i}$, the pair (S_i, S_j) satisfies the Whitney conditions (A) and (B).

8.2. **Brill-Noether stratification is Whitney.** Recall that C is a genus $2n+1$ smooth hyperelliptic curve. For each $0 \leq r \leq n$, denote

$$W^r_{g-1}(C)^\circ := W^r_{g-1}(C) - W^{r+1}_{g-1}(C),$$

which is a connected smooth algebraic variety, and parametrizes degree $g-1$ line bundles with exactly $r+1$ independent sections. The subvariety $\mathrm{Jac}(C) - \Theta$ is also smooth and parametrizes degree $g-1$ lined bundles with no sections. The Brill-Noether stratification of $\mathrm{Jac}(C)$ is defined to be

$$\mathrm{Jac}(C) = (\mathrm{Jac}(C) - \Theta) \sqcup \bigsqcup_{0\leq r\leq n} W^r_{g-1}(C)^\circ.$$

Proposition 8.8. *The Brill-Noether stratification of* $\mathrm{Jac}(C)$ *is a Whitney stratification.*

Proof. Note that $\overline{W^i_{g-1}(C)^\circ} = W^i_{g-1}(C)$ and for $i < j$, we have

$$W^j_{g-1}(C)^\circ \subseteq W^j_{g-1}(C) \subseteq W^i_{g-1}(C).$$

We also have $\overline{\mathrm{Jac}(C) - \Theta} = \mathrm{Jac}(C)$. By Definition 8.7, it suffices to show that for each $i < j$, the pair $(W^i_{g-1}(C)^\circ, W^j_{g-1}(C)^\circ)$ satisfies the Whitney conditions, and the same holds for the pair $(\mathrm{Jac}(C) - \Theta, W^i_{g-1}(C)^\circ)$. To apply Lemma 8.6, we need to understand the conormal bundles of the Brill-Noether strata. Recall that for each $0 \leq r \leq n$, the Abel-Jacobi map $\delta_{(g-1-2r)/2} = \delta_{n-r}$ induces an isomorphism

$$\delta_{(g-1-2r)/2} : U_{g-1-2r} \xrightarrow{\sim} W^r_{g-1}(C)^\circ, \quad D \mapsto \mathcal{O}_C(D) \otimes rg^1_2, \tag{28}$$

where U_{g-1-2r} is defined in Notation 4.1 and is the open subset of C_{g-1-2r} consisting of divisors D such that $h^0(C, \mathcal{O}_C(D)) = 1$. By Lemma 4.7, for any $D \in U_{g-1-2r}$ and $L := \mathcal{O}_C(D) \otimes rg^1_2$, we have

$$(N^*_{W^r_{g-1}(C)^\circ|\,\mathrm{Jac}(C)})_L = (N^*_{\delta_{(g-1-2r)/2}})_D = H^0(C, \omega_C(-D)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$$

where the last isomorphism is induced by $h : C \to \mathbf{P}^1$, the hyperelliptic map determined by the unique g^1_2 and h_*D is the degree $g-1-2r$ divisor defined in Notation 4.3.

For each $i < j$, let $\{L_k\} \subseteq W^i_{g-1}(C)^\circ$ be a sequence of line bundles converging to $L \in W^j_{g-1}(C)^\circ$. Using the isomorphism (28), we can write

$$L_k = \mathcal{O}_C(D_k) \otimes ig^1_2, \quad L = \mathcal{O}_C(D) \otimes jg^1_2$$

such that $D_k \in U_{g-1-2i}$ and $D \in U_{g-1-2j}$. From the discussion above, we know that

$$\begin{aligned}(N^*_{W^i_{g-1}(C)^\circ|\,\mathrm{Jac}(C)})_{L_k} &= H^0(C, \omega_C(-D_k)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D_k)) \\ (N^*_{W^j_{g-1}(C)^\circ|\,\mathrm{Jac}(C)})_L &= H^0(C, \omega_C(-D)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))\end{aligned}$$

If we denote $\overline{D} := \lim_{k\to\infty} D_k \in C_{g-1-2i}$ to be the limit divisor, since $\lim_{k\to\infty} L_k = L$, we see that $\overline{D}$ is an effective subdivisor of D. Therefore,

$$\begin{aligned}\lim_{k\to\infty} (N^*_{W^i_{g-1}(C)^\circ|\,\mathrm{Jac}(C)})_{L_k} =& H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*\overline{D})) \\ &\supseteq H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)) = (N^*_{W^j_{g-1}(C)^\circ|\,\mathrm{Jac}(C)})_L,\end{aligned}$$

where the first equality uses the fact that $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k)) = 0$ for any $k \geq 0$ and hence we can take limits. This verifies the Whitney condition (A) for the pair $(W^i_{g-1}(C)^\circ, W^j_{g-1}(C)^\circ)$, by going to the dual spaces. Now by Lemma 8.6, in order to prove the Whitney condition (B), we just need to show that there exists a constant A such that

$$d(H^0(C, \omega_C(-\overline{D})), H^0(C, \omega_C(-D_k))) \leq A \cdot d(L, L_k) = d(\overline{D}, D_k),$$

where the distance function on the left is induced by an inner product on the vector space $H^0(C, \omega_C)$ and the distance function on the right is induced by the Euclidean norm on a neighborhood of $\overline{D}$ in C_{g-1-2i}. Since the hyperelliptic map $h : C \to \mathbf{P}^1$ is either a local isomorphism (off the branch locus) or locally of the form $t \mapsto t^2$ (on the branch locus), we can push everything down to $\mathbf{P}^1$; there, it sufficies to prove that

$$d(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*\overline{D})), H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D_k))) \leq A \cdot d(h_*\overline{D}, h_*D_k),$$

which follows from the interpretation of the space $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$ as the space of degree $g-1$ homogeneous polynomials vanishing along the divisor h_*D and explicit computations.[1]

For the pair $(\mathrm{Jac}(C) - \Theta, W^i_{g-1}(C)^\circ)$, Condition (W) is vacuous because $\mathrm{Jac}(C)$ is a complex manifold (using the property (1) of the distance function in Fact 8.3). $\square$

9. Questions and open problems

This section is devoted to some questions and open problems.

The log resolution of the hyperelliptic theta divisor is rather intricate. To have a better understanding of it, we ask

Question 9.1. Is there a modular interpretation of the log resolution in Theorem A?

Let C be a Brill-Noether general curve. The Brill-Noether varieties $W^r_{g-1}(C)$ behave like generic determinantal varieties. It is natural to ask for an extension of our results:

Problem 9.2. Prove that Theorem A and Proposition C hold for such a curve C.

Appendix A. Reducedness of $W^r_d(C)$

In this appendix, we provide the proof of the following result, due to the lack of suitable references. A similar argument can be found in [3].

Proposition A.1. *Let C be a smooth hyperelliptic curve of genus g. Let $d, r \in \mathbb{N}$ be integers such that $0 \leq r \leq d \leq g$. Then the Brill-Noether variety $W^r_d(C)$ is reduced.*

We recall the following result saying that reducedness can be checked on the level of tangent cones.

Lemma A.2. *Let $Z \subseteq X$ be a closed subscheme of a smooth variety X and $x \in Z$ be a closed point. If the tangent cone $TC_xZ \subseteq T_xX$ is reduced, then Z is reduced in an open neighborhood of x.*

Proof. Equip the tangent cone TC_xZ with its natural scheme structure, then there is a flat specialization of (a neighborhood of x in) Z to TC_xZ. The desired result follows from the fact that reducedness is an open condition in flat families, c.f. [7, Theorem 12.1.1 (vii)]. $\square$

By Lemma A.2, Proposition A.1 is reduced to the following

Lemma A.3. *The tangent cone $TC_LW^r_d(C)$ is reduced for any $L \in W^r_d(C)$.*

Proof. To simplify the notation, we denote $W^r_d = W^r_d(C)$. Fix $\mathcal{L}$ a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$ and let $\mathrm{pr}_2 : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ be the second projection. Let $L \in W^r_d$ be a line bundle of degree d and assume $h^0(L) = s+1$ with $s \geq r$. In a neighborhood of L in $\mathrm{Pic}^d(C)$, we can produce a minimal complex computing W^r_d, by a variant of the method in [1, Chapter IV, §3]. Note that we can always pick a point $p \in C$ such that

[1] Botong Wang pointed out that one can view this as a Lipschitz property of the map between compact manifolds $\mathrm{Sym}^{g-1-2i}\mathbf{P}^1 \to \mathrm{Grass}(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1)), 2i)$ which sends E to $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-E))$.

$h^1(L(p)) = h^1(L) - 1$ and $H^0(L) = H^0(L(p))$. Iterating this, we can pick an effective divisor D of degree $h^1(L) = g - d + s$ with the property that $H^1(L(D)) = 0$ and in the short exact sequence

$$0 \to L \to L(D) \to L(D) \otimes \mathcal{O}_D \to 0,$$

the induced connecting map $H^0(D, L(D) \otimes \mathcal{O}_D) \to H^1(C, L)$ is an isomorphism (equivalently, $H^0(C, L) \to H^0(C, L(D))$ is an isomorphism). Denote by $\mathcal{D} = \mathrm{pr}_2^* D$ the effective divisor on $C \times \mathrm{Pic}^d(C)$. Then on some neighborhood of the point L, we have a short exact sequence

$$0 \to \mathrm{pr}_{2,*}\mathcal{L}(\mathcal{D}) \to \mathrm{pr}_{2,*}(\mathcal{L}(\mathcal{D}) \otimes \mathcal{O}_{\mathcal{D}}) \to R^1\mathrm{pr}_{2,*}\mathcal{L} \to 0,$$

where $R^1\mathrm{pr}_{2,*}\mathcal{L}(\mathcal{D})$ vanishes on the neighborhood in question. Here $\mathrm{pr}_{2,*}\mathcal{L} = 0$ because it is torsion-free and vanishes at a general point in the neighborhood of L. This gives us a presentation

$$0 \to E^0 \xrightarrow{A} E^1 \to R^1\mathrm{pr}_{2,*}\mathcal{L} \to 0$$

where E^0 and E^1 are vector bundles of rank $h^0(L) = s+1$, respectively $h^1(L) = g-d+s$. Moreover, the differential, viewed as a matrix A, vanishes at the point L. Let A_1 be the linear part of A; that has entries in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal at L.

Now W^r_d is defined, in a neighborhood of the point L, by the vanishing of all the $(s-r+1) \times (s-r+1)$ minors of A. Because for $L' \in W^r_d$, the condition is

$$\begin{aligned} h^0(L') \geq r+1 &\Leftrightarrow h^1(L') \geq g-d+r \\ &\Leftrightarrow \mathrm{rank}(A)_{L'} \leq (g-d+s) - (g-d+r) = (s-r). \end{aligned}$$

It follows from the tangent cone theorem in the generic vanishing theory (c.f. [6, Theorem 4]) that one has the following containments:

$$\mathcal{I}_1 \subseteq \mathcal{I}_{TC_L W^r_d} \subseteq \sqrt{\mathcal{I}_{TC_L W^r_d}},$$

where the first ideal is generated by all the $(s-r+1) \times (s-r+1)$ minors of A_1. If one knows that the first ideal $\mathcal{I}_1$ is a radical ideal, and that both $\mathcal{I}_1$ and $\sqrt{\mathcal{I}_{TC_L W^r_d}}$ define the same conical subset in $T_L\mathrm{Pic}^d(C)$ (forgetting about the scheme structure), then the tangent cone $TC_L W^r_d$ is reduced.

Since $W^d_r \cong W_{d-2r}$ as sets, one has

$$\dim TC_L W^r_d = \dim W^r_d = d - 2r.$$

By the discussion above, it suffices to show that the $(s-r+1) \times (s-r+1)$ minors of the matrix A_1 defines a reduced, irreducible subscheme of $T_L\mathrm{Pic}^d(C)$ of dimension $d-2r$. This boils down to the following two claims.

Claim 1: The matrix A_1 is a Hankel/Catalecticant matrix, i.e.

$$A_1 = \begin{pmatrix} x_1 & x_2 & \cdots & x_{g-d+s} \\ x_2 & x_3 & \cdots & x_{g-d+s+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s+1} & \cdots & \cdots & x_{g-d+2s} \end{pmatrix}$$

up to a change of local coordinates.

Proof of Claim 1: We learned this argument from Nero Budur, see [3, Proposition 5.17]. By [1], the matrix A_1 is the one given by the map

$$H^0(L) \to H^1(L) \otimes H^0(\omega_C),$$

which is equivalent to the Petri map

$$\pi_L : H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C).$$

Since C is hyperelliptic and $h^0(L) = s+1$, we can write

$$L = sg_2^1 + p_1 + \cdots + p_{d-2s}$$
$$\omega_C \otimes L^{-1} = (g-1-s-(d-2s))g_2^1 + q_1 + \cdots + q_{d-2s},$$

where $p_i + q_i$ is a hyperelliptic pair for each $1 \le i \le d-2s$ and no two p_i lie in the same fiber of the hyperelliptic involution $C \to \mathbf{P}^1$. Then the Petri map corresponds to

$$H^0(\mathbf{P}^1, \mathcal{O}(s)) \otimes H^0(\mathbf{P}^1, \mathcal{O}(g-1-d+s)) \to H^0(\mathbf{P}^1, \mathcal{O}(g-1-d+2s)) \to H^0(\mathbf{P}^1, \mathcal{O}(g-1))$$

The last map is the tensor product with the section $\eta \in H^0(\mathbf{P}^1, \mathcal{O}(d-2s))$, where η is the product of all linear forms defining the image of p_i in $\mathbf{P}^1$ for $1 \le i \le d-2s$. Write $V = H^0(\mathbf{P}^1, \mathcal{O}(1))$, then the Petri map is the natural multiplication map

$$\mathrm{Sym}^s V \otimes \mathrm{Sym}^{g-1-d+s} V \to \mathrm{Sym}^{g-1-d+2s} V,$$

which clearly gives a Catalecticant matrix since $\dim V = 2$.

Claim 2: Let $C_{v,w}$ be a $v \times w$ Catalecticant matrix with $v \ge w$, i.e.

$$C_{v,w} = \begin{pmatrix} x_1 & x_2 & \cdots & x_w \\ x_2 & x_3 & \cdots & x_{w+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_v & \cdots & \cdots & x_{v+w-1} \end{pmatrix}$$

Then for $k < w$, the ideals of $(k+1) \times (k+1)$ minors of $C_{v,w}$ defines a reduced irreducible subscheme Z of dimension $2k$ in $\mathbb{C}^{v+w-1}$.

Proof of Claim 2: We use notations in [5]. Let $M = \mathrm{Cat}(v,w) \subseteq \mathbf{P}\mathbb{C}^{vw}$ be the Catalecticant space, which is of dimension $v+w-2$ (c.f. [5, Page 561]). Let M_k be the subscheme of matrices of rank $\le k$ in M, the linear space corresponds to all the minors of $C_{v,w}$ of size $k+1$. By [5, Proposition 4.3], one has

$$\mathrm{codim}_M M_k = v + w - 1 - 2k,$$

and M_k is the k-secant variety of a rational normal curve. Thus

$$\dim M_k = \dim M - (v+w-1-2k) = (v+w-2) - (v+w-1-2k) = 2k-1$$

and M_k is irreducible. Moreover, it is pointed out by the author after [5, Proposition 4.3] that M_k is reduced. Therefore the corresponding space $Z \subseteq \mathbb{C}^{v+w-1}$ is reduced, irreducible and has dimension $2k$.

Now we can finish the proof of this lemma. If $d \ge g-1$, then $s - r < s+1 \le g-d+s$; if $d = g$, then we can assume $r \ge 1$ ($W_g^0(C)$ is reduced by a theorem of Kempf) and still get $s - r < s = g-d+s \le s+1$. Therefore we can apply Claim 1 and 2 to obtain that the $(s-r+1) \times (s-r+1)$ minors of the matrix A_1 defines a reduced, irreducible subscheme in $T_L\mathrm{Pic}^d(C) = \mathbb{C}^g$ of dimension $2(s-r)+(d-2s)$ (because only the variables $x_1, \cdots, x_{g-d+2s}$ show up in the matrix A_1 and the other variables provide an additional $d-2s$ dimensions). This gives what we want and therefore we finish the proof that $TC_LW_d^r$ is reduced.

As a consequence, $W_d^r(C)$ is reduced for any $0 \le r \le d \le g$.

□

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[2] A. Bertram. Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space. *J. Differential Geom.*, 35(2):429–469, 1992.

[3] N. Budur and A.-K. Doan. Deformations with cohomology constraints of stable vector bundles on curves. *in preparation*, 2023.
[4] L. Ein. Normal sheaves of linear systems on curves. In *Algebraic geometry: Sundance 1988*, volume 116 of *Contemp. Math.*, pages 9–18. Amer. Math. Soc., Providence, RI, 1991.
[5] D. Eisenbud. Linear sections of determinantal varieties. *Amer. J. Math.*, 110(3):541–575, 1988.
[6] M. Green and R. Lazarsfeld. Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. *Invent. Math.*, 90(2):389–407, 1987.
[7] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
[8] A. A. Johnson. *Multiplier ideals of determinantal ideals.* ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–University of Michigan.
[9] T.-C. Kuo. The ratio test for analytic Whitney stratifications. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, Lecture Notes in Mathematics, Vol. 192, pages 141–149. Springer, Berlin, 1971.
[10] I. Vainsencher. Complete collineations and blowing up determinantal ideals. *Math. Ann.*, 267(3):417–432, 1984.
[11] J.-L. Verdier. Stratifications de Whitney et théorème de Bertini-Sard. *Invent. Math.*, 36:295–312, 1976.

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