ON THE LOCUS OF LIMIT HODGE CLASSES

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Abstract. We introduce a “locus of limit Hodge classes” that also takes into account integral classes that become Hodge classes “in the limit”. More precisely, given a polarized variation of integral Hodge structure of weight zero on a Zariski-open subset of a complex manifold, we construct a canonical analytic space that parametrizes limits of integral classes; the extended locus of Hodge classes is an analytic subspace that contains the usual locus of Hodge classes, but is finite and proper over the base manifold. The construction uses Saito’s theory of mixed Hodge modules and a generalization of the main technical result of Cattani, Deligne, and Kaplan. We study the properties of the resulting analytic space in the case of the family of hyperplane sections of an odd-dimensional smooth projective variety.

A. Introduction

1. Summary. The purpose of this paper is to investigate some global questions about limit Hodge classes, by which I mean integral cohomology classes in a family of projective complex manifolds – and, more generally, in a polarized variation of integral Hodge structure – that become Hodge classes “in the limit”. There are two natural ways to construct a *locus of limit Hodge classes* that contains the usual locus of Hodge classes as a (not necessarily dense) open subset; our main theorem is that the resulting analytic spaces have the same good properties as the locus of Hodge classes itself. In one case, this follows from the work of Cattani, Deligne, and Kaplan; in the other, from a generalization of their main technical result.

2. The locus of Hodge classes. The motivation for looking at limit Hodge classes comes from a specific geometric example: the universal family of hyperplane sections of a Calabi-Yau threefold, or more generally, of any odd-dimensional projective complex manifold. Nevertheless, in constructing the locus of limit Hodge classes and in studying its properties, it will be convenient to work with arbitrary polarized variations of integral Hodge structure. So let $\mathcal{H}$ be a polarized variation of integral Hodge structure of weight zero, defined on a Zariski-open subset X_0 of a complex manifold X. The assumption about the weight is purely for convenience: if $\mathcal{H}$ has even weight $2k$, we can always replace it by the Tate twist $\mathcal{H}(k)$, which has weight zero. We denote by $\mathcal{H}_{\mathbb{Z}}$ the underlying local system of free $\mathbb{Z}$-modules, by $F^p\mathcal{H}$ the Hodge bundles, and by $Q\colon \mathcal{H}_{\mathbb{R}} \otimes \mathcal{H}_{\mathbb{R}} \to \mathbb{R}$ the real bilinear form giving the polarization. At each point $x \in X_0$, we thus get a polarized Hodge structure of weight zero on $\mathcal{H}_{\mathbb{Z},x}$, with Hodge filtration $F^\bullet\mathcal{H}_x$ and polarization $Q_x\colon \mathcal{H}_{\mathbb{R},x} \otimes \mathcal{H}_{\mathbb{R},x} \to \mathbb{R}$.

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Let us first recall the definition of the locus of Hodge classes [CDK95]. The local system $\mathcal{H}_{\mathbb{Z}}$ determines a (not necessarily connected) covering space

$$E(\mathcal{H}_{\mathbb{Z}}) \to X_0,$$

whose sheaf of holomorphic sections is isomorphic to $\mathcal{H}_{\mathbb{Z}}$. The points of $E(\mathcal{H}_{\mathbb{Z}})$ are pairs (x, h), with $h \in \mathcal{H}_{\mathbb{Z},x}$ a class in the fiber over the point $x \in X_0$.

Definition. The *locus of Hodge classes* of $\mathcal{H}$ is the set

$$\operatorname{Hdg}(\mathcal{H}) = \bigl\{ (x,h) \in E(\mathcal{H}_{\mathbb{Z}}) \bigm| h \in \mathcal{H}_{\mathbb{Z},x} \cap F^0\mathcal{H}_x \text{ is a Hodge class} \bigr\};$$

it is a closed analytic subspace of the complex manifold $E(\mathcal{H}_{\mathbb{Z}})$.

The analytic structure on $\operatorname{Hdg}(\mathcal{H})$ arises as follows: because $\mathcal{H}_{\mathbb{Z}}$ is a subsheaf of $\mathcal{H}$, the complex manifold $E(\mathcal{H}_{\mathbb{Z}})$ is naturally embedded into the holomorphic vector bundle $B(\mathcal{H})$, and $\operatorname{Hdg}(\mathcal{H})$ is the intersection of $E(\mathcal{H}_{\mathbb{Z}})$ with the holomorphic subbundle $B(F^0\mathcal{H})$. When X_0 is a smooth complex algebraic variety, and when $\mathcal{H}$ comes from the cohomology of a family of smooth projective varieties over X_0, the Hodge conjecture predicts that $\operatorname{Hdg}(\mathcal{H})$ should be a countable union of algebraic varieties. Cattani, Deligne, and Kaplan, in their famous article [CDK95] about the locus of Hodge classes, were able to prove this *without assuming the Hodge conjecture.*

Theorem (Cattani, Deligne, Kaplan)**.** *If $\mathcal{H}$ is a polarized variation of integral Hodge structure on a smooth complex algebraic variety X_0, then $\operatorname{Hdg}(\mathcal{H})$ is a countable union of algebraic varieties.*

This remarkable result is a consequence of Chow's theorem and the following more precise theorem about Hodge classes with bounded self-intersection number, valid on any complex manifold X. Fix an integer $K \geq 0$, and consider the subset

$$\operatorname{Hdg}_{\leq K}(\mathcal{H}) = \bigl\{ (x,h) \in \operatorname{Hdg}(\mathcal{H}) \bigm| Q_x(h,h) \leq K \bigr\}$$

consisting of Hodge classes whose self-intersection number is bounded by K. It is not hard to show that the projection from $\operatorname{Hdg}_{\leq K}(\mathcal{H})$ to X_0 is *finite* (= a proper holomorphic mapping with finite fibers); what Cattani, Deligne, and Kaplan proved is that this finiteness property still holds over the larger complex manifold X.

Theorem 2.1 (Cattani, Deligne, Kaplan)**.** *For every $K \geq 0$, it is possible to extend $\operatorname{Hdg}_{\leq K}(\mathcal{H})$ to an analytic space that is finite over X.*

This raises the question of whether there is a canonical way to extend $\operatorname{Hdg}(\mathcal{H})$ to an analytic space over X, and, if yes, of whether the points of the extension have any Hodge-theoretic meaning. As we will see below, the answer to both questions is yes: there is a good notion of "limit Hodge class", and the locus of limit Hodge classes $\operatorname{Hdg}(\mathcal{H}, X)$ is a countable union of analytic subspaces $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$, each finite over X. Note that a limit Hodge class is not necessarily the limit of a sequence of Hodge classes, and so the usual locus of Hodge clasess $\operatorname{Hdg}(\mathcal{H})$ need not be dense in the locus of limit Hodge classes $\operatorname{Hdg}(\mathcal{H}, X)$.

3. The case of a Hodge structure. To motivate the construction, let us first look at the case of a single Hodge structure H. We assume that H is polarized and integral of weight zero; we denote the underlying $\mathbb{Z}$-module by $H_{\mathbb{Z}}$; the polarization by Q; and the Hodge filtration by $F^\bullet H$. Let $\operatorname{Hdg}(H) = H_{\mathbb{Z}} \cap F^0 H$ be the set of Hodge classes in H. According to the bilinear relations, a class $h \in H_{\mathbb{Z}}$ is Hodge exactly when it is perpendicular (under Q) to the space $F^1 H$; this says that $\operatorname{Hdg}(H)$ is precisely the kernel of the linear mapping

$$\varphi\colon H_{\mathbb{Z}} \to (F^1 H)^*, \quad h \mapsto Q(h, -).$$

At first, it may seem that φ is not good for much else, because its image is not a nice subset of $(F^1 H)^*$. In fact, the dimension of the vector space $F^1 H$ can be much smaller than the rank of $H_{\mathbb{Z}}$, and so φ will typically have dense image. But it turns out that the restriction of φ to the subset

$$H_{\mathbb{Z}}(K) = \left\{ h \in H_{\mathbb{Z}} \;\middle|\; |Q(h,h)| \leq K \right\}$$

is well-behaved. The idea of bounding the self-intersection number of the integral classes already occurs in the paper by Cattani, Deligne, and Kaplan. To back up this claim, we have the following lemma; note that the estimate in the proof will also play a role in the analysis later on.

Lemma 3.1. *The mapping $\varphi\colon H_{\mathbb{Z}}(K) \to (F^1 H)^*$ is finite and proper, and its image is a discrete subset of the vector space $(F^1 H)^*$.*

Proof. We have to show that the preimage of any bounded subset of $(F^1 H)^*$ is finite. It will be convenient to measure things in the *Hodge norm*: if

$$h = \sum_p h^{p,-p}, \quad \text{with } h^{p,-p} \in F^p H \cap \overline{F^{-p} H},$$

is the Hodge decomposition of a vector $h \in H$, then its Hodge norm is

$$\|h\|_H^2 = \sum_p \|h^{p,-p}\|_H^2 = \sum_p (-1)^p Q\Big(h^{p,-p}, \overline{h^{p,-p}}\Big).$$

Now suppose that $h \in H_{\mathbb{Z}}$ satisfies $|Q(h,h)| \leq K$ and $\|\varphi(h)\|_H \leq R$; it will be enough to prove that $\|h\|_H$ is bounded by a quantity depending only on K and R. The assumption on $\varphi(h)$ means that $|Q(h,v)| \leq R\|v\|_H$ for every $v \in F^1 H$. If we apply this inequality to the vector

$$v = \sum_{p \geq 1} (-1)^p h^{p,-p},$$

we find that $\|v\|_H^2 = |Q(h,v)| \leq R\|v\|_H$, and hence that

$$\sum_{p \geq 1} \|h^{p,-p}\|_H^2 = \|v\|_H^2 \leq R^2.$$

Because h is invariant under conjugation, it follows that $\|h\|_H^2 \leq \|h^{0,0}\|_H^2 + 2R^2$. This leads to the conclusion that $\|h\|_H^2 \leq K + 4R^2$, because

$$Q(h,h) = \|h^{0,0}\|_H^2 + \sum_{p \neq 0} (-1)^p \|h^{p,-p}\|_H^2 \leq K.$$

In particular, there are only finitely many possibilities for $h \in H_{\mathbb{Z}}$, which means that φ is a finite mapping, and that the image of φ is a discrete subset of $(F^1 H)^*$. ∎

4. The general case. Now let us return to the general case. As in [CDK95], it is not actually necessary to assume that X is projective; we shall therefore consider a polarized variation of integral Hodge structure $\mathcal{H}$ of weight zero, defined on a Zariski-open subset X_0 of an arbitrary complex manifold X. By performing the construction in §3 at every point of X_0, we obtain a holomorphic mapping

$$\varphi\colon E(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H});$$

here $T(F^1\mathcal{H}) = \operatorname{Spec}\bigl(\operatorname{Sym} F^1\mathcal{H}\bigr)$ is the holomorphic vector bundle on X_0 whose sheaf of holomorphic sections is $(F^1\mathcal{H})^*$. The locus of Hodge classes $\operatorname{Hdg}(\mathcal{H})$ is then exactly the preimage of the zero section in $T(F^1\mathcal{H})$. For every $K \geq 0$, we consider the submanifold

$$E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) = \bigl\{ (x,h) \in E(\mathcal{H}_{\mathbb{Z}}) \bigm| \lvert Q_x(h,h)\rvert \leq K \bigr\}.$$

It is a union of connected components of the covering space $E(\mathcal{H}_{\mathbb{Z}})$, because the quantity $Q_x(h,h)$ is obviously constant on each connected component. More or less directly by Lemma 3.1, the holomorphic mapping

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$$

is finite and proper, with complex-analytic image; moreover, one can show that the mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to the normalization of the image is a finite covering space. For the details, please consult §11 below.

To construct an extension of $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to an analytic space over X, we use the theory of Hodge modules [Sai90]. Let M be the polarized Hodge module of weight $\dim X$ with strict support X, canonically associated with $\mathcal{H}$. We denote the underlying filtered left $\mathscr{D}$-module by the symbol $(\mathcal{M}, F_{\bullet}\mathcal{M})$. The point is that

$$\mathcal{M}\big\vert_{X_0} \simeq \mathcal{H} \quad \text{and} \quad F_k\mathcal{M}\big\vert_{X_0} \simeq F^{-k}\mathcal{H};$$

in particular, the coherent sheaf $F_{-1}\mathcal{M}$ is an extension of the Hodge bundle $F^1\mathcal{H}$ to a coherent sheaf of $\mathscr{O}_X$-modules. Now consider the holomorphic mapping

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M}),$$

where the analytic space on the right-hand side is defined as before as the spectrum of the symmetric algebra of the coherent sheaf $F_{-1}\mathcal{M}$. We have already seen that $\varphi\bigl(E_{\leq K}(\mathcal{H}_{\mathbb{Z}})\bigr)$ is an analytic subset of $T(F^1\mathcal{H})$; since we are interested in limits of integral classes, we shall extend it to the larger space $T(F_{-1}\mathcal{M})$ by taking the closure. The main result of the paper is that the closure remains analytic.

Theorem 4.1. *The closure of $\varphi\bigl(E_{\leq K}(\mathcal{H}_{\mathbb{Z}})\bigr)$ is an analytic subset of $T(F_{-1}\mathcal{M})$.*

The proof consists of two steps: (1) We reduce the problem to the special case where $X \setminus X_0$ is a divisor with normal crossings and $\mathcal{H}_{\mathbb{Z}}$ has unipotent local monodromy; this reduction is similar to [Sch12a]. (2) In that case, we prove the theorem by a careful local analysis, using the theory of degenerating variations of Hodge structure. In fact, we deduce the theorem from a strengthening of the main technical result of Cattani, Deligne, and Kaplan, which we prove by adapting the method introduced in [CDK95]. Rather than just indicating the necessary changes to their argument, I have chosen to write out a complete proof; I hope that this will make Chapter D useful also to those readers who are only interested in the locus of Hodge classes and the theorem of Cattani, Deligne, and Kaplan.

Once Theorem 4.1 is proved, it makes sense to consider the normalization of the closure of $\varphi\bigl(E_{\leq K}(\mathcal{H}_{\mathbb{Z}})\bigr)$. The mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to its image in the normalization is a finite covering space; it can therefore be extended in a canonical way to a finite branched covering by appealing to the *Fortsetzungssatz* of Grauert and Remmert.

Theorem 4.2. *There is a normal analytic space* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *containing the complex manifold* $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *as a dense open subset, and a finite holomorphic mapping*

$$\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M}),$$

whose restriction to $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *agrees with* φ. *Moreover,* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *and* $\tilde{\varphi}$ *are unique up to isomorphism.*

Since each $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ is a union of connected components of the covering space $E(\mathcal{H}_{\mathbb{Z}})$, we can take the union over all the $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$; this operation is well-defined because of the uniqueness statement in the theorem. In this way, we get a normal analytic space $\overline{E(\mathcal{H}_{\mathbb{Z}})}$, and a holomorphic mapping

$$\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})} \to T(F_{-1}\mathcal{M})$$

with discrete fibers that extends φ. Now the preimage of the zero section in $T(F_{-1}\mathcal{M})$ gives us the desired compactification for the locus of Hodge classes.

Definition 4.3. The *extended locus of Hodge classes* $\widetilde{\operatorname{Hdg}}(\mathcal{H})$ is the closed analytic subscheme $\tilde{\varphi}^{-1}(0) \subseteq \overline{E(\mathcal{H}_{\mathbb{Z}})}$; by construction, it contains the locus of Hodge classes.

Note that when X is projective, Chow's theorem implies that $\widetilde{\operatorname{Hdg}}(\mathcal{H})$ is a countable union of projective schemes, each finite over its image in X.

5. The family of hyperplane sections. The construction above can be applied to the family of hyperplane sections of a smooth projective variety of odd dimension. In this case, one has a good description of the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet)$ in terms of residues [Sch12b], and it is possible to say more about the space $\overline{E(\mathcal{H}_{\mathbb{Z}})}$. The fact that $F_{-1}\mathcal{M}$ is the quotient of an ample vector bundle leads to the following result; it was predicted by Clemens several years ago.

Theorem 5.1. *The analytic space* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *is holomorphically convex. Every compact analytic subset of dimension* ≥ 1 *lies inside the extended locus of Hodge classes.*

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B. The locus of limit Hodge classes

7. Limit Hodge classes in dimension one. We begin by defining limit Hodge classes in dimension one, where we can use the theory of limit mixed Hodge structures. Suppose then that $\mathcal{H}$ is a polarized integral variation of Hodge structure of weight zero on the punctured unit disk

$$\Delta^* = \{ t \in \mathbb{C} \mid 0 < |t| < 1 \}.$$

Let $H_{\mathbb{Z}}$ denote the space of global sections of the pullback of $\mathcal{H}_{\mathbb{Z}}$ to the universal covering space of Δ^*. The monodromy operator $T \colon H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ is quasi-unipotent; write $T = T_s T_u$, with T_s semisimple and T_u unipotent, and let $N = \log T_u$ be the logarithm of the unipotent part. According to Schmid's results in [Sch73], the $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ carries a *limit mixed Hodge structure*

$$\bigl(W(N), F\bigr),$$

and T_s is an endomorphism of this mixed Hodge structure. Choosing different coordinates on Δ replaces F by a filtration of the form $e^{wN}F$ with $w \in \mathbb{C}$, which means that the induced mixed Hodge structure on $\ker N \subseteq H_{\mathbb{Q}}$ is independent of the choice of coordinates. It therefore makes sense to consider integral Hodge classes $h \in H_{\mathbb{Z}} \cap \ker N \cap F^0$ in this mixed Hodge structure. Because $Th = T_s e^N h = T_s h$, the orbit of h under the $\mathbb{Z}$-action induced by T is finite and consists entirely of Hodge classes. The real number $Q(h,h)$ is called the *self-intersection number* of h.

Lemma 7.1. *If $h \in H_{\mathbb{Z}} \cap \ker N \cap F^0$ is nonzero, then $Q(h,h) > 0$.*

Proof. To relate the self-intersection number of h to that of a Hodge class in the usual sense, we recall that the limit mixed Hodge structure is polarized by the pairing $Q \colon H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}$ and the nilpotent operator N; in particular, the induced Hodge structure of weight zero on

$$\ker\Bigl(N \colon W_0(N)/W_{-1}(N) \to W_{-2}(N)/W_{-3}(N)\Bigr)$$

is polarized by the induced pairing $\tilde{Q}$. Since $\ker N \subseteq W_0(N)$, we can let $\tilde{h}$ be the image of $h \in H_{\mathbb{Z}} \cap \ker N \cap F^0$ under the projection

$$\ker N \to \ker\Bigl(N \colon W_0(N)/W_{-1}(N) \to W_{-2}(N)/W_{-3}(N)\Bigr);$$

Then $\tilde{h}$ is a Hodge class, and so $Q(h,h) = \tilde{Q}(\tilde{h},\tilde{h}) \geq 0$; moreover, equality only happens when $\tilde{h} = 0$, or in other words, when $h \in H_{\mathbb{Z}} \cap W_{-1}(N) \cap F^0 = \{0\}$. □

In the covering space $E(\mathcal{H}_{\mathbb{Z}})$ of Δ^* determined by the local system, the connected component containing the point h is a d-sheeted covering of Δ^*, where d is the smallest positive integer with $T^d h = h$. This covering can be completed to a branched covering of the entire disk by adding one point, which naturally corresponds to the $\mathbb{Z}$-orbit of h. This suggests that we should consider all the Hodge classes in the orbit of h as being part of the same "limit Hodge class".

Definition 7.2. The elements of the quotient

$$\bigl(H_{\mathbb{Z}} \cap \ker N \cap F^0\bigr)/\mathbb{Z}$$

are called *limit Hodge classes* for $\mathcal{H}$ at the point $0 \in \Delta$.

The quotient $H_{\mathbb{Z}}/\mathbb{Z}$ parametrizes the connected components of the covering space $E(\mathcal{H}_{\mathbb{Z}})$, with the subset $(H_{\mathbb{Z}} \cap \ker N)/\mathbb{Z}$ corresponding to those components that are finite over Δ^*. The set of limit Hodge classes is therefore naturally a subset of the set of connected components of $E(\mathcal{H}_{\mathbb{Z}})$.

8. The cosheaf of connected components. Returning to the general case, a limit Hodge class for $\mathcal{H}$ at a point $x \in X$ should be something like a class in the fiber of $\mathcal{H}_{\mathbb{Z}}$ over a nearby point of X_0 that comes from a limit Hodge class on some holomorphic arc through the point x. To get a definition that is independent of which nearby point we chooose, and to allow for the possibility that the same limit Hodge class may appear on different arcs through the same point, we clearly need to take the action of the local fundamental group into account. For that reason, we shall first give a more intrinsic definition of classes in a nearby fiber of $\mathcal{H}_{\mathbb{Z}}$.

Since Hodge structures play no role here, let us suppose for the time being that $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf of free $\mathbb{Z}$-modules of finite rank on a dense Zariski-open subset X_0 of a complex manifold X. Denote by $p\colon E(\mathcal{H}_{\mathbb{Z}}) \to X_0$ the resulting covering space. For every open set $U \subseteq X$, define

$$\mathcal{C}(U) = \pi_0\bigl(p^{-1}(U \cap X_0)\bigr) \tag{8.1}$$

to be the set of connected components of $E(\mathcal{H}_{\mathbb{Z}})$ over the open set $U \cap X_0$, with the convention that $\mathcal{C}(\emptyset) = \emptyset$. For every pair of open sets $U \subseteq V$, one has a mapping $\mathcal{C}(U) \to \mathcal{C}(V)$, and so $\mathcal{C}$ is a covariant functor from the category of open sets in X to the category of sets; we shall see below that it is an example of a "cosheaf". We can now define the set of local components at a point $x \in X$ as

$$\mathcal{C}_x = \lim_{U \ni x} \mathcal{C}(U),$$

where the projective limit is over all neighborhoods of the given point; when $x \in X_0$, this is just a different name for the stalk $\mathcal{H}_{\mathbb{Z},x}$. The following lemma justifies thinking of $\mathcal{C}_x$ as the set of components over a small neighborhood of x.

Lemma 8.2. *If U is a good neighborhood of $x \in X$ with regard to the subspace $X \setminus X_0$, then the extension mapping $\mathcal{C}_x \to \mathcal{C}(U)$ is bijective.*

Proof. Recall that Prill [Pri67, Chapter B] calls a neighborhood U good with regard to $X \setminus X_0$ if there is a neighborhood basis consisting of open sets V such that $V \cap X_0$ is a deformation retract of $U \cap X_0$. Since we can compute the projective limit in the definition of $\mathcal{C}_x$ along such a basis, the bijectivity of $\mathcal{C}_x \to \mathcal{C}(U)$ is obvious. □

One can also interpret $\mathcal{C}(U)$ in terms of the fiber of the local system at a nearby point $x_0 \in U \cap X_0$. Indeed, two elements of $\mathcal{H}_{\mathbb{Z},x_0}$ belong to the same connected component of $p^{-1}(U \cap X_0)$ if and only if they lie in the same orbit under the action of the fundamental group $\pi_1(U \cap X_0, x_0)$; this means that we have a bijection

$$\mathcal{H}_{\mathbb{Z},x_0}/\pi_1(U \cap X_0, x_0) \simeq \mathcal{C}(U).$$

By restricting our attention to good neighborhoods, we thus get an interpretation for the elements of the costalk $\mathcal{C}_x$ at a point $x \in X$ that agrees with the intuitive notion we started from.

We shall now explain how to use the information in $\mathcal{C}$ to extend the covering space $E(\mathcal{H}_{\mathbb{Z}})$ to a topological branched covering of X. For that purpose, it will be convenient to borrow some of the terminology from the theory of cosheaves; for a good summary, one can consult [Woo09, Appendix B]. Recall that a *pre-cosheaf* of sets on X is a covariant functor $\mathcal{F}$ from the category of open subsets of X to the category of sets; the elements of $\mathcal{F}(U)$ are called *cosections*, and for $U \subseteq V$, the mappings $\mathcal{F}(U) \to \mathcal{F}(V)$ are called *extension mappings*. A pre-cosheaf $\mathcal{F}$ is called a *cosheaf* if, for every open covering

$$U = \bigcup_{i \in I} U_i,$$

the set $\mathcal{F}(U)$ is the colimit of the diagram

$$\coprod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \rightrightarrows \coprod_{i \in I} \mathcal{F}(U_i)$$

in the category of sets; concretely, two elements in $\mathcal{F}(U_i)$ and $\mathcal{F}(U_j)$ get identified in $\mathcal{F}(U)$ if they are both extensions of the same element in $\mathcal{F}(U_i \cap U_j)$.

Example 8.3. The pre-cosheaf $\mathcal{C}$ from above is a cosheaf. More generally, any continuous mapping $f\colon Y \to X$ gives rise to a pre-cosheaf $\mathcal{C}_f$ on X, by setting

$$\mathcal{C}_f(U) = \pi_0\big(f^{-1}(U)\big)$$

for $U \subseteq X$ open; if Y is locally connected, this is a cosheaf.

[Need to use “spatial” cosheaves, because of erratum to Woolf’s paper]

Conversely, one can build a locally connected topological space from any cosheaf. Let $\mathcal{F}$ be a cosheaf on a topological space X. As a set, the *display space*

$$D(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$$

is the disjoint union of all the costalks; there is an obvious mapping $p\colon D(\mathcal{F}) \to X$. The topology on $D(\mathcal{F})$ is generated by a basis consisting of the sets

$$B(U, \alpha) = \big\{ \beta \in p^{-1}(U) \bigm| \text{the mapping } \mathcal{F}_{p(\beta)} \to \mathcal{F}(U) \text{ takes } \beta \text{ to } \alpha \big\}$$

for $U \subseteq X$ open and $\alpha \in \mathcal{F}(U)$. It is easy to see that the projection $p\colon D(\mathcal{F}) \to X$ is continuous, and that the topology on $D(\mathcal{F})$ is Hausdorff if and only if X itself is a Hausdorff space. The following result is proved in [Woo09, Corollary B.4].

Lemma 8.4. *If $\mathcal{F}$ is a cosheaf, $D(\mathcal{F})$ is locally connected, and the natural mapping*

$$p^{-1}(x) \to \lim_{U \ni x} \pi_0\big(p^{-1}(U)\big)$$

is a bijection for every $x \in X$.

More succinctly, the lemma says that we have an isomorphism of cosheaves $\mathcal{C}_p \simeq \mathcal{F}$. From a more functorial point of view, the display space construction is a right adjoint to the construction in Example 8.3. We shall prove this only in the special case that is needed below, and refer the interested reader to [Woo09, Proposition B.2 and Proposition B.5] for the general case.

Proposition 8.5. *Let $f\colon Y \to X$ be a continuous mapping, Y locally connected. Any morphism of cosheaves $\mathcal{C}_f \to \mathcal{F}$ induces a continuous mapping $g\colon Y \to D(\mathcal{F})$ with $p \circ g = f$.*

Proof. At each point $x \in X$, the composition

$$f^{-1}(x) \to \lim_{U \ni x} \pi_0\big(f^{-1}(U)\big) \to \mathcal{F}_x$$

gives us a mapping from $f^{-1}(x)$ to $p^{-1}(x)$; putting these together, we obtain the desired mapping $g\colon Y \to D(\mathcal{F})$. To prove that g is continuous, observe that $g^{-1}\big(B(U,\alpha)\big)$ is the union of all those connected components of $f^{-1}(U)$ that are sent to α by the mapping $\mathcal{C}_f(U) \to \mathcal{F}(U)$; this is an open subset of Y because we are assuming that Y is locally connected. □

Now let us return to the cosheaf $\mathcal{C}$ defined in (8.1). We shall use the notation

$$p \colon D(\mathcal{H}_{\mathbb{Z}}) \to X$$

for the display space of $\mathcal{C}$. Since the local system $\mathcal{H}_{\mathbb{Z}}$ is defined on $X_0 \subseteq X$, it is easy to see that $p^{-1}(X_0)$ is isomorphic to the covering space $E(\mathcal{H}_{\mathbb{Z}})$, and therefore naturally a complex manifold. The cosheaf $\mathcal{C}$ is even constructible in the following sense.

Proposition 8.6. *In any Whitney stratification of X such that $X \setminus X_0$ is a union of strata, the restriction of $p \colon D(\mathcal{H}_{\mathbb{Z}}) \to X$ to any stratum is a covering space.*

Proof. The point is that Whitney stratifications are locally topologically trivial along strata. Fix a Whitney stratification of X in which $X \setminus X_0$ is a union of strata, and let $S \subseteq X$ be an arbitrary stratum. Any point $x_0 \in S$ has an open neighborhood $U \subseteq X$ that is homeomorphic, via a stratum preserving homeomorphism, to the product of $\mathbb{C}^{\dim S}$ and the open cone over the link $L(x_0)$ of the stratum at the point. Now U is clearly a good neighborhood of every $x \in U \cap S$, and so $\mathcal{C}_x \to \mathcal{C}(U)$ is bijective for every $x \in U \cap S$. From this, one concludes easily that

$$p^{-1}(U \cap S) = \bigsqcup_{\alpha \in \mathcal{C}(U)} p^{-1}(U \cap S) \cap B(U, \alpha)$$

is the disjoint union of open subsets that are homeomorphic to $U \cap S$; consequently, $p^{-1}(S)$ is a covering space of S. This also means that $p^{-1}(S)$ is actually a complex manifold and that the restriction of p is holomorphic. □

Now suppose that $f \colon Y \to X$ is a holomorphic mapping such that $Y_0 = f^{-1}(X_0)$ is dense in Y. For the local system $f^{-1}\mathcal{H}_{\mathbb{Z}}$ on Y_0, we have

$$E(f^{-1}\mathcal{H}_{\mathbb{Z}}) = Y_0 \times_{X_0} E(\mathcal{H}_{\mathbb{Z}}).$$

Let $p \colon D(f^{-1}\mathcal{H}_{\mathbb{Z}}) \to Y$ denote the display space of the resulting cosheaf on Y.

Proposition 8.7. *The projection from $E(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $E(\mathcal{H}_{\mathbb{Z}})$ extends uniquely to a continuous mapping from $D(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $D(\mathcal{H}_{\mathbb{Z}})$, making the diagram*

$$\begin{array}{ccc} D(f^{-1}\mathcal{H}_{\mathbb{Z}}) & \longrightarrow & D(\mathcal{H}_{\mathbb{Z}}) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

commutative.

Proof. For every open set $U \subseteq X$, the natural projection from $E(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $E(\mathcal{H}_{\mathbb{Z}})$ sends each connected component of $(f \circ p)^{-1}(U \cap X_0)$ into a connected component of $p^{-1}(U \cap X_0)$. This gives us a mapping

$$\mathcal{C}_{f \circ p}(U) = \pi_0\big((f \circ p)^{-1}(U \cap X_0)\big) \to \pi_0\big(p^{-1}(U \cap X_0)\big) = \mathcal{C}(U),$$

and in fact a morphism of cosheaves $\mathcal{C}_{f \circ p} \to \mathcal{C}$. By Proposition 8.5, this morphism induces a continuous mapping from $D(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $D(\mathcal{H}_{\mathbb{Z}})$; a look at the construction shows that it agrees over Y_0 with the projection from $E(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $E(\mathcal{H}_{\mathbb{Z}})$. □

Here is another useful property of the display space construction.

Proposition 8.8. *Suppose that we have a commutative diagram*

$$\begin{array}{ccc} E(\mathcal{H}_{\mathbb{Z}}) & \hookrightarrow & W \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ X_0 & \hookrightarrow & X \end{array}$$

such that $E(\mathcal{H}_{\mathbb{Z}})$ is dense in the normal analytic space W. Then the inclusion of $E(\mathcal{H}_{\mathbb{Z}})$ into $D(\mathcal{H}_{\mathbb{Z}})$ factors uniquely through W.

Proof. For every open set $U \subseteq X$, the natural mapping

$$\mathcal{C}(U) = \pi_0\big(p^{-1}(U \cap X_0)\big) \to \pi_0\big(q^{-1}(U)\big) = \mathcal{C}_q(U)$$

is bijective because W is normal and $E(\mathcal{H}_{\mathbb{Z}}) \subseteq W$ is dense. This means that the cosheaf $\mathcal{C}_q$ is isomorphic to $\mathcal{C}$; now Proposition 8.5 gives us the desired continuous mapping from W to $D(\mathcal{H}_{\mathbb{Z}})$. □

9. Limit Hodge classes. We are now ready to define limit Hodge classes. Let $\mathcal{H}$ be a polarized variation of integral Hodge structure of weight zero, defined on a dense Zariski-open subset X_0 of a complex manifold X. Let $\mathcal{C}$ be the cosheaf of connected components of the covering space $E(\mathcal{H}_{\mathbb{Z}})$, introduced in (8.1), and

$$\mathcal{C}_x = \lim_{U \ni x} \mathcal{C}(U) = \lim_{U \ni x} \pi_0\big(p^{-1}(U \cap X_0)\big)$$

its costalk at the point $x \in X$. As before, we denote by

$$p \colon D(\mathcal{H}_{\mathbb{Z}}) \to X$$

the display space of $\mathcal{C}$; it is a locally connected Hausdorff space with the property that $p^{-1}(X_0)$ is isomorphic to $E(\mathcal{H}_{\mathbb{Z}})$. Since the function $(x, h) \mapsto Q_x(h, h)$ is locally constant on $E(\mathcal{H}_{\mathbb{Z}})$, it extends uniquely to a locally constant continuous real-valued function on $D(\mathcal{H}_{\mathbb{Z}})$; the *intersection number* of an element in $\mathcal{C}_x$ is by definition the value of this function. Note that the subset

$$D_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \subseteq D(\mathcal{H}_{\mathbb{Z}})$$

of points with intersection number at most K is a union of connected components of $D(\mathcal{H}_{\mathbb{Z}})$, and therefore both open and closed.

To define limit Hodge classes, suppose that we have a holomorphic arc $\gamma \colon \Delta \to X$ with $\gamma(0) = x$ and $\gamma(\Delta^*) \subseteq X_0$. Proposition 8.7 gives us a commutative diagram

$$\begin{array}{ccc} D(\gamma^{-1}\mathcal{H}_{\mathbb{Z}}) & \longrightarrow & D(\mathcal{H}_{\mathbb{Z}}) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{\ \gamma\ } & X. \end{array}$$

The discussion in §7 applies to the variation of Hodge structure $\gamma^{-1}\mathcal{H}$. It shows that $p^{-1}(0) \simeq H_{\mathbb{Z}}/\mathbb{Z}$, where $H_{\mathbb{Z}}$ stands for the space of global sections of $\gamma^{-1}\mathcal{H}_{\mathbb{Z}}$ on the universal covering space of the punctured disk, and the $\mathbb{Z}$-action comes from the monodromy operator $T = T_s e^N$. Since we also have $p^{-1}(x) = \mathcal{C}_x$, we thus get a well-defined mapping

$$\big(H_{\mathbb{Z}} \cap \ker N \cap F^0\big)/\mathbb{Z} \to \mathcal{C}_x \tag{9.1}$$

from the set of limit Hodge classes for $\gamma^{-1}\mathcal{H}$ at the point $0 \in \Delta$, in the sense of Definition 7.2, to the costalk of $\mathcal{C}$ at the point $x \in X$.

Definition 9.2. An element of the costalk $\mathcal{C}_x$ is called a *limit Hodge class* if there is a germ of a holomorphic arc $\gamma\colon (\Delta, 0) \to (X, x)$ with $\gamma(\Delta^*) \subseteq X_0$, such that the given element belongs to the image of the mapping in (9.1).

At points $x \in X_0$, the costalk $\mathcal{C}_x$ is isomorphic to the fiber of the local system $\mathcal{H}_{\mathbb{Z}}$, and the above definition specializes to the usual definition of Hodge classes in the pure Hodge structure of weight zero on $\mathcal{H}_{\mathbb{Z},x}$. In this special case, the set of limit Hodge classes forms a group; but in general, it does not make sense to add two limit Hodge classes.

Definition 9.3. We shall denote by $\operatorname{Hdg}(\mathcal{H}, X) \subseteq D(\mathcal{H}_{\mathbb{Z}})$ the set of all limit Hodge classes, and by $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$ its intersection with $D_{\leq K}(\mathcal{H}_{\mathbb{Z}})$.

With the topology induced from the display space, $\operatorname{Hdg}(\mathcal{H}, X)$ is a Hausdorff space, and each $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$ is both closed and open. The projection

$$p\colon \operatorname{Hdg}(\mathcal{H}, X) \to X$$

is continuous, and the open subset $p^{-1}(X_0)$ is isomorphic, as a topological space, to the locus of Hodge classes $\operatorname{Hdg}(\mathcal{H})$. We shall see in the next section that $\operatorname{Hdg}(\mathcal{H}, X)$ is in fact an analytic space, too.

10. The locus of limit Hodge classes. The purpose of this section is to prove the following theorem.

Theorem 10.1. *The set of limit Hodge classes* $\operatorname{Hdg}(\mathcal{H}, X)$ *can be given the structure of an analytic space over* X*, in such a way that for every* $K \geq 0$*, the subspace of limit Hodge classes of self-intersection number at most* K *is finite over* X*.*

The idea of the proof is to reduce the problem to the normal crossing case, which is dealt with in Chapter D. We begin by choosing a proper holomorphic mapping $f\colon Y \to X$ with the following properties: (1) the complement of $Y_0 = f^{-1}(X_0)$ is a divisor with normal crossing singularities; (2) the restriction $f_0\colon Y_0 \to X_0$ is a finite covering space; (3) at points of $Y \setminus Y_0$, the local monodromy of $f^{-1}\mathcal{H}_{\mathbb{Z}}$ is unipotent. Here is one way of constructing such a mapping. Let $f_0\colon Y_0 \to X_0$ be a finite covering space on which the monodromy representation of the local system $\mathcal{H}_{\mathbb{Z}}$ becomes trivial modulo the prime number 3. By the *Fortsetzungssatz* of Grauert and Remmert [GPR94, VI.3.3], such a covering space extends in a unique way to a finite branched covering of X; now let $f\colon Y \to X$ be an embedded resolution of singularities that leaves Y_0 unchanged and makes the complement of Y_0 into a normal crossing divisor. Then any local monodromy transformation of $f^{-1}\mathcal{H}_{\mathbb{Z}}$ is quasi-unipotent and congruent to the identity modulo 3, and therefore unipotent [Sch08, Lemma on p. 25].

Let us assume for now that we already know the conclusion of Theorem 10.1 for $\operatorname{Hdg}(f^{-1}\mathcal{H}, Y)$; the proof in this special case is the topic of Chapter D. By Proposition 8.7, the projection from $E(f^{-1}\mathcal{H}_{\mathbb{Z}})$ to $E(\mathcal{H}_{\mathbb{Z}})$ extends to a continuous mapping between display spaces, giving us a commutative diagram

$$\begin{array}{ccc} D(f^{-1}\mathcal{H}_{\mathbb{Z}}) & \xrightarrow{\ g\ } & D(\mathcal{H}_{\mathbb{Z}}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ f\ } & X. \end{array}$$

Since Y_0 is finite over X_0, it is easy to see that g takes limit Hodge classes for $f^{-1}\mathcal{H}$ at the point $y \in Y$ to limit Hodge classes for $\mathcal{H}$ at the point $f(y) \in X$.

Proposition 10.2. *The induced continuous mapping*

$$g\colon \operatorname{Hdg}(f^{-1}\mathcal{H}, Y) \to \operatorname{Hdg}(\mathcal{H}, X)$$

is surjective and proper, and therefore a quotient mapping.

Proof. Surjectivity follows from the properness of f. Indeed, a limit Hodge class for $\mathcal{H}$ at a point $x \in X$ comes from a germ of a holomorphic arc $\gamma\colon (\Delta, 0) \to (X, x)$ with $\gamma(\Delta^*) \subseteq X_0$. Since Y_0 is a finite covering space of X_0 and f is proper, we can find some $d \geq 1$ such that $\gamma \circ t^d$ lifts to a holomorphic arc on Y; more precisely, for some point $y \in f^{-1}(x)$, we get a commutative diagram

$$\begin{array}{ccc} (\Delta, 0) & \xrightarrow{\tilde{\gamma}} & (Y, y) \\ \big\downarrow{\scriptstyle t^d} & & \big\downarrow{\scriptstyle f} \\ (\Delta, 0) & \xrightarrow{\gamma} & (X, x). \end{array}$$

Since the set of limit Hodge classes for $\tilde{\gamma}^{-1}\mathcal{H}$ is the same as that for $\gamma^{-1}\mathcal{H}$, the surjectivity of g follows.

It remains to prove properness of g. Granting Theorem 10.1 in this special case, $\operatorname{Hdg}_{\leq K}(f^{-1}\mathcal{H}, Y)$ is proper over Y, hence proper over X; from this, it follows that the restriction of g defines a proper mapping

$$\operatorname{Hdg}_{\leq K}(f^{-1}\mathcal{H}, Y) \to \operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$$

for every $K \geq 0$. Since each $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$ is both closed and open in $\operatorname{Hdg}(\mathcal{H}, X)$, this is enough to conclude that g itself is proper. □

The proposition tells us that, as a topological space, $\operatorname{Hdg}(\mathcal{H}, X)$ is a quotient of the analytic space $\operatorname{Hdg}(f^{-1}\mathcal{H}, Y)$; what we have to show is that this quotient is itself an analytic space. In general, this can be a difficult problem – but in this particular case, it turns out to be doable. Evidently, the first thing we should prove is that the equivalence relation

$$R \subseteq \operatorname{Hdg}(f^{-1}\mathcal{H}, Y) \times \operatorname{Hdg}(f^{-1}\mathcal{H}, Y)$$

defining the quotient is a closed analytic subset of the product. Since two points are equivalent if and only if they have the same image in $D(\mathcal{H}_{\mathbb{Z}})$, it is clear that

$$\begin{aligned} R &= \operatorname{Hdg}(f^{-1}\mathcal{H}, Y) \times_{D(\mathcal{H}_{\mathbb{Z}})} \operatorname{Hdg}(f^{-1}\mathcal{H}, Y) \\ &\subseteq \operatorname{Hdg}(f^{-1}\mathcal{H}, Y) \times_X \operatorname{Hdg}(f^{-1}\mathcal{H}, Y); \end{aligned}$$

note that the fiber product over X is closed analytic because $p \circ g = f \circ p$ is a holomorphic mapping from $\operatorname{Hdg}(f^{-1}\mathcal{H}, Y)$ to X.

Proposition 10.3. *The equivalence relation R is analytic.*

Proof. Since g is continuous and $D(\mathcal{H}_{\mathbb{Z}})$ is Hausdorff, R is closed. To prove that R is analytic, we only need to show that it is a union of locally closed analytic subsets. As in Proposition 8.6, choose a Whitney stratification of X in which $X \setminus X_0$ is a union of strata. For any stratum $S \subseteq X$, the preimage $p^{-1}(S)$ is a covering space of S, and therefore a complex manifold. Since $p \circ g = f \circ p$ is holomorphic, this implies that the restriction of g to the locally closed analytic subset $(p \circ g)^{-1}(S)$ is also holomorphic. But then the part of R that lies over $S \subseteq X$ is a locally closed analytic subset of the fiber product, and so we get the result. □

We conclude that $\mathrm{Hdg}(\mathcal{H}, X)$ is the quotient of the analytic space $\mathrm{Hdg}(f^{-1}\mathcal{H}, Y)$ by the analytic equivalence relation R. This is of course still not enough to say that $\mathrm{Hdg}(\mathcal{H}, X)$ is an analytic space. Quotients by *finite* analytic equivalence relations, however, are always analytic spaces, and together with the finiteness result in Theorem 10.1, this is enough to complete the proof.

Proof of Theorem 10.1. Fix a real number $K \geq 0$. As we have seen, $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$ is the quotient of the analytic space $\mathrm{Hdg}_{\leq K}(f^{-1}\mathcal{H}, Y)$ by the analytic equivalence relation R. Granting Theorem 10.1 in the special case that is proved in Chapter D, the holomorphic mapping

$$p \circ g\colon \mathrm{Hdg}_{\leq K}(f^{-1}\mathcal{H}, Y) \to X$$

is proper, and it is easy to see from the definition of the equivalence relation R that the connected components of the fibers are each contained in a single equivalence class. This means that if we pass to the Stein factorization of $p \circ g$, and denote by

$$q_K \colon Z_K \to X$$

the resulting finite holomorphic mapping, then R descends to an analytic equivalence relation $R_K \subseteq Z_K \times Z_K$, and as a topological space, $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X) \simeq Z_K/R_K$.

Now the two projections from R_K to Z_K are finite holomorphic mappings, because R_K is contained in the fiber product $Z_K \times_X Z_K$, which has this property. As proved in [KK83, Proposition 49 A.13], quotients by finite analytic equivalence relations always exist in the category of complex spaces; therefore $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$ has the structure of an analytic space over X that makes the projection p into a finite holomorphic mapping. It is easy to see that if we perform the same construction for a larger real number $L \geq K$, then the resulting analytic structure on the closed and open subset $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X) \subseteq \mathrm{Hdg}_{\leq L}(\mathcal{H}, X)$ is the same. Since

$$\mathrm{Hdg}(\mathcal{H}, X) = \bigcup_{K \geq 0} \mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$$

is the union of the closed and open subsets $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$, this ends the proof. □

To conclude this section, let us show that the analytic structure on $\mathrm{Hdg}(\mathcal{H}, X)$ is independent of the choice of $f\colon Y \to X$. Since any two such holomorphic mappings can be dominated by a third, the problem reduces to the following special case.

Proposition 10.4. *Suppose that $X \setminus X_0$ is a divisor with normal crossing singularities and that the local monodromy of $\mathcal{H}_{\mathbb{Z}}$ at points of $X \setminus X_0$ is unipotent. Then the analytic structure on* $\mathrm{Hdg}(\mathcal{H}, X)$ *is independent of the choice of $f\colon Y \to X$.*

Proof. Let $\tilde{\mathcal{H}}$ denote the canonical extension of the flat bundle underlying $\mathcal{H}$. In this situation, $f^*\tilde{\mathcal{H}}$ is the canonical extension of the flat bundle underlying $f^{-1}\mathcal{H}$ □

C. A variant of the construction

11. Setup and basic properties. Let X be a complex manifold, $Z \subseteq X$ an analytic subset, and $\mathcal{H}$ a polarized variation of integral Hodge structure on $X_0 = X \setminus Z$. We denote by $\mathcal{H}_{\mathbb{Z}}$ the underlying local system of free $\mathbb{Z}$-modules, and by $Q\colon \mathcal{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}} \to \mathbb{Q}(0)$ the bilinear form giving the polarization.

Now let $E(\mathcal{H}_{\mathbb{Z}})$ be the étalé space of the local system $\mathcal{H}_{\mathbb{Z}}$; it is a (usually disconnected) covering space of the complex manifold X_0. Point of $E(\mathcal{H}_{\mathbb{Z}})$ may be thought of as pairs (x, h), where $x \in X_0$ and $h \in \mathcal{H}_{\mathbb{Z},x}$ is a class in the stalk. As in the introduction, we define

$$E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) = \left\{ (x, h) \in E(\mathcal{H}_{\mathbb{Z}}) \;\middle|\; |Q_x(h, h)| \leq K \right\}$$

for every $K \geq 0$; note that it is a union of connected components of $E(\mathcal{H}_{\mathbb{Z}})$, because the function $(x, h) \mapsto Q_x(h, h)$ is constant on each connected component. Let $T(\mathcal{H}) = \operatorname{Spec}(\operatorname{Sym} \mathcal{H}^*)$ be the vector bundle with sheaf of sections $\mathcal{H}^*$; we similarly define $T(F^1\mathcal{H})$.

We first describe in more detail how the holomorphic mapping $\varphi \colon E(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$ is constructed. The pairing Q induces an injective morphism of sheaves

$$\mathcal{H}_{\mathbb{Z}} \hookrightarrow \mathcal{H}^*, \quad h \mapsto Q(h, -);$$

it is injective because Q is nondegenerate. As in [Sch12a, Section 2.6], this morphism gives rise to a holomorphic mapping

$$E(\mathcal{H}_{\mathbb{Z}}) \hookrightarrow T(\mathcal{H}),$$

which embeds the complex manifold $E(\mathcal{H}_{\mathbb{Z}})$ into the holomorphic vector bundle $T(\mathcal{H})$. From now on, we identify $E(\mathcal{H}_{\mathbb{Z}})$ with a complex submanifold of $T(\mathcal{H})$. We obtain $\varphi \colon E(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$ by composing with the projection $q \colon T(\mathcal{H}) \to T(F^1\mathcal{H})$.

Now fix some $K \geq 0$. We already know from the result about Hodge structures in Lemma 3.1 that $\varphi \colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$ has finite fibers; the purpose of this section is to understand its global properties. The following diagram shows all the relevant mappings:

$$\begin{array}{ccc} & \nearrow & T(\mathcal{H}) \\ & & \downarrow{\scriptstyle q} \\ E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) & \xrightarrow{\varphi} & T(F^1\mathcal{H}) \\ & \searrow^{\pi} & \downarrow \\ & & X_0. \end{array}$$

The polarization defines a hermitian metric on the holomorphic vector bundle associated with $\mathcal{H}$, the so-called *Hodge metric*. It induces hermitian metrics on the two bundles $T(\mathcal{H})$ and $T(F^1\mathcal{H})$. Let $B_r(\mathcal{H}) \subseteq T(\mathcal{H})$ denote the closed tube of radius $r > 0$ around the zero section. The proof of Lemma 3.1 shows that

$$\varphi^{-1}\big(B_r(\mathcal{H})\big) \subseteq B_{\sqrt{K+4r^2}}\big(F^1\mathcal{H}\big);$$

in particular, the general discussion in §22 applies to our situation. We summarize the results in the following proposition.

Proposition 11.1. *The holomorphic mapping $\varphi \colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$ is finite, and its image is a closed analytic subset of $T(F^1\mathcal{H})$. Moreover, the induced mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to the normalization of the image is a finite covering space.*

Proof. This is proved in §22 below. □

12. Analyticity of the closure. In this section, we prove Theorem 4.1 in general. We denote by M the polarized Hodge module of weight $\dim X$ with strict support X, canonically associated with $\mathcal{H}$ by the equivalence of categories in [Sai90, Theorem 3.21]. Let $(\mathcal{M}, F_{\bullet}\mathcal{M})$ denote the underlying filtered regular holonomic $\mathscr{D}_X$-module. By construction, the restriction of $F_{-1}\mathcal{M}$ to the open subset X_0 is isomorphic to $F^1\mathcal{H}$. The analytic space $T(F_{-1}\mathcal{M})$ therefore contains an open subset isomorphic to the vector bundle $T(F^1\mathcal{H})$. We denote by

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$$

the resulting holomorphic mapping.

Theorem 12.1. *The closure of the image of the holomorphic mapping*

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$$

is an analytic subset of $T(F_{-1}\mathcal{M})$*.*

Proof. There is a proper holomorphic mapping $f\colon Y \to X$, whose restriction to $Y_0 = f^{-1}(X_0)$ is a finite covering space, such that $D = f^{-1}(Z)$ is a divisor with normal crossings, and such that the local monodromy of $f_0^*\mathcal{H}$ at every point of D is unipotent. To construct f, we first take an embedded resolution of singularities of (X, Z). According to [Sch73, Lemma 4.5], the pullback of $\mathcal{H}$ has quasi-unipotent local monodromy at every point of the preimage of Z; after a finite branched covering and a further resolution of singularities, we arrive at the stated situation.

Now let M' denote the polarized Hodge module of weight $\dim Y$ with strict support Y, associated with $\mathcal{H}' = f_0^*\mathcal{H}$. According to [Sch12a, Lemma 2.21], there is a canonical morphism

$$F_{-1}\mathcal{M}' \to f^*F_{-1}\mathcal{M},$$

whose restriction to Y_0 is an isomorphism. We then have the following commutative diagram of holomorphic mappings:

$$\begin{array}{ccccc}
E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) & \xleftarrow{p_1} & E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \times_X Y & = & E(\mathcal{H}_{\mathbb{Z}})'(K) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi\times\mathrm{id}} & & \downarrow{\scriptstyle \varphi'} \\
T(F_{-1}\mathcal{M}) & \xleftarrow{p_1} & T(F_{-1}\mathcal{M}) \times_X Y & \xrightarrow{g} & T(F_{-1}\mathcal{M}')
\end{array}$$

By Theorem 14.3, the closure of the image of φ' is analytic. The same is therefore true for $\varphi \times \mathrm{id}$, because g is an isomorphism over Y_0. Because f is proper, the result for φ now follows from Remmert's proper mapping theorem [GPR94, III.4.3]. □

13. Extension of the finite mapping. We are now ready to prove the main result, namely that $\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$ can be extended to a finite mapping.

Theorem 13.1. *There is a normal analytic space* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *containing the complex manifold* $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *as a dense open subset, and a finite holomorphic mapping*

$$\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M}),$$

whose restriction to $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *agrees with* φ*. Moreover,* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *and* $\tilde{\varphi}$ *are unique up to isomorphism.*

Proof. The closure of the image of φ is an analytic subset of $T(F_{-1}\mathcal{M})$ according to Theorem 12.1. Let W denote its normalization; according to Proposition 11.1, the induced mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to W is a finite covering space over its image. The *Fortsetzungssatz* of Grauert and Remmert [GPR94, VI.3.3] shows that it extends in a unique way to a finite branched covering of W. If we define $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ to be the analytic space in this covering, and $\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M})$ to be the induced holomorphic mapping, then all the requirements are fulfilled. The last assertion follows from the uniqueness statement in [GPR94, VI.3.3]. □

Recall from Definition 4.3 that the *extended locus of Hodge classes* $\widetilde{\mathrm{Hdg}}(\mathcal{H})$ is the preimage, under $\tilde{\varphi}$, of the zero section of $T(F_{-1}\mathcal{M})$ under $\tilde{\varphi}$. It is therefore a (possibly not reduced) closed analytic subspace of $\overline{E(\mathcal{H}_{\mathbb{Z}})}$.

Corollary 13.2. *The extended locus of Hodge classes* $\widetilde{\mathrm{Hdg}}(\mathcal{H})$ *contains the usual locus of Hodge classes* $\mathrm{Hdg}(\mathcal{H})$*. For every* $K \geq 0$*, the intersection* $\widetilde{\mathrm{Hdg}}(\mathcal{H}) \cap \overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *is finite and proper over its image in* X*.*

Proof. The first assertion is clear from the construction, because $\mathrm{Hdg}(\mathcal{H})$ is by definition the preimage, under φ, of the zero section in $T(F^1\mathcal{H})$. The second assertion follows from the fact that $\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M})$ is finite. □

Note that the extended locus of Hodge classes $\widetilde{\mathrm{Hdg}}(\mathcal{H})$ is canonically associated with the original polarized variation of Hodge structure $\mathcal{H}$ on X_0. The reason is that the polarized Hodge module M and its underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ are uniquely determined by $\mathcal{H}$; according to Theorem 13.1, the same is true for the holomorphic mapping $\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M})$. In this sense, $\widetilde{\mathrm{Hdg}}(\mathcal{H})$ is a canonical extension of $\mathrm{Hdg}(\mathcal{H})$ to an analytic space over X with good properties.

Corollary 13.3. *Let* $\mathcal{H}$ *be a polarized variation of Hodge structure of weight zero, defined on a Zariski-open subset* X_0 *of a smooth projective variety* X*. Then* $\widetilde{\mathrm{Hdg}}(\mathcal{H})$ *is a countable union of projective schemes, each finite over its image in* X*.*

Proof. This follows from Chow's theorem by noting that the pullback of an ample line bundle under a finite holomorphic mapping remains ample. □

D. The normal crossing case

14. Introduction. In this chapter, we treat the case of variations of Hodge structure with unipotent local monodromy on the complement of a normal crossing divisor. We use the same notation as in the introduction, namely X is a complex manifold, $D \subseteq X$ a divisor with normal crossing singularities, and $X_0 = X \setminus D$ its open complement. Suppose that $\mathcal{H}$ is a polarized variation of integral Hodge structure of weight zero on X_0 whose local monodromy at each point of D is unipotent. We write $F^\bullet\mathcal{H}$ for the Hodge filtration on the locally free sheaf $\mathcal{H}$, and

$$\pi\colon B(\mathcal{H}) \to X_0$$

for the corresponding holomorphic vector bundle; then $B(F^p\mathcal{H})$ is the subbundle corresponding to $F^p\mathcal{H}$. As in the introduction, the underlying local system of free $\mathbb{Z}$-modules $\mathcal{H}_{\mathbb{Z}}$ determines a covering space $E(\mathcal{H}_{\mathbb{Z}})$ of X_0; note that $E(\mathcal{H}_{\mathbb{Z}})$ is naturally a complex submanifold of $B(\mathcal{H})$, because $\mathcal{H}_{\mathbb{Z}}$ is a subsheaf of $\mathcal{H}$. The locus of Hodge classes is then exactly the intersection

$$\mathrm{Hdg}(\mathcal{H}) = E(\mathcal{H}_{\mathbb{Z}}) \cap B(F^0\mathcal{H}) \subseteq B(\mathcal{H});$$

as such, it is an analytic subspace. For every $K \geq 0$, the set

$$E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) = \left\{ (x, h) \in E(\mathcal{H}_{\mathbb{Z}}) \;\middle|\; Q_x(h, h) \leq K \right\} \subseteq E(\mathcal{H}_{\mathbb{Z}})$$

is a union of connected components of $E(\mathcal{H}_{\mathbb{Z}})$, and we have

$$\mathrm{Hdg}_{\leq K}(\mathcal{H}) = E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \cap B(F^0\mathcal{H}) \subseteq B(\mathcal{H}).$$

To construct the locus of limit Hodge classes in this setting, let $\tilde{\mathcal{H}}$ denote the canonical extension of $(\mathcal{H}, \nabla)$ to a locally free sheaf on X [Del70, Proposition 5.2]. If we again use the notation $\pi\colon B(\tilde{\mathcal{H}}) \to X$ for the corresponding holomorphic vector bundle, then $\pi^{-1}(X_0) = B(\mathcal{H})$. Since we are interested in limits of sequences of integral classes, we take the closure of $E(\mathcal{H}_{\mathbb{Z}})$ inside this larger ambient space.

Theorem 14.1. *The topological closure of $E(\mathcal{H}_{\mathbb{Z}})$ inside $B(\tilde{\mathcal{H}})$ is an an analytic subspace; in particular, every point in the closure lies on the image of some holomorphic arc $f\colon \Delta \to B(\tilde{\mathcal{H}})$ with $f(\Delta^*) \subseteq E(\mathcal{H}_{\mathbb{Z}})$.*

The proof is basically an exercise in linear algebra; it does not use the fact that the local system $\mathcal{H}_{\mathbb{Z}}$ comes from a variation of Hodge structure. Note that while $E(\mathcal{H}_{\mathbb{Z}})$ is a covering space of X_0, its closure may have fibers of positive dimension over the boundary divisor $D = X \setminus X_0$. If we think of points in $\overline{E(\mathcal{H}_{\mathbb{Z}})}$ as limits of sequences of integral classes, what this means is that different ways of approaching a point on D can lead to different limits.

Now let $\overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$ denote the normalization of the analytic space $\overline{E(\mathcal{H}_{\mathbb{Z}})}$; by construction, it comes with a finite mapping

$$\nu\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu} \to B(\tilde{\mathcal{H}})$$

that is an isomorphism over the preimage of $X_0 = X \setminus D$. Recall that the Hodge filtration $F^{\bullet}\mathcal{H}$ extends to a filtration $F^{\bullet}\tilde{\mathcal{H}}$ by locally free subsheaves [Sch73, §4]; in particular, $B(F^0\tilde{\mathcal{H}})$ is a holomorphic subbundle of $B(\tilde{\mathcal{H}})$. Its preimage

$$\mathrm{Hdg}(\mathcal{H}, X) = \nu^{-1}\Big(B(F^0\tilde{\mathcal{H}})\Big) \subseteq \overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$$

is therefore an analytic subspace that coincides, over X_0, with the usual locus of Hodge classes $\mathrm{Hdg}(\mathcal{H})$. As above, we have

$$\mathrm{Hdg}(\mathcal{H}, X) = \bigcup_{K=0}^{\infty} \mathrm{Hdg}_{\leq K}(\mathcal{H}, X),$$

where $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$ is defined by intersecting $\mathrm{Hdg}(\mathcal{H}, X)$ with the normalization of the closure of $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$, which is a union of connected components of $\overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$. The following theorem justifies calling $\mathrm{Hdg}(\mathcal{H}, X)$ the *locus of limit Hodge classes.*

Theorem 14.2. *Let X and $\mathcal{H}$ be as above.*

(a) *The points of the analytic space $\mathrm{Hdg}(\mathcal{H}, X)$ are in one-to-one correspondence with limit Hodge classes for $\mathcal{H}$ on X, as defined in §9.*
(b) *For every $K \geq 0$, the natural mapping from $\mathrm{Hdg}_{\leq K}(\mathcal{H}, X)$ to X is finite (= proper with finite fibers).*

The content of Theorem 14.2 is that the locus of limit Hodge classes is well-behaved, provided one imposes a bound on the self-intersection number. Evidently, Theorem 14.2 is inspired by the work of Cattani, Deligne, and Kaplan – in fact, while the theorem itself does not appear in [CDK95], we shall see that it can be deduced from some of the technical results proved there. It is also worth noting that although Cattani, Deligne, and Kaplan did not consider limit Hodge classes as such, the statement of [CDK95, Theorem 2.16] strongly suggests that something like Theorem 14.2 ought to be true.

As explained in the introduction, there are certain cases (such as families of hypersurfaces) where a different construction seems more natural. Recall that $\mathcal{H}$ extends uniquely to a polarized Hodge module

$$M \in \mathrm{HM}_X(X, \dim X)$$

with strict support X. The underlying regular holonomic left $\mathscr{D}_X$-module $\mathcal{M}$ is the minimal extension of the flat bundle $(\mathcal{H}, \nabla)$, and the coherent $\mathscr{O}_X$-modules $F_p\mathcal{M}$ are extensions of the locally free sheaves $F^{-p}\mathcal{H}$. We can define an analytic space

$$T(F_{-1}\mathcal{M}) = \operatorname{Spec}_X\big(\operatorname{Sym}_{\mathscr{O}_X} F_{-1}\mathcal{M}\big)$$

by taking the spectrum of the symmetric algebra of $F_{-1}\mathcal{M}$; over X_0, it restricts to the holomorphic vector bundle corresponding to $(F^1\mathcal{H})^*$. The polarization induces a holomorphic mapping

$$\varphi\colon E(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M}),$$

and as above, we are interested in the closure of the image. The second important result in this chapter is that the closure is analytic, provided we again impose a bound on the self-intersection number.

Theorem 14.3. *The topological closure of the image of the holomorphic mapping*

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$$

is a complex-analytic subspace of $T(F_{-1}\mathcal{M})$.

This theorem is far stronger than Theorem 14.1, because convergence in the space $T(F_{-1}\mathcal{M})$ gives us much less control over a sequence of integral classes than convergence in $B(\tilde{\mathcal{H}})$. In fact, Theorem 14.3 does not follow from the results of Cattani, Deligne, and Kaplan: to prove it, we have to establish a more powerful version of [CDK95, Theorem 2.16]. Using some facts from complex analysis, one can then construct an extension of $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ that is still finite over $T(F_{-1}\mathcal{M})$.

Corollary 14.4. *There is a normal analytic space* $\widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *containing the complex manifold* $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *as a dense open subset, and a finite holomorphic mapping*

$$\tilde{\varphi}\colon \widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$$

whose restriction to $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ *agrees with* φ*; both are unique up to isomorphism.*

The uniqueness statement in Corollary 14.4 means that if we define

$$\widetilde{E}(\mathcal{H}_{\mathbb{Z}}) = \varinjlim_{K \in \mathbb{N}} \widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}}),$$

then $\widetilde{E}(\mathcal{H}_{\mathbb{Z}})$ is a normal analytic space with countably many connected components; by construction, it comes with a holomorphic mapping

$$\tilde{\varphi}\colon \widetilde{E}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$$

whose fibers are discrete, and whose restriction to each subset $\widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ is finite and proper. The preimage of the zero section in $T(F_{-1}\mathcal{M})$ therefore gives a second compactification

$$\widetilde{\operatorname{Hdg}}(\mathcal{H}, X) = \tilde{\varphi}^{-1}(0) \subseteq \widetilde{E}(\mathcal{H}_{\mathbb{Z}})$$

for the locus of Hodge classes, with the same finiteness properties as the locus of limit Hodge classes.

Concerning the relationship between the two constructions, we have the following result. It would be interesting to know more – but at present, I do not even have a guess as to what the image of λ might be.

Corollary 14.5. *There is a unique holomorphic mapping*

$$\lambda\colon \widetilde{E}(\mathcal{H}_{\mathbb{Z}}) \to \overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$$

whose restriction to $E(\mathcal{H}_{\mathbb{Z}})$ *is the identity.*

In particular, we get a finite mapping from $\widetilde{\mathrm{Hdg}}(\mathcal{H}, X)$ to the locus of limit Hodge classes. Note that this mapping is generally not surjective; there is also no good reason to think that it should be injective.

15. Review of the local theory. Since both Theorem 14.2 and Theorem 14.3 are basically local statements, we shall begin by reviewing the local theory of polarized variations of Hodge structure on the complement of a normal crossing divisor [Sch73, Kas85, CKS86]. Fortunately, Cattani and Kaplan have written a beautiful survey article, where they describe all the major results [CK89]. Rather than citing the original sources, I will only quote from this article.

Let Δ^n, with coordinates $s = (s_1, \dots, s_n)$, be the product of n copies of the unit disk; then $(\Delta^*)^n$ is the complement of the divisor defined by $s_1 \cdots s_n = 0$. Let $\mathcal{H}$ be a polarized variation of integral Hodge structure of weight zero on $(\Delta^*)^n$; we assume that the underlying local system of free $\mathbb{Z}$-modules $\mathcal{H}_{\mathbb{Z}}$ has unipotent monodromy around each of the divisors $s_j = 0$. Let $\mathbb{H}^n$, with coordinates $z = (z_1, \dots, z_n)$, be the product of n copies of the upper half-plane; the holomorphic mapping

$$\mathbb{H}^n \to (\Delta^*)^n, \quad z \mapsto \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}\right)$$

makes it into the universal covering space of $(\Delta^*)^n$. If we pull back the local system $\mathcal{H}_{\mathbb{Z}}$ to $\mathbb{H}^n$, it becomes trivial; let $H_{\mathbb{Z}}$ denote the free $\mathbb{Z}$-module of its global sections, and $Q\colon H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}$ the symmetric bilinear form coming from the polarization on $\mathcal{H}$. By assumption, the monodromy transformation around $s_j = 0$ is of the form e^{N_j}, where N_j is a nilpotent endomorphism of $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ that satisfies $Q(N_j h_1, h_2) + Q(h_1, N_j h_2) = 0$. It is clear that $N_1, \dots, N_n$ commute.

We now review the description of $\mathcal{H}$ that results from the work of Cattani, Kaplan, and Schmid. Let $\check{D}$ denote the parameter space for filtrations $F = F^{\bullet} H_{\mathbb{C}}$ that satisfy $Q(F^p, \overline{F^q}) = 0$ whenever $p+q > 0$; let $D \subseteq \check{D}$ denote the subset of those F that define a polarized Hodge structure on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ with polarization Q. Recall that $\check{D}$ is a closed subvariety of a flag variety, and that the so-called *period domain* D is an open subset of $\check{D}$.

The variation of Hodge structure $\mathcal{H}$ can be lifted to a period mapping

$$\Phi\colon \mathbb{H}^n \to D$$

which is holomorphic and horizontal. It is known that every element of the cone

$$C(N_1, \dots, N_n) = \left\{ a_1 N_1 + \dots + a_n N_n \;\middle|\; a_1, \dots, a_n > 0 \right\}$$

defines the same monodromy weight filtration [CK89, Theorem 2.3]; we denote this common filtration by $W = W(N_1, \dots, N_n)$. In the limit, $\mathcal{H}$ determines another filtration $F \in \check{D}$ for which the pair (W, F) is a mixed Hodge structure on $H_{\mathbb{C}}$, polarized by Q and every element of $C(N_1, \dots, N_n)$. According to the nilpotent orbit theorem [CK89, Theorem 2.1], the period mapping is approximated (with good bounds on the degree of approximation) by the associated *nilpotent orbit*

$$\Phi_{nil}\colon \mathbb{H}^n \to \check{D}, \quad \Phi_{nil}(z) = e^{\sum z_j N_j} F. \tag{15.1}$$

One can use the mixed Hodge structure (W, F) to express $\Phi(z)$ in terms of the nilpotent orbit and additional holomorphic data on Δ^n. Denote by

$$\mathfrak{g} = \left\{ X \in \operatorname{End}(H_{\mathbb{C}}) \;\middle|\; Q(Xh_1, h_2) + Q(h_1, Xh_2) = 0 \right\}$$

the Lie algebra of infinitesimal isometries of Q. The mixed Hodge structure (W, F) determines a decomposition of $H_{\mathbb{C}}$ with the following properties:

$$H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}, \qquad W_w = \bigoplus_{p+q \leq w} I^{p,q}, \qquad F^k = \bigoplus_{p \geq k} I^{p,q},$$

A formula for the subspaces $I^{p,q}$ can be found in [CK89, (1.12)]. The decomposition leads to a corresponding decomposition of the Lie algebra

$$\mathfrak{g} = \bigoplus_{p,q} \mathfrak{g}^{p,q},$$

with $\mathfrak{g}^{p,q}$ consisting of those operators X that satisfy $X(I^{a,b}) \subseteq I^{a+p,b+q}$ for every $a, b \in \mathbb{Z}$. In this notation, we have $N_1, \dots, N_n \in \mathfrak{g}^{-1,-1}$; moreover, the restriction of Q to the subspace $I^{p,q} \otimes I^{p',q'}$ is nondegenerate for $p' + p = q' + q = 0$, and zero otherwise.

The more precise version of the nilpotent orbit theorem [CK89, Theorem 2.8] is that the period mapping of $\mathcal{H}$ can be put into the normal form

$$\Phi \colon \mathbb{H}^n \to D, \quad \Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F, \tag{15.2}$$

for a unique holomorphic mapping

$$\Gamma \colon \Delta^n \to \bigoplus_{p \leq -1} \mathfrak{g}^{p,q}$$

with $\Gamma(0) = 0$. When we write $\Gamma(s)$, it is of course understood that $s_j = e^{2\pi i z_j}$ for every $j = 1, \dots, n$. The horizontality of the period mapping has the following very useful consequence [CK89, Proposition 2.6].

Proposition 15.3. *Let $\Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F$ be the normal form of a period mapping on $\mathbb{H}^n$. Then for every $j = 1, \dots, n$, the commutator*

$$\bigl[N_j, e^{\Gamma(s)}\bigr] = N_j e^{\Gamma(s)} - e^{\Gamma(s)} N_j$$

vanishes along the divisor $s_j = 0$.

16. Local description of the problem. The presentation of the period mapping in (15.2) is very convenient for describing the canonical extension of $(\mathcal{H}, \nabla)$ geometrically. With the conventions about the fundamental group in [CK89, (1.8)], the étalé space $E(\mathcal{H}_{\mathbb{Z}})$ of the local system $\mathcal{H}_{\mathbb{Z}}$ is the quotient of $\mathbb{H}^n \times H_{\mathbb{Z}}$ by the following $\mathbb{Z}^n$-action:

$$a \cdot (z, h) = \bigl(z + a, e^{\sum a_j N_j} h\bigr) \qquad \text{for } a \in \mathbb{Z}^n \text{ and } (z, h) \in \mathbb{H}^n \times H_{\mathbb{Z}} \tag{16.1}$$

Since we are only considering classes of bounded self-intersection, we define for any integer $K \geq 0$ the set

$$H_{\mathbb{Z}}(K) = \bigl\{\, h \in H_{\mathbb{Z}} \bigm| \lvert Q(h,h) \rvert \leq K \,\bigr\}.$$

Then $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ is the quotient of $\mathbb{H}^n \times H_{\mathbb{Z}}(K)$ by the action in (16.1). As above, let $(\mathcal{H}, \nabla)$ be the flat bundle on $(\Delta^*)^n$ underlying the variation of Hodge structure. It admits a canonical extension to locally free sheaf $\tilde{\mathcal{H}}$ on Δ^n, on which the connection has a logarithmic pole along each of the divisors $s_j = 0$ with nilpotent residue [Del70, Proposition 5.2]. Explicitly, for each $v \in H_{\mathbb{C}}$, the holomorphic mapping

$$\mathbb{H}^n \to H_{\mathbb{C}}, \quad z \mapsto e^{\sum z_j N_j} v \tag{16.2}$$

descends to a holomorphic section of $\mathcal{H}$ on $(\Delta^*)^n$, and $\tilde{\mathcal{H}}$ is the locally free subsheaf of $j_*\mathcal{H}$ generated by all such sections [CK89, (2.2)]. This means that

$$B(\tilde{\mathcal{H}}) \simeq \Delta^n \times H_{\mathbb{C}},$$

and with respect to this isomorphism, $E(\mathcal{H}_{\mathbb{Z}})$ is identified with the image of the holomorphic mapping

$$\varepsilon \colon \mathbb{H}^n \times H_{\mathbb{Z}} \to \Delta^n \times H_{\mathbb{C}}, \quad \varepsilon(z,h) = \Big(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-\sum z_j N_j} h\Big). \tag{16.3}$$

If we use the above trivialization of the bundle $B(\tilde{\mathcal{H}})$, then (15.2) tells us that the subbundle $B(F^p\tilde{\mathcal{H}})$ is exactly the image of the holomorphic mapping

$$\Delta^n \times F^p \to \Delta^n \times H_{\mathbb{C}}, \quad (s,v) \mapsto \Big(s, e^{\Gamma(s)} v\Big).$$

From this point of view, it is easy to see the difference between Φ and the nilpotent orbit Φ_{nil} in (15.1): it is the difference between the above embedding of $\Delta^n \times F^p$ and the obvious embedding induced by $F^p \subseteq H_{\mathbb{C}}$.

We close this section by describing the extension of $\mathcal{H}$ to a polarized Hodge module on Δ^n. Let us denote this extension by M, and let $(\mathcal{M}, F_\bullet\mathcal{M})$ be the filtered regular holonomic left $\mathscr{D}$-module underlying M. Then $\mathcal{M}$ is simply the $\mathscr{D}$-submodule of $j_*\mathcal{H}$ generated by $\tilde{\mathcal{H}}$, and the filtration on $\mathcal{M}$ is given by

$$F_k\mathcal{M} = \sum_{j\geq 0} F_j\mathscr{D}_{\Delta^n} \cdot F^{j-k}\tilde{\mathcal{H}}. \tag{16.4}$$

It satisfies $F_j\mathscr{D}_{\Delta^n} \cdot F_k\mathcal{M} \subseteq F_{j+k}\mathcal{M}$, and each $F_k\mathcal{M}$ is a coherent sheaf on Δ^n whose restriction to $(\Delta^*)^n$ agrees with $F^{-k}\mathcal{H}$. This is a translation of Saito's results in [Sai90, §3.10]; note that Saito is working with right $\mathscr{D}$-modules. For the purposes of Theorem 14.3, the important point is that $F_{-1}\mathcal{M}$ has more sections than $F^1\tilde{\mathcal{H}}$; the following lemma exhibits the ones that we will use.

Lemma 16.5. *For any vector $v \in F^2$, and any index $1 \leq k \leq n$, the formula*

$$\sigma_{v,k}(z) = e^{\sum z_j N_j} e^{\Gamma(s)} \frac{N_k v}{s_k}$$

defines a holomorphic section of the coherent sheaf $F_{-1}\mathcal{M}$ on Δ^n.

Proof. It is clear from the description above that

$$\sigma_v \colon \mathbb{H}^n \to H_{\mathbb{C}}, \quad \sigma_v(z) = e^{\sum z_j N_j} e^{\Gamma(s)} v$$

defines a holomorphic section of $F^2\tilde{\mathcal{H}}$ for every $v \in F^2$; consequently, σ_v is also a holomorphic section of $F_{-2}\mathcal{M}$. By [CK89, (2.7)], the horizontality of the period mapping is equivalent to

$$d\Big(e^{\sum z_j N_j} e^{\Gamma(s)}\Big) = e^{\sum z_j N_j} e^{\Gamma(s)} \left(d\,\Gamma_{-1}(s) + \sum_{j=1}^n N_j dz_j\right),$$

where $\Gamma_{-1}(s)$ is the sum of all the $\mathfrak{g}^{p,q}$-components of $\Gamma(s)$ with $p = -1$. Using this identity and the fact that $s_k = e^{2\pi i z_k}$, we compute that

$$\begin{aligned} \frac{\partial}{\partial s_k}\sigma_v(z) &= e^{\sum z_j N_j} e^{\Gamma(s)} \left(\frac{\partial \Gamma_{-1}(s)}{\partial s_k} + \frac{N_k}{2\pi i s_k}\right) v \\ &= e^{\sum z_j N_j} e^{\Gamma(s)} \frac{\partial \Gamma_{-1}(s)}{\partial s_k} v + \frac{1}{2\pi i}\sigma_{v,k}(z). \end{aligned}$$

This section belongs to $F_{-1}\mathcal{M}$ by virtue of (16.4); we now obtain the result by noting that $\Gamma_{-1}(s) \cdot v$ is a holomorphic mapping from Δ^n into F^1. $\square$

17. Two results about sequences of Hodge classes. The theorem of Cattani, Deligne, and Kaplan about the locus of Hodge classes with bounded self-intersection number rests mainly on the following technical result [CDK95, Theorem 2.16].

Theorem 17.1 (Cattani, Deligne, Kaplan)**.** *Consider a sequence of points*

$$\big(z(m), h(m)\big) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K)$$

with $x_j(m) = \operatorname{Re} z_j(m)$ *bounded and* $y_j(m) = \operatorname{Im} z_j(m)$ *going to infinity for every* $j = 1, \dots, n$*. If every* $h(m)$ *is a Hodge class in the Hodge structure induced by the filtration* $\Phi\big(z(m)\big)$*, then there is a subsequence*[1] *with the following properties:*

(a) The sequence $h(m)$ *is constant, equal to some* $h \in H_{\mathbb{Z}}(K)$*.*
(b) One has $(a_1 N_1 + \dots + a_n N_n)h = 0$ *for certain positive integers* $a_1, \dots, a_n$*; in particular,* $h \in W_0$*.*
(c) There is a vector $w \in \mathbb{C}^n$ *such that*

$$\lim_{m \to \infty} e^{-\sum z_j(m) N_j} h(m) = e^{-\sum w_j N_j} h.$$

(d) Lastly, h *is a Hodge class in the mixed Hodge structure* $\left(W_0, e^{\sum w_j N_j} F\right)$*.*

For practical reasons, they actually prove a more general statement, in which the assumption $h(m) \in \Phi^0\big(z(m)\big)$ is replaced by the condition that

$$h(m) \equiv b(m) \mod \Phi^0\big(z(m)\big), \tag{17.2}$$

where $b(m) \in H_{\mathbb{C}}$ is a sequence of exponentially small error terms; more precisely, the norm of $b(m)$ should be in $O\big(e^{-\alpha \max_j y_j(m)}\big)$ for some fixed $\alpha > 0$. Theorem 17.1 gives enough control over the asymptotic behavior of sequences of Hodge classes to conclude that the locus of Hodge classes extends analytically over Δ^n. It is also sufficient for proving Theorem 14.2 about the locus of limit Hodge classes.

Interestingly, trying to prove Theorem 14.3 also leads to the relation in (17.2), but with a much weaker condition on the sequence of error terms. The reason is that when we consider a sequence for which $\tilde{\varphi}\big(z(m), h(m)\big)$ converges in $T(F_{-1}\mathcal{M})$, we are somehow controlling the distance from $h(m)$ to the subspace $\Phi^0\big(z(m)\big)$, and so we can expect to get something like (17.2) with a bound on the sequence $b(m)$. The precise condition that emerges from Lemma 16.5 is the following.

Definition 17.3. Fix an inner product on $H_{\mathbb{C}}$ and let $\|-\|$ denote the corresponding norm. A sequence of vectors $b(m) \in H_{\mathbb{C}}$ is called *harmless* with respect to $y(m)$ if there is a positive real number $\alpha > 0$ such that the quantity

$$\|b(m)\| + \sum_{k=1}^{n} e^{\alpha y_k(m)} \|N_k b(m)\|$$

remains bounded as $m \to \infty$. It is called *exponentially small* with respect to $y(m)$ if, for some $\alpha > 0$, the quantity $e^{\alpha \max_j y_j(m)} \|b(m)\|$ remains bounded as $m \to \infty$.

[1] To simplify the notation, we always denote a subsequence of a sequence by the same symbol.

In other words, a sequence $b(m) \in H_{\mathbb{C}}$ is harmless if and only if it is bounded and $\|N_k b(m)\|$ is in $O(e^{-\alpha y_k(m)})$ for every $k = 1, \dots, n$; it is exponentially small if $\|b(m)\|$ is actually in $O(e^{-\alpha \max_j y_j(m)})$. The proof of Theorem 14.3 is based on the following generalization of Theorem 17.1.

Theorem 17.4. *Suppose we are given a sequence of points*

$$\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K)$$

with $x_j(m) = \operatorname{Re} z_j(m)$ bounded and $y_j(m) = \operatorname{Im} z_j(m)$ going to infinity for every $j = 1, \dots, n$. Also suppose that

$$h(m) \equiv b(m) \mod \Phi^0\bigl(z(m)\bigr)$$

for a sequence of vectors $b(m) \in H_{\mathbb{C}}$ that is harmless with respect to $y(m)$. Then there exists a subsequence with the following properties:

(a) *The sequence $h(m)$ is constant, equal to some $h \in H_{\mathbb{Z}}(K)$.*
(b) *One has $(a_1 N_1 + \dots + a_n N_n)h = 0$ for certain positive integers $a_1, \dots, a_n$; in particular, $h \in W_0$.*
(c) *There is a vector $w \in \mathbb{C}^n$ such that*
$$\lim_{m \to \infty} e^{-\sum z_j(m) N_j} h(m) = e^{-\sum w_j N_j} h.$$
(d) *The sequence $b(m)$ converges to a limit $b \in H_{\mathbb{C}}$, and one has*
$$h \equiv b \mod e^{\sum w_j N_j} F^0$$
as well as $N_1 b = \dots = N_n b = 0$.

In order not to interrupt the flow of the argument, I decided to devote a separate Chapter E to the proof of Theorem 17.4.

18. Proof of Theorem 14.1. In this section, we prove that the topological closure of $E(\mathcal{H}_{\mathbb{Z}})$ is an analytic subspace of the vector bundle $B(\tilde{\mathcal{H}})$. This is of course a purely local problem: it suffices to show that, for any given point $x \in X$, the closure of $E(\mathcal{H}_{\mathbb{Z}})$ is analytic in a neighborhood of $\pi^{-1}(x) \subseteq B(\tilde{\mathcal{H}})$. After choosing suitable local coordinates, we may therefore assume that $X = \Delta^n$, with coordinates $s = (s_1, \dots, s_n)$, and that D is the divisor given by $s_1 \cdots s_k = 0$. Because $E(\mathcal{H}_{\mathbb{Z}})$ is already closed in $B(\mathcal{H})$, we are free to enlarge the divisor D; hence it is enough to treat the case $k = n$, where $X_0 = (\Delta^*)^n$. Now recall from (16.3) that $B(\tilde{\mathcal{H}}) \simeq \Delta^n \times H_{\mathbb{C}}$ and that $E(\mathcal{H}_{\mathbb{Z}})$ is the image of the holomorphic mapping

$$\varepsilon \colon \mathbb{H}^n \times H_{\mathbb{Z}} \to \Delta^n \times H_{\mathbb{C}}, \quad \varepsilon(z, h) = \Bigl(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-\sum z_j N_j} h\Bigr).$$

To prove Theorem 14.1, we have to show that $\overline{E(\mathcal{H}_{\mathbb{Z}})}$ is analytic in a neighborhood of every limit point of $E(\mathcal{H}_{\mathbb{Z}})$ that lies over the origin in Δ^n. This is basically just a problem in linear algebra that can be solved without knowing that the local system $\mathcal{H}_{\mathbb{Z}}$ comes from a polarized variation of Hodge structure.

As a first step, we describe all possible limit points of $E(\mathcal{H}_{\mathbb{Z}})$. In the natural stratification of Δ^n, the strata are indexed by subsets $J \subseteq \{1, 2, \dots, n\}$; the stratum corresponding to J is the set

$$\Delta_J^n = \bigl\{\, s \in \Delta^n \bigm| s_j = 0 \text{ if and only if } j \in J \,\bigr\}.$$

This notation makes the following result easier to state.

Proposition 18.1. *A point $(s,v) \in \Delta_J^n \times H_{\mathbb{C}}$ belongs to the topological closure $\overline{E(\mathcal{H}_{\mathbb{Z}})}$ if and only if the following two conditions are satisfied:*

(1) There is a vector $w \in \mathbb{C}^n$ and an integral class $h \in H_{\mathbb{Z}}$ such that
$$v = e^{-\sum w_j N_j} h$$
and such that $s_j = e^{2\pi i w_j}$ for every $j \notin J$.

(2) One has $\sum a_j N_j h = 0$ for some positive integers $\{a_j\}_{j \in J}$.

Proof. Suppose that $v = e^{-\sum w_j N_j} h$ with $w \in \mathbb{C}^n$ and $h \in H_{\mathbb{Z}}$ as above. If we put $a_j = 0$ for $j \notin J$ and denote the resulting vector by $a \in \mathbb{N}^n$, we obtain
$$v = e^{-\sum_{j \in J} (ita_j + w_j) N_j} h$$
for every $t \in \mathbb{R}$; but then it is easy to see that
$$(s,v) = \lim_{t \to \infty} \varepsilon\bigl(w + ita, h\bigr) \in \overline{E(\mathcal{H}_{\mathbb{Z}})}.$$
To prove the converse, let us consider an arbitrary point $(s,v) \in \Delta_J^n \times H_{\mathbb{C}}$ that also belongs to $\overline{E(\mathcal{H}_{\mathbb{Z}})}$. It is the limit of a sequence in $E(\mathcal{H}_{\mathbb{Z}})$; we can therefore choose a sequence $\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times H_{\mathbb{Z}}$ with bounded real parts $\operatorname{Re} z(m)$ such that
$$\lim_{m \to \infty} \Bigl(e^{2\pi i z_1(m)}, \dots, e^{2\pi i z_n(m)}, e^{-\sum z_j(m) N_j} h(m)\Bigr) = (s_1, \dots, s_n, v).$$
As in (26.1), it will be convenient to expand the sequence $z(m)$ according to the rate of growth of its imaginary parts. After passing to a subsequence, we can find a partition $J = J_1 \sqcup J_2 \sqcup \dots \sqcup J_d$, an $n \times d$-matrix A with nonnegative entries that satisfy $a_{j,k} \neq 0$ if and only if $j \in J_1 \sqcup \dots \sqcup J_k$, and two sequences $t(m) \in \mathbb{R}^d$ and $w(m) \in \mathbb{C}^n$, which together have the following three properties:
$$z(m) = iAt(m) + w(m),$$
the sequence $w(m)$ converges to a limit $w \in \mathbb{C}^n$, and the ratios
$$\frac{t_1(m)}{t_2(m)}, \frac{t_2(m)}{t_3(m)}, \dots, \frac{t_d(m)}{1} \tag{18.2}$$
are going to infinity. An easy calculation shows that $s_j = e^{2\pi i w_j}$ for $j \notin J$. If we again define
$$T_k = \sum_{j=1}^n a_{j,k} N_j,$$
then $T_1, \dots, T_d$ are commuting nilpotent operators, and
$$v = \lim_{m \to \infty} e^{-\sum z_j(m) N_j} h(m) = \lim_{m \to \infty} e^{-\sum w_j(m) N_j} e^{-i \sum_{k=1}^d t_k(m) T_k} h(m).$$
Since $w(m)$ converges to w, it follows that the sequence of vectors
$$e^{-i \sum_{k=1}^d t_k(m) T_k} h(m) \in H_{\mathbb{C}}$$
is also convergent. We are going to deduce from this that the sequence $h(m)$ is eventually constant, and that the constant value $h \in H_{\mathbb{Z}}$ satisfies $T_1 h = \cdots T_d h = 0$. To that end, we define, for every multi-index $\alpha \in \mathbb{N}^n$, a nilpotent operator
$$N^\alpha = N_1^{\alpha_1} \cdots N_n^{\alpha_n}.$$
Note that $N^\alpha h(m) = 0$ whenever $|\alpha| = \alpha_1 + \dots + \alpha_n$ is sufficiently large; we can therefore use induction to show that $N^\alpha h(m)$ is eventually constant for every multi-index α, and that the constant value $h^\alpha \in H_{\mathbb{Q}}$ satisfies $T_1 h^\alpha = \dots = T_d h^\alpha = 0$.

Suppose that for some integer $\ell \geq 0$, we already know this for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq \ell + 1$. Take an arbitrary multi-index α of length $|\alpha| = \ell$. From the convergence of the sequence

$$N^\alpha e^{-i\sum_{k=1}^d t_k(m)T_k} h(m) = N^\alpha h(m) - i \sum_{k=1}^d t_k(m) T_k N^\alpha h(m)$$

we deduce that the sequence of its real parts $N^\alpha h(m)$ must be eventually constant (because it lies in a discrete subset of $H_\mathbb{Q}$). Denote the constant value by $h^\alpha \in H_\mathbb{Q}$. Then the convergence of the sequence of imaginary parts

$$\sum_{k=1}^d t_k(m) T_k N^\alpha h(m) = \sum_{k=1}^d t_k(m) T_k h^\alpha$$

implies that $T_1 h^\alpha = \cdots = T_k h^\alpha = 0$ by virtue of (18.2).

The conclusion (for $\ell = 0$) is that the sequence $h(m)$ is eventually constant, and that the constant value $h \in H_\mathbb{Z}$ satisfies $T_1 h = \cdots = T_d h = 0$. Since

$$v = \lim_{m\to\infty} e^{-\sum z_j(m)N_j} h(m) = \lim_{m\to\infty} e^{-\sum w_j(m)N_j} h = e^{-\sum w_j N_j} h,$$

we get the first assertion. The second one follows from the identity $T_d h = 0$ by noting that T_d is a positive linear combination of the N_j with $j \in J$. □

One consequence of this result is that $\overline{E(\mathcal{H}_\mathbb{Z})}$ has, at least locally, a well-defined set of irreducible components; moreover, they are locally finite, as one would expect for an analytic space. The irreducible components are the closed sets

$$C(h) = \overline{\varepsilon_h(\mathbb{H}^n)} \subseteq \Delta^n \times H_\mathbb{C}, \tag{18.3}$$

where $\varepsilon_h = \varepsilon(-, h) \colon \mathbb{H}^n \to \Delta^n \times H_\mathbb{C}$ for $h \in H_\mathbb{Z}$. It is clear that the set $C(h)$ only depends on the $\mathbb{Z}^n$-orbit $\left\{ e^{\sum a_j N_j} h \mid a \in \mathbb{Z}^n \right\}$ of h inside $H_\mathbb{Z}$.

Corollary 18.4. *We have*

$$\overline{E(\mathcal{H}_\mathbb{Z})} = \bigcup_{h \in H_\mathbb{Z}} C(h),$$

and the family of closed sets $C(h)$ is locally finite on $\Delta^n \times H_\mathbb{C}$.

Proof. Clearly, $C(h) \subseteq \overline{E(\mathcal{H}_\mathbb{Z})}$ for every $h \in H_\mathbb{Z}$; conversely, Proposition 18.1 shows that any point in the closure of $E(\mathcal{H}_\mathbb{Z})$ belongs to one of the closed sets $C(h)$. To prove the local finiteness, we have to show that every point $(s, v) \in \Delta^n \times H_\mathbb{C}$ has an open neighborhood that meets only finitely many distinct sets $C(h)$. If this was not the case, we could construct a sequence $\big(z(m), h(m)\big) \in \mathbb{H}^n \times H_\mathbb{Z}$ with bounded real parts $\operatorname{Re} z(m)$ such that

$$(s, v) = \lim_{m\to\infty} \varepsilon\big(z(m), h(m)\big),$$

and such that no two of the sets $C\big(h(m)\big)$ are the same. But this obviously contradicts the fact – established during the proof of Proposition 18.1 – that a subsequence of the sequence $h(m)$ must be constant. □

To prove that $\overline{E(\mathcal{H}_\mathbb{Z})}$ is analytic, it is now enough to show that each of the closed sets $C(h)$ is analytic. This we do by finding a set of holomorphic equations. For the sake of convenience, we shall allow ourselves to replace $\mathbb{H}^n$ by $\mathbb{C}^n$ in this problem; of course, this has no effect on what happens over Δ^n.

Proposition 18.5. *For every $h \in H_{\mathbb{Z}}$, the topological closure of the image of*

$$\mathbb{C}^n \to \mathbb{C}^n \times H_{\mathbb{C}}, \quad z \mapsto \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-\sum z_j N_j} h\right), \tag{18.6}$$

is an analytic subspace of $\mathbb{C}^n \times H_{\mathbb{C}}$.

Proof. Let $S(h) = \left\{ a \in \mathbb{Z}^n \mid \sum a_j N_j h = 0 \right\}$ be the stabilizer of h; note that the quotient $\mathbb{Z}^n / S(h)$ is a free $\mathbb{Z}$-module of some rank $0 \leq r \leq n$, since it embeds into $H_{\mathbb{Q}}$. We can thus find a matrix $A \in \mathrm{SL}_n(\mathbb{Z})$ whose last $n - r$ columns give a basis for $S(h) \subseteq \mathbb{Z}^n$. If we introduce new coordinates $(z_1', \dots, z_n') \in \mathbb{C}^n$ by defining

$$z_j = \sum_{k=1}^n a_{j,k} z_k' \quad \text{and} \quad N_k' = \sum_{j=1}^n a_{j,k} N_j,$$

we have $z_1 N_1 + \cdots + z_n N_n = z_1' N_1' + \cdots + z_n' N_n'$. The vectors $N_1' h, \dots, N_r' h \in H_{\mathbb{Q}}$ are linearly independent, while $N_{r+1}' h = \cdots = N_n' h = 0$. The mapping in (18.6) therefore has the same image as

$$\mathbb{C}^n \to \mathbb{C}^n \times H_{\mathbb{C}}, \quad z' \mapsto \left(\prod_{k=1}^n e^{2\pi i a_{1,k} z_k'}, \dots, \prod_{k=1}^n e^{2\pi i a_{n,k} z_k'}, e^{-\sum z_k' N_k'} h \right). \tag{18.7}$$

We are now going to find a set of holomorphic equations that define the closure of the image. These equations will be of two kinds: the ones coming from the last coordinate in (18.7) will be polynomials, whereas the others will involve exponential functions. We first consider the polynomial mapping

$$\mathbb{C}^r \to H_{\mathbb{C}}, \quad (z_1', \dots, z_r') \mapsto e^{-\sum z_k' N_k'} h$$

Since the vectors $N_1' h, \dots, N_r' h \in H_{\mathbb{Q}}$ are linearly independent, it has a left inverse by Lemma 18.11; more precisely, there are polynomial mappings

$$p_\ell \colon H_{\mathbb{C}} \to \mathbb{C}$$

with the property that $p_\ell\big(e^{-\sum z_k' N_k'} h\big) = z_\ell'$ on all of $\mathbb{C}^r$. Now a vector $v \in H_{\mathbb{C}}$ is of the form $e^{-\sum z_k' N_k'} h$ if and only if it satisfies the system of polynomial equations

$$v = e^{-\sum p_k(v)' N_k'} h. \tag{18.8}$$

In that case, we also have $z_k' = p_k(v)$ for $k = 1, \dots, r$, and therefore

$$s_j = \prod_{k=1}^n e^{2\pi i a_{j,k} z_k'} = \prod_{k=1}^r e^{2\pi i a_{j,k} p_k(v)} \cdot \prod_{k=r+1}^n e^{2\pi i a_{j,k} z_k'}.$$

The shape of these formulas suggests looking at the monomial mapping

$$(\mathbb{C}^*)^{n-r} \to \mathbb{C}^n, \quad (u_{r+1}, \dots, u_n) \mapsto \left(\prod_{k=r+1}^n u_k^{a_{1,k}}, \dots, \prod_{k=r+1}^n u_k^{a_{n,k}} \right). \tag{18.9}$$

It is well-known that the topological closure of the image is an algebraic variety; in fact, it is a (not necessarily normal) affine toric variety [Stu97]. More precisely, let

$$\Gamma = \mathbb{N}\langle a_1, \dots, a_n \rangle \subseteq \mathbb{Z}^{n-r}$$

be the semigroup generated by the vectors $a_i = (a_{i,r+1}, \dots, a_{i,n}) \in \mathbb{Z}^{n-r}$. If

$$\mathbb{C}[\Gamma] = \bigoplus_{a \in \Gamma} \mathbb{C} \chi^a$$

denotes the group algebra of Γ, with multiplication given by $\chi^a \cdot \chi^b = \chi^{a+b}$, then the closure of the image of (18.9) is the affine algebraic variety

$$\operatorname{Spec} \mathbb{C}[\Gamma].$$

It embeds into $\mathbb{C}^n$, because Γ is generated by $a_1, \dots, a_n$; the corresponding ideal $I_\Gamma \subseteq \mathbb{C}[s_1, \dots, s_n]$ in the polynomial ring is generated by the binomial equations

$$\prod_{u_i > 0} s_i^{u_i} = \prod_{u_i < 0} s_i^{-u_i},$$

where $u \in \mathbb{Z}^n$ runs over all integer solutions of the equation $u_1 a_1 + \cdots + u_n a_n = 0$. The conclusion is that any point $(s, v) \in \mathbb{C}^n \times H_{\mathbb{C}}$ in the image of (18.7) also satisfies the system of holomorphic equations

$$f\left(s_1 \prod_{k=1}^r e^{-2\pi i a_{1,k} p_k(v)}, \dots, s_n \prod_{k=1}^r e^{-2\pi i a_{n,k} p_k(v)}\right) = 0 \quad \text{for all } f \in I_\Gamma. \tag{18.10}$$

Since it is easy to see from the construction that every solution of the equations in (18.8) and (18.10) belongs to the closure of the image, the assertion is proved. □

Lemma 18.11. *Let $h \in H_{\mathbb{C}}$. If the vectors $N_1 h, \dots, N_n h \in H_{\mathbb{C}}$ are linearly independent, then there are polynomial mappings $p_k \colon H_{\mathbb{C}} \to \mathbb{C}$ such that*

$$p_k\big(e^{-\sum z_j N_j} h\big) = z_k$$

for every $z \in \mathbb{C}^n$ and every $k = 1, \dots, n$.

Proof. The proof is by induction on $n \geq 0$. To shorten the notation, set $v = e^{-\sum z_j N_j} h$. Among all multi-indices $\alpha \in \mathbb{N}^n$ with $N^\alpha h \neq 0$, select one of maximal length; after some reordering, we may assume that $\alpha_n \geq 1$. We have

$$N^{\alpha - e_n} v = \big(\mathrm{id} - z_1 N_1 - \cdots - z_{n-1} N_{n-1}\big) N^{\alpha - e_n} h - z_n N^\alpha h,$$

and because $N^\alpha h \neq 0$, we can solve for z_n in the form

$$z_n = c_1 z_1 + \cdots + c_{n-1} z_{n-1} + q(v),$$

where $q \colon H_{\mathbb{C}} \to \mathbb{C}$ is an affine linear form. Now

$$v = e^{-\sum z_j (N_j + c_j N_n)} e^{-q(v) N_n} h,$$

and by induction, $z_1, \dots, z_{n-1}$ are given by polynomials in the coordinates of the vector $e^{q(v) N_n} v$. This shows that there are polynomial mappings $p_k \colon H_{\mathbb{C}} \to \mathbb{C}$ with $p_k(v) = z_k$ for $k = 1, \dots, n-1$; but then

$$p_n = c_1 p_1 + \cdots + c_{n-1} p_{n-1} + q$$

is also a polynomial mapping and satisfies $p_n(v) = z_n$. □

In defining the locus of limit Hodge classes, we actually work with the normalization of $\overline{E(\mathcal{H}_{\mathbb{Z}})}$. The following result describes what the normalization looks like.

Lemma 18.12. *The normalization mapping $\nu \colon \overline{E(\mathcal{H}_{\mathbb{Z}})}^\nu \to B(\tilde{\mathcal{H}})$ separates the local analytically irreducible components...*

Proof. The statement is of a local nature, and so with the same choice of coordinate system as above, it suffices to analyze what happens at a point of the form $(0, v) \in \Delta^n \times H_{\mathbb{C}}$. By general properties of normalization, ν separates the different local irreducible components $C(h)$; it is therefore enough to prove that the fiber of

$$\nu \colon C(h)^\nu \to C(h)$$

over any $(0, v) \in C(h)$ consists of exactly one point. During the proof of Proposition 18.5, we described $C(h)$ in terms of a closed embedding and a (not necessarily normal) affine toric variety of dimension $n - r$; recall that $(0, v) \in C(h)$ is only possible if the subgroup $S(h)$ contains a vector with positive entries. Looking back at the arguments that we used to prove Proposition 18.5, we see that in order to verify the assertion about ν, it suffices to understand what happens for the normalization of the closure of the image of the monomial mapping in (18.9).

We shall now describe the normalization in algebraic terms, following [Stu97, §2]. Recall that the closure of the image is the affine algebraic variety $\operatorname{Spec} \mathbb{C}[\Gamma]$, where $\Gamma = \mathbb{N}\langle a_1, \dots, a_n \rangle$ is the semigroup generated by the vectors $a_1, \dots, a_n \in \mathbb{Z}^{n-r}$. These vectors generate $\mathbb{Z}^{n-r}$ as a group because $A \in \mathrm{SL}_n(\mathbb{Z})$, and so Γ is contained inside the larger semigroup

$$\tilde{\Gamma} = \mathbb{Z}^{n-r} \cap \operatorname{Cone}(a_1, \dots, a_n),$$

of all lattice points in the convex cone spanned by $a_1, \dots, a_n$. With this notation, the normalization of $\operatorname{Spec} \mathbb{C}[\Gamma]$ is given by

$$\operatorname{Spec} \mathbb{C}[\tilde{\Gamma}] \to \operatorname{Spec} \mathbb{C}[\Gamma].$$

Since we are assuming that there is a point $(0, v) \in C(h)$, the embedding of $\operatorname{Spec} \mathbb{C}[\Gamma]$ into $\mathbb{C}^n$ goes through the origin; algebraically, this means that we have a well-defined $\mathbb{C}$-algebra homomorphism

$$o \colon \mathbb{C}[\Gamma] \to \mathbb{C}$$

that annihilates all the elements $\chi^a \in \mathbb{C}[\Gamma]$ with $0 \neq a \in \Gamma$. Now points of $\operatorname{Spec} \mathbb{C}[\tilde{\Gamma}]$ in the fiber over the origin are in one-to-one correspondence with $\mathbb{C}$-algebra homomorphisms

$$p \colon \mathbb{C}[\tilde{\Gamma}] \to \mathbb{C}$$

whose restriction to the subring $\mathbb{C}[\Gamma]$ is equal to o. It is easy to see that every nonzero $b \in \tilde{\Gamma}$ can be written as a positive rational linear combination of $a_1, \dots, a_n$, which means that $mb \in \Gamma$ for some $m \geq 1$. But then

$$\bigl(p(\chi^b)\bigr)^m = p(\chi^{mb}) = o(\chi^{mb}) = 0,$$

hence $p(\chi^b) = 0$, and so p is uniquely determined by o. □

19. Proof of Theorem 14.2. In this section, we show that the locus of limit Hodge classes still satisfies the theorem of Cattani, Deligne, and Kaplan. Fix a an integer $K \geq 0$, and denote by

$$\nu \colon \overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})}^{\nu} \to B(\tilde{\mathcal{H}})$$

the normalization of the closure of $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ inside the holomorphic vector bundle $B(\tilde{\mathcal{H}})$; recall from Theorem 14.1 that the closure is an analytic subspace. Let

$$\operatorname{Hdg}_{\leq K}(\mathcal{H}, X) = \nu^{-1}\Bigl(B(F^0 \tilde{\mathcal{H}})\Bigr) \subseteq \overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})}^{\nu}$$

be the locus of limit Hodge classes with self-intersection number bounded by K. Our goal is to show that the projection from $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$ to X is a finite mapping. Since $\operatorname{Hdg}_{\leq K}(\mathcal{H}, X)$ is by construction finite over the analytic space

$$\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}}) \subseteq B(\tilde{\mathcal{H}}),$$

it suffices to prove that the restriction of $\pi \colon B(\tilde{\mathcal{H}}) \to X$ to this subspace is finite. This is again a local problem, and so we may assume that $X = \Delta^n$ and that D is the divisor given by $s_1 \cdots s_n = 0$; we may also restrict our attention to what happens near the fiber $\pi^{-1}(0)$. In the notation of §15, we have $B(\tilde{\mathcal{H}}) \simeq \Delta^n \times H_{\mathbb{C}}$, the subbundle $B(F^0\tilde{\mathcal{H}})$ is the image of the holomorphic mapping

$$\Delta^n \times F^0 \to \Delta^n \times H_{\mathbb{C}}, \quad (s, v) \mapsto \left(s, e^{\Gamma(s)} v\right),$$

and $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})}$ is the closure of the image of

$$\varepsilon \colon \mathbb{H}^n \times H_{\mathbb{Z}}(K) \to \Delta^n \times H_{\mathbb{C}}, \quad \varepsilon(z, h) = \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}, e^{-\sum z_j N_j} h\right).$$

As a first step towards proving Theorem 14.2, we will show that the projection to Δ^n has finite fibers. This is somewhat unexpected, given that $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})}$ can have fibers of dimension up to $n-1$. Because of how we choose the coordinate system, it is enough to show that the fiber over $0 \in \Delta^n$ contains only finitely many points.

Proposition 19.1. *The intersection* $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}}) \cap \pi^{-1}(0)$ *is a finite set.*

Proof. According to Proposition 18.1, any point $(0, v) \in \overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap \pi^{-1}(0)$ is of the form $v = e^{-\sum w_j N_j} h$ for some $w \in \mathbb{C}^n$ and some $h \in H_{\mathbb{Z}}(K)$; moreover, one has $Nh = 0$ for some $N \in C(N_1, \dots, N_n)$, and therefore $h \in W_0$. If the point belongs to the subbundle $B(F^0\tilde{\mathcal{H}})$, then $e^{-\sum w_j N_j} h \in F^0$, which means that

$$h \in W_0 \cap e^{\sum w_j N_j} F^0$$

is a Hodge class in the mixed Hodge structure $\left(W, e^{\sum w_j N_j} F\right)$. Thus, $h \in I^{0,0}$, where $H_{\mathbb{C}} = \bigoplus I^{p,q}$ is Deligne's decomposition of this mixed Hodge structure.

It follows that h uniquely determines v. To see why, suppose that $v' = e^{-\sum w'_j N_j} h$ also satisfies $(0, v') \in B(F^0\tilde{\mathcal{H}})$. As before, we have

$$e^{-\sum (w'_j - w_j) N_j} h \in e^{\sum w_j N_j} F^0 = \bigoplus_{p \geq 0} I^{p,q};$$

now the left-hand side is an element of $I^{0,0} \oplus I^{-1,-1} \oplus \cdots$, and therefore equal to h, and this gives $v' = v$. Since there are only countably many choices for $h \in H_{\mathbb{Z}}(K)$, this observation already tells us that $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}}) \cap \pi^{-1}(0)$ is discrete.

To show that the intersection is finite, we shall appeal to the technical result in Theorem 17.1. Suppose that $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}}) \cap \pi^{-1}(0)$ was an infinite set. Then we could find a sequence of distinct elements $h(m) \in H_{\mathbb{Z}}(K)$, a sequence of vectors $w(m) \in \mathbb{C}^n$, and a sequence $a(m) \in \mathbb{N}^n$ with positive entries, such that

$$h(m) \in e^{\sum w_j(m) N_j} F^0 \quad \text{and} \quad \sum_{j=1}^n a_j(m) N_j h(m) = 0.$$

Now let $\Phi_{nil} \colon \mathbb{H}^n \to \check{D}$ be the associated nilpotent orbit as in (15.1), and choose a sequence of positive real numbers $t(m) \in \mathbb{R}$ that are large enough to ensure that $z(m) = w(m) + ia(m)t(m) \in \mathbb{H}^n$ and $\Phi_{nil}\bigl(z(m)\bigr) \in D$. Then each

$$h(m) \in H_{\mathbb{Z}}(K) \cap \Phi_{nil}^0\bigl(z(m)\bigr)$$

is an integral Hodge class in the Hodge structure $\Phi_{nil}\bigl(z(m)\bigr)$, and so the sequence $\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K)$ satisfies the assumptions of Theorem 17.1. Since $h(m)$ does not contain a constant subsequence, this is a contradiction. □

The next task is to prove that $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}})$ is proper over Δ^n; here it is convenient to use the formulation of properness in terms of sequences. In preparation for the proof, let us fix a subset $J \subseteq \{1, 2, \dots, n\}$ and let us describe the points of the intersection that lie over the stratum Δ_J^n. According to Proposition 18.1, a point $(s, v) \in \Delta_J^n \times H_{\mathbb{C}}$ belongs to $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})}$ if and only if

$$v = e^{-\sum w_j N_j} h \quad \text{and} \quad s_j = e^{2\pi i w_j} \text{ for } j \notin J$$

for some $w \in \mathbb{C}^n$ and some $h \in H_{\mathbb{Z}}(K)$ with $\sum_{j \in J} a_j N_j h = 0$. If the point in question also belongs to $B(F^0\tilde{\mathcal{H}})$, then $v \in e^{\Gamma_J(s)} F^0$; here $\Gamma_J(s)$ is obtained from $\Gamma(s)$ by setting to zero all the variables s_j with $j \in J$. In other words, we have

$$h \in e^{\sum w_j N_j} e^{\Gamma_J(s)} F^0;$$

By [CK89, Theorem 2.8], $\Phi_J(z) = e^{\sum z_j N_j} e^{\Gamma_J(s)} F$ is again the period mapping of a polarized variation of integral Hodge structure of weight zero. Set $a_j = 0$ for $j \notin J$; then we have

$$h \in e^{\sum (w_j + ita_j) N_j} e^{\Gamma_J(s)} F^0$$

for every $t \in \mathbb{R}$, which means exactly that

$$h \in H_{\mathbb{Z}}(K) \cap \Phi_J^0(w + ita)$$

is a Hodge class at the point $w + ita$ for sufficiently large t. This observation reduces the proof of properness to another application of Theorem 17.1.

Proposition 19.2. *The restriction of the projection $\pi \colon B(\tilde{\mathcal{H}}) \to \Delta^n$ to the analytic subspace $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}})$ is a proper mapping.*

Proof. It suffices to show that any sequence of points in $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}})$ whose projection to Δ^n converges to the origin must have a convergent subsequence. After passing to a subsequence, we may assume without loss of generality that our sequence $\bigl(s(m), v(m)\bigr)$ lies entirely over the stratum Δ_J^n for some $J \subseteq \{1, 2, \dots, n\}$. As explained above, there is a sequence $w(m) \in \mathbb{C}^n$ with bounded real parts $\operatorname{Re} w(m)$ such that

$$v(m) = e^{-\sum w_j(m) N_j} h(m) \quad \text{and} \quad s_j(m) = e^{2\pi i w_j(m)} \text{ for } j \notin J,$$

and two further sequences $a(m) \in \mathbb{N}^n$ and $t(m) \in \mathbb{R}$ such that

$$h(m) \in H_{\mathbb{Z}}(K) \cap \Phi_J^0\bigl(w(m) + it(m)a(m)\bigr).$$

By applying Theorem 17.1 to the period mapping Φ_J, we obtain the existence of a subsequence with the following properties: $h(m) = h$ is constant, in the kernel of some element of $C(N_1, \dots, N_n)$, and there is a vector $w \in \mathbb{C}^n$ such that

$$e^{-\sum w_j N_j} h = \lim_{m \to \infty} e^{-\sum (w_j(m) + it(m) a_j(m)) N_j} h(m) = \lim_{m \to \infty} v(m).$$

But this is saying that $\big(s(m), h(m)\big)$ converges to the point

$$\big(0, e^{-\sum w_j N_j} h\big),$$

which again belongs to $\overline{E_{\leq K}(\mathcal{H}_{\mathbb{Z}})} \cap B(F^0\tilde{\mathcal{H}})$ according to Proposition 18.1. □

To complete the proof of Theorem 14.2, it remains to relate the points of the analytic space $\operatorname{Hdg}(\mathcal{H}, \Delta^n)$ to our definition of limit Hodge classes in §9.

Proposition 19.3. *The points of* $\operatorname{Hdg}(\mathcal{H}, \Delta^n)$ *are in one-to-one correspondence with limit Hodge classes for* $\mathcal{H}$ *on* Δ^n*.*

Proof. We begin by collecting a few facts that we established earlier. By our choice of coordinate system, it is again enough to consider what happens over the origin in Δ^n. Recall from Corollary 18.4 that the irreducible components of $\overline{E(\mathcal{H}_{\mathbb{Z}})} \subseteq \Delta^n \times H_{\mathbb{C}}$ are the closed sets $C(h)$, and that $C(h_1) = C(h_2)$ if and only if $h_1, h_2 \in H_{\mathbb{Z}}$ belong to the same orbit of the $\mathbb{Z}^n$-action. By Proposition 18.1, having a point $(0, v) \in C(h)$ requires that $Nh = 0$ for some $N \in C(N_1, \dots, N_n)$, and then $v = e^{-\sum w_j N_j} h$ for some $w \in \mathbb{C}^n$. As we have seen during the proof of Proposition 19.1, $(0, v) \in B(F^0\tilde{\mathcal{H}})$ is equivalent to

$$h \in e^{\sum w_j N_j} F^0,$$

and in that case, v is uniquely determined by h. Lastly, we proved in Lemma 18.12 that $\nu \colon \overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu} \to B(\tilde{\mathcal{H}})$ pulls apart the different irreducible components $C(h)$, but that the normalization of each $C(h)$ contains exactly one point over $(0, v) \in C(h)$.

The conclusion is that we have a one-to-one correspondence between points in $\operatorname{Hdg}(\mathcal{H}, \Delta^n)$ that lie over the origin in Δ^n and elements of the finite set

$$\mathbb{L} = \left\{ [h] \in H_{\mathbb{Z}}/\mathbb{Z}^n \;\middle|\; \begin{array}{c} h \in \ker N \cap e^{\sum w_j N_j} F^0, \text{ for some} \\ w \in \mathbb{C}^n \text{ and } N \in C(N_1, \dots, N_n) \end{array} \right\}$$

To conclude the proof, it only remains to identify the elements of $\mathbb{L}$ with limit Hodge classes for $\mathcal{H}$ at the point $0 \in \Delta^n$. In the notion of §9, we have

$$M_0(\mathcal{H}_{\mathbb{Z}}) = H_{\mathbb{Z}}/\mathbb{Z}^n.$$

Suppose first that $[h] \in H_{\mathbb{Z}}/\mathbb{Z}^n$ is a limit Hodge class. By definition, this means that there is a germ of a holomorphic arc

$$f \colon (\Delta, 0) \to (\Delta^n, 0)$$

with $f(\Delta^*) \subseteq (\Delta^*)^n$ and $f(0) = 0$, such that some branch of h is a limit Hodge class for $f^{-1}\mathcal{H}$ at the point $0 \in \Delta$. After choosing a coordinate t on Δ, we can write $f_j(t) = t^{a_j} g_j(t)$ for a positive integer a_j and a holomorphic function g_j with $g_j(0) \neq 0$. In order to identify the space of global sections of $f^{-1}\mathcal{H}_{\mathbb{Z}}$ on the universal covering space of Δ^* with $H_{\mathbb{Z}}$, we also need to choose a lifting of f to a holomorphic mapping $\mathbb{H} \to \mathbb{H}^n$; then the logarithm of the monodromy of $f^{-1}\mathcal{H}$ is equal to $N = a_1 N_1 + \dots + a_n N_n$, and the limit mixed Hodge structure has the form

$$\big(W(N), e^{\sum w_j N_j} F\big) = \big(W, e^{\sum w_j N_j} F\big)$$

with $e^{2\pi i w_j} = g_j(0)$. For at least one choice of lifting, we get

$$h \in H_{\mathbb{Z}} \cap \ker N \cap e^{\sum w_j N_j} F,$$

and so $[h] \in \mathbb{L}$.

Conversely, suppose that we start from an element $[h] \in \mathbb{L}$. Since $h \in H_{\mathbb{Z}}$, we can find positive integers $a_1, \dots, a_n$ such that $h \in \ker(a_1 N_1 + \dots + a_n N_n)$. For small enough $r > 0$, consider the holomorphic arc

$$\Delta_r \to \Delta^n, \quad t \mapsto \bigl(t^{a_1} e^{2\pi i w_1}, \dots, t^{a_n} e^{2\pi i w_n}\bigr)$$

and the pullback of the variation of Hodge structure $\mathcal{H}$ to Δ_r^*. The logarithm of the monodromy is $a_1 N_1 + \dots + a_n N_n$, and the limit mixed Hodge structure is

$$\bigl(W(a_1 N_1 + \dots + a_n N_n), e^{\sum w_j N_j} F\bigr) = \bigl(W, e^{\sum w_j N_j} F\bigr).$$

Since $h \in H_{\mathbb{Z}} \cap \ker(a_1 N_1 + \dots + a_n N_n) \cap e^{\sum w_j N_j} F^0$, we conclude that $[h]$ is a limit Hodge class for $\mathcal{H}$ at the point $0 \in \Delta^n$. □

20. Proof of Theorem 14.3. The purpose of this section is to deduce Theorem 14.3 from the technical result in Theorem 17.4.

It suffices to show that the closure of the image of $\varphi \colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$ is analytic in a neighborhood of any given point in $T(F_{-1}\mathcal{M})$. After choosing local coordinates, we may therefore assume without loss of generality that $X = \Delta^n$ and $X_0 = (\Delta^*)^n$, and consider the behavior of the closure over the origin. Using the notation introduced in §15, we have the following commutative diagram:

(20.1)
$$\begin{array}{ccccc} & & \tilde{\varphi} & & \\ \mathbb{H}^n \times H_{\mathbb{Z}}(K) & \xrightarrow{\varepsilon} & E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) & \xrightarrow{\varphi} & T(F_{-1}\mathcal{M}) \\ & & \big\downarrow & & \big\downarrow p \\ & & B(\tilde{\mathcal{H}}) & \xrightarrow{Q} & T(F^1\tilde{\mathcal{H}}) \end{array}$$

The first step towards understanding the topological closure of the image of φ is to see what happens when a sequence of points in $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ has bounded image in $T(F_{-1}\mathcal{M})$. Suppose then that we are given a sequence

$$\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K)$$

with the property that $\tilde{\varphi}\bigl(z(m), h(m)\bigr)$ remains bounded inside the space $T(F_{-1}\mathcal{M})$, while $s_j(m) = e^{2\pi i z_j(m)}$ goes to zero for every $j = 1, \dots, n$. This means that the sequence of imaginary parts $y_j(m) = \operatorname{Im} z_j(m)$ is going to infinity; using the relation in (16.1), we can furthermore arrange that the sequence of real parts $x_j(m) = \operatorname{Re} z_j(m)$ remains bounded.

Observe now that $h \in H_{\mathbb{Z}}$ satisfies $\tilde{\varphi}(z, h) = 0$ if and only if $h \in \Phi^0(z)$, which suggests that the boundedness of $\tilde{\varphi}\bigl(z(m), h(m)\bigr)$ should give us some control over the distance from $h(m)$ to the subspace $\Phi^0\bigl(z(m)\bigr)$. The following result makes this idea precise, using the notion of a harmless sequence introduced in Definition 17.3.

Proposition 20.2. *If the sequence $\tilde{\varphi}\bigl(z(m), h(m)\bigr) \in T(F_{-1}\mathcal{M})$ is bounded, then*

(20.3)
$$h(m) \equiv b(m) \mod \Phi^0\bigl(z(m)\bigr),$$

for a sequence of vectors $b(m) \in H_{\mathbb{C}}$ that is harmless with respect to $\operatorname{Im} z(m)$.

Proof. We are going to use the collection of holomorphic sections

$$\sigma_v(z) = e^{\sum z_j N_j} e^{\Gamma(s)} v \quad (\text{for } v \in F^1)$$

$$\sigma_{v,k}(z) = e^{\sum z_j N_j} e^{\Gamma(s)} \frac{N_k v}{s_k} \quad (\text{for } v \in F^2)$$

of the coherent sheaf $F_{-1}\mathcal{M}$; see Lemma 16.5 for details. Define the auxiliary sequence of vectors

$$h'(m) = e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} h(m) \in H_{\mathbb{C}}.$$

Using Deligne's decomposition $H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ of the mixed Hodge structure (W, F), we also define

$$h'(m)_{-1} = \sum_{p \leq -1} h'(m)^{p,q} \in \bigoplus_{p \leq -1} I^{p,q}.$$

We have $h'(m) \equiv h'(m)_{-1}$ modulo $F^0 = \bigoplus_{p \geq 0} I^{p,q}$, and therefore (20.3) holds with

$$b(m) = e^{\sum z_j(m) N_j} e^{\Gamma(s(m))} h'(m)_{-1} \in H_{\mathbb{C}}.$$

It remains to show that $b(m)$ is harmless with respect to the sequence of imaginary parts $y(m) = \operatorname{Im} z(m)$. By assumption, the sequence of complex numbers

$$Q\Big(h(m), \sigma_v\big(z(m)\big)\Big) = Q\left(h(m), e^{\sum z_j(m) N_j} e^{\Gamma(s(m))} v\right) = Q\big(h'(m), v\big)$$

is bounded for every $v \in F^1$. Since the pairing Q is nondegenerate and compatible with Deligne's decomposition, we conclude that $\|h'(m)_{-1}\|$ is bounded. Likewise, the boundedness of the sequence

$$Q\Big(h(m), \sigma_{v,k}\big(z(m)\big)\Big) = -Q\left(\frac{N_k h'(m)}{s_k(m)}, v\right)$$

for every $v \in F^2$ implies that $\|N_k h'(m)_{-1}\|$ is in $O(e^{-2\pi y_k(m)})$. Combining both observations, we find that the sequence $h'(m)_{-1}$ is harmless (for $\alpha = 2\pi$). But then $b(m)$ is also harmless (for any $\alpha < 2\pi$) by the elementary Lemma 24.3 below. □

We can therefore apply the technical result in Theorem 17.4 to the given sequence; the conclusion is that there is a subsequence $\big(z(m), h(m)\big) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K)$ with the following properties:

(a) The sequence $h(m)$ is constant and equal to $h \in H_{\mathbb{Z}}(K)$.
(b) One has $(a_1 N_1 + \dots + a_n N_n) h = 0$ for certain positive integers $a_1, \dots, a_n$; in particular, $h \in W_0$.
(c) There is a vector $w \in \mathbb{C}^n$ such that

$$\lim_{m \to \infty} e^{-\sum z_j(m) N_j} h(m) = e^{-\sum w_j N_j} h.$$

(d) The sequence $b(m)$ converges to a limit $b \in H_{\mathbb{C}}$, and one has

$$h \equiv b \mod e^{\sum w_j N_j} F^0$$

as well as $N_1 b = \dots = N_n b = 0$.

One consequence is that our original sequence $h(m) \in H_{\mathbb{Z}}$ can only take finitely many values: otherwise, we could pass to a subsequence in which all values are different, and this would obviously contradict (a). If we define, for each $h \in H_{\mathbb{Z}}$, the holomorphic mapping

$$\tilde{\varphi}_h = \tilde{\varphi}(-, h) \colon \mathbb{H}^n \to T(F_{-1}\mathcal{M}),$$

then what this says is that any bounded subset of $T(F_{-1}\mathcal{M})$ can intersect only finitely many of the sets $\tilde{\varphi}_h(\mathbb{H}^n)$. Moreover, unless h also satisfies the conditions in (b) and (d), the set $\tilde{\varphi}_h(\mathbb{H}^n)$ does not have any limit points that lie over the origin in Δ^n. If we apply N_k to the congruence in (d), we obtain $N_k h \in e^{\sum w_j N_j} F^{-1}$;

likewise, (b) implies that $N_k h \in W_{-2}$. These observations reduce the proof of Theorem 14.3 to the following statement.

Proposition 20.4. *Let $h \in H_{\mathbb{Z}}$ be an element with $N_k h \in W_{-2} \cap e^{\sum w_j N_j} F^{-1}$ for all $k = 1, \dots, n$. Then the topological closure of the image of*

$$\tilde{\varphi}_h = \tilde{\varphi}(-, h) \colon \mathbb{H}^n \to T(F_{-1}\mathcal{M})$$

is an analytic subset of $T(F_{-1}\mathcal{M})$.

Proof. As in (20.1), we denote by $p \colon T(F_{-1}\mathcal{M}) \to T(F^1\tilde{\mathcal{H}})$ the holomorphic mapping induced by the inclusion $F^1\tilde{\mathcal{H}} \hookrightarrow F_{-1}\mathcal{M}$; note that it is an isomorphism over $(\Delta^*)^n$. We are going to prove the stronger result that the image of $p \circ \tilde{\varphi}_h$ has an analytic closure in the vector bundle $T(F^1\tilde{\mathcal{H}})$. This is enough to conclude the proof, because the image of $\tilde{\varphi}_h$ is then contained in the closed analytic subset

$$p^{-1}\Bigl(\overline{(p \circ \tilde{\varphi}_h)(\mathbb{H}^n)}\Bigr).$$

As p is an isomorphism over $(\Delta^*)^n$, it follows that the closure of $\tilde{\varphi}_h(\mathbb{H}^n)$ is also analytic – in fact, it has to be a connected component of the above set.

The pairing Q giving the polarization on $\mathcal{H}$ induces a surjective morphism $Q \colon \tilde{\mathcal{H}} \to \mathcal{H}om(F^1\tilde{\mathcal{H}}, \mathscr{O}_X)$, and hence a surjective morphism of vector bundles

$$Q \colon B(\tilde{\mathcal{H}}) \to T(F^1\tilde{\mathcal{H}})$$

whose kernel is the subbundle $B(F^0\tilde{\mathcal{H}})$. Using the notation from (18.3), we shall deduce the assertion about the image of $p \circ \tilde{\varphi}_h$ from the following stronger claim: if $h \in H_{\mathbb{Z}}$ is such that $N_k h \in W_{-2} \cap e^{\sum w_j N_j} F^{-1}$ for every $k = 1, \dots, n$, then the restriction of Q to the closed analytic subset $C(h)$ is a holomorphic embedding. This evidently completes the proof, because it shows that the closure of the image of $p \circ \tilde{\varphi}_h$ is isomorphic to $C(h)$.

The concrete description of $F^1\tilde{\mathcal{H}}$ in §15 shows that $T(F^1\tilde{\mathcal{H}}) \simeq \Delta^n \times \operatorname{Hom}(F^1, \mathbb{C})$. Using this isomorphism, the mapping $p \circ \tilde{\varphi}_h$ is given in coordinates by the formula

$$\mathbb{H}^n \to \Delta^n \times \operatorname{Hom}(F^1, \mathbb{C}), \quad z \mapsto \Bigl(s, v \mapsto Q\bigl(h, e^{\sum z_j N_j} e^{\Gamma(s)} v\bigr)\Bigr). \tag{20.5}$$

As usual, the relation $s_j = e^{2\pi i z_j}$ is implicit in the notation. Let $H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ be Deligne's decomposition of the mixed Hodge structure $\bigl(W, e^{\sum w_j N_j} F\bigr)$. Since

$$e^{\sum z_j N_j} e^{\Gamma(s)} = e^{\sum (z_j - w_j) N_j} \Bigl(e^{\sum w_j N_j} e^{\Gamma(s)} e^{-\sum w_j N_j}\Bigr) e^{\sum w_j N_j},$$

we may replace F by $e^{\sum w_j N_j} F$ and $e^{\Gamma(s)}$ by the expression in parentheses, and assume without essential loss of generality that $w = 0$. We then have $N_k h \in I^{-1,-1}$ for every $k = 1, \dots, n$. Under the isomorphism

$$\bigoplus_{p \leq -1} I^{p,q} \simeq \operatorname{Hom}(F^1, \mathbb{C})$$

induced by Q, the linear functional $v \mapsto Q\bigl(h, e^{\sum z_j N_j} e^{\Gamma(s)} v\bigr)$ corresponds to

$$\sum_{p \leq -1} \Bigl(e^{-\Gamma(s)} e^{-\sum z_j N_j} h\Bigr)^{p,q} = \sum_{p \leq -1} \Bigl(e^{-\Gamma(s)} h\Bigr)^{p,q} + e^{-\Gamma(s)} \Bigl(e^{-\sum z_j N_j} - \mathrm{id}\Bigr) h; \tag{20.6}$$

here we have used that $N_k h \in I^{-1,-1}$ and that $\Gamma(s) \in \bigoplus_{p \leq -1} \mathfrak{g}^{p,q}$. Now recall that $C(h)$ was defined as the closure of the image of the holomorphic mapping

$$\mathbb{H} \to \Delta^n \times H_{\mathbb{C}}, \quad z \mapsto \Bigl(s, e^{-\sum z_j N_j} h\Bigr).$$

Solving (20.6) for the term $e^{-\sum z_j N_j} h$, we find that

$$e^{-\sum z_j N_j} h = h - e^{\Gamma(s)} \sum_{p \leq -1} \Bigl(e^{-\Gamma(s)} h\Bigr)^{p,q} + e^{\Gamma(s)} \sum_{p \leq -1} \Bigl(e^{-\Gamma(s)} e^{-\sum z_j N_j} h\Bigr)^{p,q}.$$

Since h is fixed and $\Gamma(s)$ is holomorphic on Δ^n, the formula on the right-hand side defines a holomorphic mapping $\Delta^n \times \operatorname{Hom}(F^1, \mathbb{C}) \to \Delta^n \times H_{\mathbb{C}}$ whose composition with Q is equal to the identity on $C(h)$, as claimed above. □

21. Proof of Corollary 14.4 and Corollary 14.5. Fix some $K \geq 0$. We already know from the result about Hodge structures in Lemma 3.1 that $\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$ has finite fibers; the purpose of this section is to understand its global properties. The following diagram shows all the relevant mappings:

$$\begin{array}{ccc} & & T(\mathcal{H}) \\ & \nearrow & \downarrow q \\ E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) & \xrightarrow{\varphi} & T(F^1\mathcal{H}) \\ & \searrow^{\pi} & \downarrow \\ & & X_0. \end{array}$$

The polarization defines a hermitian metric on the holomorphic vector bundle associated with $\mathcal{H}$, the so-called *Hodge metric*; it induces hermitian metrics on the two bundles $T(\mathcal{H})$ and $T(F^1\mathcal{H})$. Let $B_r(\mathcal{H}) \subseteq T(\mathcal{H})$ denote the closed tube of radius $r > 0$ around the zero section. The proof of Lemma 3.1 shows that

$$\varphi^{-1}\bigl(B_r(\mathcal{H})\bigr) \subseteq B_{\sqrt{K+4r^2}}\bigl(F^1\mathcal{H}\bigr);$$

in particular, the discussion in §22 below applies to our situation. It follows that

$$\varphi\colon E_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F^1\mathcal{H})$$

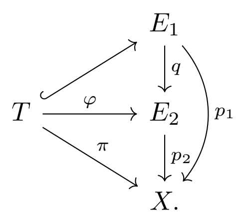
is finite, and that the induced mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to the normalization of the image is a finite covering space. To construct an extension that is finite over the larger analytic space $T(F_{-1}\mathcal{M})$, we can now argue as follows.

Proof of Corollary 14.4. The closure of the image of φ is an analytic subset of $T(F_{-1}\mathcal{M})$ according to Theorem 14.3. Let W denote its normalization; as we have just seen, the induced mapping from $E_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to W is a finite covering space over its image. The *Fortsetzungssatz* of Grauert and Remmert [GPR94, VI.3.3] shows that it extends in a unique way to a finite branched covering of W. If we define $\widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}})$ to be the analytic space in this covering, and $\tilde{\varphi}\colon \widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}}) \to T(F_{-1}\mathcal{M})$ to be the induced holomorphic mapping, then all the requirements are fulfilled. The final assertion follows from the uniqueness statement in [GPR94, VI.3.3]. □

Now let us revisit the relationship between the two constructions.

Proof of Corollary 14.5. Recall that we defined $\widetilde{E}(\mathcal{H}_{\mathbb{Z}})$ as the limit, over all $K \in \mathbb{N}$, of the normal analytic spaces $\widetilde{E}_{\leq K}(\mathcal{H}_{\mathbb{Z}})$; it is therefore itself normal and contains the complex manifold $E(\mathcal{H}_{\mathbb{Z}})$ an a dense open subset. Since the same is true for $\overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$, it is clear that there can be at most one holomorphic mapping with the asserted properties; consequently, it is enough to construct such a mapping locally. After choosing suitable local coordinates, we may therefore assume that we are working in a neighborhood of the origin in Δ^n. The local irreducible components of the analytic space $\overline{E(\mathcal{H}_{\mathbb{Z}})}^{\nu}$ are the normalizations of the closed subsets $C(h)$, for $h \in H_{\mathbb{Z}}$. As we have seen during the proof of Proposition 20.4, each local irreducible component of $\widetilde{E}(\mathcal{H}_{\mathbb{Z}})$ maps holomorphically to some $C(h)$; the desired result follows from this by passing the normalizations. $\square$

22. Auxiliary results about a class of covering spaces. In this section, we consider the following general situation. Let X be a complex manifold, and suppose that we have a surjective mapping $q\colon E_1 \to E_2$ between two holomorphic vector bundles on X. We assume that E_1 has a hermitian metric h_1, and we endow E_2 with the induced hermitian metric h_2. Lastly, we shall assume that we have a complex submanifold $T \hookrightarrow E_1$, with the property that $\pi\colon T \to X$ is a (possibly disconnected) covering space. We denote by $\varphi\colon T \to E_2$ the induced holomorphic mapping; see also the diagram below.



For any real number $r > 0$, we denote by $B_r(E_j)$ the closed tube of radius r around the zero section in the vector bundle E_j. We assume the following condition:

(22.1) For every $r > 0$, there exists $R > 0$ with $T \cap \varphi^{-1}\bigl(B_r(E_2)\bigr) \subseteq B_R(E_1)$.

Lemma 22.2. *If* (22.1) *holds, then* $\varphi\colon T \to E_2$ *is a finite mapping.*

Proof. Recall that a holomorphic mapping is called *finite* if it is closed and has finite fibers [GPR94, I.2.4]; an equivalent condition is that the mapping is proper and has finite fibers. Let us first show that φ is proper. Given an arbitrary compact subset $K \subseteq E_2$, we can find $r > 0$ such that $K \subseteq B_r(E_2)$. According to (22.1), the preimage $\varphi^{-1}(K)$ is contained in $B_R(E_1)$ for some $R > 0$; because it is closed, it must be compact. Now it is easy to show that φ has finite fibers: the fibers of φ are contained in the fibers of π, which are discrete because $\pi\colon T \to X$ is a covering space; being compact, they must therefore be finite sets. $\square$

Corollary 22.3. *The image of* φ *is an analytic subset of* E_2.

Proof. This follows from the finite mapping theorem [GPR94, I.8.2], which is a special case of Remmert's proper mapping theorem. $\square$

Of course, φ is still a local biholomorphism; but the images of different sheets of the covering space T may intersect in E_2. This picture suggests the following result about the normalization of $\varphi(T)$.

Lemma 22.4. *The normalization of $\varphi(T)$ is a complex manifold, and the induced mapping from T to the normalization is a finite covering space.*

Proof. Let Y denote the normalization of $\varphi(T)$; for the construction, see [GPR94, I.14.9]. Because T is a complex manifold, we obtain a factorization

$$\begin{array}{ccccc} & & \overset{\varphi}{\frown} & & \\ T & \xrightarrow{f} & Y & \xrightarrow{\nu} & E_2 \\ & \searrow^{\pi} & & & \downarrow{\scriptstyle p_2} \\ & & & & X; \end{array}$$

note that f is again a finite mapping. According to [GPR94, I.13.1], Y is locally irreducible; now [GPR94, I.10.14] implies that $f\colon T \to Y$ is open. Since $\pi\colon T \to X$ is a covering space, this is enough to guarantee that Y is again a complex manifold, and that $f\colon T \to Y$ is a finite covering space. $\square$

E. Asymptotic behavior of sequences of Hodge classes

23. Introduction. Cattani, Deligne, and Kaplan prove Theorem 17.1 by using the theory of degenerating variations of Hodge structure, especially the multi-variable SL(2)-orbit theorem [CKS86]. Roughly speaking, the argument is by induction on the dimension of the smallest analytic subset of Δ^n containing the image of the sequence $z(m)$; the description of period mappings in [CKS86] lends itself very well to such an approach. A subtle point is that the assumption $h(m) \in H_{\mathbb{Z}}$ is needed in many places: it ensures that certain terms that would only be going to zero when $h(m) \in H_{\mathbb{R}}$ are actually equal to zero after passing to a subsequence.

We are going to prove Theorem 17.4 by adapting the method of Cattani, Deligne, and Kaplan; to do that, we have to deal with the problem that the sequence of error terms $b(m)$ is no longer exponentially small. The proof is contained in §24 to §30; rather than giving an abstract description of the argument here, I have decided to include, in §25, a careful discussion of the special case $n = 1$. All the interesting features of the general case are already present there, but without the added complications of having several nilpotent operators $N_1, \dots, N_n$ and several variables $z_1(m), \dots, z_n(m)$. Hopefully, this will help the reader understand the proof in the general case.

24. Properties of harmless sequences. This section contains a few elementary results about harmless sequences that will be useful later. Fix a sequence $y(m) \in \mathbb{R}^n$ with the property that $y_1(m) \geq y_2(m) \geq \cdots \geq y_n(m) \geq 1$ for all $m \in \mathbb{N}$; to simplify the notation, we put $y_{n+1}(m) = 1$. Of course, we can always get into this situation by reordering the coordinates in Definition 17.3, so there is no loss of generality. First, we prove the following structure theorem for harmless sequences.

Proposition 24.1. *A harmless sequence can always be written in the form*

$$b(m) = b_0(m) + b_1(m) + \cdots + b_n(m),$$

where $b_k(m) \in \ker N_1 \cap \cdots \cap \ker N_k$ and $\|b_k(m)\|$ is in $O(e^{-\alpha y_{k+1}(m)})$.

In other words, $b_0(m)$ is of size $e^{-\alpha y_1(m)}$; $b_1(m)$ is in the kernel of N_1 and of size $e^{-\alpha y_2(m)}$; and so on, down to $b_n(m)$, which is in the kernel of all the N_j and bounded. The proof is based on the following simple result from linear algebra.

Lemma 24.2. *Let $T\colon V \to V$ be a linear operator on a finite-dimensional vector space. Then every $v \in V$ can be written in the form $v = v_0 + v_1$, where $Tv_1 = 0$ and $\|v_0\| \leq C\|Tv\|$, for a constant C that depends only on V, T, and $\|-\|$.*

Proof. Recall that $\|-\|$ comes from an inner product on V. By projecting to $\ker T$, we get $v = v_0 + v_1$, with $Tv_1 = 0$ and $v_0 \perp \ker T$. In particular, $Tv = Tv_0$. Now

$$T\colon (\ker T)^{\perp} \to \operatorname{im} T$$

is an isomorphism, and therefore has an inverse S. We then get

$$\|v_0\| = \|S(Tv)\| \leq C\|Tv\|,$$

for a constant C that depends only on V, T, and the choice of norm. □

Proof of Proposition 24.1. Since $N_1, \dots, N_n$ commute with each other, we can use the lemma and induction. First, we apply the lemma to $V = H_{\mathbb{C}}$ and $T = N_1$; this gives $b(m) = b_0(m) + b'(m)$, with $N_1 b'(m) = 0$ and $\|b_0(m)\|$ in $O(e^{-\alpha y_1(m)})$. In the next step, we apply the lemma to $V = \ker N_1$ and $T = N_2$ to decompose the sequence $b'(m) = b_1(m) + b''(m)$, remembering that the norm of $N_2 b'(m) = N_2 b(m) - N_2 b_0(m)$ is still in $O(e^{-\alpha y_2(m)})$; etc. □

We also need to know that harmless sequences are preserved when we apply certain operators; this fact will be used during the proof of Proposition 20.2.

Lemma 24.3. *Let $\Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F$ be the normal form of a period mapping. If $b(m) \in H_{\mathbb{C}}$ is harmless/exponentially small with respect to $\operatorname{Im} z(m)$, then so are*

$$e^{\sum z_j(m) N_j} b(m) \quad \textit{and} \quad e^{\Gamma(s(m))} b(m),$$

provided that $\operatorname{Im} z_1(m), \dots, \operatorname{Im} z_n(m)$ are going to infinity.

Proof. Both assertions are clear when $b(m)$ is exponentially small; let us therefore assume that $b(m) \in H_{\mathbb{C}}$ is a harmless sequence. Since the operator $e^{\sum z_j N_j}$ is polynomial in $z_1, \dots, z_n$, whereas $\|N_j b(m)\|$ is in $O(e^{-\alpha \operatorname{Im} z_j(m)})$, it is not hard to see that $e^{\sum z_j(m) N_j} b(m)$ is again harmless (for a slightly smaller value of α). On the other hand, the operator $e^{\Gamma(s)}$ is holomorphic on Δ^n, and therefore bounded; moreover, Proposition 15.3 shows that the norm of

$$N_j e^{\Gamma(s(m))} b(m) - e^{\Gamma(s(m))} N_j b(m)$$

is bounded by a constant multiple of $|s_j(m)| = e^{-2\pi \operatorname{Im} z_j(m)}$. This is clearly enough to conclude that $e^{\Gamma(s(m))} b(m)$ is a harmless sequence, too. □

Next, we consider the case when the sequence $h(m)$ belongs to certain subspaces. For any subset $J \subseteq \{1, \dots, n\}$, we let $W(J)$ denote the weight filtration of the cone

$$C(J) = \left\{ \sum_{j \in J} a_j N_j \;\middle|\; a_j > 0 \text{ for every } j \in J \right\}.$$

We would like to know that when $h(m) \in W_w(J)$, we can also take $b(m) \in W_w(J)$. This requires the following assumption on the period mapping.

Definition 24.4. Let $J \subseteq \{1, \dots, n\}$ be a subset of the index set. We say that $\Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F$ is a *nilpotent orbit in the variables* $\{s_j\}_{j \in J}$ if

$$\frac{\partial \Gamma(s)}{\partial s_j} = 0 \quad \text{for every } j \in J;$$

in other words, if $\Gamma(s)$ does not depend on the variables $\{s_j\}_{j\in J}$.

The point is that $\Gamma(s)$ then commutes with N_j for $j \in J$ (by Proposition 15.3), and therefore preserves the weight filtration $W(J)$. Note that nilpotent orbits in the usual sense are the special case when $J = \{1, \dots, n\}$.

Lemma 24.5. *Suppose that $h(m) \equiv b(m) \mod \Phi^0\bigl(z(m)\bigr)$, with $b(m) \in H_{\mathbb{C}}$ harmless/exponentially small. If $h(m) \in W_w(J)$, and if $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j\in J}$, then one can arrange that $b(m) \in W_w(J)$ as well.*

Proof. Define the auxiliary sequence of vectors

$$h'(m) = e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} h(m) \in H_{\mathbb{C}}.$$

In fact, we have $h'(m) \in W_w(J)$: the reason is that $\Gamma(s)$ commutes with N_j for $j \in J$ (by Proposition 15.3), and therefore preserves the weight filtration $W(J)$. Now consider Deligne's decomposition

$$H_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

of the mixed Hodge structure (W, F). Because each N_j is a $(-1,-1)$-morphism, $W_w(J)$ is a mixed Hodge substructure of (W, F), and therefore compatible with Deligne's decomposition. This means that if we define

$$h'(m)_{-1} = \sum_{p\leq -1} h'(m)^{p,q} \in \bigoplus_{p\leq -1} I^{p,q},$$

then $h'(m)_{-1} \in W_w(J)$; moreover, we have

$$h'(m) \equiv h'(m)_{-1} \mod F^0 = \bigoplus_{p\geq 0} I^{p,q}.$$

By construction, $h'(m)$ is congruent, modulo F^0, to the sequence of vectors

$$b'(m) = e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} b(m) \in H_{\mathbb{C}},$$

which shows that $h'(m)_{-1} = b'(m)_{-1}$. Now $b'(m)$, and therefore also its projection $b'(m)_{-1}$, is again harmless/exponentially small by Lemma 24.3. Consequently,

$$c(m) = e^{\sum z_j(m) N_j} e^{\Gamma(s(m))} h'(m)_{-1} \in W_w(J)$$

is a harmless/exponentially small sequence with $h(m) \equiv c(m) \mod \Phi^0\bigl(z(m)\bigr)$. □

We end this section with a word of caution. During the proof of Theorem 17.4, the fact that $b(m) \notin H_{\mathbb{R}}$ causes some trouble. If being harmless was preserved by the Hodge decomposition for the Hodge structure $\Phi\bigl(z(m)\bigr)$, we could easily arrange that $b(m) \in H_{\mathbb{R}}$. Unfortunately, this is not the case.

Example 24.6. Let $n = 1$, and consider the special case where $H_{\mathbb{C}} = I^{1,1} \oplus I^{-1,-1}$ splits over $\mathbb{R}$. If $b \in I^{-1,-1}$, then $Nb = 0$, and so the constant sequence $b(m) = b$ is harmless for any choice of $y(m)$. Now consider the Hodge decomposition of b in the pure Hodge structure of weight zero

$$e^{iyN} F^0 \oplus e^{-iyN} \overline{F^1}.$$

A short calculation gives

$$b = e^{iyN} \frac{N^+ b}{2iy} - e^{-iyN} \frac{N^+ b}{2iy} = \left(\frac{b}{2} + \frac{N^+ b}{2iy}\right) + \left(\frac{b}{2} - \frac{N^+ b}{2iy}\right)$$

and neither of the two components is harmless with respect to $y = y(m)$. The best one can say is that, even after applying the operator yN, they remain bounded; this is consistent with [Sch12a, Proposition 24.3].

25. Proof in the one-dimensional case. In this section, we prove Theorem 17.4 in the special case $n = 1$. This case is technically easier, because it avoids the complications coming from the presence of several variables and several nilpotent operators. Because many key features of the proof are the same as in the general case, it may be useful to understand them first in this special case.

Suppose then that $\big(z(m), h(m)\big) \in \mathbb{H} \times H_{\mathbb{Z}}(K)$ is a sequence of the type considered in Theorem 17.4. Fix an inner product on the space $H_{\mathbb{C}}$, and denote by $\|-\|$ the corresponding norm. By Proposition 24.1, we can arrange that

$$h(m) \equiv b(m) = b_0(m) + b_1(m) \mod \Phi^0\big(z(m)\big),$$

with $\|b_0(m)\|$ in $O(e^{-\alpha y(m)})$, and $b_1(m) \in \ker N$ bounded; passing to a subsequence, we may therefore assume that the limit

$$b = \lim_{m\to\infty} b(m) = \lim_{m\to\infty} b_1(m) \in \ker N$$

exists. Our goal is to prove that, after taking a further subsequence, $h(m)$ becomes constant, and that the constant value satisfies $Nh = 0$ and $h \equiv b \mod F^0$.

We first introduce some notation. Let $(W, \hat{F})$ denote the $\mathbb{R}$-split mixed Hodge structure canonically associated with (W, F) by the SL(2)-orbit theorem [CK89, Corollary 3.15]. If $Y \in \mathfrak{g}_{\mathbb{R}}$ denotes the corresponding splitting, the eigenspaces $E_\ell(Y)$ define a real grading of the weight filtration W, meaning that

$$W_k = \bigoplus_{\ell \le k} E_\ell(Y).$$

To simplify some of the arguments below, we shall choose the inner product on $H_{\mathbb{C}}$ in such a way that this decomposition is orthogonal. The most important tool in the proof will be the following sequence of real operators:

$$e(m) = \exp\left(\frac{1}{2}\log y(m)\cdot Y\right) \in \operatorname{End}(H_{\mathbb{R}})$$

Note that $e(m)$ acts as multiplication by $y(m)^{\ell/2}$ on the subspace $E_\ell(Y)$, and preserves the filtration $\hat{F}$. Because $[Y, N] = -2N$, we have

$$e(m)N = \frac{1}{y(m)} N e(m). \tag{25.1}$$

Since the sequence of real parts $x(m)$ is bounded, [CK89, Theorem 4.8] shows that

$$F_\sharp \underset{\text{def}}{=} e^{iN}\hat{F} = \lim_{m\to\infty} e(m)\Phi\big(z(m)\big) \in D. \tag{25.2}$$

The filtration $F_\sharp$ has two important properties: on the one hand, it belongs to D, and therefore defines a polarized Hodge structure of weight zero on $H_{\mathbb{C}}$; on the other hand, the pair $(W, F_\sharp)$ is a mixed Hodge structure.

We divide the proof of the theorem (in the case $n = 1$) into six steps; each of the six steps will appear again in a similar form during the proof of the general case.

Step 1. We prove that $b(m)$ and $h(m)$ are bounded with respect to the Hodge norm. This will also show that $\|h(m)\|$ grows at most polynomially in $y(m)$.

Lemma 25.3. *The two sequences* $\|b(m)\|_{\Phi(z(m))}$ *and* $\|h(m)\|_{\Phi(z(m))}$ *are bounded.*

Proof. Recall that the Hodge norm of a vector $h \in H_{\mathbb{C}}$ with respect to the polarized Hodge structure $\Phi(z) \in D$ is defined as

$$\|h\|^2_{\Phi(z)} = \sum_{p\in\mathbb{Z}} \|h^{p,-p}\|^2_{\Phi(z)} = \sum_{p\in\mathbb{Z}} (-1)^p Q\big(h^{p,-p}, \overline{h^{p,-p}}\big),$$

where $h = \sum h^{p,-p}$ is the Hodge decomposition of h in $\Phi(z)$. We begin the proof by observing that the sequence $e(m)b(m) = e(m)b_0(m) + e(m)b_1(m)$ is bounded, for the following reason. On the one hand, $b_1(m) \in \ker N \subseteq W_0$ implies that

$$b_1(m) = \sum_{\ell \le 0} b_1(m)_\ell \in \bigoplus_{\ell \le 0} E_\ell(Y);$$

consequently, the boundedness of $b_1(m)$ implies the boundedness of

$$e(m)b_1(m) = \sum_{\ell \le 0} y(m)^{\ell/2} b_1(m)_\ell.$$

On the other hand, the term $e(m)b_0(m)$ is going to zero, because $\|b_0(m)\|$ is in $O(e^{-\alpha y(m)})$, whereas $e(m)$ grows at most polynomially in $y(m)$. Because $e(m)$ is a real operator, we have

$$\|b(m)\|_{\Phi(z(m))} = \|e(m)b(m)\|_{e(m)\Phi(z(m))},$$

which is bounded by virtue of (25.2). Now let

$$h(m) = \sum_{p\in\mathbb{Z}} h(m)^{p,-p}$$

denote the Hodge decomposition of $h(m)$ in the Hodge structure $\Phi(z(m))$. The difference $h(m) - b(m)$ is an element of $\Phi^0(z(m))$, and for $p \le -1$,

$$\|h(m)^{p,-p}\|_{\Phi(z(m))} = \|b(m)^{p,-p}\|_{\Phi(z(m))}$$

is bounded. Recalling that $h(m) \in H_{\mathbb{Z}}(K)$, we now have

$$\begin{aligned} \|h(m)\|^2_{\Phi(z(m))} &= Q\big(h(m), h(m)\big) + \sum_{p\neq 0} \big(1 - (-1)^p\big) \|h(m)^{p,-p}\|^2_{\Phi(z(m))} \\ &\le K + 4\|b(m)\|^2_{\Phi(z(m))}, \end{aligned}$$

and so the Hodge norm of $h(m)$ is bounded, too. □

Step 2. Next, we reduce the problem to the case of a nilpotent orbit. Lemma 25.3 gives the boundedness of the sequence $e(m)h(m)$. Since $e(m)^{-1}$ is polynomial in $y(m)$, it follows that $\|h(m)\|$ grows at most like a fixed power of $y(m)$. We have

$$e^{z(m)N} e^{-\Gamma(s(m))} e^{-z(m)N} \big(h(m) - b(m)\big) \in e^{z(m)N} F^0,$$

and by using Lemma 24.3 and the bounds on $\|h(m)\|$ and $\|b(m)\|$, we see that $h(m)$ is congruent to a harmless sequence modulo $e^{z(m)N}F^0$. We may therefore assume without loss of generality that $\Phi(z) = e^{zN}F$ is a nilpotent orbit. Note that we only have $\Phi(z) \in D$ when the imaginary part of $z \in \mathbb{H}$ is sufficiently large; this does not cause any problems because $y(m) = \operatorname{Im} z(m)$ is going to infinity anyway.

Step 3. We exploit the boundedness of $e(m)h(m)$ to prove that $h(m) \in W_0$. By passing to a subsequence, we can arrange that there is a limit

$$v = \lim_{m\to\infty} e(m)h(m) \in H_{\mathbb{R}}.$$

With respect to the eigenspace decomposition of Y, we have

$$e(m)h(m) = \sum_{\ell\in\mathbb{Z}} y(m)^{\ell/2} h(m)_\ell,$$

and so $h(m)_\ell$ is going to zero when $\ell \geq 1$, and is bounded when $\ell = 0$. Let $\ell \in \mathbb{Z}$ be the largest index such that $h(m)_\ell \neq 0$ along a subsequence. The projection from $E_\ell(Y)$ to gr_ℓ^W is an isomorphism, and because $h(m) \in H_{\mathbb{Z}}$, it follows that $h(m)_\ell$ lies in a discrete subset of $E_\ell(Y)$. This is only possible if $\ell \leq 0$, and hence $h(m) \in W_0$; moreover, the component $h(m)_0$ takes values in a finite set. After passing to a subsequence, we may therefore assume that

$$h(m) \equiv h_0 \mod W_{-1},$$

where $h_0 \in E_0(Y)$ is constant.

Step 4. We prove that $Nh_0 = 0$. Consider again the decomposition

$$b_1(m) = \sum_{\ell\leq 0} b_1(m)_\ell,$$

where $b_1(m)_\ell \in E_\ell(Y) \cap \ker N$; note that all summands are bounded, and that only terms with $\ell \leq 0$ appear because $b_1(m) \in \ker N \subseteq W_0$. Consequently,

$$e(m)b_1(m) = b_1(m)_0 + \sum_{\ell\leq -1} y(m)^{\ell/2} b_1(m)_\ell$$

has the same limit as $b_1(m)_0 \in E_0(Y) \cap \ker N$. If we now look back at

$$e(m)h(m) \equiv e(m)b_0(m) + e(m)b_1(m) \mod e(m)\Phi^0\big(z(m)\big),$$

we find that all terms converge individually, and hence that

$$v = \lim_{m\to\infty} e(m)h(m) \equiv \lim_{m\to\infty} b_1(m)_0 \mod F_\sharp^0.$$

To show that $Nv = 0$, we apply the following version of [CDK95, Proposition 3.10] to the $\mathbb{R}$-split mixed Hodge structure $(W, \hat{F})$, recalling that $F_\sharp = e^{iN}\hat{F}$. This result is, in a sense, an asymptotic form of Theorem 17.4.

Lemma 25.4. *Let (W, F) be an $\mathbb{R}$-split mixed Hodge structure on $H_{\mathbb{C}}$, and let N be a real $(-1,-1)$-morphism of (W, F). If a vector $h \in W_{2\ell} \cap H_{\mathbb{R}}$ satisfies*

$$h \equiv b \mod e^{iN} F^\ell$$

for some $b \in E_{2\ell}(Y) \cap \ker N$, then $h \in E_{2\ell}(Y) \cap \ker N$.

Proof. If we apply e^{-iN} to both sides and use the fact that $Nb = 0$, we get

$$e^{-iN} h \in E_{2\ell}(Y) + W_{2\ell} \cap F^\ell.$$

Thus $Nh \in W_{2\ell-2} \cap F^\ell \cap H_{\mathbb{R}}$, which can only happen if $Nh = 0$, because (W, F) is a mixed Hodge structure. But then $Yh - 2\ell h \in W_{2\ell-1} \cap F^\ell \cap H_{\mathbb{R}}$, and so $Yh = 2\ell h$ for the same reason. Alternatively, one can use the decomposition into the subspaces $I^{p,q} = W_{p+q} \cap F^p \cap \overline{F^q}$, which is preserved by Y. □

Consequently, $v \in E_0(Y) \cap \ker N$. We can now project the congruence

$$e(m)h(m) \equiv h_0 \mod W_{-1}$$

to the subspace $E_0(Y)$ to conclude that $v = h_0$, and hence that $Nh_0 = 0$.

Step 5. Next, we shall argue that $Nh(m) = 0$. Since $Nh_0 = 0$, we already know that $Nh(m) \in W_{-3}$. In addition, we have $Nb_1(m) = 0$, and so

$$e(m)Nh(m) \equiv e(m)Nb_0(m) \mod Ne(m)\Phi^0\big(z(m)\big);$$

here we have used (25.1) to interchange N and $e(m)$.

We claim that $\|e(m)Nh(m)\|$ is bounded by a constant multiple of $\|e(m)Nb_0(m)\|$, and therefore in $O(e^{-\alpha y(m)})$. If not, then the ratios

$$\frac{\|e(m)Nb_0(m)\|}{\|e(m)Nh(m)\|}$$

would be going to zero; after passing to a subsequence, the unit vectors

$$\frac{e(m)Nh(m)}{\|e(m)Nh(m)\|} \in W_{-3} \cap H_{\mathbb{R}}$$

would then converge to a unit vector in $W_{-3} \cap NF_\sharp^0 \cap H_{\mathbb{R}} \subseteq W_{-3} \cap F_\sharp^{-1} \cap H_{\mathbb{R}}$; but this is not possible because $(W, F_\sharp)$ is a mixed Hodge structure. Consequently, $\|e(m)Nh(m)\|$ is in $O(e^{-\alpha y(m)})$; because $e(m)^{-1}$ only grows like a power of $y(m)$, the same is true for the norm of $Nh(m)$. But these vectors lie in a discrete set, and so $Nh(m) = 0$ after passing to a subsequence.

Step 6. To finish the proof, we have to show that $h(m)$ is bounded. Choose a point $w \in \mathbb{H}$ with sufficiently large imaginary part to ensure that $\Phi(w) = e^{wN}F \in D$; this filtration defines a polarized Hodge structure of weight zero on $H_{\mathbb{C}}$. Since $\Phi(z)$ is a nilpotent orbit and $Nh(m) = 0$, we then have

$$h(m) = e^{wN - z(m)N}h(m) \equiv e^{wN - z(m)N}b_0(m) + b_1(m) \mod \Phi^0(w);$$

note that both terms on the right-hand side are bounded. This relation shows that, with respect to the Hodge structure $\Phi(w)$, all the Hodge components $h(m)^{p,-p}$ with $p \leq -1$ are bounded. Because $h(m) \in H_{\mathbb{Z}}(K)$, we conclude as in Lemma 25.3 that $h(m)$ is bounded in the Hodge norm for $\Phi(w)$, and therefore bounded. After passing to a subsequence, the sequence $h(m)$ becomes constant. We now have

$$h(m) \equiv b(m) \mod F^0,$$

and by taking limits, we obtain the desired relation $h \equiv b \mod F^0$. This completes the proof of Theorem 17.4 in the case $n = 1$.

26. Setup in the general case. In the remaining sections, we shall prove Theorem 17.4 in general, by adapting the method in [CDK95, Section 4] to our setting. The proof uses many results from the theory of degenerating variations of Hodge structures; these will be introduced in the appropriate places. Thoughout the discussion, we fix an inner product on $H_{\mathbb{C}}$, and denote by $\|-\|$ the corresponding norm; we shall make a more specific choice later on.

To explain the idea of the proof, let us first consider a sequence $z(m) \in \mathbb{H}^n$, with the property that the real parts $x_j(m) = \operatorname{Re} z_j(m)$ are bounded, and the imaginary parts $y_j(m) = \operatorname{Im} z_j(m)$ are going to infinity. In the course of the argument, it will often be necessary to pass to a subsequence; to reduce clutter, we shall use the same notation for the subsequence. A new feature of the general case is that we no longer have a unique scale on which we can measure the rate of growth of a sequence; the reason is that $y_1(m), \dots, y_n(m)$ may be going to infinity at different rates. The most efficient way to deal with this problem is as follows.

Following [CDK95, (4.1.3)], we expand the sequence $z(m)$ according to the rate of growth of its imaginary parts. After passing to a subsequence, we can find an integer $1 \leq d \leq n$ and an $n \times d$-matrix A with nonnegative real entries, such that

$$z(m) = iAt(m) + w(m). \tag{26.1}$$

Here $w(m) \in \mathbb{H}^n$ is a convergent sequence with $\Phi\bigl(\lim_{m\to\infty} w(m)\bigr) \in D$; and the sequence of vectors $t(m) \in \mathbb{R}^d$ has the property that all the ratios

$$\frac{t_1(m)}{t_2(m)}, \frac{t_2(m)}{t_3(m)}, \dots, \frac{t_d(m)}{t_{d+1}(m)}$$

are going to infinity. (To avoid having to deal with special cases, we always define $t_{d+1}(m) = 1$.) Moreover, we can partition the index set

$$\{1, 2, \dots, n\} = J_1 \sqcup J_2 \sqcup \dots \sqcup J_d$$

in such a way that $a_{j,k} \neq 0$ if and only if $j \in J_1 \sqcup \dots \sqcup J_k$. By construction, $|s_j(m)|$ is in $O(e^{-\alpha t_k(m)})$ for every $j \in J_k$. We define new operators

$$T_k = \sum_{j=1}^n a_{j,k} N_j \in C(J_1 \sqcup \dots \sqcup J_k) \tag{26.2}$$

and have the identity

$$\sum_{j=1}^n z_j(m) N_j = \sum_{k=1}^d i t_k(m) T_k + \sum_{j=1}^n w_j(m) N_j,$$

Now suppose that $b(m) \in H_{\mathbb{C}}$ is a harmless sequence with respect to $\operatorname{Im} z(m)$. By definition, there is some $\alpha > 0$ such that

$$\|b(m)\| + \sum_{j=1}^n e^{\alpha y_j(m)} \|N_j b(m)\|$$

is bounded; it is easy to see that the same is true (with a different $\alpha > 0$) for

$$\|b(m)\| + \sum_{k=1}^d e^{\alpha t_j(m)} \|T_j b(m)\|.$$

From now on, we shall use the expression *harmless* to refer to sequences with this property. We also say that a sequence $b(m)$ is *exponentially small* if $\|b(m)\|$ is in $O(e^{-\alpha t_1(m)})$ for some $\alpha > 0$; of course, all exponentially small sequences are harmless. We are going to prove the following version of Theorem 17.4.

Theorem 26.3. *Suppose we are given a sequence of points*

$$\bigl(z(m), h(m)\bigr) \in \mathbb{H}^n \times H_{\mathbb{Z}}(K),$$

where $z(m) = iAt(m) + w(m)$ is as above, and $h(m) \equiv b(m) \mod \Phi^0\bigl(z(m)\bigr)$ for a sequence of vectors $b(m) \in H_{\mathbb{C}}$ that is harmless with respect to $t(m)$. Then after passing to a subsequence, $h(m)$ becomes constant, and $T_1 h(m) = \dots = T_d h(m) = 0$.

Proof that Theorem 26.3 implies Theorem 17.4. Suppose we are given a sequence of points $\bigl(z(m), h(m)\bigr)$ as in Theorem 17.4. After replacing it by a subsequence, we may clearly assume that $b(m)$ converges to a vector $b \in H_{\mathbb{C}}$; since $b(m)$ was harmless, we have $N_1 b = \dots = N_n b = 0$. As explained above, we can choose a subsequence along which $z(m) = iAt(m) + w(m)$; after passing to a further subsequence, $h(m)$ is constant and $T_1 h(m) = \dots = T_d h(m) = 0$. Let $h \in H_{\mathbb{Z}}(K)$ denote the constant value. We have

$$T_d h = \sum_{j=1}^{n} a_{j,d} N_j h = 0$$

for positive real numbers $a_{1,d}, \dots, a_{n,d}$. Since $N_j h \in H_{\mathbb{Q}}$, we can then obviously find positive integers $a_1, \dots, a_n$ with the property that $a_1 N_1 h + \dots + a_n N_n h = 0$. At the same time,

$$\lim_{m\to\infty} e^{-\sum z_j(m) N_j} h(m) = \lim_{m\to\infty} e^{-\sum w_j(m) N_j} h = e^{-\sum w_j N_j} h,$$

where $w \in \mathbb{C}^n$ is the limit of the sequence $w(m)$. Finally,

$$e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} h(m) \equiv e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j} b(m) \mod F^0,$$

and since Γ vanishes at the origin and $b(m)$ is harmless, the left-hand side converges to $e^{-\sum w_j N_j} h$ and the right-hand side to b. □

The proof of Theorem 26.3 is organized as follows. In §27, we introduce a common $\mathbb{Z}^d$-grading for the weight filtrations of $T_1, \dots, T_d$, and a corresponding sequence of operators $e(m) \in \operatorname{End}(H_{\mathbb{R}})$, and show that the boundedness of $Q\bigl(h(m), h(m)\bigr)$ is equivalent to the boundedness of the sequence $e(m)h(m) \in H_{\mathbb{R}}$. In §28, we show that the subquotients of the weight filtration $W(T_1)$ again satisfy the assumptions of the theorem. In §29, we explain how the boundedness of $e(m)h(m)$ can be used to control the position of the sequence $h(m)$ with respect to the above $\mathbb{Z}^d$-grading. The actual proof of the theorem will be given in §30.

27. Boundedness results. The purpose of this section is to translate the boundedness of $Q\bigl(h(m), h(m)\bigr)$ into a more manageable condition. Since the existence of an integral structure is not important here, we consider an arbitrary sequence of real vectors $h(m) \in H_{\mathbb{R}}$, subject only to the condition that

(27.1) $$h(m) \equiv b(m) \mod \Phi^0\bigl(z(m)\bigr)$$

for a harmless sequence $b(m) \in H_{\mathbb{C}}$. By Proposition 24.1, we have

$$b(m) = b_0(m) + b_1(m) + \dots + b_d(m),$$

where $b_k(m) \in \ker T_1 \cap \dots \cap \ker T_k$ is in $O(e^{-\alpha t_{k+1}(m)})$. By passing to a subsequence, we can also arrange that $b_d(m)$ converges to an element of $H_{\mathbb{C}}$.

Proposition 27.2. *Given* (27.1)*, the following statements are equivalent:*

1. *The sequence $Q\bigl(h(m), h(m)\bigr)$ is bounded*
2. *The sequence of Hodge norms $\|h(m)\|_{\Phi(z(m))}$ is bounded.*
3. *The sequence $e(m)h(m) \in H_{\mathbb{R}}$ is bounded.*

If any of them is satisfied, $\|h(m)\|$ is in $O(t_1(m)^N)$ for some $N \in \mathbb{N}$.

The proof is based on the existence of a $\mathbb{Z}^d$-grading on $H_{\mathbb{C}}$ with good properties. Before we can define it, we have to recall a few results from the theory of degenerating variations of Hodge structure. Let $W^k = W(T_k)$ denote the weight filtration of the nilpotent operator T_k; it agrees with that of the cone $C(J_1 \sqcup \dots \sqcup J_k)$, and is therefore defined over $\mathbb{Q}$, even though T_k is only defined over $\mathbb{R}$. The multi-variable SL(2)-orbit theorem [CK89, Theorem 4.3] associates with

$$\big(W, F, T_1, \dots, T_d\big)$$

a sequence of mutually commuting splittings $Y_1, \dots, Y_d \in \operatorname{End}(H_{\mathbb{R}})$. Their common eigenspaces define a real $\mathbb{Z}^d$-grading

$$H_{\mathbb{C}} = \bigoplus_{\ell \in \mathbb{Z}^d} H_{\mathbb{C}}^{(\ell_1, \dots, \ell_d)} \tag{27.3}$$

of the vector space $H_{\mathbb{C}}$ (and also of $H_{\mathbb{R}}$), with the property that

$$W_w^k = \bigoplus_{\ell_1 + \dots + \ell_k \leq w} H_{\mathbb{C}}^{(\ell_1, \dots, \ell_d)}.$$

Given a vector $h \in H_{\mathbb{C}}$, we denote its component in the subspace $H_{\mathbb{C}}^{(\ell_1, \dots, \ell_d)}$ by the symbol $h^{(\ell_1, \dots, \ell_d)}$. To simplify some arguments below, we shall assume that the norm $\|-\|$ comes from an inner product for which the decomposition is orthogonal. As in the one-variable case, we then define a sequence of operators

$$e(m) = \exp\left(\frac{1}{2}\sum_{k=1}^{d} t_k(m) Y_k\right) \in \operatorname{End}(H_{\mathbb{R}}); \tag{27.4}$$

note that $e(m)$ acts on the subspace $H_{\mathbb{C}}^{(\ell_1, \dots, \ell_d)}$ as multiplication by

$$\begin{aligned} &t_1(m)^{\ell_1/2} t_2(m)^{\ell_2/2} \cdots t_d(m)^{\ell_d/2} \\ &\qquad = \left(\frac{t_1(m)}{t_2(m)}\right)^{\ell_1/2} \left(\frac{t_2(m)}{t_3(m)}\right)^{(\ell_1+\ell_2)/2} \cdots \left(\frac{t_d(m)}{t_{d+1}(m)}\right)^{(\ell_1+\dots+\ell_d)/2} \end{aligned} \tag{27.5}$$

What makes these operators useful is that the filtrations $e(m)\Phi\big(z(m)\big)$ have a well-defined limit, which is again a polarized Hodge structure. In other words,

$$F_\sharp = \lim_{m\to\infty} e(m)\Phi\big(z(m)\big) \in D. \tag{27.6}$$

This is explained in [CK89, Theorem 4.8], and depends on the fact that $w(m)$ is bounded and all the ratios $t_k(m)/t_{k+1}(m)$ are going to infinity. This is one reason for using an expansion of the form $z(m) = iAt(m) + w(m)$.

The multi-variable SL(2)-orbit theorem gives some additional information about the filtration $F_\sharp$. According to [CK89, Theorem 4.3], there are nilpotent operators $\hat{T}_k \in C(J_1 \sqcup \dots \sqcup J_k)$, with the property that

$$[Y_j, \hat{T}_k] = \begin{cases} -2\hat{T}_k & \text{if } j = k, \\ 0 & \text{otherwise;} \end{cases}$$

note that $\hat{T}_1 = T_1$. In this notation, each of the d pairs

$$\big(W^k, e^{-i(\hat{T}_1 + \dots + \hat{T}_k)} F_\sharp\big)$$

defines an $\mathbb{R}$-split mixed Hodge structure on $H_{\mathbb{C}}$, whose associated grading is given by $Y_1 + \dots + Y_k$ [CK89, Theorem 4.3]. In particular, every $(W^k, F_\sharp)$ is itself a mixed Hodge structure; this fact will be important later.

We now turn to the proof of Proposition 27.2. As in the one-variable case, we first study the effect of the operator $e(m)$ on the harmless sequence $b(m)$.

Lemma 27.7. *Suppose that $b(m) \in H_{\mathbb{C}}$ is a harmless sequence. Then*

$$\lim_{m\to\infty} e(m)b(m) = \lim_{m\to\infty} b_d(m)^{(0,\dots,0)},$$

and the limit belongs to $H_{\mathbb{C}}^{(0,\dots,0)} \cap \ker T_1 \cap \dots \cap \ker T_d$.

Proof. Since $b(m)$ is harmless with respect to $t(m)$, it is easy to see that $e(m)b(m)$ is bounded. Indeed, we have

$$e(m)b(m) = e(m)b_0(m) + e(m)b_1(m) + \dots + e(m)b_d(m).$$

Now $\|b_k(m)\|$ is in $O(e^{-\alpha t_{k+1}(m)})$, and so the same is true for each of the components in the decomposition

$$b_k(m) = \sum_{\ell\in\mathbb{Z}^d} b_k(m)^{(\ell_1,\dots,\ell_d)}.$$

On the other hand, $b_k(m)$ is in $\ker T_1 \cap \dots \cap \ker T_k \subseteq W_0^1 \cap \dots \cap W_0^k$; this means that $b_k(m)^{(\ell_1,\dots,\ell_d)} = 0$ unless $\ell_1 \leq 0$, $\ell_1 + \ell_2 \leq 0$, and so on up to $\ell_1 + \dots + \ell_k \leq 0$. It follows from this and (27.5) that

$$e(m)b_k(m) = \sum_{\ell\in\mathbb{Z}^d} t_1(m)^{\ell_1/2} \cdots t_d(m)^{\ell_d/2} \cdot b_k(m)^{(\ell_1,\dots,\ell_d)}$$

is going to zero for $k = 0, \dots, d-1$, and converges for $k = d$. This implies the asserted formula for the limit. □

Proof of Proposition 27.2. By the previous lemma, the sequence $e(m)b(m)$ converges, and so

$$\|b(m)\|_{\Phi(z(m))} = \|e(m)b(m)\|_{e(m)\Phi(z(m))}$$

is bounded by virtue of (27.6). With respect to the Hodge structure $\Phi\bigl(z(m)\bigr)$,

$$\overline{h(m)^{-p,p}} = h(m)^{p,-p} = b(m)^{p,-p}$$

for every $p \leq -1$. This gives us a bound on the difference

$$\|h(m)\|^2_{\Phi(z(m))} - Q\bigl(h(m), h(m)\bigr) = \sum_{p\neq 0} \bigl(1 - (-1)^p\bigr) \|h(m)^{p,-p}\|^2_{\Phi(z(m))};$$

the boundedness of $Q\bigl(h(m), h(m)\bigr)$ is therefore equivalent to the boundedness of $\|h(m)\|_{\Phi(z(m))}$. Because we also have

$$\|e(m)h(m)\|_{e(m)\Phi(z(m))} = \|h(m)\|_{\Phi(z(m))},$$

both conditions are equivalent to the boundedness of the sequence $e(m)h(m)$. The last assertion follows from the fact that the operator $e(m)^{-1}$ depends polynomially on $t_1(m), \dots, t_d(m)$. □

28. Mechanism of the induction. The proof of Theorem 26.3 is by induction on $d \geq 1$. One situation where we can potentially apply the inductive hypothesis is for a subquotient of the form

$$\tilde{H}_{\mathbb{C}} = \operatorname{gr}_{\ell_1}^{W^1} = W_{\ell_1}^1 / W_{\ell_1 - 1}^1.$$

In this section, we show that if the period mapping $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$, in the sense of Definition 24.4, the quotient again supports a polarized variation of integral Hodge structure of weight ℓ_1.

Denote by $\tilde{F}$ the filtration on $\tilde{H}_{\mathbb{C}}$ induced by F. We also write $\tilde{N}_j$, $\tilde{T}_k$, and $\tilde{Y}_k$ for the operators induced by N_j, T_k, and Y_k respectively, and $\tilde{W}^k$ for the filtration induced by W^k; then

$$\tilde{W}^k = W(\tilde{T}_k)[-\ell_1] = W(\tilde{T}_2, \dots, \tilde{T}_k)[-\ell_1]$$

because W^k is the relative weight filtration of T_k on W^1 by [CK89, Theorem 2.9]. By construction, the operators $\tilde{T}_1$ and $\tilde{N}_j$ with $j \in J_1$ are equal to zero; moreover, $\tilde{Y}_1$ is multiplication by the integer ℓ_1. The operators $\tilde{Y}_2, \dots, \tilde{Y}_d$ define a $\mathbb{Z}^{d-1}$-grading on $\tilde{H}_{\mathbb{C}}$, which is compatible with the $\mathbb{Z}^d$-grading on $H_{\mathbb{C}}$; in fact, the projection

$$H_{\mathbb{C}}^{(\ell_1, \ell_2, \dots, \ell_d)} \to \tilde{H}_{\mathbb{C}}^{(\ell_2, \dots, \ell_d)}$$

is an isomorphism. As in (27.4), we define a sequence of operators

$$\tilde{e}(m) = \exp\left(\frac{1}{2} \sum_{k=2}^{d} t_k(m) \tilde{Y}_k\right) \in \operatorname{End}(\tilde{H}_{\mathbb{R}}).$$

Finally, let $\|-\|$ denote the norm on $\tilde{H}_{\mathbb{C}}$ induced by the isomorphism $E_{\ell_1}(Y_1) \simeq \tilde{H}_{\mathbb{C}}$.

Proposition 28.1. *Suppose that $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$. Then the induced period mapping*

$$\tilde{\Phi}(z) = e^{\sum_{j \notin J_1} z_j \tilde{N}_j} e^{\tilde{\Gamma}(s)} \tilde{F}$$

defines a polarizable variation of integral Hodge structure of weight ℓ_1 on $\tilde{H}_{\mathbb{C}}$.

Proof. We first explain how $\tilde{\Gamma}(s)$ is defined. For $j \in J_1$, the operator $\Gamma(s)$ does not depend on s_j, and therefore commutes with N_j by Proposition 15.3. Consequently, $\Gamma(s)$ preserves the weight filtration $W^1 = W(J_1)$, and induces an operator $\tilde{\Gamma}(s)$ on $\tilde{H}_{\mathbb{C}}$. By [CK89, Proposition 2.10], the pair

$$\left(W^1, e^{\sum_{j \notin J_1} z_j N_j} e^{\Gamma(s)} F\right)$$

is a mixed Hodge structure, polarized by the form Q and every element of the cone $C(J_1)$, in the sense of [CK89, Definition 1.16]. In particular, $\tilde{\Phi}(z)$ gives a Hodge structure of weight ℓ_1 on $\tilde{H}_{\mathbb{C}}$ for every $z \in \mathbb{H}^n$ with sufficiently large imaginary parts. To construct a polarization $\tilde{Q} \colon \tilde{H}_{\mathbb{Q}} \otimes \tilde{H}_{\mathbb{Q}} \to \mathbb{Q}$, we fix an arbitrary rational element $N \in C(J_1)$ and use the Lefschetz decomposition [CK89, (1.11)]

$$\tilde{H}_{\mathbb{Q}} = \bigoplus_{k \geq 0} N^k P_{\ell_1 + 2k}(N).$$

The decomposition is orthogonal with respect to the bilinear form induced by Q; if we define $\tilde{Q}$ on the subspace $N^k P_{\ell_1 + 2k}(N)$ by the formula

$$\tilde{Q}(h_1, h_2) = (-1)^k Q(h_1, N^{\ell_1} h_2),$$

then [CK89, Definition 1.16] shows that $\tilde{Q}$ polarizes the Hodge structure $\tilde{\Phi}(z)$. Since we get an integral structure $\tilde{H}_{\mathbb{Z}}$ by taking the image of $H_{\mathbb{Z}}$, it follows that $\tilde{\Phi}(z)$ is the period mapping of a variation of integral Hodge structure of weight ℓ_1 on $\tilde{H}_{\mathbb{C}}$, polarized by the form $\tilde{Q}$. □

We note that this construction reduces the value of d, in the following sense.

Corollary 28.2. *Notation being as above, $\tilde{\Phi}\big(z(m)\big)$ only depends on*

$$\tilde{z}(m) = iA\tilde{t}(m) + w(m),$$

where $\tilde{t}(m) = \big(0, t_2(m), \dots, t_d(m)\big)$, and A and $w(m)$ are as in (26.1).

Proof. Because $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j\in J_1}$, it is clear that $\tilde{\Phi}(z)$ only depends on the variables $\{z_j\}_{j\notin J_1}$; but $z_j(m) = \tilde{z}_j(m)$ for $j \notin J_1$. □

The period mapping $\tilde{\Phi}$ determines a mixed Hodge structure and various operators and splittings on $\tilde{H}_{\mathbb{C}}$, and our next goal is to show that they are induced by the ones on $H_{\mathbb{C}}$. Because $\tilde{\Phi}$ has weight ℓ_1, the limiting mixed Hodge structure is

$$\big(W(\tilde{T}_2, \dots, \tilde{T}_d)[-\ell_1], \tilde{F}\big) = \big(\tilde{W}^d, \tilde{F}\big).$$

As before, the multi-variable SL(2)-orbit theorem [CK89, Theorem 4.3] associates with the data $\big(\tilde{W}^d, \tilde{F}, \tilde{T}_2, \dots, \tilde{T}_d\big)$ a sequence of $d-1$ commuting splittings of $\tilde{H}_{\mathbb{R}}$.

Proposition 28.3. *These splittings are equal to $\tilde{Y}_2, \dots, \tilde{Y}_d$. The $\mathbb{Z}^{d-1}$-grading of $\tilde{H}_{\mathbb{C}}$ and the sequence of operators $\tilde{e}(m)$ thus have the same properties as in §27.*

Proof. Before we prove the assertion, let us quickly recall the construction of the splittings $Y_1, \dots, Y_d$ from [CK89, §4]. Given a mixed Hodge structure (W, F), there is a canonical way to get an $\mathbb{R}$-split mixed Hodge structure $(W, F_0) = \sigma(W, F)$ with the same weight filtration [CK89, Theorem 3.15]. Its Hodge filtration is given by

$$F_0 = e^{\xi} e^{-i\delta} F,$$

where $\delta \in L_{\mathbb{R}}^{-1,-1}(W, F)$ is the unique element such that $(W, e^{-i\delta} F)$ is an $\mathbb{R}$-split mixed Hodge structure [CK89, Theorem 1.15], and ξ is a universal non-commutative polynomial in the components $\delta^{p,q}$. This $\mathbb{R}$-split mixed Hodge structure defines a semisimple endomorphism $Y_0 \in \operatorname{End}(H_{\mathbb{R}})$, which acts as multiplication by $p+q$ on the subspace $I^{p,q}(W, F_0)$ in Deligne's decomposition. Note that if (W, F) is polarized by a bilinear form Q and a nilpotent operator N, in the sense of [CK89, Definition 1.16], the splitting Y_0 is automatically an infinitesimal isometry of Q.

Now the operators $Y_1, \dots, Y_d, \hat{T}_1, \dots, \hat{T}_d$ and the filtration $F_\sharp$ in the multi-variable SL(2)-orbit theorem are obtained by the following recursive procedure. Consider the $\mathbb{R}$-split mixed Hodge structure

$$\big(W^d, F_{(d)}\big) = \sigma\big(W^d, F\big),$$

and denote by $Y_{(d)} \in \operatorname{End}(H_{\mathbb{R}})$ its associated splitting. The pair $\big(W^{d-1}, e^{iT_d} F_{(d)}\big)$ is again a mixed Hodge structure on $H_{\mathbb{C}}$; define

$$\big(W^{d-1}, F_{(d-1)}\big) = \sigma\big(W^{d-1}, e^{iT_d} F_{(d)}\big)$$

and denote by $Y_{(d-1)} \in \operatorname{End}(H_{\mathbb{R}})$ the associated splitting. Continuing in this manner, one has for $k = 1, \dots, d-1$ an $\mathbb{R}$-split mixed Hodge structure

$$\big(W^k, F_{(k)}\big) = \sigma\big(W^k, e^{iT_{k+1}} F_{(k+1)}\big),$$

and a corresponding splitting $Y_{(k)} \in \operatorname{End}(H_{\mathbb{R}})$. Since each of these mixed Hodge structures is polarized by Q and any element in the cone $C(J_1 \sqcup \cdots \sqcup J_k)$, the operators $Y_{(1)}, \dots, Y_{(d)}$ are infinitesimal isometries of Q. Now define $Y_1, \dots, Y_n$ by asking that $Y_{(k)} = Y_1 + \cdots + Y_k$; also set $\hat{T}_1 = T_1$, and for $k = 2, \dots, d$, let $\hat{T}_k$ be the component of T_k in $\ker \operatorname{ad} Y_1 \cap \cdots \cap \ker \operatorname{ad} Y_{k-1}$. Then the point of [CK89, Theorem 4.3] is that $Y_1, \dots, Y_d$ commute, and that the filtration

$$F_\sharp = e^{i(\hat{T}_1 + \cdots + \hat{T}_k)} F_{(k)}$$

is independent of k.

The splittings $\tilde{Y}_2, \dots, \tilde{Y}_d \in \operatorname{End}(\tilde{H}_{\mathbb{R}})$ are obtained by applying the same procedure to $\big(\tilde{W}^d, \tilde{F}, \tilde{T}_2, \dots, \tilde{T}_d\big)$. To prove the assertion, it is enough to show that at each stage, the $\mathbb{R}$-split mixed Hodge structure on $\tilde{H}_{\mathbb{C}}$ is induced by the one on $H_{\mathbb{C}}$. For $k = 2, \dots, d$, let $\tilde{F}_{(k)}$ be the filtration of $\tilde{H}_{\mathbb{C}}$ induced by $F_{(k)}$. Then we have

$$\sigma\big(\tilde{W}^d, \tilde{F}\big) = \big(\tilde{W}^d, \tilde{F}_{(d)}\big),$$

for the following reason: $\delta \in L_{\mathbb{R}}^{-1,-1}(W^d, F)$ commutes with T_1, hence preserves W^1; the induced operator $\tilde{\delta}$ belongs to $L_{\mathbb{R}}^{-1,-1}\big(\tilde{W}^d, \tilde{F}\big)$ and must therefore be equal to the one in [CK89, Theorem 1.15]; and ξ is given by a universal non-commutative polynomial in the $\delta^{p,q}$. Taking the shift in weight into account, it follows that the corresponding splitting is $\tilde{Y}_{(d)} - \ell_1 = \tilde{Y}_2 + \cdots + \tilde{Y}_d$. The same argument proves that

$$\sigma\big(\tilde{W}^k, e^{i\tilde{T}_{k+1}} \tilde{F}_{(k+1)}\big) = \big(\tilde{W}^k, \tilde{F}_{(k)}\big),$$

with corresponding splitting $\tilde{Y}_{(k)} - \ell_1 = \tilde{Y}_2 + \cdots + \tilde{Y}_k$. This is obviously enough to conclude the proof. □

Now suppose that $\ell_1 = 0$, so that we are again dealing with a polarized variation of integral Hodge structure of weight zero. Suppose we have a sequence $h(m) \in H_{\mathbb{R}}$ with

$$h(m) \equiv b(m) \mod \Phi^0\big(z(m)\big)$$

for a harmless sequence $b(m) \in H_{\mathbb{C}}$. Let $\tilde{h}(m) \in \tilde{H}_{\mathbb{R}}$ denote the image of $h(m)$; note that $\tilde{h}(m) \in \tilde{H}_{\mathbb{Z}}$ if the initial sequence satisfies $h(m) \in H_{\mathbb{Z}}$. Lemma 24.5 allows us to assume that the harmless sequence $b(m)$ lies in $W^1_{\ell_1}$; consequently,

$$\tilde{h}(m) \equiv \tilde{b}(m) \mod \tilde{\Phi}^0\big(z(m)\big)$$

for a harmless sequence $\tilde{b}(m) \in \tilde{H}_{\mathbb{C}}$. The sequence $\tilde{h}(m)$ automatically inherits the following boundedness property from $h(m)$.

Lemma 28.4. *If $Q\big(h(m), h(m)\big)$ is bounded, then $\tilde{Q}\big(\tilde{h}(m), \tilde{h}(m)\big)$ is also bounded.*

Proof. By Proposition 27.2, the assertion is equivalent to the boundedness of the sequence $\tilde{e}(m)\tilde{h}(m)$; note that this requires $\ell_1 = 0$. But clearly

$$\|\tilde{e}(m)\tilde{h}(m)\| \leq t_1(m)^{-\ell_1/2} \|e(m)h(m)\|,$$

which is bounded as long as $\ell_1 \geq 0$. □

29. Position relative to the $\mathbb{Z}^d$-grading. In the proof of Theorem 17.4 for $n = 1$, one of the key steps was to use the boundedness of the sequence $e(m)h(m)$ to show that a subsequence of $h(m)$ has to lie in W_0. Here we investigate the general case of this problem, namely how to conclude from the boundedness of $e(m)h(m)$ that certain components of $h(m)$ with respect to the $\mathbb{Z}^d$-grading in (27.3) have to vanish. We give a slightly streamlined version of [CDK95, Lemma 4.4 and Lemma 4.5].

Since we are going to argue by induction on $n \geq 1$, we relax the condition on the weight and only assume that $\Phi(z)$ is the period mapping of a polarized variation of integral Hodge structure of weight $w \geq 0$. We also fix a sequence $z(m) = iAt(m) + w(m)$ as in (26.1), and consider on $H_{\mathbb{C}}$ the $\mathbb{Z}^d$-grading defined by $Y_1, \dots, Y_d$.

Definition 29.1. The *position* of a sequence $h(m) \in H_{\mathbb{R}}$ relative to the $\mathbb{Z}^d$-grading is the largest multi-index $(\ell_1, \dots, \ell_d) \in \mathbb{Z}^d$ in the lexicographic ordering with the property that $h(m)^{(\ell_1,\dots,\ell_d)} \neq 0$ for infinitely many $m \in \mathbb{N}$.

Now suppose we are given a sequence $h(m) \in H_{\mathbb{R}}$ that is in the position $(\ell_1, \dots, \ell_d)$ relative to the $\mathbb{Z}^d$-grading defined by $Y_1, \dots, Y_d$. Assume moreover that

$$\|h(m)^{(\ell_1,\dots,\ell_d)}\| \geq \varepsilon$$

for a positive constant $\varepsilon > 0$; this replaces the condition that $h(m) \in H_{\mathbb{Z}}$. Our goal is to show that if $t_1(m)^{w/2} \cdot e(m)h(m)$ is bounded, then the sequence $h(m)$ must be in the position $(-w, 0, \dots, 0)$ with respect to the $\mathbb{Z}^d$-grading.

Proposition 29.2. *Suppose that we have $h(m) \equiv b(m) \mod \Phi^0\big(z(m)\big)$ for an exponentially small sequence $b(m) \in H_{\mathbb{C}}$. If $t_1(m)^w \|e(m)h(m)\|^2$ is bounded, then $w + \ell_1 = \ell_2 = \cdots = \ell_d = 0$.*

Proof. Since $b(m)$ is exponentially small with respect to $t(m)$, whereas $e(m)$ grows at most polynomially in $t_1(m)$, it is clear that $\|e(m)b(m)\|$ is in $O(e^{-\alpha t_1(m)})$ for some $\alpha > 0$. Now the key observation is that the ratios

$$\frac{\|e(m)b(m)\|^2}{\|e(m)h(m)\|^2}$$

are going to zero. Indeed, $\|e(m)b(m)\|^2$ is in $O(e^{-2\alpha t_1(m)})$, whereas $\|e(m)h(m)\|^2$ is bounded from below by

$$t_1(m)^{\ell_1} \cdots t_d(m)^{\ell_d} \|h(m)^{(\ell_1,\dots,\ell_d)}\|^2 \geq t_1(m)^{\ell_1} \cdots t_d(m)^{\ell_d} \cdot \varepsilon^2.$$

The unit vectors $\|e(m)h(m)\|^{-1} \cdot e(m)h(m)$ therefore converge to a unit vector in $W^1_{w+\ell_1} \cap F^0_\sharp \cap H_{\mathbb{R}}$, and so $w + \ell_1 \geq 0$ because $(W^1, F_\sharp)$ is a mixed Hodge structure.

Because of the bound on $e(m)h(m)$, we know that $\|h(m)\|$ grows at most like a fixed power of $t_1(m)$. We can therefore assume that $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$ (by Lemma 29.3 below), and that $b(m) \in W^1_{\ell_1}$ (by Lemma 24.5). We now project the sequence to $\tilde{H}_{\mathbb{C}} = \operatorname{gr}^{W^1}_{w+\ell_1}$, which carries a polarized variation of Hodge structure of weight $w + \ell_1$ by Proposition 28.1. The new sequence $\tilde{h}(m)$ is in the position $(\ell_2, \dots, \ell_d)$ relative to the $\mathbb{Z}^{d-1}$-grading on $\tilde{H}_{\mathbb{C}}$, and congruent to an exponentially small sequence modulo $\tilde{\Phi}^0\big(z(m)\big)$. Moreover, the expression

$$t_2(m)^{w+\ell_1} \|\tilde{e}(m)\tilde{h}(m)\|^2 \leq t_1(m)^{w+\ell_1} \|\tilde{e}(m)\tilde{h}(m)\|^2 \leq t_1(m)^w \|e(m)h(m)\|^2$$

is bounded (because $w + \ell_1 \geq 0$), and we still have

$$\|\tilde{h}(m)^{(\ell_2,\dots,\ell_d)}\| = \|h(m)^{(\ell_1,\dots,\ell_d)}\| \geq \varepsilon.$$

By induction, $w + \ell_1 + \ell_2 = \ell_3 = \dots = \ell_d = 0$. But now we get

$$t_1(m)^w \|e(m)h(m)\|^2 \geq \varepsilon^2 \cdot t_1(m)^{w+\ell_1} t_2(m)^{\ell_2} \cdots t_d(m)^{\ell_n} = \varepsilon^2 \left(\frac{t_1(m)}{t_2(m)}\right)^{w+\ell_1}.$$

This can only be bounded if $w + \ell_1 \leq 0$; since we already know that $w + \ell_1 \geq 0$, we conclude that $w + \ell_1 = 0$ and $\ell_2 = 0$. □

The following lemma was used during the proof; it will make another appearance when we prove Theorem 26.3. We put the period mapping into the standard form $\Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F$. Let $\Gamma_1(s)$ denote the result of setting all the variables $\{s_j\}_{j \in J_1}$ in $\Gamma(s)$ to zero, and define $\Phi_1(z) = e^{\sum z_j N_j} e^{\Gamma_1(s)} F$, which is still a period mapping by [CK89, p. 76], but now also a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$.

Lemma 29.3. *Suppose that $h(m)$ is congruent, modulo $\Phi^0\big(z(m)\big)$, to a sequence that is harmless/exponentially small with respect to $t(m)$. Then the same is true modulo $\Phi_1^0\big(z(m)\big)$, provided that $\|h(m)\|$ is in $O(t_1(m)^N)$ for some $N \in \mathbb{N}$.*

Proof. Suppose that $h(m) \equiv b(m) \mod \Phi^0\big(z(m)\big)$, with $b(m)$ either harmless or exponentially small. We have

$$e^{\sum z_j N_j} e^{\Gamma(s)} = \left(e^{\sum z_j N_j} e^{\Gamma(s)} e^{-\Gamma_1(s)} e^{-\sum z_j N_j}\right) e^{\sum z_j N_j} e^{\Gamma_1(s)},$$

and because $|s_j(m)|$ is in $O(e^{-\alpha t_1(m)})$ for $j \in J_1$, Proposition 15.3 shows that the sequence of operators

$$\Delta(m) = \left(e^{\sum z_j(m) N_j} e^{\Gamma_1(s(m))} e^{-\Gamma(s(m))} e^{-\sum z_j(m) N_j}\right) - \mathrm{id}$$

is in $O(e^{-\alpha t_1(m)})$ with respect to the operator norm. We therefore have

$$h(m) \equiv b(m) + \Delta(m)\big(b(m) - h(m)\big) \mod e^{\sum z_j(m) N_j} e^{\Gamma_1(s(m))} F^0,$$

and because $\|b(m)\|$ is bounded and $\|h(m)\|$ is in $O(t_1(m)^N)$, the extra term on the right-hand side is exponentially small with respect to $t(m)$. □

30. Proof in the general case. We now prove Theorem 26.3 by induction on $d \geq 1$. As in the one-variable case, the argument can be divided into six steps.

Step 1. To get started, we have to prove that the sequence $h(m) \in H_{\mathbb{Z}}(K)$ is bounded in the Hodge norm at the point $\Phi\big(z(m)\big)$. This follows immediately from Proposition 27.2. As in the one-variable case, we will later use only the equivalent fact that the sequence $e(m)h(m)$ is bounded. We also note that the sequence $\|h(m)\|$ grows at most like a fixed power of $t_1(m)$.

Step 2. We now reduce to the case where $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$; those are the ones that are going to zero most quickly. Recall that

$$\Phi(z) = e^{\sum z_j N_j} e^{\Gamma(s)} F;$$

let $\Gamma_1(s)$ denote the result of setting $s_j = 0$ for every $j \in J_1$. The claim is that we can replace $\Gamma(s)$ by $\Gamma_1(s)$ without affecting any of the conditions of the problem; this is proved in Lemma 29.3. After making the obvious replacements, we can therefore assume without loss of generality that the operator $\Gamma(s)$ does not depend on the variables s_j with with $j \in J_1$. In particular, $e^{\Gamma(s)}$ now commutes with T_1

by Proposition 15.3, and therefore preserves the weight filtration W^1. Note that we only have $\Phi(z) \in D$ when all the imaginary parts of $z \in \mathbb{H}^n$ are sufficiently large; after passing to a subsequence, we may assume that this is the case along our sequence $z(m)$.

Step 3. Our next goal is to show that $h(m) \in W_0^1$. As in the one-variable case, we will deduce this from the boundedness of the sequence $e(m)h(m) \in H_{\mathbb{R}}$. Let $\ell \in \mathbb{Z}^d$ be the largest index (in the lexicographic ordering) with the property that $h(m)^{(\ell_1,\dots,\ell_d)}$ is nonzero for infinitely many m. After passing to a subsequence, we therefore have $h(m) \in W_{\ell_1}^1$; its projection to $\operatorname{gr}_{\ell_1}^{W^1}$ lies in the image of $W_{\ell_1+\ell_2}^2$, and so on. Note that the projection

$$H_{\mathbb{C}}^{(\ell_1,\dots,\ell_d)} \to \operatorname{gr}_{\ell_1+\dots+\ell_d}^{W^d} \cdots \operatorname{gr}_{\ell_1+\ell_2}^{W^2} \operatorname{gr}_{\ell_1}^{W^1}$$

is an isomorphism; because $h(m) \in H_{\mathbb{Z}}$, it follows that $h(m)^{(\ell_1,\dots,\ell_d)}$ takes values in a discrete set. In particular, we have $\|h(m)^{(\ell_1,\dots,\ell_d)}\| \geq \varepsilon$ for a constant $\varepsilon > 0$.

Now suppose that $h(m) \notin W_0^1$; in other words, suppose that $\ell_1 \geq 1$. Define $\tilde{H}_{\mathbb{C}} = \operatorname{gr}_{\ell_1}^{W^1}$; according to Proposition 28.1, it again supports a polarized variation of integral Hodge structure of weight ℓ_1. Let $\tilde{h}(m)$ denote the image of $h(m)$ in $\tilde{H}_{\mathbb{C}}$. Because $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j\in J_1}$, we can use Lemma 24.5 to make sure that $b(m) \in W_{\ell_1}^1$. In the congruence

$$h(m) - \big(b_1(m) + \dots + b_d(m)\big) \equiv b_0(m) \mod \Phi^0\big(z(m)\big),$$

the term in parentheses is contained in $\ker T_1 \subseteq W_0^1$, and therefore disappears when we project to $\tilde{H}_{\mathbb{C}}$. Under the assumption that $\ell_1 \geq 1$, our sequence $\tilde{h}(m)$ is therefore exponentially close to the subspace $\tilde{\Phi}^0\big(z(m)\big)$. We can now apply Proposition 29.2 to the sequence $\tilde{h}(m)$ and the polarized variation of Hodge structure $\tilde{\Phi}(z)$ on $\tilde{H}_{\mathbb{C}}$; the result is that $\ell_1 + \ell_2 = 0$ and $\ell_3 = \dots = \ell_d = 0$. But then

$$\|e(m)h(m)\|^2 \geq \|e(m)h(m)^{(\ell_1,\dots,\ell_d)}\|^2 \geq \varepsilon^2 \left(\frac{t_1(m)}{t_2(m)}\right)^{\ell_1},$$

and since $\ell_1 \geq 1$, this inequality contradicts the boundedness of $e(m)h(m)$. Consequently, $h(m) \in W_0^1$ after all.

Step 4. Using the notation from §28, we now apply the induction hypothesis to the sequence $\big(\tilde{z}(m), \tilde{h}(m)\big)$ and the period mapping $\tilde{\Phi}(z)$ on the space $\tilde{H}_{\mathbb{C}} = \operatorname{gr}_0^{W^1}$; the construction in Proposition 28.1 shows that all the assumptions are again satisfied, but with a smaller value of d. After passing to a subsequence, $\tilde{h}(m)$ has a constant value $\tilde{h} \in \tilde{H}_{\mathbb{Z}}$, and $\tilde{T}_k \tilde{h} = 0$ for $k = 2, \dots, d$. In order to lift these results back to $H_{\mathbb{C}}$, we define

$$h_0 = \sum_{\ell_2,\dots,\ell_d} h(m)^{(0,\ell_2,\dots,\ell_d)} \in H_{\mathbb{R}};$$

note that h_0 is constant, because it projects to the constant sequence $\tilde{h}$ under the isomorphism $E_0(Y_1) \simeq \tilde{H}_{\mathbb{C}}$. We also have $h_0 \in W_0^k$ for every $k = 2, \dots, d$, because $\tilde{h} \in \tilde{W}_0^k$. The conclusion is that

$$h(m) \equiv h_0 \mod W_{-1}^1.$$

Our next task is to prove that $T_1 h_0^{(0,\dots,0)} = 0$.

If we apply the operator $e(m)$ to the congruence in (27.1), we obtain

$$e(m)h(m) \equiv e(m)b(m) \mod e(m)\Phi^0\bigl(z(m)\bigr).$$

Because $e(m)h(m)$ is bounded, and because we have already computed the limit of $e(m)b(m)$ in Lemma 27.7, we can pass to a subsequence where

$$v = \lim_{m\to\infty} e(m)h(m) \equiv \lim_{m\to\infty} b_d(m)^{(0,\dots,0)} \mod F_\sharp^0.$$

Now comes the crucial point: by Lemma 27.7, the right-hand side of the congruence is an element of $E_0(Y_k)\cap\ker T_k$ for every $k = 1,\dots,d$. Because $v \in W_0^1\cap H_{\mathbb{R}}$, we can apply Lemma 25.4 from the one-variable case to the $\mathbb{R}$-split mixed Hodge structure $\bigl(W^1, e^{-iT_1}F_\sharp\bigr)$ and conclude that $v \in E_0(Y_1)$ and $T_1 v = 0$.

On the other hand, we can project the congruence

$$e(m)h(m) \equiv e(m)h_0 \mod W_{-1}^1$$

to the subspace $E_0(Y_1)$; because $h_0 \in W_0^k$ for every $k = 1,\dots,d$, we get

$$v = \lim_{m\to\infty} e(m)h_0 = h_0^{(0,\dots,0)}.$$

In particular, we have $T_1 h_0^{(0,\dots,0)} = 0$.

Step 5. Now we show that $\|T_1 h(m)\|$ is in $O(e^{-\alpha t_1(m)})$; the method is almost the same as in the one-variable case. We have

$$e(m)T_1h(m) \equiv e(m)T_1b_0(m) \mod T_1e(m)\Phi^0\bigl(z(m)\bigr);$$

here we used the fact that $T_1e(m) = t_1(m)\cdot e(m)T_1$, because $T_1 = \hat{T}_1$ commutes with $Y_2,\dots,Y_d$ and satisfies $[Y_1,T_1] = -2T_1$. We claim that $\|e(m)T_1h(m)\|$ is bounded by a constant multiple of $\|e(m)T_1b_0(m)\|$. If not, then the ratios

$$\frac{\|e(m)T_1b_0(m)\|}{\|e(m)T_1h(m)\|}$$

are going to zero. After passing to a subsequence, the sequence of unit vectors

$$u(m) = \frac{e(m)T_1h(m)}{\|e(m)T_1h(m)\|} \in W_{-2}^1 \cap H_{\mathbb{R}} \tag{30.1}$$

converges to a unit vector $u \in W_{-2}^1 \cap F_\sharp^{-1} \cap H_{\mathbb{R}}$. Now $\bigl(W^1, e^{-iT_1}F_\sharp\bigr)$ is an $\mathbb{R}$-split mixed Hodge structure; we can therefore apply Lemma 25.4 from the one-variable case to deduce that $u \in E_{-2}(Y_1)$.

Recall that the decomposition $W_{-2}^1 = E_{-2}(Y_1)\oplus W_{-3}^1$ is orthogonal with respect to the inner product on $H_{\mathbb{C}}$. If we project the congruence

$$u(m) \equiv \frac{e(m)T_1h_0}{\|e(m)T_1h(m)\|} \mod W_{-3}^1$$

to the subspace $E_{-2}(Y_1)$, we find that

$$u = \lim_{m\to\infty} \frac{e(m)T_1h_0}{\|e(m)T_1h(m)\|}.$$

Because the right-hand side belongs to $W_{-2}^2\cap\cdots\cap W_{-2}^d$, it follows that u lies in the intersection $W_{-2}^1\cap\cdots\cap W_{-2}^d\cap F_\sharp^{-1}\cap H_{\mathbb{R}}$. We can therefore apply Lemma 25.4 again, this time to the $\mathbb{R}$-split mixed Hodge structure

$$\bigl(W^k, e^{-i(\hat{T}_1+\cdots+\hat{T}_k)}F_\sharp\bigr),$$

to show that $(Y_1 + \cdots + Y_k)u = -2u$ for every $k = 1, \cdots, d$. These relations are saying that $u \in H_{\mathbb{C}}^{(-2,0,\dots,0)}$. But if we project (30.1) to that summand and use the fact that $h(m) \equiv h_0 \mod W_{-1}^1$, we find that

$$u(m)^{(-2,0,\dots,0)} = \frac{e(m)T_1 h_0^{(0,\dots,0)}}{\|e(m)T_1 h(m)\|} = 0.$$

This forces $u = 0$, in contradiction to the fact that u is a unit vector. Consequently, $\|e(m)T_1 h(m)\|$ must be bounded by a constant multiple of $\|e(m)T_1 b_0(m)\|$, and therefore in $O(e^{-\alpha t_1(m)})$. Because $e(m)^{-1}$ grows at most like a power of $t_1(m)$, this is enough to conclude that $\|T_1 h(m)\|$ is exponentially small.

Step 6. We can now complete the proof by the method of [CDK95, 4.9]. If $d \geq 2$, we observe that

$$e^{-it_1(m)T_1} h(m) \equiv e^{-it_1(m)T_1} b_0(m) + b_1(m) + \cdots + b_d(m) \mod e^{-it_1(m)T_1}\Phi^0\big(z(m)\big). \tag{30.2}$$

Because $\|T_1 h(m)\|$ is in $O(e^{-\alpha t_1(m)})$, it follows that $h(m)$ is the sum of a harmless element and an element of $e^{-it_1(m)T_1}\Phi^0\big(z(m)\big)$. Remembering that $\Phi(z)$ is a nilpotent orbit in the variables $\{s_j\}_{j \in J_1}$, the sequence of filtrations

$$e^{-it_1(m)T_1}\Phi\big(z(m)\big) \in D$$

no longer involves either $t_1(m)$ or T_1; this means that we have managed to reduce the value of d. By induction, we can pass to a subsequence and arrange that $h(m)$ is constant and in the kernel of $T_2, \dots, T_d$. Since $T_1 h(m)$ is exponentially small, it has to be zero as well, concluding the proof in the case $d \geq 2$.

If $d = 1$, then we argue as in the one-variable case. Recall that

$$\Phi\big(z(m)\big) = e^{\sum z_j(m) N_j} F = e^{it(m)T_1} e^{\sum w_j(m) N_j} F$$

is a nilpotent orbit, with $w(m) \in \mathbb{C}^n$ convergent and $\Phi\big(w(m)\big) \in D$. The formula in (30.2) shows that the Hodge norm of $h(m)$ with respect to $\Phi\big(w(m)\big)$ is bounded. Since these Hodge filtrations lie in a compact set, $\|h(m)\|$ must be bounded; after passing to a subsequence, $h(m)$ is constant, and then $T_1 h(m) = 0$ as before. This completes the proof of Theorem 17.4 for all $n \geq 1$.

F. The universal family of hyperplane sections

31. Description of the variation of Hodge structure. The purpose of this chapter is to apply the general construction from Chapter C to the universal family of hyperplane sections of a smooth projective variety. Let X be a smooth projective variety of odd dimension $2n+1$, and let L be a very ample line bundle on X. It determines an embedding of X into the projective space $\mathbb{P}\big(H^0(X, L)\big)$. We denote by $B = \mathbb{P}\big(H^0(X, L)^*\big)$ the dual projective space; a point $b \in B$ corresponds to a hyperplane H_b, and therefore to a hyperplane section $H_b \cap X$ of X. There is a natural incidence variety

$$\mathscr{X} = \big\{ (b, x) \in B \times X \;\big|\; x \in H_b \cap X \big\};$$

it is a projective bundle over X, and therefore again a smooth projective variety of dimension $2n + \dim B$. Let $f \colon \mathscr{X} \to B$ denote the first projection, and $f_0 \colon \mathscr{X}_0 \to B_0$ its restriction to the Zariski-open subset where $H_b \cap X$ is nonsingular.

On B_0, we have a polarized variation of integral Hodge structure $\mathcal{H}$ of weight zero, obtained by taking the quotient of $R^{2n} f_{0*}\mathbb{Z}(n)$ by the constant part $H^{2n}\big(X, \mathbb{Z}(n)\big)$; note that the polarization is only defined over $\mathbb{Q}$ in general. Recall that for a smooth hyperplane section $Y = H \cap X$, the quotient

$$H^{2n}\big(Y, \mathbb{Z}(n)\big) / H^{2n}\big(X, \mathbb{Z}(n)\big)$$

is torsion-free (by the Lefschetz theorems); tensored with $\mathbb{Q}$, it becomes isomorphic to the variable part

$$\ker\Big(H^{2n}\big(Y, \mathbb{Q}(n)\big) \to H^{2n+2}\big(X, \mathbb{Q}(n+1)\big)\Big),$$

and is therefore canonically polarized by the intersection product on Y.

As usual, let M denote the polarized Hodge module of weight $\dim B$ with strict support B, associated with $\mathcal{H}$. In this situation, the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet)$ can be described concretely in terms of residues [Sch12b]. Recall that when $Y = H \cap X$ is a smooth hyperplane section, we have a residue mapping

$$\operatorname{Res}_Y \colon H^0\big(X, \Omega_X^{2n+1}(kY)\big) \to F^{2n+1-k} H^{2n}(Y, \mathbb{C}).$$

By applying this construction on each smooth hyperplane section, we can obtain sections of $\mathcal{H}$ from meromorphic $(2n+1)$-forms on $B \times X$ with poles along $\mathscr{X}$. To state the precise result, let $j \colon B_0 \hookrightarrow B$ denote the inclusion. Then $\mathcal{M}$ is a subsheaf of $j_*\mathcal{H}$, and the space of sections $H^0(U, F_k\mathcal{M})$ on an open set $U \subseteq B$ consists exactly of those $s \in H^0(U, j_*\mathcal{H})$ that satisfy

$$s(b) = \operatorname{Res}_{H_b \cap X}\Big(\omega\big|_{\{b\} \times X}\Big) \quad \text{at every point } b \in U \cap B_0$$

for some choice of meromorphic $(2n+1)$-form

$$\omega \in H^0\Big(U \times X, \Omega_{B \times X}^{2n+1}\big((n+1+k)\mathscr{X}\big)\Big).$$

In addition to this description, the following result is proved in [Sch12b, Corollary 4].

Theorem 31.1. *The coherent sheaf $F_k\mathcal{M}$ is a quotient of the ample vector bundle*

$$H^0\big(X, \Omega_X^{2n+1} \otimes L^{n+1+k}\big) \otimes \mathscr{O}_B(n+1+k),$$

and therefore globally generated.

32. Properties of the extension space. Now let us see what our general construction produces in the special case of the universal family of hyperplane sections. As before, we denote by $E(\mathcal{H}_\mathbb{Z})$ the (possibly disconnected) covering space of B_0 determined by the local system $\mathcal{H}_\mathbb{Z}$, and by $\varphi \colon E(\mathcal{H}_\mathbb{Z}) \to T(F_{-1}\mathcal{M})$ the holomorphic mapping induced by the polarization. Fix some $K \geq 0$. According to the general result in Theorem 13.1, we have a finite holomorphic mapping

$$\tilde{\varphi} \colon \overline{E(\mathcal{H}_\mathbb{Z})}(K) \to T(F_{-1}\mathcal{M})$$

from a normal analytic space $\overline{E(\mathcal{H}_\mathbb{Z})}(K)$ that contains $E_{\leq K}(\mathcal{H}_\mathbb{Z})$ as a dense open subset. Also recall from Corollary 13.3 that the intersection $\widetilde{\operatorname{Hdg}}(\mathcal{H}) \cap \overline{E(\mathcal{H}_\mathbb{Z})}(K)$ with the extended locus of Hodge classes is finite over B, and therefore again a projective scheme. Now the fact that $F_{-1}\mathcal{M}$ is a quotient of an ample vector bundle has the following interesting consequence; it was predicted by Clemens several years ago.

Theorem 32.1. *The analytic space* $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ *is holomorphically convex. Every compact analytic subset of dimension* ≥ 1 *lies inside the extended locus of Hodge classes, and is therefore necessarily a projective algebraic variety.*

Proof. For a discussion of holomorphic convexity, see [Car60]. The result in Theorem 31.1 shows that $T(F_{-1}\mathcal{M})$ embeds into the holomorphic vector bundle

$$E = T\Big(H^0\big(X, \Omega_X^{2n+1} \otimes L^n\big) \otimes \mathscr{O}_B(n)\Big).$$

Since E is the dual of an ample vector bundle, the zero section can be contracted to produce a Stein space Y; in particular, E is holomorphically convex. Because $\tilde{\varphi}\colon \overline{E(\mathcal{H}_{\mathbb{Z}})}(K) \to T(F_{-1}\mathcal{M})$ is finite, it follows that $\overline{E(\mathcal{H}_{\mathbb{Z}})}(K)$ is proper over Y, and therefore also holomorphically convex. Every compact analytic subset of positive dimension has to map into the zero section of E, and must therefore be contained in the extended locus of Hodge classes. □

G. The example of Calabi-Yau threefolds

33. Hodge loci on Calabi-Yau threefolds. The purpose of this paper is to describe the construction of the *extended locus of Hodge classes* for polarized variations of integral Hodge structure of weight zero. Before defining things more precisely, we shall consider a typical example that shows why this is an interesting problem, and what some of the issues are.

Let X be a smooth projective Calabi-Yau threefold; this means that $\Omega_X^3 \simeq \mathscr{O}_X$, and that $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$. We fix an embedding of X into projective space, with $\mathscr{O}_X(1)$ the corresponding very ample line bundle, and consider the family of hyperplane sections of X. These are parametrized by the linear system

$$B = \big|\mathscr{O}_X(1)\big|,$$

and we let $B_0 \subseteq B$ denote the open subset that corresponds to smooth hyperplane sections. Given a cohomology class $\gamma \in H^2(S, \mathbb{Z})$ on a smooth hyperplane section $S \subseteq X$, we can use parallel transport along paths in B_0 to move γ to other hyperplane sections; this operation is of course purely topological and does not preserve the Hodge decomposition. The *Hodge locus of* γ is the set

$$\big\{\, b \in B_0 \,\big|\, \gamma \text{ can be transported to a Hodge class on } S_b \,\big\}.$$

Most of these loci are non-empty: in fact, Voisin [Voi92] has proved that the union of the Hodge loci of all classes $\gamma \in H^2(S, \mathbb{Z})$ is a dense subset of B_0. Since Hodge classes on surfaces are algebraic, the Hodge locus is an algebraic subvariety of B_0; in basic terms, what we are looking at are curves (or algebraic one-cycles) on X that lie on hyperplane sections.

We observe that the expected dimension of the Hodge locus is zero. Indeed, a class $\gamma \in H^2(S, \mathbb{Z})$ is Hodge exactly when it pairs to zero against every holomorphic two-form on S; because X is a Calabi-Yau threefold, we have

$$h^0(S, \Omega_S^2) = h^0\big(X, \Omega_X^3(S)\big) - h^0(X, \Omega_X^3) = h^0\big(X, \mathscr{O}_X(1)\big) - 1 = \dim B.$$

The number of conditions is the same as the dimension of the parameter space, and the Hodge locus of γ should therefore have a "virtual" number of points; those numbers are of interest in Donaldson-Thomas theory [KMPS10]. But there are two issues that need to be dealt with:

(1) If the Hodge locus actually has finitely many points, one can of course just count them. But there may be components of positive dimension, and before one can use excess intersection theory (or some other method) to assign them a number, one has to compactify such components.
(2) An obvious idea is to take the closure of the Hodge locus inside the projective space B; but this is not the right thing to do because there are interesting limit phenomena that one cannot see in this way.

Example. Here is a typical example. Consider a family of hyperplane sections $S_t \subseteq X$, parametrized by $t \in \Delta$, with S_t smooth for $t \neq 0$, and S_0 having a single ordinary double point. In this case, $H^2(S_t, \mathbb{Z})$ contains a vanishing cycle γ_t, namely the class of an embedded two-sphere with self-intersection number $\gamma_t^2 = -2$. The vanishing cycle is not a Hodge class on S_t, but becomes one "in the limit". On the one hand, one has the limit mixed Hodge structure, which is pure of weight two in this case; γ_t is a Hodge class in this Hodge structure. On the other hand, one can blow up S_0 at the node; the exceptional divisor $E \simeq \mathbb{P}^1$ satisfies $[E]^2 = -2$, and in a sense, $[E]$ is the limit of the γ_t as $t \to 0$.

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