1

Torsion points on cohomology support loci: from D-modules to Simpson's theorem

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Abstract

We study cohomology support loci of regular holonomic $\mathscr{D}$-modules on complex abelian varieties, and obtain conditions under which each irreducible component of such a locus contains a torsion point. One case is that both the $\mathscr{D}$-module and the corresponding perverse sheaf are defined over a number field; another case is that the $\mathscr{D}$-module underlies a graded-polarizable mixed Hodge module with a $\mathbb{Z}$-structure. As a consequence, we obtain a new proof for Simpson's result that Green-Lazarsfeld sets are translates of subtori by torsion points.

1.1 Overview of the paper

1.1.1 Introduction

Let X be a projective complex manifold. In their two influential papers about the generic vanishing theorem [6, 7], Green and Lazarsfeld showed that the so-called *cohomology support loci*

$$\Sigma_m^{p,q}(X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^q\big(X, \Omega_X^p \otimes L\big) \geq m \right\},$$

are finite unions of translates of subtori of $\operatorname{Pic}^0(X)$. Beauville and Catanese [2] conjectured that the translates are always by *torsion points*, and this was proved by Simpson [20] with the help of the Gelfond-Schneider theorem from transcendental number theory. There is also a proof using positive characteristic methods by Pink and Roessler [13].

Over the past ten years, the results of Green and Lazarsfeld have been reinterpreted and generalized several times [8, 14, 18], and we now understand that they are consequences of a general theory of holonomic

$\mathscr{D}$-modules on abelian varieties. The purpose of this paper is to investigate under what conditions the result about torsion points on cohomology support loci remains true in that setting. One application is a new proof for the conjecture by Beauville and Catanese that does not use transcendental number theory or reduction to positive characteristic.

Note In a recent preprint [21], Wang extends Theorem 1.4 to polarizable Hodge modules on compact complex tori; as a corollary, he proves the conjecture of Beauville and Catanese for arbitrary compact Kähler manifolds.

1.1.2 Cohomology support loci for D-modules

Let A be a complex abelian variety, and let $\mathcal{M}$ be a regular holonomic $\mathscr{D}_A$-module; recall that a $\mathscr{D}$-module is called *holonomic* if its characteristic variety is a union of Lagrangian subvarieties of the cotangent bundle. Denoting by $A^\natural$ the moduli space of line bundles with integrable connection on A, we define the *cohomology support loci* of $\mathcal{M}$ as

$$S_m^k(A, \mathcal{M}) = \Big\{ (L, \nabla) \in A^\natural \;\Big|\; \dim \mathbf{H}^k\Big(A, \mathrm{DR}_A\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big)\Big) \geq m \Big\}$$

for $k, m \in \mathbb{Z}$. It was shown in [18, Theorem 2.2] that $S_m^k(A, \mathcal{M})$ is always a finite union of linear subvarieties of $A^\natural$, in the following sense.

Definition 1.1 A *linear subvariety* of $A^\natural$ is any subset of the form

$$(L, \nabla) \otimes \mathrm{im}\big(f^\natural \colon B^\natural \to A^\natural\big),$$

for $f \colon A \to B$ a homomorphism of abelian varieties, and (L, ∇) a point of $A^\natural$. An *arithmetic subvariety* is a linear subvariety that contains a torsion point.

Moreover, the analogue of Simpson's theorem is true for semisimple regular holonomic $\mathscr{D}$-modules of geometric origin: for such $\mathcal{M}$, every irreducible component of $S_m^k(A, \mathcal{M})$ contains a torsion point. We shall generalize this result in two directions:

1. Suppose that $\mathcal{M}$ is regular holonomic, and that both $\mathcal{M}$ and the corresponding perverse sheaf $\mathrm{DR}_A(\mathcal{M})$ are defined over a number field. We shall prove that the cohomology support loci of $\mathcal{M}$ are finite unions of arithmetic subvarieties; this is also predicted by Simpson's standard conjecture.

2. Suppose that $\mathcal{M}$ underlies a graded-polarizable mixed Hodge module with $\mathbb{Z}$-structure; for example, $\mathcal{M}$ could be the intermediate extension of a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$. We shall prove that the cohomology support loci of $\mathcal{M}$ are finite unions of arithmetic subvarieties.

1.1.3 Simpson's standard conjecture

In his article [19], Simpson proposed several conjectures about regular holonomic systems of differential equations whose monodromy representation is defined over a number field. The principal one is the so-called "standard conjecture"; restated in the language of regular holonomic $\mathscr{D}$-modules and perverse sheaves, it takes the following form.

Conjecture 1.2 *Let $\mathcal{M}$ be a regular holonomic $\mathscr{D}$-module on a smooth projective variety X, both defined over $\bar{\mathbb{Q}}$. If $\mathrm{DR}_X(\mathcal{M})$ is the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$, then $\mathcal{M}$ is of geometric origin.*

He points out that, "there is certainly no more reason to believe it is true than to believe the Hodge conjecture, and whether or not it is true, it is evidently impossible to prove with any methods which are now under consideration. However, it is an appropriate motivation for some easier particular examples, and it leads to some conjectures which might in some cases be more tractable" [19, p. 372].

In the particular example of abelian varieties, Conjecture 1.2 predicts that when $\mathcal{M}$ is a regular holonomic $\mathscr{D}$-module with the properties described in the conjecture, then the cohomology support loci of $\mathcal{M}$ should be finite unions of arithmetic subvarieties. Our first result – actually a simple consequence of [18] and an old theorem by Simpson [20] – is that this prediction is correct.

Theorem 1.3 *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$, and let $\mathcal{M}$ be a regular holonomic $\mathscr{D}_A$-module. If $\mathcal{M}$ is defined over $\bar{\mathbb{Q}}$, and if $\mathrm{DR}_A(\mathcal{M})$ is the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$, then all cohomology support loci $S^k_m(A, \mathcal{M})$ are finite unions of arithmetic subvarieties of $A^\natural$.*

Proof Let $\mathrm{Char}(A) = \mathrm{Hom}\big(\pi_1(A), \mathbb{C}^*\big)$ be the space of rank one characters of the fundamental group; for a character $\rho \in \mathrm{Char}(A)$, we denote by $\mathbb{C}_\rho$ the corresponding local system of rank one. We define the cohomology support loci of a constructible complex of $\mathbb{C}$-vector spaces

$K \in \mathrm{D}_c^b(\mathbb{C}_A)$ to be the sets

$$S_m^k(A, K) = \left\{ \rho \in \mathrm{Char}(A) \;\middle|\; \dim \mathbf{H}^k\bigl(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\bigr) \geq m \right\}.$$

The correspondence between local systems and vector bundles with integrable connection gives a biholomorphic mapping $\Phi\colon A^\natural \to \mathrm{Char}(A)$; it takes a point (L, ∇) to the local system of flat sections of ∇. According to [18, Lemma 14.1], the cohomology support loci satisfy

$$\Phi\bigl(S_m^k(A, \mathcal{M})\bigr) = S_m^k\bigl(A, \mathrm{DR}_A(\mathcal{M})\bigr).$$

Note that $\mathrm{Char}(A)$ is an affine variety defined over $\mathbb{Q}$; in our situation, $A^\natural$ is moreover a quasi-projective variety defined over $\bar{\mathbb{Q}}$, because the same is true for A. The assumptions on the $\mathscr{D}$-module $\mathcal{M}$ imply that $S_m^k(A, \mathcal{M}) \subseteq A^\natural$ is defined over $\bar{\mathbb{Q}}$, and that $S_m^k\bigl(A, \mathrm{DR}_A(\mathcal{M})\bigr) \subseteq \mathrm{Char}(A)$ is defined over $\bar{\mathbb{Q}}$. We can now use [20, Theorem 3.3] to conclude that both must be finite unions of arithmetic subvarieties. □

1.1.4 Mixed Hodge modules with $\mathbb{Z}$-structure

We now consider a much larger class of regular holonomic $\mathscr{D}_A$-modules, namely those that come from mixed Hodge modules with $\mathbb{Z}$-structure. This class includes, for example, intermediate extensions of polarizable variations of Hodge structure defined over $\mathbb{Z}$; the exact definition can be found in Definition 1.9 below.

We denote by $\mathrm{MHM}(A)$ the category of graded-polarizable mixed Hodge modules on the abelian variety A, and by $\mathrm{D}^b\,\mathrm{MHM}(A)$ its bounded derived category [17, §4]; because A is projective, every mixed Hodge module is automatically algebraic. Let

$$\mathrm{rat}\colon \mathrm{D}^b\,\mathrm{MHM}(A) \to \mathrm{D}_c^b(\mathbb{Q}_A)$$

be the functor that takes a complex of mixed Hodge modules to the underlying complex of constructible sheaves of $\mathbb{Q}$-vector spaces; then a $\mathbb{Z}$-structure on $M \in \mathrm{D}^b\,\mathrm{MHM}(A)$ is a constructible complex $E \in \mathrm{D}_c^b(\mathbb{Z}_A)$ with the property that $\mathrm{rat}\, M \simeq \mathbb{Q} \otimes_{\mathbb{Z}} E$.

To simplify the notation, we shall define the *cohomology support loci* of a complex of mixed Hodge modules $M \in \mathrm{D}^b\,\mathrm{MHM}(A)$ as

$$S_m^k(A, M) = \bigl\{ \rho \in \mathrm{Char}(A) \bigm| \dim H^k\bigl(A, \mathrm{rat}\, M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) \geq m \bigr\},$$

where $k \in \mathbb{Z}$ and $m \geq 1$. Our second result is the following structure theorem for these sets.

Theorem 1.4 *If a complex of mixed Hodge modules $M \in \mathrm{D}^b \operatorname{MHM}(A)$ admits a $\mathbb{Z}$-structure, then all of its cohomology support loci $S^k_m(A, M)$ are complete unions of arithmetic subvarieties of* $\operatorname{Char}(A)$.

Definition 1.5 A collection of arithmetic subvarieties of $\operatorname{Char}(A)$ is called *complete* if it is a finite union of subsets of the form

$$\left\{ \rho^k \;\middle|\; \gcd(k, n) = 1 \right\} \cdot \operatorname{im}\bigl(\operatorname{Char}(f) \colon \operatorname{Char}(B) \to \operatorname{Char}(A)\bigr),$$

where $\rho \in \operatorname{Char}(A)$ is a point of finite order n, and $f \colon A \to B$ is a surjective morphism of abelian varieties with connected fibers.

The proof of Theorem 1.4 occupies the remainder of the paper; it is by induction on the dimension of the abelian variety. Since we already know that the cohomology support loci are finite unions of linear subvarieties, the issue is to prove that every irreducible component contains a torsion point. Four important ingredients are the Fourier-Mukai transform for $\mathscr{D}_A$-modules [11, 15]; results about Fourier-Mukai transforms of holonomic $\mathscr{D}_A$-modules [18]; the theory of perverse sheaves with integer coefficients [3]; and of course Saito's theory of mixed Hodge modules [17]. Roughly speaking, they make it possible to deduce the assertion about torsion points from the following elementary special case: if V is a graded-polarizable variation of mixed Hodge structure on A with coefficients in $\mathbb{Z}$, and if $\mathbb{C}_\rho$ is a direct factor of $V \otimes_{\mathbb{Z}} \mathbb{C}$ for some $\rho \in \operatorname{Char}(A)$, then ρ must be a torsion character. The completeness of the set of components follows from the fact that $S^k_m(A, M)$ is defined over $\mathbb{Q}$, hence stable under the natural $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$-action; note that the $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$-orbit of a character ρ of order n consists exactly of the characters ρ^k with $\gcd(k, n) = 1$.

1.1.5 The conjecture of Beauville and Catanese

Now let X be a projective complex manifold. As a consequence of Theorem 1.4, we obtain a purely analytic proof for the conjecture of Beauville and Catanese.

Theorem 1.6 *Each $\Sigma^{p,q}_m(X)$ is a finite union of subsets of the form $L \otimes T$, where $L \in \operatorname{Pic}^0(X)$ is a point of finite order, and $T \subseteq \operatorname{Pic}^0(X)$ is a subtorus.*

Proof Inside the group $\operatorname{Char}(X)$ of rank one characters of the fundamental group of X, let $\operatorname{Char}^0(X)$ denote the connected component of the trivial character. If $f \colon X \to A$ is the Albanese morphism (for some choice of base point on X), then $\operatorname{Char}(f)\colon \operatorname{Char}(A) \to \operatorname{Char}^0(X)$ is an isomorphism. As above, we denote the local system corresponding to a character $\rho \in \operatorname{Char}(X)$ by the symbol $\mathbb{C}_\rho$. Define the auxiliary sets

$$\Sigma_m^k(X) = \left\{\, \rho \in \operatorname{Char}^0(X) \;\middle|\; \dim \mathbf{H}^k\bigl(X, \mathbb{C}_\rho\bigr) \geq m \,\right\};$$

by the same argument as in [1, Theorem 3], it suffices to prove that each $\Sigma_m^k(X)$ is a finite union of arithmetic subvarieties of $\operatorname{Char}^0(X)$. But this follows easily from Theorem 1.4. To see why, consider the complex of mixed Hodge modules $M = f_* \mathbb{Q}_X^H[\dim X] \in \mathrm{D}^b \operatorname{MHM}(A)$. The underlying constructible complex is $\operatorname{rat} M = \mathbf{R} f_* \mathbb{Q}[\dim X]$, and so

$$\Sigma_m^{k+\dim X}(X) = \operatorname{Char}(f)\bigl(S_m^k(A, M)\bigr).$$

Because $\mathbf{R} f_* \mathbb{Z}[\dim X]$ is a $\mathbb{Z}$-structure on M, the assertion is an immediate consequence of Theorem 1.4. $\square$

For some time, I thought that each $\Sigma_m^{p,q}(X)$ might perhaps also be complete in the sense of Definition 1.5, meaning a finite union of subsets of the form

$$\left\{\, L^{\otimes k} \;\middle|\; \gcd(k, n) = 1 \,\right\} \otimes T,$$

where n is the order of L. Unfortunately, this is not the case.

Example 1.7 Here is an example of a surface X where certain cohomology support loci are not complete. Let A be an elliptic curve. Choose a nontrivial character $\rho \in \operatorname{Char}(A)$ of order three, let $L = \mathbb{C}_\rho \otimes_{\mathbb{C}} \mathscr{O}_A$, and let $B \to A$ be the étale cover of degree three that trivializes ρ. The Galois group of this cover is $G = \mathbb{Z}/3\mathbb{Z}$, and if we view G as a quotient of $\pi_1(A, 0)$, then the three characters of G correspond exactly to $1, \rho, \rho^2$. Finally, let ω be a primitive third root of unity, and let E_ω be the elliptic curve with an automorphism of order three. Now G acts diagonally on the product $E_\omega \times B$, and the quotient is an isotrivial family of elliptic curves $f\colon X \to A$. Let us consider the variation of Hodge structure on the first cohomology groups of the fibers. Setting $H = H^1(E_\omega, \mathbb{Z})$, the corresponding representation of the fundamental group of A factors as

$$\pi_1(A, 0) \to G \to \operatorname{Aut}(H),$$

and is induced by the G-action on E_ω. This representation is the direct sum of the two characters ρ and ρ^2, because G acts as multiplication by ω and ω^2 on $H^{1,0}(E_\omega)$ and $H^{0,1}(E_\omega)$, respectively. For the same reason, $f_* \omega_X \simeq L$ and $R^1 f_* \omega_X \simeq \mathscr{O}_A$. Since $f^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is an isomorphism, the projection formula gives

$$\Sigma_1^{2,0}(X) = \{L^{-1}\} \quad \text{and} \quad \Sigma_1^{2,1}(X) = \{L^{-1}, \mathscr{O}_A\}.$$

We conclude that not all cohomology support loci of X are complete.

Note Although he does not state his result in quite this form, Pareschi [12, Scholium 4.3] shows that the set of positive-dimensional irreducible components of

$$V^0(\omega_X) = \Sigma_1^{\dim X,0}(X)$$

is complete, provided that X has maximal Albanese dimension.

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1.2 Preparation for the proof

1.2.1 Variations of Hodge structure

In what follows, A will always denote a complex abelian variety, and $g = \dim A$ its dimension. To prove Theorem 1.4, we have to show that certain complex numbers are roots of unity; we shall do this with the help of Kronecker's theorem, which says that if all conjugates of an algebraic integer have absolute value 1, then it is a root of unity. To motivate what follows, let us consider the simplest instance of Theorem 1.4, namely a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$.

Lemma 1.8 *If a local system with coefficients in $\mathbb{Z}$ underlies a polarizable variation of Hodge structure on A, then it is a direct sum of torsion points of* $\mathrm{Char}(A)$.

Proof The associated monodromy representation $\mu\colon \pi_1(A) \to \mathrm{GL}_n(\mathbb{Z})$, tensored by $\mathbb{C}$, is semisimple [4, §4.2]; the existence of a polarization implies that it is isomorphic to a direct sum of unitary characters of $\pi_1(A)$. Since μ is defined over $\mathbb{Z}$, the collection of these characters is preserved by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. This means that the values of each character, as well as all their conjugates, are algebraic integers of absolute value 1; by Kronecker's theorem, they must be roots of unity. It follows that μ is a direct sum of torsion characters. □

Corollary *Let V be a local system of $\mathbb{C}$-vector spaces on A. If V underlies a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$, all cohomology support loci of V are finite unions of arithmetic subvarieties.*

Proof By Lemma 1.8, we have $V \simeq \mathbb{C}_{\rho_1} \oplus \cdots \oplus \mathbb{C}_{\rho_n}$ for torsion points $\rho_1, \dots, \rho_n \in \mathrm{Char}(A)$. All cohomology support loci of V are then obviously contained in the set

$$\bigl\{\rho_1^{-1}, \dots, \rho_n^{-1}\bigr\},$$

and are therefore trivially finite unions of arithmetic subvarieties. □

1.2.2 Mixed Hodge modules with $\mathbb{Z}$-structure

We shall say that a mixed Hodge module has a $\mathbb{Z}$-structure if the underlying perverse sheaf, considered as a constructible complex with coefficients in $\mathbb{Q}$, can be obtained by extension of scalars from a constructible complex with coefficients in $\mathbb{Z}$. A typical example is the intermediate extension of a variation of Hodge structure with coefficients in $\mathbb{Z}$. To be precise, we make the following definition.

Definition 1.9 A $\mathbb{Z}$-*structure* on a complex of mixed Hodge modules

$$M \in \mathrm{D}^b \operatorname{MHM}(A)$$

is a constructible complex $E \in \mathrm{D}_c^b(\mathbb{Z}_A)$ such that $\operatorname{rat} M \simeq \mathbb{Q} \otimes_{\mathbb{Z}} E$.

The standard operations on complexes of mixed Hodge modules clearly respect $\mathbb{Z}$-structures. For instance, suppose that $M \in \mathrm{D}^b \operatorname{MHM}(A)$ has a $\mathbb{Z}$-structure, and that $f\colon A \to B$ is a homomorphism of abelian varieties; then $f_* M \in \mathrm{D}^b \operatorname{MHM}(B)$ again has a $\mathbb{Z}$-structure. The proof is straightforward:

$$\operatorname{rat}\bigl(f_* M\bigr) = \mathbf{R} f_*(\operatorname{rat} M) \simeq \mathbf{R} f_*\bigl(\mathbb{Q} \otimes_{\mathbb{Z}} E\bigr) \simeq \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{R} f_* E$$

By [3, Section 3.3] and [9], there are two natural perverse t-structures on the category $\mathrm{D}_c^b(\mathbb{Z}_A)$; after tensoring by $\mathbb{Q}$, both become equal to the usual perverse t-structure on $\mathrm{D}_c^b(\mathbb{Q}_A)$. We shall use the one corresponding to the perversity p_+; concretely, it is defined as follows:

$$E \in {}^{p_+}\mathrm{D}_c^{\leq 0}(\mathbb{Z}_A) \Longleftrightarrow \begin{cases} \text{for any stratum } S, \text{ the local system} \\ \mathcal{H}^m i_S^* E \text{ is zero if } m > -\dim S + 1, \\ \text{and } \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}^{-\dim S+1} i_S^* E = 0 \end{cases}$$

$$E \in {}^{p_+}\mathrm{D}_c^{\geq 0}(\mathbb{Z}_A) \Longleftrightarrow \begin{cases} \text{for any stratum } S, \text{ the local system} \\ \mathcal{H}^m i_S^! E \text{ is zero if } m < -\dim S, \\ \text{and } \mathcal{H}^{-\dim S} i_S^! E \text{ is torsion-free} \end{cases}$$

We can use the resulting formalism of perverse sheaves with integer coefficients to show that $\mathbb{Z}$-structures are also preserved under taking cohomology.

Lemma 1.10 *If $M \in \mathrm{D}^b \operatorname{MHM}(A)$ admits a $\mathbb{Z}$-structure, then each cohomology module $\mathcal{H}^k(M) \in \operatorname{MHM}(A)$ also admits a $\mathbb{Z}$-structure.*

Proof Let ${}^{p_+}\mathcal{H}^k(E)$ denote the p_+-perverse cohomology sheaf in degree k of the constructible complex $E \in \mathrm{D}_c^b(\mathbb{Z}_A)$. With this notation, we have

$$\operatorname{rat} \mathcal{H}^k(M) = {}^p\mathcal{H}^k(\operatorname{rat} M) \simeq {}^p\mathcal{H}^k\big(\mathbb{Q} \otimes_{\mathbb{Z}} E\big) \simeq \mathbb{Q} \otimes_{\mathbb{Z}} {}^{p_+}\mathcal{H}^k(E),$$

which gives the desired $\mathbb{Z}$-structure on $\mathcal{H}^k(M)$. □

There is also a notion of intermediate extension for local systems with integer coefficients. If $i \colon X \hookrightarrow A$ is a subvariety of A, and $j \colon U \hookrightarrow X$ is a Zariski-open subset of the smooth locus of X, then for any local system V on U with coefficients in $\mathbb{Z}$, one has a canonically defined p_+-perverse sheaf

$$i_*\big(j_{!*} V[\dim X]\big) \in {}^{p_+}\mathrm{D}_c^{\leq 0}(\mathbb{Z}_A) \cap {}^{p_+}\mathrm{D}_c^{\geq 0}(\mathbb{Z}_A).$$

After tensoring by $\mathbb{Q}$, it becomes isomorphic to the usual intermediate extension of the local system $\mathbb{Q} \otimes_{\mathbb{Z}} V$. This has the following immediate consequence.

Lemma 1.11 *Let M be a polarizable Hodge module. Suppose that M is the intermediate extension of $\mathbb{Q} \otimes_{\mathbb{Z}} V$, where V is a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$. Then M admits a $\mathbb{Z}$-structure.*

Proof In fact, $E = i_*\big(j_{!*} V[\dim X]\big)$ gives a $\mathbb{Z}$-structure on M. □

We conclude our discussion of $\mathbb{Z}$-structures by improving Lemma 1.8.

Lemma 1.12 *Let M be a mixed Hodge module with $\mathbb{Z}$-structure. Let $\rho \in \operatorname{Char}(A)$ be a character with the property that, for all $g \in \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$, the local system $\mathbb{C}_{g\rho}[\dim A]$ is a subobject of $\mathbb{C} \otimes_{\mathbb{Q}} \operatorname{rat} M$. Then ρ is a torsion point of* $\operatorname{Char}(A)$.

Proof Let $j \colon U \hookrightarrow A$ be the maximal open subset with the property that $j^*M = V[\dim A]$ for a graded-polarizable variation of mixed Hodge structure V. Consequently, $j^*\mathbb{C}_\rho$ embeds into the complex variation of mixed Hodge structure $\mathbb{C} \otimes_{\mathbb{Q}} V$. Since the variation is graded-polarizable, and since $j_* \colon \pi_1(U) \to \pi_1(A)$ is surjective, it follows that ρ must be unitary [5, §1.12]. On the other hand, we have $\operatorname{rat} M \simeq \mathbb{Q} \otimes_{\mathbb{Z}} E$ for a constructible complex E with coefficients in $\mathbb{Z}$. Then $H^{-\dim A} j^* E$ is a local system with coefficients in $\mathbb{Z}$, and $j^*\mathbb{C}_\rho$ embeds into its complexification. The values of the character ρ are therefore algebraic integers of absolute value 1. We get the same conclusion for all their conjugates, by applying the argument above to the characters $g\rho$, for $g \in \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$. Now Kronecker's theorem shows that ρ takes values in the roots of unity, and is therefore a torsion point of $\operatorname{Char}(A)$. $\square$

1.2.3 The Galois action on the space of characters

In this section, we study the natural action of $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$ on the space of characters, and observe that the cohomology support loci of a regular holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure are stable under this action.

The space of characters $\operatorname{Char}(A)$ is an affine algebraic variety, and its coordinate ring is easy to describe. We have $A = V/\Lambda$, where V is a complex vector space of dimension g, and $\Lambda \subseteq V$ is a lattice of rank $2g$; note that Λ is canonically isomorphic to the fundamental group $\pi_1(A, 0)$. For a field k, we denote by

$$k[\Lambda] = \bigoplus_{\lambda \in \Lambda} k e_\lambda$$

the group ring of Λ with coefficients in k; the product is determined by $e_\lambda e_\mu = e_{\lambda+\mu}$. As a complex algebraic variety, $\operatorname{Char}(A)$ is isomorphic to $\operatorname{Spec} \mathbb{C}[\Lambda]$; in particular, $\operatorname{Char}(A)$ can already be defined over $\operatorname{Spec} \mathbb{Q}$, and therefore carries in a natural way an action of the Galois group $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$.

Proposition 1.13 *Let $M \in \mathrm{D}^b \operatorname{MHM}(A)$ be a complex of mixed Hodge modules on a complex abelian variety A. Then all cohomology support loci of* $\operatorname{rat} M$ *are stable under the action of* $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$ *on* $\operatorname{Char}(A)$.

Proof The natural Λ-action on the group ring $k[\Lambda]$ gives rise to a local system of k-vector spaces $\mathcal{L}_{k[\Lambda]}$ on the abelian variety. The discussion in [18, Section 14] shows that the cohomology support loci of $\operatorname{rat} M$ are computed by the complex

$$\mathbf{R}p_*\big(\operatorname{rat} M \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \in \mathrm{D}^b_{coh}\big(\mathbb{C}[\Lambda]\big),$$

where $p\colon A \to pt$ denotes the morphism to a point. In the case at hand,

$$\mathbf{R}p_*\big(\operatorname{rat} M \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \simeq \mathbf{R}p_*\big(\operatorname{rat} M \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}[\Lambda]}\big) \otimes_{\mathbb{Q}[\Lambda]} \mathbb{C}[\Lambda]$$

is obtained by extension of scalars from a complex of $\mathbb{Q}[\Lambda]$-modules [18, Proposition 14.7]; this means that all cohomology support loci of M are defined over $\mathbb{Q}$, and therefore stable under the $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$-action on $\operatorname{Char}(A)$. $\square$

1.2.4 The Fourier-Mukai transform

In this section, we review a few results about Fourier-Mukai transforms of holonomic $\mathscr{D}_A$-modules from [18]. The Fourier-Mukai transform, introduced by Laumon [11] and Rothstein [15], is an equivalence of categories

$$\mathrm{FM}_A\colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural});$$

for a single coherent $\mathscr{D}_A$-module $\mathcal{M}$, it is defined by the formula

$$\mathrm{FM}_A(\mathcal{M}) = \mathbf{R}(p_2)_* \,\mathrm{DR}_{A\times A^\natural/A^\natural}\big(p_1^*\mathcal{M} \otimes (P^\natural, \nabla^\natural)\big),$$

where $(P^\natural, \nabla^\natural)$ is the universal line bundle with connection on $A \times A^\natural$.

The Fourier-Mukai transform satisfies several useful exchange formulas [11, Section 3.3]; recall that for $f\colon A \to B$ a homomorphism of abelian varieties,

$$f_+\colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{D}_B) \quad \text{and} \quad f^+\colon \mathrm{D}^b_{coh}(\mathscr{D}_B) \to \mathrm{D}^b_{coh}(\mathscr{D}_A)$$

denote, respectively, the direct image and the shifted inverse image functor, while $\mathbf{D}_A\colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{D}_A)^{opp}$ is the duality functor.

Theorem 1.14 *Let $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \in \mathrm{D}^b_{coh}(\mathscr{D}_A)$ and $\mathcal{N} \in \mathrm{D}^b_{coh}(\mathscr{D}_B)$.*

(a) For any homomorphism of abelian varieties $f\colon A \to B$, one has

$$\mathbf{L}(f^\natural)^* \,\mathrm{FM}_A(\mathcal{M}) \simeq \mathrm{FM}_B\big(f_+\mathcal{M}\big),$$
$$\mathbf{R}f^\natural_* \,\mathrm{FM}_B(\mathcal{N}) \simeq \mathrm{FM}_A\big(f^+\mathcal{N}\big).$$

(b) One has $\mathrm{FM}_A\big(\mathbf{D}_A\mathcal{M}\big) \simeq \langle -1_{A^\natural}\rangle^* \,\mathbf{R}\mathcal{H}om\big(\mathrm{FM}_A(\mathcal{M}), \mathscr{O}_{A^\natural}\big)$.

(c) Let $m\colon A \times A \to A$ be the addition morphism. Then one has

$$\mathrm{FM}_A\big(m_+(\mathcal{M}_1 \boxtimes \mathcal{M}_2)\big) \simeq \mathrm{FM}_A(\mathcal{M}_1) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{A^\natural}} \mathrm{FM}_A(\mathcal{M}_2).$$

Now let $\mathrm{D}_h^b(\mathscr{D}_A)$ be the full subcategory of $\mathrm{D}_{coh}^b(\mathscr{D}_A)$ consisting of cohomologically bounded and holonomic complexes. We already mentioned that the cohomology support loci $S_m^k(A, \mathcal{M})$ of a holonomic complex are finite unions of linear subvarieties; here is another result from [18] that will be used below.

Theorem 1.15 *Let $\mathcal{M}$ be a holonomic $\mathscr{D}_A$-module. Then $\mathrm{FM}_A(\mathcal{M}) \in \mathrm{D}_{coh}^{\geq 0}(\mathscr{O}_{A^\natural})$, and for any $\ell \geq 0$, one has $\operatorname{codim} \operatorname{Supp} \mathcal{H}^\ell \, \mathrm{FM}_A(\mathcal{M}) \geq 2\ell$.*

The precise relationship between the support of $\mathrm{FM}_A(\mathcal{M})$ and the cohomology support loci of $\mathcal{M}$ is given by the base change theorem, which implies that, for every $n \in \mathbb{Z}$, one has

$$\bigcup_{k \geq n} \operatorname{Supp} \mathcal{H}^k \, \mathrm{FM}_A(\mathcal{M}) = \bigcup_{k \geq n} S_1^k(A, \mathcal{M}) \tag{1.1}$$

In particular, the support of the Fourier-Mukai transform $\mathrm{FM}_A(\mathcal{M})$ is equal to the union of all the cohomology support loci of $\mathcal{M}$.

1.3 Proof of the theorem

Consider a complex of mixed Hodge modules $M \in \mathrm{D}^b \operatorname{MHM}(A)$ that admits a $\mathbb{Z}$-structure, and denote by $\operatorname{rat} M \in \mathrm{D}_c^b(\mathbb{Q}_A)$ the underlying complex of constructible sheaves. To prove Theorem 1.4, we have to show that that all cohomology support loci of M are complete unions of arithmetic subvarieties of $\operatorname{Char}(A)$.

1.3.1 Reduction steps

Our first task is to show that every $S_m^k(A, M)$ is a finite union of arithmetic subvarieties. The proof is by induction on the dimension of A; we may therefore assume that *the theorem is valid on every abelian variety of strictly smaller dimension.* This has several useful consequences.

Lemma 1.16 *Let $f\colon A \to B$ be a homomorphism from A to a lower-dimensional abelian variety B. Then every intersection*

$$S_m^k(A, M) \cap \operatorname{im} \operatorname{Char}(f)$$

is a finite union of arithmetic subvarieties.

Proof The complex $f_* M \in \mathrm{D}^b \operatorname{MHM}(B)$ again admits a $\mathbb{Z}$-structure. If we now tensor by points of $\operatorname{Char}(B)$ and take cohomology, we find that

$$\operatorname{Char}(f)^{-1} S_m^k(A, M) = S_m^k(B, f_* M).$$

By induction, we know that the right-hand side is a finite union of arithmetic subvarieties of $\operatorname{Char}(B)$; consequently, the same is true for the intersection $S_m^k(A, M) \cap \operatorname{im} \operatorname{Char}(f)$. □

The inductive assumption lets us to show that all positive-dimensional components of the cohomology support loci of M are arithmetic.

Lemma 1.17 *Let Z be an irreducible component of some $S_m^k(A, M)$. If $\dim Z \geq 1$, then Z is an arithmetic subvariety of $\operatorname{Char}(A)$.*

Proof Since Z is a linear subvariety, it suffices to prove that Z contains a torsion point. Now A is an abelian variety, and so we can find a surjective homomorphism $f \colon A \to B$ to an abelian variety of dimension $\dim A - \dim Z/2$, such that $Z \cap \operatorname{im} \operatorname{Char}(f)$ is a finite set of points. According to Lemma 1.16, the intersection is a finite union of arithmetic subvarieties, hence a finite set of torsion points. In particular, Z contains a torsion point, and is therefore an arithmetic subvariety of $\operatorname{Char}(A)$. □

Irreducible components that are already contained in a proper arithmetic subvariety of $\operatorname{Char}(A)$ can also be handled by induction.

Lemma 1.18 *Let Z be an irreducible component of $S_m^k(A, M)$. If Z is contained in a proper arithmetic subvariety of $\operatorname{Char}(A)$, then Z is itself an arithmetic subvariety.*

Proof It again suffices to show that Z contains a torsion point. For some $n \geq 1$, there is a torsion point of order n on the arithmetic subvariety that contains Z. After pushing forward by the multiplication-by-n morphism $\langle n \rangle \colon \operatorname{Char}(A) \to \operatorname{Char}(A)$, which corresponds to replacing M by its inverse image $\langle n \rangle^* M$ under $\langle n \rangle \colon A \to A$, we can assume that $Z \subseteq \operatorname{im} \operatorname{Char}(f)$, where $f \colon A \to B$ is a morphism to a lower-dimensional abelian variety. The assertion now follows from Lemma 1.16. □

The following result allows us to avoid cohomology in degree 0.

Lemma 1.19 *Let $M \in \operatorname{MHM}(A)$, and let Z be an irreducible component of some cohomology support locus of M. If $Z \neq \operatorname{Char}(A)$, then Z is contained in $S_m^k(A, M)$ for some $k \neq 0$ and some $m \geq 1$.*

Proof This follows easily from the fact that the Euler characteristic

$$\chi\bigl(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) = \sum_{k \in \mathbb{Z}} (-1)^k \dim H^k\bigl(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr)$$

is independent of the point $\rho \in \operatorname{Char}(A)$. □

1.3.2 Torsion points on components

Let Z be an irreducible component of some cohomology support locus of M. If $\dim Z \geq 1$, Lemma 1.17 shows that Z is an arithmetic subvariety; we may therefore assume that $Z = \{\rho\}$ consists of a single point. We have to prove that ρ has finite order in $\operatorname{Char}(A)$. There are three steps.

Step 1 We begin by reducing the problem to the case where M is a single mixed Hodge module. Each of the individual cohomology modules $\mathcal{H}^q(M) \in \operatorname{MHM}(A)$ also admits a $\mathbb{Z}$-structure (by Lemma 1.10); we know by induction that all positive-dimensional irreducible components of its cohomology support loci are arithmetic subvarieties. If ρ is contained in such a component, Lemma 1.18 proves that ρ is a torsion point; we may therefore assume that whenever there is some $p \neq 0$ such that $H^p\bigl(A, \operatorname{rat} \mathcal{H}^q(M) \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr)$ is nontrivial, ρ is an isolated point of the corresponding cohomology support locus. To exploit this fact, let us consider the spectral sequence

$$E_2^{p,q} = H^p\bigl(A, \operatorname{rat} \mathcal{H}^q(M) \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) \Longrightarrow H^{p+q}\bigl(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr).$$

If $E_2^{p,q} \neq 0$ for some $p \neq 0$, then ρ must be an isolated point in some cohomology support locus of $\mathcal{H}^q(M)$; in that case, we can replace M by the single mixed Hodge module $\mathcal{H}^q(M)$. If $E_2^{p,q} = 0$ for every $p \neq 0$, then the spectral sequence degenerates and

$$H^k\bigl(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) \simeq H^0\bigl(A, \operatorname{rat} \mathcal{H}^k(M) \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr).$$

But $\rho \in S_m^k(A, M)$ is an isolated point, and so by semi-continuity, it must also be an isolated point in $S_m^0\bigl(A, \mathcal{H}^k(M)\bigr)$; again, we can replace M by the single mixed Hodge module $\mathcal{H}^k(M)$.

Step 2 We now construct *another* mixed Hodge module with $\mathbb{Z}$-structure, such that the union of all cohomology support loci contains ρ but is not equal to $\operatorname{Char}(A)$. We can then use the inductive hypothesis to reduce the problem to the case where $\mathbb{C}_{\rho^{-1}}[\dim A]$ is a direct factor of

$\mathbb{C} \otimes_{\mathbb{Q}} \operatorname{rat} M$; because of Lemma 1.12, this will be sufficient to conclude that the character ρ has finite order.

The idea for the construction comes from a recent article by Krämer and Weissauer [10, Section 13]. Since $M \in \operatorname{MHM}(A)$ is a single mixed Hodge module, we can use Lemma 1.19 to arrange that $\rho \in S^k_m(A, M)$ is an isolated point for some $k \neq 0$ and some $m \geq 1$; to simplify the argument, we shall take the absolute value $|k|$ to be as large as possible. Now let $A \times \cdots \times A$ denote the d-fold product of A with itself, and let $m \colon A \times \cdots \times A \to A$ be the addition morphism. The d-fold exterior product $M \boxtimes \cdots \boxtimes M$ is mixed Hodge module on $A \times \cdots \times A$, and clearly inherits a $\mathbb{Z}$-structure from M. Setting

$$M_d = m_* \bigl(M \boxtimes \cdots \boxtimes M \bigr) \in \mathrm{D}^b \operatorname{MHM}(A),$$

it is easy to see from our choice of k that

$$H^{kd}\bigl(A, \operatorname{rat} M_d \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) \simeq H^k\bigl(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr)^{\otimes d}.$$

The right-hand side is nonzero, and so $\rho \in S^{kd}_{md}(A, M_d)$. By a similar spectral sequence argument as above, we must have

$$H^p\bigl(A, \operatorname{rat} \mathcal{H}^q(M_d) \otimes_{\mathbb{Q}} \mathbb{C}_\rho\bigr) \neq 0$$

for some $p, q \in \mathbb{Z}$ with $p + q = kd$ and $-g \leq p \leq g$. If we take $d > g$, this forces $q \neq 0$. In other words, we can find $q \neq 0$ such that ρ lies in some cohomology support locus of the mixed Hodge module $\mathcal{H}^q(M_d)$.

Lemma 1.20 *If $q \neq 0$, all nontrivial cohomology support loci of $\mathcal{H}^q(M_d)$ are properly contained in* $\operatorname{Char}(A)$.

Proof It suffices to prove this for the underlying regular holonomic $\mathscr{D}$-module $\mathcal{H}^q m_+(\mathcal{M} \boxtimes \cdots \boxtimes \mathcal{M})$. The properties of the Fourier-Mukai transform in Theorem 1.14 imply that

$$\operatorname{FM}_A\bigl(m_+(\mathcal{M} \boxtimes \cdots \boxtimes \mathcal{M})\bigr) \simeq \operatorname{FM}_A(\mathcal{M}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{A^\natural}} \cdots \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{A^\natural}} \operatorname{FM}_A(\mathcal{M}),$$

and all cohomology sheaves of this complex, except possibly in degree 0, are torsion sheaves (by Theorem 1.15). In the spectral sequence

$$\begin{aligned} E_2^{p,q} = \mathcal{H}^p \operatorname{FM}_A\bigl(\mathcal{H}^q m_+(\mathcal{M} \boxtimes \cdots \boxtimes \mathcal{M})\bigr) \\ \Longrightarrow \mathcal{H}^{p+q} \operatorname{FM}_A\bigl(m_+(\mathcal{M} \boxtimes \cdots \boxtimes \mathcal{M})\bigr), \end{aligned}$$

the sheaf $E_2^{p,q}$ is zero when $p < 0$, and torsion when $p > 0$, for the same reason. It follows that $E_2^{0,q}$ is also a torsion sheaf for $q \neq 0$, which proves the assertion. $\square$

Step 3 Now we can easily finish the proof. The mixed Hodge module $\mathcal{H}^q(M_d)$ again admits a $\mathbb{Z}$-structure by Lemma 1.10; by induction, all positive-dimensional irreducible components of its cohomology support loci are proper arithmetic subvarieties of $\operatorname{Char}(A)$. If ρ is contained in one of them, we are done by Lemma 1.18. After replacing M by $\mathcal{H}^q(M_d)$, we can therefore assume that, whenever $H^k\big(A, \operatorname{rat} M \otimes_{\mathbb{Q}} \mathbb{C}_\rho\big)$ is nontrivial, ρ is an isolated point of the corresponding cohomology support locus. Note that we now have this for all values of $k \in \mathbb{Z}$, including $k = 0$.

Let $\mathcal{M}$ denote the regular holonomic $\mathscr{D}$-module underlying the mixed Hodge module M. If $(L, \nabla) \in A^\natural$ is the flat line bundle corresponding to our character ρ, the assumptions on M guarantee that (L, ∇) is an isolated point in the support of $\operatorname{FM}_A(\mathcal{M})$. This means that, in the derived category, $\operatorname{FM}_A(\mathcal{M})$ has a direct factor supported on the point (L, ∇). But the Fourier-Mukai transform is an equivalence of categories, and so $\mathcal{M} \simeq \mathcal{M}' \oplus \mathcal{M}''$, where $\mathcal{M}'$ is a regular holonomic $\mathscr{D}$-module whose Fourier-Mukai transform is supported on (L, ∇). It is well-known that $\mathcal{M}'$ is the tensor product of $(L, \nabla)^{-1}$ and a unipotent flat vector bundle; in particular, $\mathcal{M}$ contains a sub-$\mathscr{D}$-module isomorphic to $(L, \nabla)^{-1}$. Equivalently, $\mathbb{C} \otimes_{\mathbb{Q}} \operatorname{rat} M$ has a subobject isomorphic to $\mathbb{C}_{\rho^{-1}}[\dim A]$. Because the cohomology support loci of M are stable under the $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$-action on $\operatorname{Char}(A)$ (by Proposition 1.13), the same is true for every conjugate $g\rho$, where $g \in \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$. We can now apply Lemma 1.12 to show that ρ must be a torsion point.

This concludes the proof that all cohomology support loci of M are finite unions of arithmetic subvarieties of $\operatorname{Char}(A)$.

1.3.3 Completeness of the set of components

We finish the proof of Theorem 1.4 by showing that each cohomology support locus of M is a *complete* union of arithmetic subvarieties of $\operatorname{Char}(A)$. The argument is based on the following simple criterion for completeness.

Lemma 1.21 *A finite union of arithmetic subvarieties of* $\operatorname{Char}(A)$ *is complete if and only if it is stable under the action by* $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$.

Proof For a point $\tau \in \operatorname{Char}(A)$ of order n, the orbit under the group $G = \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$ consists precisely of the characters τ^k with $\gcd(k, n) = 1$; consequently, a complete collection of arithmetic subvarieties is stable under the G-action. To prove the converse, let Z be a finite union of arithmetic subvarieties stable under the action by G. Let τL be one of its components; here L is a linear subvariety and $\tau \in \mathrm{Char}(A)$ a point of order n, say. Let p be any prime number with $\gcd(n,p) = 1$, and denote by $L[p]$ the set of points of order p. For any character $\rho \in L[p]$, we have $\mathrm{ord}(\tau\rho) = np$; the G-orbit of the set $\tau L[p]$ is therefore equal to

$$(G\tau) \cdot L[p] = \left\{ \tau^k \;\middle|\; \gcd(k,n) = 1 \right\} \cdot L[p].$$

Because the union of all the finite subsets $L[p]$ with $\gcd(n,p) = 1$ is dense in the linear subvariety L, it follows that

$$\left\{ \tau^k \;\middle|\; \gcd(k,n) = 1 \right\} \cdot L \subseteq Z;$$

this proves that Z is complete. □

Theorem 1.22 *Let $M \in \mathrm{D}^b \,\mathrm{MHM}(A)$ be a complex of mixed Hodge modules that admits a $\mathbb{Z}$-structure. Then all cohomology support loci of M are complete collections of arithmetic subvarieties of* $\mathrm{Char}(A)$*.*

Proof We already know that each $S^k_m(A,M)$ is a finite union of arithmetic subvarieties of $\mathrm{Char}(A)$. By Proposition 1.13, it is stable under the $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-action on $\mathrm{Char}(A)$; we can now apply Lemma 1.21 to conclude that $S^k_m(A,M)$ is complete. □

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