Hodge theory and Lagrangian fibrations on holomorphic symplectic manifolds

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1 Introduction

1. The purpose of this paper is to establish several new results about the Hodge theory of Lagrangian fibrations on (not necessarily compact) holomorphic symplectic manifolds. In particular, we prove two beautiful recent conjectures by Maulik, Shen and Yin; and we show, without using hyperkähler metrics, that every Lagrangian fibration gives rise to an action by the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (in the noncompact case) or $\mathfrak{sl}_4(\mathbb{C})$ (in the compact case).

2. The most interesting Lagrangian fibration is arguably the Hitchin fibration on the moduli space of stable Higgs bundles on a smooth projective curve of genus $g \geq 2$. It was famously used by Ngô in his proof of the fundamental lemma [Ngô10], and is also the central object in the $P = W$ conjecture by de Cataldo, Hausel, and Migliorini [dCHM12], recently proved by Maulik and Shen [MS22] and Hausel, Mellit, Minets, and Schiffmann [HMMS22]. But Lagrangian fibrations are also very useful for studying compact hyperkähler manifolds, which are compact holomorphic symplectic manifolds with a hyperkähler metric. For example, de Cataldo, Rapagnetta, and Saccà [dCRS21] used a pair of Lagrangian fibrations to compute the Hodge numbers of O'Grady's 10-dimensional sporadic compact hyperkähler manifold.

3. Let M be a holomorphic symplectic manifold of dimension $2n$ that is Kähler but not necessarily compact, and let $\pi\colon M \to B$ be a Lagrangian fibration over a complex manifold B of dimension n. The general fiber of π is an n-dimensional abelian variety, but very little is known about the singular fibers. The method we use in this paper is to apply the decomposition theorem; this produces certain perverse sheaves on the base of the Lagrangian fibration. In some cases, such as Ngô's support theorem [Ngô17], these perverse sheaves are controlled by what happens on the smooth locus, but in general, their behavior is a mystery. This mystery is the subject of the conjectures by Maulik, Shen, and Yin [SY22a, MSY23].

4. Our main result is that there is a very close relationship between two seemingly unrelated objects: the k-th perverse sheaf P_k in the decomposition theorem for π, and the derived direct image $\mathbf{R}\pi_* \Omega_M^{n+k}$ of the sheaf of holomorphic $(n+k)$-forms on M (see §22 Theorem). This is formulated – and proved – with the help of Saito's theory of Hodge modules, and the BGG correspondence (between graded modules over the symmetric and exterior algebras). On both sides, we need to take the associated graded with respect to a certain filtration: in the case of P_k, this is the Hodge filtration of P_k, viewed as a Hodge module; in the case of $\mathbf{R}\pi_* \Omega_M^{n+k}$, it is the perverse filtration coming from the decomposition theorem.

5. Along the way, we prove a relative Hard Lefschetz theorem for the action of the holomorphic symplectic form (in §15 Theorem), as well as the symmetry conjecture $G_{i,k} \cong G_{k,i}$ of Shen and Yin for the complexes $G_{i,k} = \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)[-i]$ (in §12 Conjecture). The structure that one gets on the direct sum of all the complexes $G_{i,k}$ looks somewhat like the "Hodge diamond" of a compact hyperkähler manifold, except that it has a hexagonal shape (see §16) and comes with an action by the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (see §17). One interesting aspect is that all of these structures are only visible in the derived category. Perhaps the most useful feature of the present work is that no restrictions on the singular fibers are needed: all the results below apply for example to the entire Hitchin fibration on the moduli space of stable Higgs bundles (provided that the rank and the degree are coprime).

6. One application of our main result is a different proof for the "numerical perverse = Hodge" symmetry for irreducible compact hyperkähler manifolds [SY22b] that does not rely on the existence of a hyperkähler metric (see §27 Theorem). Another application is that the Lie algebra $\mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{so}_6(\mathbb{C})$ acts on the cohomology of a compact holomorphic symplectic manifold with a Lagrangian fibration (see Chapter 10); this generalizes a result by Looijenga-Lunts [LL97, §4] and Verbitsky [Ver96], who proved this for irreducible compact hyperkähler manifolds. Once again, our proof does not rely on the existence of a hyperkähler metric.

1.1 Lagrangian fibrations

7. We now give a more detailed summary of the paper. Let M be a holomorphic symplectic manifold of dimension $2n$; we assume that M is Kähler, but we allow M to be noncompact. We denote by $\sigma \in H^0(M, \Omega^2_M)$ the holomorphic symplectic form; σ is nondegenerate, which means that it induces an isomorphism between the holomorphic tangent sheaf $\mathscr{T}_B$ and the sheaf of holomorphic 1-forms Ω^1_B. We need to add the assumption that $d\sigma = 0$; this would of course be automatic in the compact case (by Hodge theory). We also fix a Kähler form $\omega \in A^{1,1}(M)$; recall that ω is real and positive, and satisfies $d\omega = 0$. In the special case where M is a compact hyperkähler manifold, this might be the Kähler form of a hyperkähler metric; but in the noncompact case, any Kähler metric seems to be as good as any other.

8. Further, let $\pi\colon M \to B$ be a Lagrangian fibration on M. This means that B is a complex manifold of dimension n, and that π is a proper surjective holomorphic mapping whose smooth fibers are Lagrangian, in the sense that σ restricts to zero on every smooth fiber of π. For a very nice introduction to the general theory of Lagrangian fibrations, see the recent paper by Huybrechts and Mauri [HM22]. It is known that the smooth fibers are abelian varieties of dimension n. Moreover, according to a theorem by Matsushita [Mat00, Thm. 1], all fibers of π have dimension n, and σ pulls back to zero on a resolution of singularities (of the reduction) of every fiber. Let me emphasize again that B is assumed to be a complex manifold; in the special case where M is an irreducible compact hyperkähler manifold, this implies that B is isomorphic to $\mathbb{P}^n$ [Hwa08].

9. The starting point for the work by Shen and Yin [SY22a] is the following curious analogy between two completely different sets of objects. On the one hand, we have the sheaves of holomorphic forms Ω^{n+k}_M for $k = -n, \dots, n$. The rank of $\Omega^{n+k}_M \cong \bigwedge^{n+k} \Omega^1_M$ is of course $\binom{2n}{n+k}$. In addition, wedge product with the symplectic form induces an isomorphism

$$\sigma^k \colon \Omega^{n-k}_M \to \Omega^{n+k}_M.$$

On the other hand, we have a collection of perverse sheaves P_i on the base B of the Lagragian fibration, where P_i is defined as the i-th perverse cohomology sheaf of the complex

$\mathbf{R}\pi_*\mathbb{Q}_M[2n]$. The fact that all fibers of π have dimension n implies that P_i is nonzero only for $i = -n, \dots, n$. The restriction of P_i to the smooth locus of π is just the local system on the $(n+i)$-th cohomology of the fibers, which are n-dimensional abelian varieties. As it happens, $H^{n+i}(A, \mathbb{C}) \cong \bigwedge^{n+i} H^1(A, \mathbb{C})$, which means that the generic rank of P_i is also $\binom{2n}{n+i}$. The most striking part of the analogy is that wedge product with the Kähler form induces isomorphisms

$$\omega^i \colon P_{-i} \to P_i.$$

This result, called the relative Hard Lefschetz theorem, used to be known only for projective morphisms, but recent work by Mochizuki [Moc22] shows that it also holds for proper holomorphic mappings from Kähler manifolds.[1] The same is true for the decomposition theorem, which guarantees the existence of a decomposition

$$\mathbf{R}\pi_*\mathbb{Q}_M[2n] \cong \bigoplus_{i=-n}^{n} P_i[-i]$$

in the derived category. Since we have chosen a Kähler form, there is a preferred choice of decomposition, constructed by Deligne; this is described in §35.

1.2 The symmetry conjecture of Shen and Yin

10. Are the two objects Ω_M^{n+k} and P_k actually related in some way? In [SY22a], Shen and Yin proposed a conjectural symmetry that would relate at least certain complexes of coherent sheaves derived from the two objects. Their conjecture is formulated using Saito's theory of Hodge modules [Sai88, Sai90]. Recall that $\mathbb{Q}_M[2n]$ is actually a Hodge module of weight $2n$. We can adjust the weight by a Tate twist, making $\mathbb{Q}_M(n)[2n]$ a Hodge module of weight 0. Saito's version of the decomposition theorem

$$\mathbf{R}\pi_*\mathbb{Q}_M(n)[2n] \cong \bigoplus_{i=-n}^{n} P_i[-i]$$

then shows that each P_i is a Hodge module of weight i on B. We denote by $\mathcal{P}_i$ the underlying (regular holonomic) right $\mathscr{D}_B$-module, and by $F_\bullet\mathcal{P}_i$ its Hodge filtration, which is an increasing filtration by coherent $\mathscr{O}_B$-modules. The perverse sheaf P_i and the $\mathscr{D}_B$-module $\mathcal{P}_i$ are related by the Riemann-Hilbert correspondence:

$$P_i \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathrm{DR}(\mathcal{P}_i)$$

Here $\mathrm{DR}(\mathcal{P}_i)$ is the de Rham complex; since $\mathcal{P}_i$ is a right $\mathscr{D}_B$-module, the de Rham complex (which is usually called the "Spencer complex" in the $\mathscr{D}$-module literature) is the complex

$$\mathrm{DR}(\mathcal{P}_i) = \Big[\mathcal{P}_i \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \dots \to \mathcal{P}_i \otimes \mathscr{T}_B \to \mathcal{P}_i\Big].$$

It lives in cohomological degrees $-n, \dots, 0$, and the differential is induced by the multiplication map $\mathcal{P}_i \otimes \mathscr{T}_B \to \mathcal{P}_i$. The de Rham complex is filtered by the subcomplexes

$$F_k \,\mathrm{DR}(\mathcal{P}_i) = \Big[F_{k-n}\mathcal{P}_i \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \dots \to F_{k-1}\mathcal{P}_i \otimes \mathscr{T}_B \to F_k\mathcal{P}_i\Big],$$

[1] Locally on the base, Lagrangian fibrations are actually projective [Cam21].

and the graded pieces of this filtration give us several complexes of coherent $\mathscr{O}_B$-modules

$$\operatorname{gr}_k^F \operatorname{DR}(\mathcal{P}_i) = \Big[\operatorname{gr}_{k-n}^F \mathcal{P}_i \otimes \bigwedge^n \mathscr{T}_B \to \cdots \to \operatorname{gr}_{k-1}^F \mathcal{P}_i \otimes \mathscr{T}_B \to \operatorname{gr}_k^F \mathcal{P}_i\Big].$$

One consequence of Saito's theory is that one has an isomorphism (in the derived category)

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n-k] \cong \bigoplus_{i=-n}^{n} \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i)[-i]; \tag{10.1}$$

here the decomposition is induced by the one in the decomposition theorem.

11. In [SY22a], Shen and Yin introduced what they call the "perverse-Hodge complexes"

$$G_{i,k} = \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i)[-i],$$

although with a different choice of indexing.[2] These are complexes of coherent $\mathscr{O}_B$-modules on the base manifold B of the Lagrangian fibration. The two indices i and k have the following meaning:

1. The first index i records the cohomological degree, in the sense that $G_{i,k}$ is associated (over the smooth locus of π) with the $(n+i)$-th cohomology groups of the fibers.
2. The second index k records the holomorphic degree, in the sense that $G_{i,k}$ is associated with the sheaf Ω_M^{n+k} of holomorphic forms of degree $(n+k)$.

12. Over the smooth locus of π, the complexes $G_{i,k}$ can be described fairly explicitly using the Hodge bundles $\mathcal{V}^{p,q}$ in the variation of Hodge structure: in fact, the restriction of $G_{i,k}$ to the smooth locus is the complex

$$\Big[\Omega_B^k \otimes \mathcal{V}^{n,i} \to \Omega_B^{k+1} \otimes \mathcal{V}^{n-1,i+1} \to \cdots \to \Omega_B^n \otimes \mathcal{V}^{k,n+i-k}\Big].$$

Note again that the cohomological degree of each term is $n+i$, whereas the holomorphic degree is $n+k$. But the behavior of these complexes on the singular locus of π is quite mysterious. Shen and Yin proved that the two complexes $G_{i,k}$ and $G_{k,i}$ are isomorphic over the smooth locus of π, and motivated by this, they made the following bold conjecture.

Conjecture. *In the derived category of coherent $\mathscr{O}_B$-modules, one has $G_{i,k} \cong G_{k,i}$.*

Note that the complexes $G_{i,k}$ are in general *not* determined by their restriction to the smooth locus of π; there are also examples where $G_{i,k}$ and $G_{k,i}$ are isomorphic in the derived category, but not isomorphic as complexes [SY22a, §2.4]. In any case, if the conjecture by Shen and Yin is true, then one can combine it with Saito's isomorphism (10.1) to get

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n] \cong \bigoplus_{i=-n}^{n} G_{i,k}[k] \cong \bigoplus_{i=-n}^{n} G_{k,i}[k] = \bigoplus_{i=-n}^{n} \operatorname{gr}_{-i}^F \operatorname{DR}(\mathcal{P}_k),$$

which relates Ω_M^{n+k} and P_k at least in a sort of indirect way.

[2] In the notation of [SY22a, 0.2], one has $\mathcal{G}_{i,k} = G_{i-n,k-n}$.

13. The original motivation for §12 Conjecture is the "numerical perverse = Hodge" symmetry for compact hyperkähler manifolds in [SY22b, Thm. 0.2]. Its proof by Shen and Yin makes heavy use of the hyperkähler metric, and the conjecture arose in at attempt to find a more "local" explanation for this symmetry, and to extend it to Lagrangian fibrations on noncompact holomorphic symplectic manifolds. In terms of the complexes $G_{i,k}$, the main result in [SY22b] is the statement that

$$H^j(B, G_{i,k}) \cong H^j(B, G_{k,i})$$

for all $i, j, k \in \mathbb{Z}$, provided that M is an irreducible compact hyperkähler manifold. In my opinion, this was the most convincing piece of evidence for the conjecture.

14. Our first result explains the symmetry between $G_{i,k}$ and $G_{k,i}$ as coming from a new relative Hard Lefschetz theorem for the symplectic form (which, unlike the relative Hard Lefschetz theorem for the Kähler form, only holds in the derived category). The decomposition theorem gives us a decomposition

$$\pi_+(\omega_M, F_\bullet \omega_M) \cong \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i]$$

for the direct image of the $\mathscr{D}$-module ω_M (again with the filtration for which $\operatorname{gr}^F_{-n} \omega_M = \omega_M$), in the derived category of filtered $\mathscr{D}_B$-modules. Since the Kähler form ω is closed and of type $(1,1)$, it gives rise to a morphism

$$\omega \colon \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i] \to \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_{\bullet-1} \mathcal{P}_i)[2-i].$$

Let us denote by $\omega_j \colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+j}, F_{\bullet-1} \mathcal{P}_{i+j})[2-j]$ the individual components of ω with respect to this decomposition. The topmost component ω_2 accounts for the action of the Kähler form on the cohomology of the fibers of π. In these terms, the relative Hard Lefschetz theorem (for proper holomorphic mappings from Kähler manifolds) is saying that

$$\omega_2^i \colon (\mathcal{P}_{-i}, F_\bullet \mathcal{P}_{-i}) \to (\mathcal{P}_i, F_{\bullet-i} \mathcal{P}_i)$$

is an isomorphism for every $i \geq 1$. From ω_2, we also get a morphism of complexes

$$\omega_2 \colon G_{i,k} \to G_{i+2,k+1}[2],$$

and as a consequence of the relative Hard Lefschetz theorem, the induced morphism

$$\omega_2^i \colon G_{-i,k} \to G_{i,i+k}[2i]$$

is an isomorphism for every $i \geq 1$ and every $k \in \mathbb{Z}$.

15. The symplectic form σ is closed and of type $(2,0)$, and so it gives rise to a morphism

$$\sigma \colon \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i] \to \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_{\bullet-2} \mathcal{P}_i)[2-i],$$

again in the derived category of filtered $\mathscr{D}_B$-modules. As before, we denote the components of σ with respect to this decomposition by $\sigma_j \colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+j}, F_{\bullet-2} \mathcal{P}_{i+j})[2-j]$.

This time, we get $\sigma_2 = 0$ because π is a Lagrangian fibration and σ acts trivially on the cohomology of the fibers (see §37 Lemma). The first nonzero component of σ is therefore

$$\sigma_1 \colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+1}, F_{\bullet-2}\mathcal{P}_{i+1})[1].$$

From σ_1, we get another morphism (which now only exists in the derived category)

$$\sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2],$$

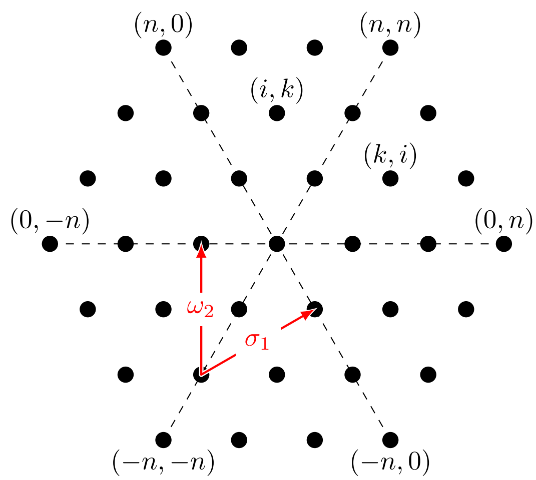
The following result might be called the "symplectic relative Hard Lefschetz theorem".

Theorem. *The induced morphism*

$$\sigma_1^k \colon G_{i,-k} \to G_{i+k,k}[2k]$$

is an isomorphism for every $k \geq 1$ and every $i \in \mathbb{Z}$.

16. The following picture may be helpful in understanding this result. The complexes $G_{i,k}$ are exact unless $-n \leq i, k \leq n$ and $-n \leq i - k \leq n$ (see §44 Lemma). We can therefore arrange them on a hexagonal grid by putting $G_{i,k}$ at the point with coordinates $i\rho + k$, where $\rho = \frac{1}{2}(-1 + \sqrt{-3})$ is a cube root of unity:



The symmetry coming from the relative Hard Lefschetz theorem for the Kähler form ω, which exchanges the two points $(-i, k)$ and $(i, i+k)$, is reflection in one of the diagonals of this hexagon; the symmetry coming from §15 Theorem, which exchanges the two points $(i, -k)$ and $(i+k, k)$, is reflection in another one.[3] These two reflections together generate the symmetric group S_3, and the reflection in the remaining diagonal that we get in this way exchanges the two points (i, k) and (k, i). This gives a conceptual proof for §12 Conjecture.

17. Note that S_3 is the Weyl group of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$; this suggests that the direct sum of all the complexes $G_{i,k}$ should form a representation of $\mathfrak{sl}_3(\mathbb{C})$. The precise statement is slightly more cumbersome because both ω_2 and σ_1 involve a shift.

[3] If we forget about the shifts that appear when applying ω_2 and σ_1

Theorem. *The two operators* $\omega_2 \colon G_{i,k} \to G_{i+2,k+1}[2]$ *and* $\sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2]$ *determine a representation of the Lie algebra* $\mathfrak{sl}_3(\mathbb{C})$ *on the object*

$$\bigoplus_{i,k=-n}^{n} G_{i,k}\Big[\lfloor \tfrac{2}{3}(i+k) \rfloor\Big],$$

in the derived category.

18. The $\mathfrak{sl}_2(\mathbb{C})$-representations determined by ω_2 and σ_1 give us two more operators

$$\mathsf{Y}_{\omega_2} \colon G_{i,k} \to G_{i-2,k-1}[-2] \quad \text{and} \quad \mathsf{Y}_{\sigma_1} \colon G_{i,k} \to G_{i-1,k-2}[-2].$$

We prove §17 Theorem by verifying that the four operators ω_2, σ_1, Y_{ω_2}, and Y_{σ_1} satisfy the Serre relations for the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. (This is similar to the $\mathfrak{sl}_2(\mathbb{C})$-action on the cohomology of a compact Kähler manifold, which is also described in terms of generators and relations.) Since ω and σ commute as 2-forms on M, it is easy to see that $[\omega_2, \sigma_1] = 0$. The nonobvious part of the Serre relations is that $[\mathsf{Y}_{\omega_2}, \mathsf{Y}_{\sigma_1}] = 0$. This sort of identity may be familiar from the work of Looijenga and Lunts [LL97, Sec. 4], specifically from the computation of the total Lie algebra acting on the cohomology of an irreducible compact hyperkähler manifold. That said, the proof is completely different in our case, because we do not have the hyperkähler metric (or harmonic forms) to work with.

19. The nontrivial Serre relation also explains very nicely the analogy between the relative Hard Lefschetz theorems for ω_2 and σ_1 that we had observed in §9.

Corollary. *The isomorphism* $G_{i,k} \cong G_{k,i}$ *can be chosen in such a way that it interchanges the action of* $\omega_2 \colon G_{i,k} \to G_{i+2,k+1}[2]$ *and* $\sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2]$.

Concretely, this is saying that the reflection along the third diagonal in the picture in §16 also interchanges the two arrows marked ω_2 and σ_1. This is a general fact about representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

1.3 Relating holomorphic forms and perverse sheaves

20. We will deduce §15 Theorem from a much more precise relationship between Ω_M^{n+i} and P_i. The main result, stated below, is a sharpening of another conjecture by Maulik, Shen and Yin [MSY23], who had proposed relating the two objects with the help of the Fourier-Mukai transform (whose existence for arbitrary Lagrangian fibrations is unfortunately still a conjecture). It is formulated using the BGG correspondence [BGG78, EFS03], which relates graded modules over the symmetric algebra $\mathcal{S}_B = \operatorname{Sym}(\mathscr{T}_B)$ and graded modules over the algebra $\Omega_B = \bigoplus_j \Omega_B^j$ of holomorphic forms on B.[4] More precisely, the BGG correspondence gives an equivalence between derived categories

$$\mathbf{R}_B \colon D^b_{coh} G(\mathcal{S}_B) \to D^b_{coh} G(\Omega_B).$$

From the filtered $\mathscr{D}_B$-module $(\mathcal{P}_i, F_\bullet \mathcal{P}_i)$, we obtain the graded $\mathcal{S}_B$-module $\operatorname{gr}^F_\bullet \mathcal{P}_i$, and under the BGG correspondence, this goes to the complex of graded Ω_B-modules

$$\mathbf{R}_B\big(\operatorname{gr}^F_\bullet \mathcal{P}_i\big) = \bigoplus_{k=-n}^{n} \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)[k] = \bigoplus_{k=-n}^{n} G_{i,k}[i+k].$$

[4]One can think of the BGG correspondence as being a linearization of the Fourier-Mukai transform.

Here the grading in the direct sum is by k, and the Ω_B-module structure is induced by the natural morphism of complexes

$$\Omega_B^j \otimes \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i) \to \operatorname{gr}_{-k-j}^F \operatorname{DR}(\mathcal{P}_i)[j]$$

that contracts forms against vector fields. The content of the BGG correspondence is that one can recover the graded $\mathcal{S}_B$-module $\operatorname{gr}_\bullet^F \mathcal{P}_i$ from this object.

21. It turns out that one can also construct a complex of graded Ω_B-modules from Ω_M^{n+k}. If we rewrite (10.1) in terms of the complexes $G_{i,k}$, it becomes

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n] \cong \bigoplus_{i=-n}^{n} G_{i,k}[k].$$

The filtration by increasing i might be called the "perverse filtration", because it is induced by the usual perverse filtration in the decomposition theorem. Now the derived pushforward $\mathbf{R}\pi_* \mathscr{O}_M$ clearly acts on this complex; it preserves the perverse filtration, but not the grading in the above isomorphism. According to a theorem of Matsushita [Mat05], $\mathbf{R}\pi_* \mathscr{O}_M$ is formal and closely related to Ω_B, in the sense that

$$\mathbf{R}\pi_* \mathscr{O}_M \cong \bigoplus_{j=0}^{n} R^j \pi_* \mathscr{O}_M[-j] \cong \bigoplus_{j=0}^{n} \Omega_B^j[-j].$$

From $\mathbf{R}\pi_* \Omega_M^{n+k}[n]$, we obtain a complex of graded Ω_B-modules by the following procedure. First, we take the associated graded with respect to the perverse filtration; this gives

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n] \cong \bigoplus_{i=-n}^{n} G_{i,k}[k]$$

the structure of a graded module over $\bigoplus_j \Omega_B^j[-j]$. Then we turn it into a complex of graded modules over Ω_B by adding suitable shifts; in this way, we arrive at

$$\bigoplus_{i=-n}^{n} G_{i,k}[i+k].$$

Here the summand $G_{i,k}$ has degree i with respect to the grading, and the Ω_B-module structure is induced by the collection of morphisms

$$R^j \pi_* \mathscr{O}_M \otimes \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i) \to \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_{i+j})$$

together with Matsushita's theorem (see §49 Theorem).

22. The next result, which is really the main result of the paper, is that the two complexes of graded Ω_B-modules derived from P_i and Ω_M^{n+i} are isomorphic in the derived category.

Theorem. *The two complexes*

$$\bigoplus_{k=-n}^{n} G_{i,k}[i+k] \quad \text{and} \quad \bigoplus_{k=-n}^{n} G_{k,i}[i+k]$$

with their respective gradings and Ω_B-module structures, are isomorphic in $D^b_{coh} G(\Omega_B)$.

According to this theorem, P_i and Ω_M^{n+i} are related in the following way. Starting from the filtered $\mathscr{D}_B$-module $(\mathcal{P}_i, F_\bullet \mathcal{P}_i)$, we first take the associated graded with respect to the Hodge filtration, and then we apply the BGG correspondence to produce a complex of graded Ω_B-modules. This complex is isomorphic to the complex that we get from $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$ by taking the associated graded with respect to the perverse filtration, and then using Matsushita's theorem to convert the action by $\mathbf{R}\pi_* \mathscr{O}_M$ into an action by Ω_B.

23. Note the very striking symmetry in this procedure: on one side, we start from a perverse sheaf and take the associated graded with respect to the Hodge filtration (which involves holomorphic forms); on the other side, we start from the sheaf of holomorphic forms and take the associated graded with respect to the perverse filtration. In fact, the perverse sheaf P_i is one of the graded pieces of the complex $\mathbf{R}\pi_* \mathbb{Q}_M(n)[2n]$ with respect to the perverse filtration, and the sheaf Ω_M^{n+i} is one of the graded pieces of the de Rham complex $\mathrm{DR}(\omega_M)$ with respect to the Hodge filtration. The main result is therefore saying that

$$\mathrm{gr}^F \circ \mathrm{gr}^P \cong \mathrm{gr}^P \circ \mathrm{gr}^F;$$

in words, taking the associated graded with respect to the Hodge filtration commutes with taking the associated graded with respect to the perverse filtration.

1.4 An outline of the proof

24. Let us a give a very brief outline of the proof, with references to the relevant sections of the paper. From the fact that $\sigma^k \colon \Omega_M^{n-k} \to \Omega_M^{n+k}$ is an isomorphism for every $k \geq 1$, we first deduce the following theorem (which is §57 Theorem below).

Theorem. *The following three statements are equivalent:*

(a) The complexes $G_{i,k}$ and $G_{k,i}$ are isomorphic in the derived category (for all i, k).

(b) For every $k = -n, \dots, n$, there is an isomorphism (in the derived category)

$$\bigoplus_{i=-n}^{n} G_{i,k}[i+k] \cong \bigoplus_{i=-n}^{n} G_{k,i}[i+k].$$

(c) For every $k \geq 1$, the morphism $\sigma_1^k \colon G_{i,-k} \to G_{i,k}[2k]$ is an isomorphism.

Along the way, we give a short proof for Matsushita's theorem (in Chapter 4), because it serves as a nice introduction to the general method. This reduces the proof of the symplectic relative Hard Lefschetz theorem to establishing the isomorphism in (b).

25. The heart of the matter is the proof of §22 Theorem. This is a result about the relationship between $\mathrm{gr}_\bullet^F \mathcal{P}_i$ and $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$ for each $i = -n, \dots n$ separately – but it turns out that by combining all these objects into one, we get some additional structure that we can exploit. With that idea in mind, we consider the direct sum

$$\bigoplus_{i=-n}^{n} \mathrm{gr}_\bullet^F \mathcal{P}_i[-i], \tag{25.1}$$

which lives in the derived category of graded modules over $\mathcal{S}_B = \operatorname{Sym}(\mathscr{T}_B)$. Under the BGG correspondence (see Chapter 7), this goes to the complex

$$G = \bigoplus_{i,k=-n}^{n} G_{i,k}[k] = \mathbf{R}_B \left(\bigoplus_{i=-n}^{n} \operatorname{gr}_{\bullet}^{F} \mathcal{P}_i[-i] \right).$$

Our starting point is a concrete description of G in terms of smooth differential forms. Saito's version of the decomposition theorem gives us an isomorphism between (25.1) and a certain complex of graded $\mathcal{S}_B$-modules $\operatorname{gr}_{\bullet}^{F} \mathcal{C}_\pi$ (see §34). If we apply the BGG correspondence to this isomorphism, we obtain an isomorphism (in the derived category)

$$G \cong (M, d),$$

where (M, d) is the complex of graded Ω_B-modules with

$$M_k^i = \pi_* \mathcal{A}_M^{n+k,n+i} \quad \text{and} \quad d = (-1)^k \bar{\partial}.$$

The module structure on (M, d) is the obvious one: a local section β of Ω_B^j acts as wedge product with the pullback $\pi^*\beta$.

26. The next idea is to transform this module structure into a different one with the help of the symplectic form σ and the Kähler form ω. This is based on the following simple construction (which is also behind Matsushita's theorem). Suppose that $\beta \in H^0(U, \Omega_B^1)$ is a holomorphic 1-form, defined on an open subset $U \subseteq B$. Using the symplectic form, we can transform the holomorphic 1-form $\pi^*\beta$ into a holomorphic vector field

$$v(\beta) \in H^0\big(\pi^{-1}(U), \mathscr{T}_M\big), \quad \pi^*\beta = v(\beta) \lrcorner \sigma,$$

where $\lrcorner$ means contraction with a vector field. Using the Kähler form, we can further transform the vector field $v(\beta)$ into a $\bar{\partial}$-closed $(0,1)$-form

$$f(\beta) \in A^{0,1}\big(\pi^{-1}(U)\big), \quad f(\beta) = -v(\beta) \lrcorner \omega.$$

All three of these objects act on the complex (M, d), either by wedge product or by contraction. We use the reflection operator (or Weil element) coming from the symplectic form σ to show that G is isomorphic to the auxiliary object

$$G_v = \bigoplus_{i,k=-n}^{n} G_{i,-k}[-k],$$

in a way that exchanges the action by $\beta \in \Omega_B^1$ and the action of the vector field $v(\beta)$. We then use the Weil operator coming from the relative Hard Lefschetz theorem for the Kähler form ω to show that G_v is in turn isomorphic to

$$G_f = \bigoplus_{i,k=-n}^{n} G_{i,k}[i+k],$$

in a way that exchanges the action by the vector field $v(\beta)$ and the action of the $(0,1)$-form $f(\beta)$. §22 Theorem follows from the isomorphism $G \cong G_f$ by a careful analysis of the BGG correspondence and some general facts about Hodge modules.

1.5 Some applications

27. One application is a new proof for the "numerical perverse = Hodge" symmetry [SY22b] for compact holomorphic symplectic manifolds. Our proof does *not* use the existence of a hyperkähler metric on M; in return, we need to assume that B is a complex manifold.

Theorem (Shen-Yin)**.** *Let M be a holomorphic symplectic manifold that is compact and Kähler. If $\pi\colon M \to B$ is a Lagrangian fibration whose base B is a complex manifold, then*

$$H^j(B, G_{i,k}) \cong H^j(B, G_{k,i}) \quad \textit{for all } i, j, k \in \mathbb{Z}.$$

28. Looijenga-Lunts [LL97, §4] and Verbitsky [Ver96] showed that cohomology of a compact hyperkähler manifold with a Lagrangian fibration carries an action by the Lie algebra $\mathfrak{so}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C})$. We prove a generalization of this result to Lagrangian fibrations on compact holomorphic symplectic manifolds.

Theorem. *Let M be a holomorphic symplectic compact Kähler manifold, and let $\pi\colon M \to B$ be a Lagrangian fibration over a compact Kähler manifold B. In this situation, the cohomology of M is a representation of the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$.*

More precisely, the weight spaces of this representation are

$$H^{i,j,k} = H^{i+j}(B, G_{j,k}) = H^j\big(B, \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)\big),$$

and the representation on

$$\bigoplus_{j,k=-n}^{n} H^{n+k,n+j}(M) \cong \bigoplus_{i,j,k=-n}^{n} H^{i,j,k}$$

is built from the two operators ω_2 and σ_1 and the action by a Kähler form on B. Besides the symplectic relative Hard Lefschetz theorem, our proof relies on some identities among differential forms on M; this is entirely different from Verbitsky's proof, which needs a hyperkähler metric and the theory of harmonic forms.

29. Another (quite elementary) byproduct is the following bound on the supports in the decomposition theorem for Lagrangian fibrations (see §47 Proposition); this only relies on the the fact the Lagrangian fibrations are equidimensional.

Proposition. *In the decomposition by strict support of the Hodge module P_i, the support of every summand has dimension $\geq |i|$.*

For example, consider the decomposition by strict support of the Hodge module P_{-n+1}, whose restriction to the smooth locus of π is the variation of Hodge structure on $H^1(M_b, \mathbb{Q})$, where $M_b = \pi^{-1}(b)$. The summand with strict support B is the intersection complex of the variation of Hodge structure. The proposition is telling us that all other summands in the decomposition by strict support must be supported on *divisors* in B.

2 Hodge modules and the decomposition theorem

30. In this chapter, we review a few relevant results about Hodge modules and introduce the main objects of study. For a short overview of Hodge modules, one can look at Saito's "Introduction to Mixed Hodge Modules" [Sai89] or my more recent survey paper [Sch20]; for more details, there are Saito's original papers [Sai88, Sai90], as well as the "Mixed Hodge Module Project" by Sabbah and myself [SS16].

31. Recall that a Hodge module on a complex manifold B has three components: a perverse sheaf P with coefficients in $\mathbb{Q}$; a right $\mathscr{D}_B$-module $\mathcal{P}$; and a good filtration $F_\bullet \mathcal{P}$ by coherent $\mathscr{O}_B$-modules. The three components are related by the Riemann-Hilbert correspondence: the precise requirement is that

$$\mathrm{DR}(\mathcal{P}) \cong P \otimes_{\mathbb{Q}} \mathbb{C}$$

are isomorphic as perverse sheaves with coefficients in $\mathbb{C}$. The de Rham complex (or Spencer complex) of the right $\mathscr{D}$-module $\mathcal{P}$ is the complex

$$\mathrm{DR}(\mathcal{P}) = \Big[\mathcal{P} \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \cdots \to \mathcal{P} \otimes \mathscr{T}_B \to \mathcal{P}\Big]$$

which lives in cohomological degrees $-n, \dots, 0$, where $n = \dim B$. The differential in the de Rham complex is given by the (local) formula

$$\delta \colon \mathcal{P} \otimes \textstyle\bigwedge^k \mathscr{T}_B \to \mathcal{P} \otimes \bigwedge^{k-1} \mathscr{T}_B, \quad \delta(s \otimes \partial_J) = \displaystyle\sum_{j=1}^n \mathrm{sgn}(J, j) \cdot s\partial_j \otimes \partial_{J \setminus \{j\}}.$$

Here the notation is as follows. Let $t_1, \dots, t_n$ be local holomorphic coordinates on B, and denote by $\partial_j = \partial/\partial t_j$ the resulting holomorphic vector fields. For any subset $J \subseteq \{1, \dots, n\}$, we list the elements in increasing order as $j_1 < \cdots < j_\ell$, and then define

$$\partial_J = \partial_{j_1} \wedge \cdots \wedge \partial_{j_\ell}$$

with the convention that this expression equals 1 when J is empty. We also define

$$\mathrm{sgn}(J, j) = \begin{cases} (-1)^{k-1} & \text{if } j = j_k, \\ 0 & \text{if } j \notin J. \end{cases}$$

Note that we are always using Deligne's Koszul sign rule, according to which swapping two elements of degrees p and q leads to a sign $(-1)^{pq}$; this is the reason for the factor $\mathrm{sgn}(J, j)$.

32. The de Rham complex $\mathrm{DR}(\mathcal{P})$ is filtered by the subcomplexes

$$F_k \, \mathrm{DR}(\mathcal{P}) = \Big[F_{k-n}\mathcal{P} \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \cdots \to F_{k-1}\mathcal{P} \otimes \mathscr{T}_B \to F_k\mathcal{P}\Big].$$

The graded pieces of this filtration give us a collection of complexes of coherent $\mathscr{O}_B$-modules

$$\mathrm{gr}_k^F \, \mathrm{DR}(\mathcal{P}) = \Big[\mathrm{gr}_{k-n}^F \mathcal{P} \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \cdots \to \mathrm{gr}_{k-1}^F \mathcal{P} \otimes \mathscr{T}_B \to \mathrm{gr}_k^F \mathcal{P}\Big].$$

Since the rational structure on P is mostly irrelevant for our purposes, we generally work with the underlying filtered $\mathscr{D}$-module $(\mathcal{P}, F_\bullet \mathcal{P})$.

33. As in the introduction, let M be a holomorphic symplectic complex manifold of dimension $2n$, and let $\pi \colon M \to B$ be a Lagrangian fibration. Then $\mathbb{Q}_M[2n]$ is a Hodge module of weight $2n$ on M, and so the Tate twist $\mathbb{Q}_M(n)[2n]$ has weight 0. The underlying filtered $\mathscr{D}$-module is ω_M, with the filtration for which $\mathrm{gr}_{-n}^F \omega_M = \omega_M$. Since we are interested in the cohomology of the fibers, we now apply the decomposition theorem [Sai88, Thm. 5.3.1]; this holds for proper holomorphic mappings from Kähler manifolds by recent work of Mochizuki [Moc22]. According to the decomposition theorem, the direct image decomposes as

$$\mathbf{R}\pi_* \mathbb{Q}_M(n)[2n] \cong \bigoplus_{i=-n}^{n} P_i[-i],$$

where each P_i is a polarizable Hodge module of weight i on the complex manifold B. Note that P_i can only be nonzero for $i = -n, \dots, n$; this is because all fibers of π have dimension n by [Mat00, Thm. 1]. On the open subset of B over which π is submersive, the fibers of π are n-dimensional abelian varieties, and P_i is just the variation of Hodge structure on the $(n+i)$-th cohomology of the fibers. If we write $(\mathcal{P}_i, F_\bullet \mathcal{P}_i)$ for the filtered $\mathscr{D}$-module underlying P_i, we get an induced decomposition

$$\pi_+\big(\omega_M, F_\bullet \omega_M\big) \cong \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i].$$

in the derived category of filtered right $\mathscr{D}_B$-modules.

34. We are going to need a more concrete description for the direct image of $(\omega_M, F_\bullet \omega_M)$ in terms of smooth forms. This is easily derived from Saito's formalism of induced $\mathscr{D}$-modules [Sai88, §2.1]. Let $\mathcal{A}_M^{p,q}$ be the sheaf of smooth (p,q)-forms on M; this is a fine sheaf, and therefore acyclic for the functor π_*. Let $\mathcal{C}_\pi$ be the complex of filtered right $\mathscr{D}_B$-modules whose i-th term is the filtered right $\mathscr{D}_B$-module

$$\mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_* \mathcal{A}_M^{n+p,n+q} \otimes_{\mathscr{O}_M} (\mathscr{D}_B, F_{\bullet+p} \mathscr{D}_B),$$

and whose differential is given (in local coordinates) by the formula

$$d_\pi \colon \mathcal{C}_\pi^i \to \mathcal{C}_\pi^{i+1}, \quad d_\pi\big(\alpha \otimes D\big) = d\alpha \otimes D + \sum_{j=1}^{n} \pi^*(dt_j) \wedge \alpha \otimes \partial_j D,$$

where $\partial_j = \partial/\partial t_j$. The indexing is done is such a way that $(\mathcal{P}_i, F_\bullet \mathcal{P}_i)$ is exactly the i-th cohomology module of the complex $\mathcal{C}_\pi$; the decomposition theorem tells us that

$$\mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i] \tag{34.1}$$

in the derived category of filtered right $\mathscr{D}_B$-modules. (More precisely, this is a consequence of the strictness property for the direct image of the underlying filtered $\mathscr{D}$-modules.) Passing to the associated graded objects, we get an isomorphism

$$\operatorname{gr}_\bullet^F \mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i]$$

in the derived category of graded modules over $\operatorname{gr}_\bullet^F \mathscr{D}_B \cong \operatorname{Sym}^\bullet(\mathscr{T}_B)$. The left-hand side is the complex with terms

$$\operatorname{gr}_\bullet^F \mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_* \mathcal{A}_M^{n+p,n+q} \otimes_{\mathscr{O}_M} \operatorname{Sym}^{\bullet+p}(\mathscr{T}_B),$$

and with differential (again in local coordinates)

$$d_\pi\big(\alpha \otimes P\big) = \bar{\partial}\alpha \otimes P + \sum_{j=1}^{n} \pi^*(dt_j) \wedge \alpha \otimes \partial_j P.$$

Compare this with Laumon's description [Lau83, Constr. 2.3.3] of the associated graded of the direct image in the derived category of filtered $\mathscr{D}$-modules.

35. Let $\omega \in A^{1,1}(M)$ be a Kähler form on M. Once we have made this choice, there is a preferred decomposition in the decomposition theorem, constructed by Deligne [Del94]; with considerable understatement, Deligne calls it "less bad" than the others. This works as follows. Since $d\omega = 0$, the Kähler form induces a morphism of complexes

$$\omega\colon \mathcal{C}_\pi \to \mathcal{C}_\pi(-1)[2]$$

that increases the cohomological degree by 2 and decreases the degree with respect to the filtration by 1. For any choice of decomposition in the decomposition theorem, the isomorphism in (34.1) lets us break up

$$\omega\colon \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i] \to \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_{\bullet-1} \mathcal{P}_i)[2-i]$$

into a finite sum $\omega = \omega_2 + \omega_1 + \omega_0 + \cdots$, where each component ω_j is a morphism

$$\omega_j\colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+j}, F_{\bullet-1} \mathcal{P}_{i+j})[2-j]$$

in the derived category of filtered right $\mathscr{D}_B$-modules. According to the relative Hard Lefschetz theorem, the morphism

$$\omega_2^i\colon (\mathcal{P}_{-i}, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_i, F_{\bullet-i} \mathcal{P}_i)$$

is an isomorphism for every $i \geq 1$. This means that we get a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the direct sum

$$\bigoplus_{i=-n}^{n} \mathcal{P}_i.$$

If $\mathsf{H}, \mathsf{X}, \mathsf{Y} \in \mathfrak{sl}_2(\mathbb{C})$ denote the three standard generators, then X acts as ω_2 and H acts as multiplication by the integer i on the summand $\mathcal{P}_i$. Deligne proves that there is a unique choice of decomposition for which the components ω_j with $j \leq 1$ are *primitive*, meaning that they commute with the operator Y in the $\mathfrak{sl}_2(\mathbb{C})$-representation. Since the weight of a primitive element (with respect to $\operatorname{ad}\mathsf{Y}$) must be ≤ 0, it follows that $\omega_1 = 0$. In general, Deligne's decomposition tends to eliminate unwanted components in the decomposition of various operators; we will exploit this effect later on.

36. Let us now turn our attention to the symplectic form $\sigma \in H^0(M, \Omega_M^2)$. Since we are assuming that $d\sigma = 0$, the symplectic form also induces a morphism of complexes

$$\sigma\colon \mathcal{C}_\pi \to \mathcal{C}_\pi(-2)[2]$$

that increases the cohomological degree by 2 and decreases the degree with respect to the filtration by 2. Using (34.1), we again get a decomposition of

$$\sigma\colon \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i] \to \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_{\bullet-2} \mathcal{P}_i)[2-i]$$

into a finite sum $\sigma = \sigma_2 + \sigma_1 + \sigma_0 + \cdots$, where each component σ_j is now a morphism

$$\sigma_j\colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+j}, F_{\bullet-2} \mathcal{P}_{i+j})[2-j].$$

37. The Lagrangian condition implies the vanishing of the topmost component σ_2.

Lemma. *We have* $\sigma_2 = 0$.

Proof. Since there is no shift, $\sigma_2 \colon \mathcal{P}_i \to \mathcal{P}_{i+2}$ is a morphism of right $\mathscr{D}_B$-modules. Both $\mathscr{D}$-modules underlie polarizable Hodge modules on B, and therefore admit a decomposition by strict support [Sai88, §5.1]. It is then enough to show that σ_2 vanishes on every summand in this decomposition; the reason is that morphisms of $\mathscr{D}$-modules respect the decomposition by strict support. If we take one of the summands of the Hodge module P_i, say with strict support $Z \subseteq B$, then on a dense open subset of Z, it comes from a variation of Hodge structure of weight $i - \dim Z$ [Sai88, Lem. 5.1.10]. The strict support condition then means that we only have to check that the restriction of σ_2 to a general point $b \in Z$ is zero. Let $i_b \colon \{b\} \to B$ be the inclusion. By proper base change for Hodge modules, we have

$$i_b^* \mathbf{R}\pi_* \mathbb{Q}_M(n)[2n] \cong \mathbf{R}\pi_* \mathbb{Q}_{M_b}(n)[2n],$$

where $M_b = \pi^{-1}(b)$; consequently, $H^{-\dim Z} i_b^* P_i$, which is a Hodge structure of weight $i - \dim Z$, is isomorphic to a direct summand in $H^{2n+i-\dim Z}(M_b, \mathbb{Q})(n)$. If we let $\tilde{M}_b \to M_b$ be a resolution of singularities, then for weight reasons, the composition

$$H^{-\dim Z} i_B^* P_i \to H^{2n+i-\dim Z}(M_b, \mathbb{Q})(n) \to H^{2n+i-\dim Z}(\tilde{M}_b, \mathbb{Q})(n)$$

is injective. This reduces the problem to showing that the pullback of σ to $\tilde{M}_b$ is trivial; but this follows from the fact that $\pi \colon M \to B$ is Lagrangian, according to a theorem by Matsushita [Mat00, Thm. 1]. ☐

38. Since we are using Deligne's decomposition, we can say a bit more about the other components of $\sigma = \sigma_1 + \sigma_0 + \cdots$. This is not really going to play a role in what follows, but the same kind of proof will appear later on.

Lemma. *We have* $[\omega_2, \sigma_1] = 0$*, and the components* σ_j *with* $j \leq 0$ *are primitive (with respect to the representation of* $\mathfrak{sl}_2(\mathbb{C})$ *determined by* ω_2*).*

Proof. Since the two forms ω and σ commute, we get $[\omega, \sigma] = 0$. Decomposing this relation by degree, we find that $[\omega_2, \sigma_1] = 0$; likewise, $[\omega_2, \sigma_0] = 0$, and because σ_0 has weight 0 (in the $\mathfrak{sl}_2(\mathbb{C})$-representation), this means that σ_0 is primitive. We can therefore assume by induction that $\sigma_0, \dots, \sigma_{-k+1}$ are primitive for some $k \geq 1$. Let us prove that σ_{-k} is also primitive. From the relation $[\omega, \sigma] = 0$, we get

$$[\omega_2, \sigma_{-k}] + [\omega_0, \sigma_{-k+2}] + \cdots + [\omega_{-k+3}, \sigma_1] = 0.$$

We know that $(\operatorname{ad} \omega_2)^{j+1} \omega_{-j} = 0$ for all $j \geq 0$, due to the fact that ω_{-j} is primitive; similarly, $(\operatorname{ad} \omega_2)^{j+1} \sigma_{-j} = 0$ for $j = 0, \dots, k-1$. Now $\operatorname{ad} \omega_2$ is a derivation, and so

$$-(\operatorname{ad} \omega_2)^{k+1} \sigma_{-k} = (\operatorname{ad} \omega_2)^k [\omega_0, \sigma_{-k+2}] + \cdots + (\operatorname{ad} \omega_2)^k [\omega_{-k+3}, \sigma_1] = 0.$$

Since σ_{-k} has weight $-k$, this proves that it is primitive. ☐

39. We are going to need two other facts about Hodge modules. The first is the compatibility of the de Rham complex with direct images [Sai88, §2.3.7]. It says that

$$\mathbf{R}\pi_* \operatorname{gr}^F_{-k} \operatorname{DR}(\omega_M) \cong \bigoplus_{i=-n}^{n} \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)[-i],$$

where we give ω_M the filtration for which $\operatorname{gr}^F_{-n} \omega_M = \omega_M$, in accordance with the Tate twist in $\mathbb{Q}_M(n)[2n]$. Since we have $\operatorname{gr}^F_{-k} \operatorname{DR}(\omega_M) = \Omega_M^{n+k}[n-k]$, it follows that

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n-k] \cong \bigoplus_{i=-n}^{n} \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)[-i]. \tag{39.1}$$

40. The second fact about Hodge modules concerns duality. Let $\mathbf{D}_B$ denote the duality functor on Hodge modules. The polarization on $\mathbb{Q}_M(n)[2n]$ induces an isomorphism $P_{-i} \cong \mathbf{D}_B(P_i)$ between Hodge modules of weight i. On the level of filtered $\mathscr{D}$-modules, this gives us an isomorphism of right $\mathscr{D}_B$-modules

$$\mathcal{P}_{-i} \cong \omega_B \otimes \mathbf{R}\mathcal{H}om_{\mathscr{D}_B}\big(\mathcal{P}_i, \mathscr{D}_B\big)[n],$$

compatible with the filtrations on both sides. Passing to the associated graded modules over $\operatorname{gr}^F_\bullet \mathscr{D}_B \cong \operatorname{Sym}(\mathscr{T}_B)$, we get

$$\operatorname{gr}^F_\bullet \mathcal{P}_{-i} \cong \omega_B \otimes \mathbf{R}\mathcal{H}om_{\operatorname{Sym}(\mathscr{T}_B)}\Big(\operatorname{gr}^F_\bullet \mathcal{P}_i, \operatorname{Sym}(\mathscr{T}_B)\Big)[n],$$

where sections of $\operatorname{Sym}^j(\mathscr{T}_B)$ act with an extra factor of $(-1)^j$ on the right-hand side (due to the sign in the conversion from left to right $\mathscr{D}$-modules). The important fact, which is hidden inside the definition of the duality functor for Hodge modules, is that $\operatorname{gr}^F_\bullet \mathcal{P}_i$ is an n-dimensional Cohen-Macaulay module over $\operatorname{Sym}^\bullet(\mathscr{T}_B)$ [Sai88, Lem 5.1.13]. In geometric terms, $\operatorname{gr}^F_\bullet \mathcal{P}_i$ gives a coherent sheaf on the cotangent bundle T^*B, whose support is the (n-dimensional) characteristic variety of the $\mathscr{D}$-module $\mathcal{P}_i$, and the statement is that this sheaf is Cohen-Macaulay. This is one of the special properties of Hodge modules, and the proof of §22 Theorem would not work without this fact.

3 Basic properties of the complexes $G_{i,k}$

41. The conjecture by Shen and Yin is about the complexes of coherent $\mathscr{O}_B$-modules

$$G_{i,k} = \operatorname{gr}^F_{-k} \operatorname{DR}(\mathcal{P}_i)[-i].$$

In this chapter, we are going to look at the basic properties of these complexes. The results that we prove here only rely on the fact that M and B are complex manifolds and that all fibers of $\pi\colon M \to B$ have the same dimension.[5]

42. The first observation is that the $G_{i,k}$ are related to the direct image of Ω_M^{n+k} under the Lagrangian fibration $\pi\colon M \to B$. Indeed, if we rewrite (39.1) in these terms, we get

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n-k] \cong \bigoplus_{i=-n}^{n} G_{i,k}. \tag{42.1}$$

Since P_i can only be nonzero for $-n \leq i \leq n$, and since Ω_M^{n+k} can only be nonzero for $-n \leq k \leq n$, it follows that $G_{i,k} = 0$ unless $-n \leq i, k \leq n$.

[5] I thank Junliang Shen for pointing this out to me.

43. Next, let us see what duality can tell us about the complexes $G_{i,k}$. From the fact that $\mathbf{D}_B(P_i) \cong P_{-i}$ are isomorphic as Hodge modules, we get an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\Big(\mathrm{gr}_k^F \mathrm{DR}(\mathcal{P}_i), \omega_B[n]\Big) \cong \mathrm{gr}_{-k}^F \mathrm{DR}(\mathcal{P}_{-i}),$$

and therefore an isomorphism between $G_{-i,-k}$ and the Grothendieck dual of $G_{i,k}$:

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\big(G_{i,k}, \omega_B[n]\big) \cong G_{-i,-k}; \tag{43.1}$$

44. The isomorphism in (42.1) is also good for computing the amplitude of the complexes $G_{i,k}$, which is in agreement with §12 Conjecture. We know that all fibers of the Lagrangian fibration $\pi\colon M \to B$ have dimension n, and so the left-hand side of (42.1) is concentrated in degrees $\{-n+k, \dots, k\}$. The same thing is therefore true for the individual summands $G_{i,k}$. On the other hand, we have $F_{-n-1}\mathcal{P}_i = 0$, and so all nonzero terms in

$$G_{i,k} = \Big[\mathrm{gr}_{-k-n}^F \mathcal{P}_i \otimes \textstyle\bigwedge^n \mathscr{T}_B \to \cdots \to \mathrm{gr}_{-k-1}^F \mathcal{P}_i \otimes \mathscr{T}_B \to \mathrm{gr}_{-k}^F \mathcal{P}_i\Big][-i] \tag{44.1}$$

live in cohomological degrees $\big\{-n+\max(i,i+k), \dots, i\big\}$. Taken together with (43.1), these simple observations prove the following lemma.

Lemma. *The complex $G_{i,k}$ is concentrated in degrees*

$$\big\{-n+\max(i,k,i+k), \dots, \min(i,k,i+k)\big\}.$$

In particular, it is exact unless $-n \leq i-k \leq n$.

The bound on the amplitude is symmetric in i and k, as predicted by §12 Conjecture. Note that the complex $G_{i,k}$ is exact unless $|i| \leq n$ and $|k| \leq n$ and $|i-k| \leq n$. This is the reason for the hexagonal shape of the drawing in §16.

45. One nice consequence of the lemma is a sharp bound on the generation level of the Hodge filtration on P_i; this is very hard to come by in general. The complex $\mathrm{gr}_{-k}^F \mathrm{DR}(\mathcal{P}_i) = G_{i,k}[i]$ has no cohomology in degree 0 provided that $k-i<0$, and this means that

$$\mathrm{gr}_{p-1}^F \mathcal{P}_i \otimes \mathscr{T}_B \to \mathrm{gr}_p^F \mathcal{P}_i$$

is surjective for $p \geq -i+1$. The Hodge filtration on the $\mathscr{D}$-module $\mathcal{P}_i$ is therefore generated in degree $-i$; in symbols, $F_{-i+j}\mathcal{P}_i = F_{-i}\mathcal{P}_i \cdot F_j\mathscr{D}_B$ for $j \geq 0$. In Saito's terminology [Sai09], this is saying that the generation level of P_i is $\leq -i$. (The bound is of course achieved over the smooth locus of π, since the smooth fibers are n-dimensional abelian varieties.)

46. Another consequence is that we can compute the projective amplitude of $G_{i,k}$.

Lemma. *On every Stein open subset of B, the complex $G_{i,k}$ is isomorphic (in the derived category) to a complex of locally free $\mathscr{O}_B$-modules concentrated in degrees*

$$\big\{-n+\max(i,k,i+k), \dots, \min(i,k,i+k)\big\}.$$

Proof. After restricting to the open subset in question, we may assume that B is a Stein manifold. In particular, every coherent $\mathscr{O}_B$-module has a bounded resolution by locally free $\mathscr{O}_B$-modules. In the derived category, the complex $G_{i,k}$ is therefore isomorphic to a bounded complex $\mathscr{E}^\bullet$ of locally free $\mathscr{O}_B$-modules, where $\mathscr{E}^j = 0$ for $j > \min(i,k,i+k)$. The dual complex $\mathcal{H}om_{\mathscr{O}_B}(\mathscr{E}^{-\bullet}, \omega_B)$ computes $\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\bigl(G_{i,k}, \omega_B\bigr) \cong G_{-i,-k}[-n]$, and according to §44 Lemma, the complex $G_{-i,-k}[-n]$ is concentrated in degrees

$$\bigl\{-\min(i,k,i+k), \dots, n - \max(i,k,i+k)\bigr\}.$$

After truncating the complex $\mathcal{H}om_{\mathscr{O}_B}(\mathscr{E}^{-\bullet}, \omega_B)$ in degrees $\leq n - \max(i,k,i+k)$, it becomes a complex of locally free $\mathscr{O}_B$-modules in degrees

$$\bigl\{-\min(i,k,i+k), \dots, n - \max(i,k,i+k)\bigr\}.$$

We now get the result for the original complex $G_{i,k}$ by dualizing again. □

Taken together with (42.1), this tells us that $\mathbf{R}\pi_*\Omega_M^{n+k}$ is isomorphic, on every Stein open subset of B, to a complex of locally free $\mathscr{O}_B$-modules in degrees $\{0, \dots, n\}$. Again, this is obviously true over the smooth locus of π; the surprising thing is that it continues to be true on the locus where the fibers are singular.

47. Recall that any polarizable Hodge module admits, even locally on B, a decomposition by strict support [Sai88, §5.1]. The bound on the amplitude of the complexes $G_{i,k}$ puts the following unexpected restriction on the structure of the Hodge modules P_i.

Proposition. *In the decomposition by strict support of the Hodge module P_i, the support of every summand has dimension $\geq |i|$.*

Proof. Since $P_{-i} \cong P_i(i)$ by the relative Hard Lefschetz theorem, we may assume without loss of generality that $i \geq 0$. Let P be one of the summands in the decomposition of P_i by strict support. For any $k \geq i$, consider the complex $\operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P})$. It is supported on $\operatorname{Supp} P$ and lives in degrees ≤ 0, and so the dual complex

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\bigl(\operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}), \omega_B\bigr)$$

is concentrated in degrees $\geq r$, where $r = \operatorname{codim}_B(\operatorname{Supp} P)$. It is also a direct summand in

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\bigl(\operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i), \omega_B\bigr) = \mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\bigl(G_{i,k}, \omega_B[n]\bigr)[-n-i] \cong G_{-i,-k}[-n-i],$$

and by §44 Lemma, it is therefore concentrated in degrees $\leq n-k$. Now if we had $r > n-i$, then the complex $\operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P})$ would be exact for every $k \geq i$, and so $\operatorname{gr}_{-k}^F \mathcal{P} = 0$ for $k \geq i$, and therefore $F_{-i}\mathcal{P} = 0$. But this would say that $\mathcal{P} = 0$, because the Hodge filtration on $\mathcal{P}_i$ (and therefore on $\mathcal{P}$) is generated in degree $-i$. Since this is impossible, we get $r \leq n-i$. □

48. We end this chapter by recording the effect of the Kähler form ω and the symplectic form σ on the complexes $G_{i,k}$. We already said in the introduction that the relative Hard Lefschetz theorem can be interpreted as a symmetry among the complexes $G_{i,k}$. From $\omega_2 \colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+2}, F_{\bullet-1}\mathcal{P}_{i+2})$, we obtain a morphism of complexes

$$\omega_2 \colon G_{i,k} \to G_{i+2,k+1}[2],$$

and the relative Hard Lefschetz theorem implies that

$$\omega_2^i \colon G_{-i,k} \to G_{i,i+k}[2i] \tag{48.1}$$

is an isomorphism for every $i \geq 1$. Similarly, $\sigma_1 \colon (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_{i+1}, F_{\bullet-2}\mathcal{P}_{i+1})[1]$ gives us a morphism (in the derived category)

$$\sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2],$$

and the "symplectic relative Hard Lefschetz theorem" (in §15 Theorem) claims that

$$\sigma_1^k \colon G_{i,-k} \to G_{i+k,k}[2k] \tag{48.2}$$

is also an isomorphism (in the derived category). We will prove this in Chapter 5.

4 Matsushita's theorem

49. The fact that $\sigma^k \colon \Omega_M^{n-k} \to \Omega_M^{n+k}$ is an isomorphism for every $k \geq 1$ gives us at least some information about the morphisms $\sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2]$. On its own, this is not strong enough to prove §15 Theorem, but it does lead to a rather short proof for the following theorem by Matsushita [Mat05, Thm. 1.3], mentioned in the introduction.

Theorem (Matsushita)**.** *Let $\pi \colon M \to B$ be a Lagrangian fibration on a holomorphic symplectic manifold of dimension $\dim M = 2n$. If M is Kähler, one has*

$$\mathbf{R}\pi_* \mathscr{O}_M \cong \bigoplus_{i=0}^{n} \Omega_B^i[-i].$$

In this chapter, we explain the proof of Matsushita's result, to demonstrate in an interesting special case how to use the symplectic form σ.

50. Since M is holomorphic symplectic, σ^n gives an isomorphism between $\mathscr{O}_M$ and the canonical bundle Ω_M^{2n}. From (39.1) with $k = n$, we therefore get

$$\mathbf{R}\pi_* \mathscr{O}_M \cong \mathbf{R}\pi_* \Omega_M^{2n} \cong \bigoplus_{i=-n}^{n} \operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{P}_i)[-i].$$

In the special case where M and B are projective complex manifolds, this kind of result was first proved by Kollár [Kol86b]. In order to prove §49 Theorem, it is then enough to show that $\operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{P}_i) \cong \Omega_B^i$. (Note that $\operatorname{gr}_{-n}^F \operatorname{DR}(\mathcal{P}_i) = \operatorname{gr}_{-n}^F \mathcal{P}_i$ is actually a sheaf, due to the fact that $F_{-n-1}\mathcal{P}_i = 0$.)

51. We will deduce this from the isomorphism $\sigma^k \colon \Omega_M^{n-k} \to \Omega_M^{n+k}$. Recall from (42.1) that

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n-k] \cong \bigoplus_{i=-n}^{n} G_{i,k}.$$

If we take cohomology in the lowest possible degree, this gives

$$\pi_* \Omega_M^{n+k} \cong \bigoplus_{i=0}^{n-k} \mathcal{H}^{k-n} G_{-i,k}$$

for every $k \geq 0$, due to the bound on the amplitude of the complex $G_{i,k}$ in §44 Lemma. In a similar manner, we find (still for $k \geq 0$) that

$$\pi_* \Omega_M^{n-k} \cong \bigoplus_{i=k}^{n} \mathcal{H}^{-k-n} G_{-i,-k}.$$

From the fact that $\sigma^k \colon \Omega_M^{n-k} \to \Omega_M^{n+k}$ is an isomorphism, we now deduce that

$$\sigma^k \colon \bigoplus_{i=k}^{n} \mathcal{H}^{-k-n} G_{-i,-k} \to \bigoplus_{i=0}^{n-k} \mathcal{H}^{k-n} G_{-i,k} \tag{51.1}$$

must be an isomorphism for every $k \geq 1$.

52. Let us look at how this isomorphism acts on the summand with $i = n$. Recall from §37 Lemma that $\sigma = \sigma_1 + \sigma_0 + \cdots$, where the individual components are morphisms

$$\sigma_j \colon G_{i,k} \to G_{i+j,k+2}[2-j]$$

in the derived category. By expanding $\sigma^k = (\sigma_1 + \sigma_0 + \cdots)^k$, we find that the component of highest degree is exactly the morphism

$$\sigma_1^k \colon \mathcal{H}^{-k-n} G_{-n,-k} \to \mathcal{H}^{k-n} G_{-n+k,k}. \tag{52.1}$$

All the other components of σ^k go into the remaining summands (with $i < n-k$) of the second sum. This puts us into the situation of the following abstract lemma.

Lemma. *Let A, B, C, D be objects in an abelian category. Suppose that we have morphisms $f \colon A \to C$, $g \colon B \to C$, and $h \colon B \to D$, such that*

$$A \oplus B \xrightarrow{\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}} C \oplus D$$

is an isomorphism. Then f is injective, $C \cong A \oplus \operatorname{coker}(f)$ and $B \cong \operatorname{coker}(f) \oplus D$.

Proof. It is easy to see that f is injective and that the composition

$$B \to C \oplus D \to \operatorname{coker}(f) \oplus D$$

is an isomorphism. Now the composition

$$\operatorname{coker}(f) \longrightarrow B \xrightarrow{g} C \longrightarrow \operatorname{coker}(f)$$

is the identity, and this gives us the desired splitting $C \cong A \oplus \operatorname{coker}(f)$. □

53. We apply this lemma to the isomorphism in (51.1), letting A be the n-th summand in the first sum and C the $(n-k)$-th summand in the second sum. The result is that the morphism in (52.1) is injective, and that we have a direct sum decomposition

$$\mathcal{H}^{k-n} G_{-n+k,k} \cong \mathcal{H}^{-k-n} G_{-n,-k} \oplus \operatorname{coker}(\sigma_1^k). \tag{53.1}$$

Now P_n is just the Hodge module $\mathbb{Q}_B[n]$, and so the underlying right $\mathscr{D}$-module is $\mathcal{P}_n \cong \omega_B$, with the filtration for which $\operatorname{gr}_{-n}^F \mathcal{P}_n \cong \omega_B$. Together with the relative Hard Lefschetz isomorphism in (48.1), this gives

$$G_{-n,-k} \cong G_{n,n-k}[2n] = \operatorname{gr}_{k-n}^F \operatorname{DR}(\mathcal{P}_n)[n] \cong \Omega_B^{n-k}[n+k],$$

and therefore $\mathcal{H}^{-k-n} G_{-n,-k} \cong \Omega_B^{n-k}$. For the same reason, we have

$$G_{-n+k,k} \cong G_{n-k,n}[2n-2k]$$

and so $\mathcal{H}^{k-n} G_{-n+k,k} \cong \mathrm{gr}^F_{-n} \mathcal{P}_{n-k} \cong R^{n-k}\pi_* \omega_B$. This is a torsion-free coherent sheaf [Kol86a, Thm. 2.1] of generic rank $\binom{n}{n-k}$. The reason is that over the smooth locus of the Lagrangian fibration, P_i comes from the variation of Hodge structure (of weight $i-n$) on the cohomology groups $H^{n+i}(M_b, \mathbb{Q})(n)$, and $\mathrm{gr}^F_{-n} \mathcal{P}_i$ is the tensor product of ω_B with the Hodge bundle whose fibers are the subspaces $H^{n,i}(M_b)$ in the Hodge decomposition. Since M_b is a compact complex torus (and in fact an abelian variety) of dimension n, this subspace has dimension $\binom{n}{i}$.

54. Now the first and the second term in (53.1) have the same generic rank, and because the left-hand side is torsion-free, it follows that $\mathrm{coker}(\sigma_1^k) = 0$, which means that $\mathrm{gr}^F_{-n} \mathcal{P}_{n-k} \cong \Omega_B^{n-k}$, as required for the proof of §49 Theorem. So we see that Matsushita's theorem holds mostly for formal reasons: the only geometric facts that we used are that π is equidimensional and that the generic fiber of π is a compact complex torus of dimension n. (Along the way, we also proved the case $k = n$ of §15 Theorem.)

55. By a similar method, one can still do the next case of §15 Theorem, namely that

$$\sigma_1^{n-1} \colon G_{i,-n+1} \to G_{i+n-1,n-1}[2n-2]$$

is an isomorphism for every $i \in \mathbb{Z}$. As before, it turns out that the two complexes can differ only by the addition of a single locally free sheaf on B; and from the geometric fact that the general fiber of π is a compact complex torus of dimension n, one can deduce that the rank of this locally free sheaf must be zero. But in all other cases, just knowing what happens over the smooth locus of π is not enough.

5 Proof of the symmetry conjecture

56. In this chapter, we investigate the relationship between §12 Conjecture (which predicts that $G_{i,k} \cong G_{k,i}$) and the "symplectic relative Hard Lefschetz theorem" in §15 Theorem. Perhaps surprisingly, it turns out that the two statements are basically equivalent. The proof below relies a more careful analysis of the isomorphism between $\mathbf{R}\pi_* \Omega_M^{n-k}$ and $\mathbf{R}\pi_* \Omega_M^{n+k}$. We are going to need the following small lemma, which generalizes §52 Lemma to decompositions in the derived category.

Lemma. *Let A, B, C, D be objects in the derived category of an abelian category. Suppose that we have morphisms $f \colon A \to C$, $g \colon B \to C$, and $h \colon B \to D$, such that*

$$A \oplus B \xrightarrow{\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}} C \oplus D$$

is an isomorphism. Then $C \cong A \oplus \mathrm{Cone}(f)$ and $B \cong \mathrm{Cone}(f) \oplus D$.

Proof. Recall that the mapping cone $\mathrm{Cone}(f)$ sits in a distinguished triangle

$$A \xrightarrow{f} C \longrightarrow \mathrm{Cone}(f) \longrightarrow A[1].$$

The assumptions imply that $f \colon A \to C$ is injective on cohomology, and so

$$0 \to H^i(A) \to H^i(C) \to H^i\big(\mathrm{Cone}(f)\big) \to 0$$

is short exact for every $i \in \mathbb{Z}$. Now consider the composition

$$B \to C \oplus D \to \operatorname{Cone}(f) \oplus D.$$

It is easy to see that this induces isomorphisms on cohomology, using the short exact sequence and the fact that $H^i(A) \oplus H^i(B) \cong H^i(C) \oplus H^i(D)$. Therefore $B \cong \operatorname{Cone}(f) \oplus D$, because we are in the derived category. The composition

$$\operatorname{Cone}(f) \longrightarrow B \xrightarrow{g} C \longrightarrow \operatorname{Cone}(f)$$

is the identity, and this again gives us the desired splitting $C \cong A \oplus \operatorname{Cone}(f)$. □

57. The following result relates §12 Conjecture and §15 Theorem. Since

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n] \cong \bigoplus_{i=-n}^{n} G_{i,k}[k],$$

this will also be useful later on, when we prove §22 Theorem.

Theorem. *The following three statements are equivalent:*

(a) The complexes $G_{i,k}$ and $G_{k,i}$ are isomorphic in the derived category (for all i, k).

(b) For every $k = -n, \dots, n$, there is an isomorphism (in the derived category)

$$\bigoplus_{i=-n}^{n} G_{i,k}[i+k] \cong \bigoplus_{i=-n}^{n} G_{k,i}[i+k].$$

(c) For every $k \geq 1$, the morphism $\sigma_1^k \colon G_{i,-k} \to G_{i+k,k}[2k]$ is an isomorphism.

Note that we are not making any assumptions about the isomorphisms in (a) or (b), only that they exist (in the derived category). Obviously, the statement in (a) implies the statement in (b). As explained in §16, the statement in (c), together with the relative Hard Lefschetz isomorphism $G_{-i,k} \cong G_{i,i+k}[2i]$, implies the statement in (a); in more detail,

$$G_{i,k} \cong G_{-i,k-i}[-2i] \cong G_{-k,i-k}[-2k] \cong G_{k,i}$$

where the first and third isomorphism come from the relative Hard Lefschetz theorem, and the second isomorphism from (c). Now we only need to show that the statement in (b) is strong enough to prove the one in (c).

58. We are going to prove, by descending induction on $0 \leq i \leq n$, that

$$\sigma_1^k \colon G_{-i,-k} \to G_{-i+k,k}[2k] \quad \text{and} \quad \sigma_1^k \colon G_{i-k,-k} \to G_{i,k}[2k]$$

are isomorphisms for every $1 \leq k \leq n$. (The two statements are equivalent to each other by duality.) Our starting point is the isomorphism between the two complexes

$$\mathbf{R}\pi_* \Omega_M^{n-k}[n+k] \cong \bigoplus_{i=-n}^{n-k} G_{i,-k} \quad \text{and} \quad \mathbf{R}\pi_* \Omega_M^{n+k}[n+k] \cong \bigoplus_{i=k-n}^{n} G_{i,k}[2k]$$

induced by $\sigma^k \colon \Omega_M^{n-k} \to \Omega_M^{n+k}$, for $1 \leq k \leq n$. Consequently,

$$\sigma^k \colon \bigoplus_{i=-n}^{n-k} G_{i,-k} \to \bigoplus_{i=-n+k}^{n} G_{i,k}[2k] \tag{58.1}$$

is an isomorphism (in the derived category); the range of the summation is controlled by §44 Lemma. As in the proof of Matsushita's theorem (in Chapter 4), we can expand $\sigma^k = (\sigma_1 + \sigma_0 + \cdots)^k$ into components; the component of highest degree is exactly the morphism $\sigma_1^k \colon G_{i,-k} \to G_{i+k,k}[2k]$, and the components of lower degree map $G_{i,-k}$ into those summands in the second sum whose first index is $\leq i+k-1$.

59. For the sake of clarity, let me write $\mu_{i,-k} \colon G_{i,-k} \to G_{i+k,k}[2k]$ for the action by σ_1^k on the i-th summand in the first sum. Suppose by induction that we have already proved the statement we want for all $i \in \{d+1, \dots, n\}$. Using §56 Lemma, we can delete those terms from the isomorphism in (58.1), and still get an isomorphism

$$\bigoplus_{i=-d}^{d-k} G_{i,-k} \to \bigoplus_{i=k-d}^{d} G_{i,k}[2k];$$

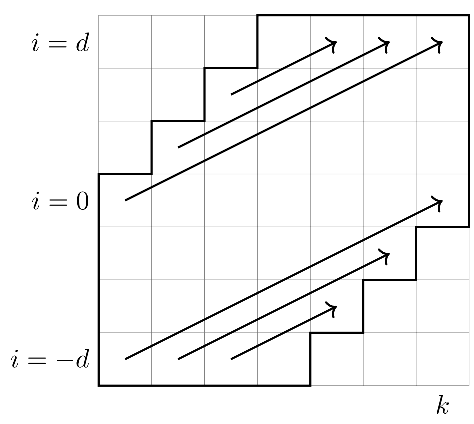
here the morphism is still basically σ^k, except that we have to remove those components that go into summands with $i < k-d$ in the second sum. We now apply §56 Lemma again, taking $A = G_{-d,-k}$ and B the rest of the first sum, $C = G_{k-d,k}[2k]$ and D the rest of the second sum. The conclusion is that we have (for every $k = 0, \dots, n$)

$$G_{k-d,k}[2k] \cong G_{-d,-k} \oplus \operatorname{Cone}(\mu_{-d,-k}). \tag{59.1}$$

By a similar argument, applied to the term with $i = d-k$, we also get

$$G_{d-k,-k} \cong G_{d,k}[2k] \oplus \operatorname{Cone}(\mu_{d-k,-k}). \tag{59.2}$$

Here is a schematic picture of the relevant morphisms:



60. Using the relative Hard Lefschetz isomorphism, we can rewrite (59.1) as

$$G_{d-k,d}[d-k] \cong G_{d,d-k}[d-k] \oplus \operatorname{Cone}(\mu_{-d,-k})[-d-k].$$

Now we take the sum over all $0 \leq k \leq n$ to produce an isomorphism

$$\bigoplus_{k=d-n}^{d} G_{k,d}[k] \cong \bigoplus_{k=d-n}^{d} G_{d,k}[k] \oplus \bigoplus_{k=0}^{n} \operatorname{Cone}(\mu_{-d,-k})[-d-k]. \tag{60.1}$$

Before we can apply the statement in (b), we have to find a way to add the missing terms. When $k < d-n$, both complexes $G_{k,d}$ and $G_{d,k}$ are exact (by §44 Lemma). When $k > d$, the inductive hypothesis, the relative Hard Lefschetz theorem, and the isomorphism in (59.2) combine to give us a chain of isomorphisms

$$G_{k,d} \cong G_{k-d,-d}[-2d] \cong G_{d-k,-k}[-2k] \cong G_{d,k} \oplus \operatorname{Cone}(\mu_{d-k,-k})[-2k].$$

If we add these terms to the sum in (60.1), and shift both sides by d, we finally arrive at an isomorphism (in the derived category)

$$\bigoplus_{k=-n}^{n} G_{k,d}[k+d] \cong \bigoplus_{k=-n}^{n} G_{d,k}[k+d] \oplus \bigoplus_{k=0}^{n} \operatorname{Cone}(\mu_{-d,-k})[-k] \oplus \bigoplus_{k=d+1}^{n} \operatorname{Cone}(\mu_{d-k,-k})[d-k].$$

Since the first and second term are isomorphic by hypothesis, this is only possible if $\operatorname{Cone}(\mu_{-d,-k}) = 0$ for every $0 \leq k \leq n$, which means exactly that

$$\sigma_1^k \colon G_{-d,-k} \to G_{k-d,k}[2k]$$

is an isomorphism for every $k = 0, \dots, n$. By duality, this shows that $G_{d-k,-k} \cong G_{d,k}[2k]$; but then (59.2) gives $\operatorname{Cone}(\mu_{d-k,-k}) \cong 0$, and consequently,

$$\sigma_1^k \colon G_{d-k,-k} \to G_{d,k}[2k]$$

is an isomorphism as well. This completes the induction, and hence the proof that the statement in part (b) of §57 Theorem implies the one in (c).

6 Lagrangian fibrations and differential forms

61. This chapter contains some general considerations about 1-forms and vector fields on Lagrangian fibrations, related to Matsushita's isomorphism $R^1\pi_*\mathscr{O}_M \cong \Omega_B^1$. The idea is that the symplectic form and the Kähler form together allow us to transform holomorphic 1-forms on the base B of the Lagrangian fibration into $\bar{\partial}$-closed $(0,1)$-forms on the holomorphic symplectic manifold M. This construction is needed for the proof of §22 Theorem.

62. The symplectic form $\sigma \in H^0(M, \Omega_M^2)$ gives

$$\bigoplus_{k=-n}^{n} \Omega_M^{n+k}$$

the structure of a module over the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Following one of several competing conventions, we denote the three standard generators of $\mathfrak{sl}_2(\mathbb{C})$ by the letters

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relations are $[\mathsf{H},\mathsf{X}]=2\mathsf{X}$, $[\mathsf{H},\mathsf{Y}]=-2\mathsf{Y}$, and $[\mathsf{X},\mathsf{Y}]=\mathsf{H}$. In our specific representation, the semisimple element H acts as multiplication by the integer k on the summand Ω_M^{n+k}, and the nilpotent element X acts as wedge product with the holomorphic 2-form σ. By general theory, this can be lifted to a representation of the Lie group $\mathrm{SL}_2(\mathbb{C})$, and the symmetry between Ω_M^{n+k} and Ω_M^{n-k} is best expressed with the help of the Weil element

$$\mathsf{w}=e^{\mathsf{X}}e^{-\mathsf{Y}}e^{\mathsf{X}}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\in\mathrm{SL}_2(\mathbb{C}).$$

The Weil element functions as a sort of linear algebra version of the Hodge $*$-operator (from the Hodge theory of compact Kähler manifolds). It has the property, easily checked by a small computation with 2×2-matrices, that

$$\mathsf{w}\mathsf{H}\mathsf{w}^{-1}=-\mathsf{H},\quad \mathsf{w}\mathsf{X}\mathsf{w}^{-1}=-\mathsf{Y},\quad \mathsf{w}\mathsf{Y}\mathsf{w}^{-1}=-\mathsf{X}.$$

The resulting isomorphism between Ω_M^{n-k} and Ω_M^{n+k} works as follows. Suppose that $\alpha\in\Omega_M^{n-k}$ is primitive, meaning that $\sigma^{k+1}\wedge\alpha=0$. Then for every $0\leq j\leq k$, one has

$$\mathsf{w}\left(\frac{1}{j!}\sigma^j\wedge\alpha\right)=\frac{(-1)^j}{(k-j)!}\sigma^{k-j}\wedge\alpha.$$

In particular, w acts on the primitive summand of Ω_M^{n-k} as multiplication by $\frac{1}{k!}\sigma^k$; on the other summands in the Lefschetz decomposition, it still acts basically as σ^k, but with certain rational coefficients that make the whole operator behave better than just σ^k by itself. We will see right away why this is useful.

63. Let $\beta\in H^0(U,\Omega_B^1)$ be a holomorphic 1-form, defined on some open subset $U\subseteq B$. In order to keep the notation from getting complicated, let us agree (for the purposes of this chapter) to replace the Lagrangian fibration $\pi\colon M\to B$ by its restriction $\pi\colon\pi^{-1}(U)\to U$, so that we can work with $\beta\in H^0(B,\Omega_B^1)$ and do not have to say "restricted to U" all the time. We may also consider the pullback $\pi^*\beta$ as a smooth $(1,0)$-form that satisfies $\bar\partial(\pi^*\beta)=0$. Because the symplectic form σ is nondegenerate, we get an associated holomorphic vector field $\xi\in H^0(M,\mathscr{T}_M)$, defined by the formula $\pi^*\beta=\xi\lrcorner\sigma=\sigma(\xi,-)$, where the symbol $\lrcorner$ means contraction by a vector field.

64. The Weil element induces an isomorphism

$$\mathsf{w}_\sigma\colon A^{n-k,q}(M)\to A^{n+k,q}(M)$$

between the two spaces of smooth forms, and the action by w_σ exchanges wedge product with $\pi^*\beta$ and contraction with the associated vector field ξ.

Lemma. *For every $k,q\in\mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccc} A^{n-k,q}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k,q}(M)\\ \downarrow{\scriptstyle \pi^*\beta\wedge} & & \downarrow{\scriptstyle \xi\lrcorner}\\ A^{n-k+1,q}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k-1,q}(M)\end{array}$$

Proof. Suppose first that $\alpha\in A^{n-k,q}(M)$ is primitive, so that $\sigma^{k+1}\wedge\alpha=0$. Then

$$\xi\lrcorner\mathsf{w}_\sigma(\alpha)=\frac{1}{k!}\xi\lrcorner(\sigma^k\wedge\alpha)=\frac{1}{(k-1)!}\sigma^{k-1}\wedge\pi^*\beta\wedge\alpha+\frac{1}{k!}\sigma^k\wedge(\xi\lrcorner\alpha).$$

We still have $\sigma^{k+1} \wedge \pi^*\beta \wedge \alpha = 0$, and therefore

$$\pi^*\beta \wedge \alpha = \gamma_0 + \sigma \wedge \gamma_1,$$

where $\gamma_0 \in A^{n-k+1,q}(M)$ and $\gamma_1 \in A^{n-k-1,q}(M)$ are both primitive. In particular, we have $\sigma^k \wedge \pi^*\beta \wedge \alpha = \sigma^{k+1} \wedge \gamma_1$. If we contract the identity $\sigma^{k+1} \wedge \alpha = 0$ with the holomorphic vector field ξ, we obtain

$$(k+1)\sigma^k \wedge \pi^*\beta \wedge \alpha + \sigma^{k+1} \wedge (\xi \lrcorner \alpha) = 0,$$

and since $\sigma^{k+1} \colon A^{n-k-1,q}(M) \to A^{n+k+1,q}(M)$ is an isomorphism, it follows that $\xi \lrcorner \alpha = -(k+1)\gamma_1$. We can now substitute into the identity from above and get

$$\begin{aligned} \xi \lrcorner \mathsf{w}_\sigma(\alpha) = \frac{1}{(k-1)!}\sigma^{k-1} \wedge \gamma_0 + \frac{1}{(k-1)!}\sigma^k \wedge \gamma_1 - \frac{k+1}{k!}\sigma^k \wedge \gamma_1 \\ \frac{1}{(k-1)!}\sigma^{k-1} \wedge \gamma_0 - \frac{1}{k!}\sigma^k \wedge \gamma_1 = \mathsf{w}_\sigma(\pi^*\beta \wedge \alpha). \end{aligned}$$

This calculation explains why we need the Weil element (instead of just σ^k).

The general case is almost the same. Because of the Lefschetz decomposition, it is enough to consider forms of the shape $\frac{1}{j!}\sigma^j \wedge \alpha$ where $\alpha \in A^{n-k,q}(M)$ is primitive and $0 \leq j \leq k$. As before, $\pi^*\beta \wedge \alpha = \gamma_0 + \sigma \wedge \gamma_1$ and $\xi \lrcorner \alpha = -(k+1)\gamma_1$. This gives us

$$\begin{aligned} \xi \lrcorner \mathsf{w}_\sigma\left(\frac{1}{j!}\sigma^j \wedge \alpha\right) &= \frac{(-1)^j}{(k-j-1)!}\sigma^{k-j-1} \wedge \pi^*\beta \wedge \alpha + \frac{(-1)^j}{(k-j)!}\sigma^{k-j} \wedge (\xi \lrcorner \alpha) \\ &= \frac{(-1)^j}{(k-j-1)!}\sigma^{k-j-1} \wedge \gamma_0 + \frac{(-1)^{j+1}(j+1)}{(k-j)!}\sigma^{k-j} \wedge \gamma_1. \end{aligned}$$

On the other hand, the relation $\pi^*\beta \wedge \alpha = \gamma_0 + \sigma \wedge \gamma_1$ implies that

$$\begin{aligned} \mathsf{w}_\sigma\left(\pi^*\beta \wedge \frac{1}{j!}\sigma^j \wedge \alpha\right) &= \mathsf{w}_\sigma\left(\frac{1}{j!}\sigma^j \wedge \gamma_0 + (j+1)\frac{1}{(j+1)!}\sigma^{j+1} \wedge \gamma_1\right) \\ &= \frac{(-1)^j}{(k-1-j)!}\sigma^{k-1-j} \wedge \gamma_0 + (j+1)\frac{(-1)^j}{(k-j)!}\sigma^{k-j} \wedge \gamma_1. \end{aligned}$$

The two expressions match, and so the diagram commutes. □

65. Since β comes from B, the holomorphic vector field ξ is tangent to the fibers of π. To see why, let $x \in M$ be a point, and assume that the fiber $M_b = \pi^{-1}(b)$ is smooth at x, where $b = \pi(x)$. Since π is a Lagrangian fibration, the tangent space $T_x M_b$ is a maximal isotropic subspace of $T_x M$. For any holomorphic tangent vector $\eta \in T_x M_b$, we have

$$\sigma(\xi, \eta) = (\pi^*\beta)(\eta) = 0,$$

because $\pi^*\beta$ vanishes along the fibers of π. By maximality, we must have $\xi \in T_x M_b$, and so ξ is indeed tangent to the fiber at the point x.

66. From the holomorphic vector field ξ, we can further construct a $(0,1)$-form on M with the help of the Kähler form ω. We define $\theta = -\xi \lrcorner \omega = -\omega(\xi, -)$; the minus sign is justified by §67 Lemma below. As one would expect, the $(0,1)$-form θ is $\bar{\partial}$-closed, and therefore defines a class in $H^1(M, \mathscr{O}_M)$.

Lemma. *We have* $\bar{\partial}\theta = 0$.

Proof. This is easiest if we use the explicit formula for the exterior derivative. According to this formula, for any two smooth $(0,1)$-vector fields λ and μ, one has

$$\big(\bar{\partial}\theta\big)(\lambda, \mu) = \lambda \cdot \theta(\mu) - \mu \cdot \theta(\lambda) - \theta\big([\lambda, \mu]\big) = \mu \cdot \omega(\xi, \lambda) - \lambda \cdot \omega(\xi, \mu) + \omega\big(\xi, [\lambda, \mu]\big).$$

Because $d\omega = 0$, we have

$$\xi \cdot \omega(\lambda, \mu) + \lambda \cdot \omega(\mu, \xi) + \mu \cdot \omega(\xi, \lambda) + \omega\big(\xi, [\lambda, \mu]\big) + \omega\big(\lambda, [\mu, \xi]\big) + \omega\big(\mu, [\xi, \lambda]\big) = 0,$$

and so we can rewrite the right-hand side in the form

$$\omega\big(\lambda, [\xi, \mu]\big) - \omega\big(\mu, [\xi, \lambda]\big) - \xi \cdot \omega(\lambda, \mu).$$

The third term vanishes because ω has type $(1,1)$, and the other two terms vanish because ξ is holomorphic. Therefore $\bar{\partial}\theta = 0$. □

67. The Kähler form ω determines another representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and the resulting Weil element induces an isomorphism

$$\mathsf{w}_\omega \colon A^{n-p,n-q}(M) \to A^{n+q,n+p}(M)$$

between the two spaces of smooth forms. This time, it exchanges contraction with the vector field ξ and wedge product with the $(0,1)$-form $\theta = -\xi \lrcorner \omega$.

Lemma. *For every $p, q \in \mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccc} A^{n-p,n-q}(M) & \xrightarrow{\mathsf{w}_\omega} & A^{n+q,n+p}(M) \\ \big\downarrow \xi \lrcorner & & \big\downarrow \theta \wedge \\ A^{n-p-1,n-q}(M) & \xrightarrow{\mathsf{w}_\omega} & A^{n+q,n+p+1}(M) \end{array}$$

Proof. Suppose first that $\alpha \in A^{n-p,n-q}(M)$ is primitive, meaning that $\omega^{k+1} \wedge \alpha = 0$, where $k = p + q \geq 0$. Contraction with ξ gives

$$0 = \xi \lrcorner (\omega^{k+1} \wedge \alpha) = -(k+1)\omega^k \wedge \theta \wedge \alpha + \omega^{k+1} \wedge (\xi \lrcorner \alpha). \tag{67.1}$$

It follows that $\omega^{k+2} \wedge (\xi \lrcorner \alpha) = 0$, and therefore $\xi \lrcorner \alpha \in A^{n-p-1,n-q}(M)$ is also primitive. Since we know how the Weil element acts on primitive forms, we then compute that

$$\mathsf{w}_\omega(\xi \lrcorner \alpha) = \frac{1}{(k+1)!}\omega^{k+1} \wedge (\xi \lrcorner \alpha) = \frac{1}{k!}\omega^k \wedge \theta \wedge \alpha = \theta \wedge \mathsf{w}_\omega(\alpha).$$

The general case is only marginally harder. Because of the Lefschetz decomposition, it is enough to consider forms of the shape $\frac{1}{j!}\omega^j \wedge \alpha$ where $\alpha \in A^{n-p,n-q}(M)$ is primitive and

$0 \leq j \leq k$. We already know that $\xi \wedge \alpha \in A^{n-p-1,n-q}(M)$ is primitive. At the same time, $\theta \wedge \alpha \in A^{n-p,n-q+1}(M)$ is annihilated by ω^{k+1}, and so it has a Lefschetz decomposition

$$\theta \wedge \alpha = \gamma_0 + \omega \wedge \gamma_1,$$

where $\gamma_0 \in A^{n-p,n-q+1}(M)$ and $\gamma_1 \in A^{n-p-1,n-q}(M)$ are primitive. From (67.1), we get

$$\omega^{k+1} \wedge (\xi \lrcorner \alpha) = (k+1)\omega^k \wedge \theta \wedge \alpha = (k+1)\omega^{k+1} \wedge \gamma_1,$$

which implies that $\xi \lrcorner \alpha = (k+1)\gamma_1$. Consequently,

$$\begin{aligned} \xi \lrcorner \frac{1}{j!}\omega^j \wedge \alpha &= -\frac{1}{(j-1)!}\omega^{j-1} \wedge \theta \wedge \alpha + \frac{1}{j!}\omega^j \wedge (\xi \lrcorner \alpha) \\ &= -\frac{1}{(j-1)!}\omega^{j-1} \wedge \gamma_0 + (k+1-j)\frac{1}{j!}\omega^j \wedge \gamma_1. \end{aligned}$$

If we now apply the Weil element w_ω, we obtain

$$\begin{aligned} \mathsf{w}_\omega \left(\xi \lrcorner \frac{1}{j!}\omega^j \wedge \alpha \right) &= \frac{(-1)^j}{(k-j)!}\omega^{k-j} \wedge \gamma_0 + \frac{(-1)^j}{(k-j)!}\omega^{k+1-j} \wedge \gamma_1 \\ &= \frac{(-1)^j}{(k-j)!}\omega^{k-j} \wedge \theta \wedge \alpha = \theta \wedge \mathsf{w}_\omega \left(\frac{1}{j!}\omega^j \wedge \alpha \right). \end{aligned}$$

This proves that the diagram is commutative. □

68. We are really interested in the operation of taking wedge product with θ. Since

$$\theta \wedge \alpha = -(\xi \lrcorner \omega) \wedge \alpha = \omega \wedge (\xi \lrcorner \alpha) - \xi \lrcorner (\omega \wedge \alpha), \tag{68.1}$$

we can realize this as the commutator of the Lefschetz operator $\omega\wedge$ and the contraction operator $\xi \lrcorner$. (This formula shows one more time why we need the minus sign in $\theta = -\xi \lrcorner \omega$.)

69. Let us now relate this construction to the decomposition theorem and to the complexes $G_{i,k}$. Let $\mathcal{S}_B^\bullet = \mathrm{Sym}^\bullet(\mathscr{T}_B)$. Recall from §34 that we have an isomorphism

$$\mathrm{gr}_\bullet^F \mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} \mathrm{gr}_\bullet^F \mathcal{P}_i[-i]$$

in the derived category of graded $\mathcal{S}_B$-modules, where the decomposition is induced by the one in the decomposition theorem. Here $\mathrm{gr}_\bullet^F \mathcal{C}_\pi$ is the complex of graded $\mathcal{S}_B$-modules with

$$\mathrm{gr}_\bullet^F \mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_* \mathcal{A}_M^{2n+p,q} \otimes_{\mathscr{O}_M} \mathcal{S}_B^{\bullet+n+p};$$

the differential in the complex is given by the formula

$$d_\pi(\alpha \otimes P) = \bar{\partial}\alpha \otimes D + \sum_{j=1}^{n} \pi^*(dt_j) \wedge \alpha \otimes \partial_j D,$$

where $t_1, \dots, t_n$ are local holomorphic coordinates on B, and $\partial_j = \partial/\partial t_j$. Since $\bar{\partial}\beta = 0$, wedge product with the holomorphic 1-form $\pi^*\beta$ defines a morphism of complexes

$$\pi^*\beta \colon \mathrm{gr}_\bullet^F \mathcal{C}_\pi \to \mathrm{gr}_{\bullet-1}^F \mathcal{C}_\pi[1]$$

that increases the cohomological degree by 1 and decreases the degree with respect to the grading by 1. This gives us a morphism (in the derived category)

$$\pi^*\beta\colon \bigoplus_{i=-n}^{n} \operatorname{gr}_{\bullet}^{F} \mathcal{P}_i[-i], \to \bigoplus_{i=-n}^{n} \operatorname{gr}_{\bullet-1}^{F} \mathcal{P}_i[1-i].$$

As usual, we break this up into components $\pi^*\beta = (\pi^*\beta)_1 + (\pi^*\beta)_0 + \cdots$, where each

$$(\pi^*\beta)_k\colon \operatorname{gr}_{\bullet}^{F} \mathcal{P}_i \to \operatorname{gr}_{\bullet-1}^{F} \mathcal{P}_{i+k}[1-k]$$

is a morphism in the derived category of graded $\mathcal{S}_B$-modules. Not surprisingly, the topmost component is zero, due to the fact that $\pi^*\beta$ vanishes on the fibers of π.

Lemma. *We have $(\pi^*\beta)_1 = 0$.*

Proof. Let us first treat the case when $d\beta = 0$. Wedge product with $\pi^*\beta$ defines a morphism

$$\pi^*\beta\colon \mathcal{C}_\pi \to \mathcal{C}_\pi[1]$$

that increases the cohomological degree by 1 and decreases the degree with respect to the filtration by 1. This follows from the fact that

$$\mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_*\mathcal{A}_M^{n+p,n+q} \otimes_{\mathscr{O}_M} (\mathscr{D}_B, F_{\bullet+p}\mathscr{D}_B),$$

and that the differential in the complex is

$$d_\pi\colon \mathcal{C}_\pi^i \to \mathcal{C}_\pi^{i+1}, \quad d_\pi\big(\alpha\otimes D\big) = d\alpha\otimes D + \sum_{j=1}^{n} \pi^*(dt_j)\wedge\alpha\otimes\partial_j D.$$

The topmost component is now a morphism $(\pi^*\beta)_1\colon (\mathcal{P}_i, F_\bullet\mathcal{P}_i) \to (\mathcal{P}_{i+1}, F_{\bullet-1}\mathcal{P}_{i+1})$. Since $\pi^*\beta$ vanishes on the fibers of π, we can then show by exactly the same argument as in §37 Lemma that $(\pi^*\beta)_1 = 0$.

To deal with the general case, we note that the problem is local on B, due to the fact that $(\pi^*\beta)_1$ is a morphism between two graded $\mathcal{S}_B$-modules. Working locally, we can choose holomorphic coordinates $t_1, \dots, t_n$, and write $\beta = \sum_{j=1}^n f_j\, dt_j$. Since the entire construction is $\mathscr{O}_B$-linear, we have $(\pi^*\beta)_1 = \sum_{j=1}^n f_j(\pi^* dt_j)_1$; but $(\pi^* dt_j)_1 = 0$ because dt_j is closed. ☐

70. We can use the properties of Deligne's decomposition to get a much better result. (For a more direct proof, see §127.)

Lemma. *For any holomorphic form $\beta\in H^0(B,\Omega_B^1)$, one has $(\pi^*\beta)_k = 0$ for all $k\neq 0$, and $(\pi^*\beta)_0$ commutes with ω_2.*

Proof. We already know that $\pi^*\beta = (\pi^*\beta)_0 + (\pi^*\beta)_{-1} + \cdots$. Recall from §35 that Deligne's decomposition has the property that $\omega = \omega_2 + \omega_0 + \omega_{-1} + \cdots$, and that the individual components ω_k with $k\leq 0$ are primitive (with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation determined by ω_2.) Since $\pi^*\beta$ and ω commute (as forms on M), the corresponding operators satisfy the relation $[\omega, \pi^*\beta] = 0$. If we expand this, we get

$$\big[\omega_2, (\pi^*\beta)_0\big] = 0, \quad \big[\omega_2, (\pi^*\beta)_{-1}\big] = 0, \quad \big[\omega_2, (\pi^*\beta)_{-2}\big] + \big[\omega_0, (\pi^*\beta)_0\big] = 0,$$

and so on. The first relation shows that $(\pi^*\beta)_0 \in \ker(\operatorname{ad}\omega_2)$; since $(\pi^*\beta)_0$ has weight 0 (with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation), it must be primitive. The second relation gives $(\pi^*\beta)_{-1} \in \ker(\operatorname{ad}\omega_2)$, and because $(\pi^*\beta)_{-1}$ has weight -1, it follows that $(\pi^*\beta)_{-1} = 0$.

By induction, we can assume that we already have $(\pi^*\beta)_{-1} = \cdots = (\pi^*\beta)_{-k+1} = 0$ for some $k \geq 2$. Let us prove that $(\pi^*\beta)_{-k} = 0$. From the relation $[\omega, \pi^*\beta] = 0$, we get

$$\big[\omega_2, (\pi^*\beta)_{-k}\big] + \big[\omega_{2-k}, (\pi^*\beta)_0\big] = 0.$$

Since ω_{2-k} is primitive of weight $-k+2$, it satisfies $(\operatorname{ad}\omega_2)^{k-1}\omega_{2-k} = 0$. This gives

$$(\operatorname{ad}\omega_2)^k(\pi^*\beta)_{-k} = -(\operatorname{ad}\omega_2)^{k-1}\big[\omega_{2-k}, (\pi^*\beta)_0\big] = -\big[(\operatorname{ad}\omega_2)^{k-1}\omega_{2-k}, (\pi^*\beta)_0\big] = 0,$$

as $(\pi^*\beta)_0 \in \ker(\operatorname{ad}\omega_2)$. Since $(\pi^*\beta)_{-k}$ has weight $-k$, it follows that $(\pi^*\beta)_{-k} = 0$. □

71. This result means concretely that the action by $\pi^*\beta$ on

$$\operatorname{gr}^F_\bullet \mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} \operatorname{gr}^F_\bullet \mathcal{P}_i[-i]$$

is diagonal (provided that we use Deligne's decomposition). The individual morphisms

$$(\pi^*\beta)_0 \colon \operatorname{gr}^F_\bullet \operatorname{DR}(\mathcal{P}_i) \to \operatorname{gr}^F_{\bullet-1} \operatorname{DR}(\mathcal{P}_i)[1]$$

are of course just given by the action of $\beta \in H^0(B, \Omega^1_B)$ on the graded pieces of the de Rham complex. Indeed, for any filtered $\mathscr{D}$-module $(\mathcal{P}, F_\bullet\mathcal{P})$, we have a morphism of complexes

$$\Omega^1_B \otimes \operatorname{gr}^F_k \operatorname{DR}(\mathcal{P}) \to \operatorname{gr}^F_{k-1} \operatorname{DR}(\mathcal{P})[1],$$

which is defined (in local coordinates) by the formula

$$\Omega^1_B \otimes \operatorname{gr}^F_{k+i} \mathcal{P} \otimes \bigwedge\nolimits^{-i} \mathscr{T}_B \to \operatorname{gr}^F_{k+i} \mathcal{P} \otimes \bigwedge\nolimits^{-i-1} \mathscr{T}_B, \quad dt_j \otimes s \otimes \partial_J \mapsto \operatorname{sgn}(J, j) s \otimes \partial_{J \setminus \{j\}}.$$

We will prove this claim in §71 below.

72. We can analyze the action of the holomorphic vector field ξ in much the same way. First, we observe that contraction with ξ induces a morphism

$$\xi \colon \operatorname{gr}^F_\bullet \mathcal{C}_\pi \to \operatorname{gr}^F_{\bullet+1} \mathcal{C}_\pi[-1].$$

To see why this is the case, let $t_1, \dots, t_n$ be local holomorphic coordinates on B, and let η_j denote the holomorphic vector field associated to the form $\pi^*(dt_j)$, so that $\pi^*(dt_j) = \eta_j \lrcorner \sigma$. Then the compatibility with the differential in the complex comes down to the identity

$$\xi \lrcorner \pi^*(dt_j) = \sigma(\eta_j, \xi) = 0,$$

which holds because both η_j and ξ are tangent to the fibers of π. We therefore get another morphism (in the derived category) that we denote by the symbol

$$\xi \colon \bigoplus_{i=-n}^{n} \operatorname{gr}^F_\bullet \mathcal{P}_i[-i], \to \bigoplus_{i=-n}^{n} \operatorname{gr}^F_{\bullet+1} \mathcal{P}_i[-i-1].$$

As before, we decompose this into components $\xi = \xi_{-1} + \xi_0 + \cdots$, where each component is a morphism $\xi_k \colon \operatorname{gr}^F_\bullet \mathcal{P}_i \to \operatorname{gr}^F_{\bullet+1} \mathcal{P}_{i+k}[-1-k]$. This time, it is much less obvious that all the components with $k \leq 0$ have to vanish.

Lemma. *We have* $\xi_k = 0$ *for every* $k \neq -1$*, and* $(\operatorname{ad}\omega_2)^2(\xi_{-1}) = 0$.

Proof. By (68.1), the commutator of the Lefschetz operator and contraction with ξ is equal to wedge product with the $(0,1)$-form θ. Since ω and θ commute (as forms on M), it follows that $\big[\omega, [\omega, \xi]\big] = 0$. After decomposing this relation by degree, we first get

$$(\operatorname{ad}\omega_2)^2(\xi_{-1}) = 0,$$

which means that ξ_{-1} is primitive (with respect to the representation of $\mathfrak{sl}_2(\mathbb{C})$ determined by ω_2). Next, we get $(\operatorname{ad}\omega_2)^2(\xi_{-2}) = 0$, and since ξ_{-2} has weight -2, it follows that $\xi_{-2} = 0$.

By induction, we may again assume that $\xi_{-2} = \cdots = \xi_{-k+1} = 0$ for some $k \geq 3$. Let us prove that $\xi_{-k} = 0$. From the relation $\big[\omega, [\omega, \xi]\big] = 0$, we obtain

$$(\operatorname{ad}\omega_2)^2(\xi_{-k}) + \operatorname{ad}\omega_2 \operatorname{ad}\omega_{3-k}(\xi_{-1}) + \operatorname{ad}\omega_{3-k} \operatorname{ad}\omega_2(\xi_{-1}) = 0.$$

Recall that ω_{3-k} is primitive of weight $3-k$, which means that $(\operatorname{ad}\omega_2)^{k-2}\omega_{3-k} = 0$. If we apply the operator $(\operatorname{ad}\omega_2)^{k-2}$ to the identity above, it becomes

$$\begin{aligned}(\operatorname{ad}\omega_2)^k(\xi_{-k}) &= -(\operatorname{ad}\omega_2)^{k-1} \operatorname{ad}\omega_{3-k}(\xi_{-1}) - (\operatorname{ad}\omega_2)^{k-2} \operatorname{ad}\omega_{3-k} \operatorname{ad}\omega_2(\xi_{-1}) \\ &= -(\operatorname{ad}\omega_2)^{k-1}\big[\omega_{3-k}, \xi_{-1}\big] - (\operatorname{ad}\omega_2)^{k-2}\big[\omega_{3-k}, \operatorname{ad}\omega_2(\xi_{-1})\big] = 0.\end{aligned}$$

In the last line, we used the fact that $\operatorname{ad}\omega_2$ is a derivation, and that $\omega_{3-k} \in \ker(\operatorname{ad}\omega_2)^{k-2}$ and $\xi_{-1} \in \ker(\operatorname{ad}\omega_2)^2$. Because ξ_{-k} has weight $-k$, this is enough to conclude that $\xi_{-k} = 0$. $\square$

73. Both results illustrate why Deligne's decomposition is "less bad" than the other possible choices in the decomposition theorem. In our setting, it has the nice effect of making a holomorphic 1-form on B act entirely in the "horizontal" direction on the summands in the decomposition theorem, whereas the associated holomorphic vector field acts entirely in the "vertical" direction (meaning along the fibers), just as one would expect from the geometry of a Lagrangian fibration. I doubt that any other choice of decomposition has this property.

74. Let us conclude this chapter by recording the effect of the various operators on the complexes $G_{i,k}$. We are going to make a slight change in the notation and assume from now on that $\beta \in H^0(U, \Omega_B^1)$ is a holomorphic 1-form, defined on an open subset $U \subseteq B$. As pointed out at the beginning of the chapter, this causes no problems, because all the arguments can be used locally on B. With the help of the symplectic form σ and the Kähler form ω, the holomorphic 1-form β gives us on the one hand a holomorphic vector field

$$v(\beta) \in H^0\big(\pi^{-1}(U), \mathscr{T}_M\big), \quad \pi^*\beta = v(\beta) \lrcorner \sigma,$$

and on the other hand a $\bar\partial$-closed $(0,1)$-form (with a minus sign)

$$f(\beta) \in A^{0,1}\big(\pi^{-1}(U)\big), \quad f(\beta) = -v(\beta) \lrcorner \omega.$$

We use the same symbols for the morphisms

$$\pi^*\beta \colon \operatorname{gr}_\bullet^F \mathcal{C}_\pi \to \operatorname{gr}_{\bullet-1}^F \mathcal{C}_\pi[1] \quad \text{and} \quad f(\beta) \colon \operatorname{gr}_\bullet^F \mathcal{C}_\pi \to \operatorname{gr}_\bullet^F \mathcal{C}_\pi[1]$$

induced by wedge product with the forms $\pi^*\beta$ and $f(\beta)$, and for the morphism

$$v(\beta) \colon \operatorname{gr}_\bullet^F \mathcal{C}_\pi \to \operatorname{gr}_{\bullet+1}^F \mathcal{C}_\pi[-1]$$

induced by contraction with the vector field $v(\beta)$. All these morphisms are of course defined only on the open set $U \subseteq B$. We saw in (68.1) that $f(\beta) = [\omega, v(\beta)]$. Because of §72 Lemma, the only nonzero component of $v(\beta)$ is $v(\beta)_{-1}$, which means that the components of $f(\beta)$ can be computed by the simple formula

$$f(\beta)_k = [\omega_{k+1}, v(\beta)].$$

In particular, the topmost component is $f(\beta)_1 = [\omega_2, v(\beta)]$.

75. From $\pi^*\beta = (\pi^*\beta)_0 \colon \operatorname{gr}_\bullet^F \mathcal{P}_i \to \operatorname{gr}_{\bullet-1}^F \mathcal{P}_i[1]$, we get a morphism of complexes

$$\pi^*\beta \colon G_{i,k} \to G_{i,k+1}[1],$$

which is given by contracting the holomorphic vector fields in the de Rham complex against the holomorphic form β. Similarly, from $v(\beta) = v(\beta)_{-1} \colon \operatorname{gr}_\bullet^F \mathcal{P}_i \to \operatorname{gr}_{\bullet+1}^F \mathcal{P}_{i-1}$, we get a second morphism of complexes

$$v(\beta) \colon G_{i,k} \to G_{i-1,k-1}[-1].$$

Lastly, we have the morphism of complexes

$$f(\beta)_1 = [\omega_2, v(\beta)] \colon G_{i,k} \to G_{i+1,k}[1].$$

All three morphisms will play an important role in the proof of §22 Theorem.

76. The relation with Matsushita's theorem (in §49 Theorem) is the following. The construction above produces a morphism of sheaves

$$\Omega_B^1 \to R^1\pi_*\mathscr{O}_M,$$

by assigning to a holomorphic 1-form $\beta \in H^0(U, \Omega_B^1)$ the image of the cohomology class of the $\bar{\partial}$-closed $(0,1)$-form $f(\beta)$ under the morphism $H^1\bigl(\pi^{-1}(U), \mathscr{O}_M\bigr) \to H^0(U, R^1\pi_*\mathscr{O}_M)$. Matsushita's theorem is then saying concretely that this morphism is an isomorphism.

7 The BGG correspondence

77. This chapter contains a brief review of the BGG correspondence [BGG78, EFS03], in a way that is convenient for our purposes. We will define everything carefully, with the correct signs, but only sketch the proofs, which mostly amount to checking that the signs in front of various terms match up correctly; the details can in any case be found in [EFS03].

78. Let B be a complex manifold of dimension n, and denote by $\mathscr{T}_B$ the holomorphic tangent sheaf. The BGG correspondence relates, on the level of the derived category, graded modules over two graded $\mathscr{O}_B$-algebras. The first is the symmetric algebra

$$\mathcal{S}_B = \operatorname{Sym}(\mathscr{T}_B) = \bigoplus_{j \in \mathbb{N}} \operatorname{Sym}^j(\mathscr{T}_B),$$

with the obvious multiplication and the natural grading in which $\operatorname{Sym}^j(\mathscr{T}_B)$ has degree j. The second is the algebra of holomorphic forms

$$\Omega_B = \bigoplus_{j=0}^{n} \Omega_B^j,$$

with multiplication given by wedge product. Note that, unlike [EFS03], we give the algebra Ω_B the naive grading in which Ω_B^j lives in degree j. If $M = M_\bullet$ is a graded module over either of these algebras, we denote by $M(d)$ the same module with the grading shifted according to the rule $M(d)_k = M_{d+k}$. In the case of Ω_B, we only work with *left* modules.

79. The central object in the BGG correspondence is the Koszul complex

$$\mathcal{S}_B(-n) \otimes \bigwedge^n \mathscr{T}_B \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{S}_B(-1) \otimes \mathscr{T}_B \xrightarrow{\delta} \mathcal{S}_B(0) \otimes \mathscr{O}_B, \tag{79.1}$$

which lives in cohomological degrees $-n, \dots, 0$, and gives a free resolution of the trivial graded $\mathcal{S}_B$-module $\mathscr{O}_B$. The differential is defined by the following compact formula:

$$\delta(s \otimes \partial_J) = \sum_{j=1}^n \operatorname{sgn}(J, j) \cdot \partial_j s \otimes \partial_{J \setminus \{j\}}.$$

Here the notation is as follows. Let $t_1, \dots, t_n$ be local holomorphic coordinates on B, and denote by $\partial_j = \partial/\partial t_j$ the resulting holomorphic vector fields. For any subset $J \subseteq \{1, \dots, n\}$, we list the elements in increasing order as $j_1 < \cdots < j_\ell$, and then define

$$\partial_J = \partial_{j_1} \wedge \cdots \wedge \partial_{j_\ell} \quad \text{and} \quad dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_\ell},$$

with the convention that both expressions equal 1 when J is empty. We also define

$$\operatorname{sgn}(J, j) = \begin{cases} (-1)^{k-1} & \text{if } j = j_k, \\ 0 & \text{if } j \notin J. \end{cases}$$

Note that we are always using Deligne's Koszul sign rule, according to which swapping two elements of degrees p and q leads to a sign $(-1)^{pq}$; this is the reason for the factor $\operatorname{sgn}(J, j)$.

80. Let us start by defining a functor $\mathbf{L}_B$ from complexes of graded Ω_B-modules to complexes of graded $\mathcal{S}_B$-modules. Let (M, d) be a complex of graded left Ω_B-modules. We can think of this concretely as a bigraded sheaf of $\mathscr{O}_B$-modules

$$M = \bigoplus_{i,k} M_k^i,$$

where i is the cohomological degree and k the degree with respect to the grading; the differential d maps each M_k^i to M_k^{i+1} and is linear over Ω_B. The idea is to send this to the induced $\mathcal{S}_B$-module $M \otimes_{\mathscr{O}_B} \mathcal{S}_B$, but where we put the summand $M_k^i \otimes \mathcal{S}_B^j$ into cohomological degree $i + k$ and in degree $j - k$ with respect to the grading.[6] More precisely, we define $\mathbf{L}_B(M, d)$ to be the complex whose i-th term is the graded $\mathcal{S}_B$-module

$$\mathbf{L}_B(M, d)^i = \bigoplus_{p+q=i} M_p^q \otimes \mathcal{S}_B(p),$$

and whose differential acts on the summand $M_p^q \otimes \mathcal{S}_B(p)$ by the formula

$$\sum_{j=1}^n dt_j \otimes \partial_j + (-1)^p d \otimes \operatorname{id}.$$

[6]The graded $\mathcal{S}_B$-modules that we are interested in come from filtered right $\mathscr{D}_B$-modules, and so they are naturally *right* $\mathcal{S}_B$-modules. Since $\mathcal{S}_B$ is commutative, we do not need to distinguish between left and right modules, but the signs work out more nicely if we put the factor $\mathcal{S}_B$ on the right.

This is the simple complex associated to the double complex with terms $M_p^q \otimes \mathcal{S}_B(p)$, with Deligne's sign rules for the differential. Since $\mathcal{S}_B$ is commutative, the differential is $\mathcal{S}_B$-linear and preserves the grading.

Example. For instance, $\mathbf{L}_B(\Omega_B)$ is the complex of graded $\mathcal{S}_B$-modules

$$\mathscr{O}_B \otimes \mathcal{S}_B(0) \to \Omega_B^1 \otimes \mathcal{S}_B(1) \to \cdots \to \Omega_B^n \otimes \mathcal{S}_B(n),$$

with differential $\alpha \otimes s \mapsto \sum_j dt_j \wedge \alpha \otimes \partial_j s$. Up to a factor of $(-1)^n$, this is just the Koszul resolution of the trivial $\mathcal{S}_B$-module ω_B, placed in cohomological degree n and with the grading shifted by n steps; in other words, $\mathbf{L}_B(\Omega_B) \cong \omega_B(n)[-n]$.

81. We denote by $G(\mathcal{S}_B)$ the category of graded $\mathcal{S}_B$-modules, and by $D^b_{coh}G(\mathcal{S}_B)$ the derived category of cohomologically bounded and coherent complexes of graded $\mathcal{S}_B$-modules; we use similar notation for Ω_B, with the understanding that modules over the noncommutative algebra Ω_B are always *left* modules. One checks that $\mathbf{L}_B$ descends to an exact functor

$$\mathbf{L}_B \colon D^b_{coh}G(\Omega_B) \to D^b_{coh}G(\mathcal{S}_B)$$

between the two derived categories. Indeed, a morphism $f \colon (M, d) \to (M', d')$ between two complexes of graded Ω_B-modules clearly induces a morphism of complexes

$$\mathbf{L}_B(f) \colon \mathbf{L}_B(M, d) \to \mathbf{L}_B(M', d').$$

The point is that if f is a quasi-isomorphism, then $\mathbf{L}_B(f)$ is also a quasi-isomorphism: when (M, d) and (M', d') are cohomologically bounded and coherent, this can easily be checked by a spectral sequence argument.

82. For later use, let us see how the complex $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$ fits into this framework.

Example. Recall that $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$ is the complex of graded $\mathcal{S}_B$-modules with terms

$$\operatorname{gr}_\bullet^F \mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_* \mathcal{A}_M^{n+p, n+q} \otimes_{\mathscr{O}_M} \mathcal{S}_B(p)$$

and with differential (written in local coordinates)

$$d_\pi\big(\alpha \otimes P\big) = \bar{\partial}\alpha \otimes P + \sum_{j=1}^n \pi^*(dt_j) \wedge \alpha \otimes \partial_j P.$$

If we compare this with the definition above, we see that this is exactly $\mathbf{L}_B(M, d)$, where (M, d) is the complex of graded Ω_B-modules with

$$M_k^i = \pi_* \mathcal{A}_M^{n+k, n+i}$$

and with differential $d = (-1)^k \bar{\partial}$. The Ω_B-module structure is the obvious one: a holomorphic form $\beta \in \Omega_B^j$ acts via wedge product with the pullback $\pi^*\beta$. This is compatible with the differential because of the factor $(-1)^k$.

83. This is a good place to prove the claim we made in §71 when we looked at holomorphic forms and the decomposition theorem: for any local section β of Ω_B^1, the morphism

$$(\pi^*\beta)_0 \colon \operatorname{gr}_\bullet^F \operatorname{DR}(\mathcal{P}_i) \to \operatorname{gr}_{\bullet-1}^F \operatorname{DR}(\mathcal{P}_i)[1]$$

comes from the action of Ω_B^1 on the de Rham complex. Let us restate the problem using the BGG correspondence. The morphism $(\pi^*\beta)_0 \colon \operatorname{gr}_\bullet^F \mathcal{P}_i \to \operatorname{gr}_{\bullet-1}^F \mathcal{P}_i[1]$ induces a morphism

$$\mathbf{L}_B\bigl(\operatorname{gr}_\bullet^F \mathcal{P}_i\bigr) \to \mathbf{L}_B\bigl(\operatorname{gr}_{\bullet-1}^F \mathcal{P}_i[1]\bigr) = \mathbf{L}_B\bigl(\operatorname{gr}_\bullet^F \mathcal{P}_i\bigr)(1),$$

and the claim is that this is just multiplication by $\beta \in \Omega_B^1$. We know from the previous paragraph that $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$ corresponds, under the BGG correspondence, to the complex (M, d); consequently, the decomposition theorem gives us an isomorphism (in the derived category)

$$(M, d) \cong \bigoplus_{i=-n}^{n} \mathbf{L}_B\bigl(\operatorname{gr}_\bullet^F \mathcal{P}_i\bigr)[-i].$$

As we have just seen, the Ω_B-module structure on (M, d) is such that $\beta \in \Omega_B^1$ acts via wedge product with $\pi^*\beta$. But this means that wedge product with $\pi^*\beta$ also gives the Ω_B-module structure on each summand $\mathbf{L}_B\bigl(\operatorname{gr}_\bullet^F \mathcal{P}_i\bigr)$, and this is exactly what we wanted to prove.

84. Next, we define a functor $\mathbf{R}_B$ from complexes of graded $\mathcal{S}_B$-modules to complexes of graded Ω_B-modules. The general idea is to take the tensor product with the Koszul complex in (79.1), but to adjust both the degree and the grading in order to get Ω_B to act correctly.

Example. Suppose we tensor a single graded $\mathcal{S}_B$-module N by the Koszul complex. This produces a collection of complexes C_k, indexed by $k \in \mathbb{Z}$, that look like this:

$$\cdots \longrightarrow N_{k-2} \otimes \textstyle\bigwedge^2 \mathscr{T}_B \xrightarrow{\delta} N_{k-1} \otimes \mathscr{T}_B \xrightarrow{\delta} N_k \otimes \mathscr{O}_B$$

If we define the action by Ω_B^1 in the obvious way as

$$\Omega_B^1 \otimes \Bigl(N_k \otimes \textstyle\bigwedge^i \mathscr{T}_B\Bigr) \to N_k \otimes \textstyle\bigwedge^{i-1} \mathscr{T}_B, \quad dt_j \otimes (s \otimes \partial_J) \mapsto \operatorname{sgn}(J, j) \cdot s \otimes \partial_{J \setminus \{j\}},$$

then a short calculation shows that we get a morphism of complexes $\Omega_B^1 \otimes C_k \to C_{k-1}[1]$, but where the differential in the second complex is $-\delta$. So both the grading and the differentials are wrong. To fix this problem, we need to work with the complexes $C_k' = C_{-k}[k]$, with differential $(-1)^k\delta$, because this makes $\Omega_B^1 \otimes C_k' \to C_{k+1}'$ behave as it should.

85. With this in mind, let (N, d) be a complex of graded $\mathcal{S}_B$-modules. We define $\mathbf{R}_B(N, d)$ as the complex whose i-th term is the graded $\mathscr{O}_B$-module

$$\mathbf{R}_B(N, d)_k^i = \bigoplus_{p+q=i} N_q^p \otimes \textstyle\bigwedge^{-q-k} \mathscr{T}_B,$$

and whose differential acts on the summand $N_q^p \otimes \bigwedge^{-q-k} \mathscr{T}_B$ by the formula

$$d \otimes \operatorname{id} + (-1)^{p+k}\delta,$$

where δ is the standard Koszul differential. This is again the simple complex associated to the double complex with terms $N_q^p \otimes \bigwedge^{-q-k} \mathscr{T}_B$; the extra $(-1)^k$ is justified by the example. Each term in the complex becomes a graded module over Ω_B through the morphism

$$\Omega_B^1 \otimes \left(N_q^p \otimes \bigwedge^{-q-k} \mathscr{T}_B\right) \to N_q^p \otimes \bigwedge^{-q-k-1} \mathscr{T}_B,$$
$$dt_j \otimes (s \otimes \partial_J) \mapsto \operatorname{sgn}(J,j) \cdot s \otimes \partial_{J \setminus \{j\}},$$

and one checks (by the same calculation as in the example) that the differential is indeed linear over Ω_B. Once again, $\mathbf{R}_B$ descends to an exact functor

$$\mathbf{R}_B \colon D^b_{coh} G(\mathcal{S}_B) \to D^b_{coh} G(\Omega_B)$$

between the two derived categories of graded modules.

Example. Let $(\mathcal{P}, F_\bullet \mathcal{P})$ be a filtered right $\mathscr{D}_B$-module. The associated graded object $\operatorname{gr}_\bullet^F \mathcal{P}$ is a graded $\mathcal{S}_B$-module. Term by term, we have

$$\mathbf{R}_B\big(\operatorname{gr}_\bullet^F \mathcal{P}\big)_k^i = \operatorname{gr}_i^F \mathcal{P} \otimes \bigwedge^{-i-k} \mathscr{T}_B,$$

and since the differential is exactly $(-1)^k \delta$, we obtain

$$\mathbf{R}_B\big(\operatorname{gr}_\bullet^F \mathcal{P}\big) = \bigoplus_{k \in \mathbb{Z}} \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P})[k],$$

where the grading is by k, and the Ω_B-module structure is defined (as above) by contraction, viewed as a morphism of complexes

$$\Omega_B^j \otimes \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P})[k] \to \operatorname{gr}_{-k-j}^F \operatorname{DR}(\mathcal{P})[k+j].$$

So the graded pieces of the de Rham complex of a filtered $\mathscr{D}$-module naturally fit into the framework of the BGG correspondence.

86. The first result is that $\mathbf{L}_B$ and $\mathbf{R}_B$ are adjoint functors.

Theorem. *We have a natural isomorphism of bifunctors*

$$\operatorname{Hom}_{D^b_{coh} G(\mathcal{S}_B)}\big(\mathbf{L}_B -, -\big) \cong \operatorname{Hom}_{D^b_{coh} G(\Omega_B)}\big(-, \mathbf{R}_B -\big),$$

which means that $(\mathbf{L}_B, \mathbf{R}_B)$ *is an adjoint pair of functors.*

Proof. Let (M, d) be a complex of graded Ω_B-modules and let (N, d) be a complex of graded $\mathcal{S}_B$-modules. A morphism of complexes of graded $\mathcal{S}_B$-modules

$$\mathbf{L}_B(M, d) \to (N, d)$$

is the same as a collection of morphisms of graded $\mathcal{S}_B$-modules

$$\mathbf{L}_B(M, d)^i = \bigoplus_{p+q=i} M_p^q \otimes \mathcal{S}_B(p) \to N^i$$

that are compatible with the differentials in the two complexes. This is equivalent to giving a collection of morphisms of $\mathscr{O}_B$-modules $f \colon M_p^q \to N_{-p}^{p+q}$, subject to the condition that

$$df(m) = (-1)^p f(dm) + \sum_{j=1}^n \partial_j f(dt_j \cdot m) \quad \text{for } m \in M_p^q.$$

From this data, we can define morphisms of $\mathscr{O}_B$-modules

$$g\colon M_k^i \to \mathbf{R}_B(N,d)_k^i = N_{-k}^{i+k} \otimes \mathscr{O}_B \oplus N_{-k-1}^{i+k+1} \otimes \mathscr{T}_B \oplus N_{-k-2}^{i+k+2} \otimes \bigwedge^2 \mathscr{T}_B \oplus \cdots$$

by the explicit formula

$$g(m) = (-1)^{ik} \sum_J (-1)^{i|J|} \varepsilon\big(|J|\big) \cdot f(dt_J \cdot m) \otimes \partial_J,$$

where $\varepsilon(\ell) = (-1)^{\ell(\ell-1)/2}$ and the summation runs over all subsets of $\{1,\dots,n\}$. A straightforward calculation shows that this is compatible with the differentials and with the action by Ω_B, and therefore defines a morphism of complexes of graded Ω_B-modules

$$(M,d) \to \mathbf{R}_B(N,d).$$

This construction is reversible, and passes to the derived category. □

87. The content of the BGG correspondence is that $\mathbf{L}_B$ is an equivalence of categories. We continue to denote by $D^b_{coh}G(\mathcal{S}_B)$ the derived category of cohomologically bounded and coherent complexes of graded $\mathcal{S}_B$-modules, and similarly for Ω_B.

Theorem. *The two functors*

$$\mathbf{L}_B\colon D^b_{coh}G(\Omega_B) \to D^b_{coh}G(\mathcal{S}_B) \quad \textit{and} \quad \mathbf{R}_B\colon D^b_{coh}G(\mathcal{S}_B) \to D^b_{coh}G(\Omega_B)$$

are equivalences of categories that are inverse to each other.

Proof. Let (M,d) be a complex of graded Ω_B-modules. The adjointness of the two functors gives a morphism $(M,d) \to \mathbf{R}_B\big(\mathbf{L}_B(M,d)\big)$. Concretely, we have

$$\mathbf{R}_B\big(\mathbf{L}_B(M,d)\big)_k^i = \bigoplus_{p+q+r=i} M_p^q \otimes \mathcal{S}_B^{p+r} \otimes \bigwedge^{-r-k} \mathscr{T}_B,$$

which looks like the simple complex associated to a triple complex with grading (p,q,r); and the differential is indeed given by the formula

$$\sum_{j=1}^n dt_j \otimes \partial_j \otimes \mathrm{id} + (-1)^p d \otimes \mathrm{id} \otimes \mathrm{id} + (-1)^{p+q+k} \mathrm{id} \otimes \delta,$$

where δ is again the standard Koszul differential. According to §86 Theorem, the morphism $M_k^i \to \mathbf{R}_B\big(\mathbf{L}_B(M,d)\big)_k^i$ is described by the formula

$$m \mapsto (-1)^{ik} \sum_J (-1)^{i|J|} \varepsilon\big(|J|\big) \cdot dt_J m \otimes 1 \otimes \partial_J.$$

The boundedness assumption ensures that the three different spectral sequences of the triple complex converge. The third spectral sequence starts from the differential δ, and the fact that the Koszul complex in (79.1) is a resolution of the trivial $\mathcal{S}_B$-module $\mathscr{O}_B$ implies that the only nonzero cohomology object is M_k^i for $(p,q,r) = (k,i,-k)$. The convergence of the spectral sequence therefore shows that $(M,d) \to \mathbf{R}_B\big(\mathbf{L}_B(M,d)\big)$ is a quasi-isomorphism. The proof in the other direction is similar. □

88. We are going to need two other facts about the BGG correspondence. The first is a simple-minded bound on the amplitude of the complex $\mathbf{L}_B(M,d)$, in terms of the amplitude of the individual complexes of $\mathscr{O}_B$-modules (M_k,d).

Theorem. *Let (M,d) be a bounded complex of graded Ω_B-modules. If $\mathcal{H}^q\big(M_p,d\big) = 0$ whenever $p+q>0$, then $\mathcal{H}^i\mathbf{L}_B(M,d) = 0$ for $i>0$.*

Proof. We view $\mathbf{L}_B(M,d)$ as a double complex with terms $M_p^q \otimes \mathcal{S}_B(p)$ and with the two commuting differentials $\sum_j dt_j \otimes \partial_j$ and $d \otimes \mathrm{id}$. The spectral sequence that starts from the differential $d\otimes \mathrm{id}$ converges because $M_p^q \otimes \mathcal{S}_B(p) = 0$ for $|q| \gg 0$. On the E_1-page, we get the graded $\mathcal{S}_B$-modules $\mathcal{H}^q(M_p,d)\otimes \mathcal{S}_B(p)$, which vanish for $p+q>0$ by assumption. The result now follows from the fact that the spectral sequence converges to the cohomology of the complex $\mathbf{L}_B(M,d)$. □

89. The second fact we need is that the BGG correspondence interacts nicely with duality. Let $\mathcal{S}_B'$ denote $\mathcal{S}_B$, but with the $\mathcal{S}_B$-module structure in which $\mathrm{Sym}^k(\mathscr{T}_B)$ act with an additional sign of $(-1)^k$. (Geometrically, this amounts to pulling back by the automorphism that acts as -1 on the fibers of the contangent bundle of B.)

Theorem. *Let (M,d) be a bounded complex of finitely generated graded Ω_B-modules. One has a natural isomorphism*

$$\mathbf{R}\mathcal{H}om_{\mathcal{S}_B}\Big(\mathbf{L}_B(M,d), \mathcal{S}_B \otimes \omega_B[n]\Big) \cong \mathbf{L}_B\Big(\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\big((M,d),\omega_B[n]\big)\Big) \otimes_{\mathcal{S}_B} \mathcal{S}_B'.$$

Proof. The boundedness assumption implies that there are only finitely many value of i and k for which $M_k^i \neq 0$; this is needed in order to avoid infinite direct sums (which do not commute with $\mathcal{H}om$). Pick a resolution for $\omega_B[n]$ by injective $\mathscr{O}_B$-modules, say

$$0 \to \omega_B \to I^{-n} \to \cdots \to I^{-1} \to I^0 \to 0;$$

we will abbreviate this by (I,d); recall that the injective dimension of ω_B is $n = \dim B$ according to [Gol75]. We can then represent $\mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\big((M,d),\omega_B[n]\big)$ by the complex of graded $\mathscr{O}_B$-modules $(\widehat{M},d)$, where

$$\widehat{M}_k^i = \bigoplus_{p+q=i} \mathcal{H}om_{\mathscr{O}_B}\big(M_{-k}^{-p}, I^q\big),$$

and where the differential is given by the formula $df = f\circ d + (-1)^p d \circ f$ for any local section $f \in \mathcal{H}om_{\mathscr{O}_B}\big(M_{-k}^{-p}, I^q\big)$. Since each M^i is a graded left Ω_B-module, the terms in the complex $(\widehat{M},d)$ are naturally *right* Ω_B-modules, but we can convert them back into left Ω_B-modules by letting Ω_B^j act with an extra factor of $(-1)^j$, meaning that

$$(dt_j f)(m) = -f\big(dt_j m\big).$$

With this convention, $(\widehat{M},d)$ is a complex of graded left Ω_B-modules, and so $\mathbf{L}_B(\widehat{M},d)$ is defined. This describes the complex on the right-hand side of the claimed isomorphism.

Now we turn to the complex $\mathbf{R}\mathcal{H}om_{\mathcal{S}_B}\big(\mathbf{L}_B(M,d), \mathcal{S}_B \otimes \omega_B[n]\big)$ on the left-hand side. It is realized by the simple complex associated to the double complex $\mathcal{H}om_{\mathcal{S}_B}\big(\mathbf{L}_B(M,d), \mathcal{S}_B \otimes I^\bullet\big)$. After some rearranging, the i-th term of the resulting simple complex comes out to be

$$\bigoplus_{p+q=i} \mathcal{H}om_{\mathcal{S}_B}\big(\mathbf{L}_B(M,d)^{-p}, \mathcal{S}_B \otimes I^q\big) = \bigoplus_{p+q=i} \widehat{M}_p^q \otimes \mathcal{S}_B(p),$$

and the differential matches up with the differential in $\mathbf{L}_B(\widehat{M},d)$, except for an extra -1 in front of the term $\sum_j dt_j \otimes \partial_j$. This is corrected by tensoring with $\mathcal{S}_B'$, whence the result. □

90. Note that $\mathcal{S}'_B \cong \mathcal{S}_B$ are isomorphic as graded $\mathcal{S}_B$-modules, and so we can remove the tensor product with $\mathcal{S}'_B$ from the statement if we like.

8 Proof of the main theorem

91. In this chapter, we give the proof of §22 Theorem; along the way, we also establish §15 Theorem (and therefore §12 Conjecture). Even though we are interested in relating $\operatorname{gr}_\bullet^F \mathcal{P}_k$ and Ω_M^{n+k} individually, it turns out that the necessary structure is only there if we look at all of these objects together. The general idea is the following. There are three different ways to make the direct sum of the complexes $G_{i,k}$ (with appropriate shifts) into a complex of graded modules over the algebra $\Omega_B = \bigoplus_j \Omega_B^j$. In the first, a local section β of Ω_B^1 acts via $\pi^*\beta$; in the second, via the associated vector field $v(\beta)$; and in the third, via the associated $(0,1)$-form $f(\beta)_1$ (in the notation of Chapter 6). These three different structures are related by the action of the symplectic form σ and the Kähler form ω, and together with the BGG correspondence and basic facts about Hodge modules, this gives us enough information to prove §22 Theorem.

92. From the Hodge modules $P_{-n}, \dots, P_n$, we get a collection of graded $\mathcal{S}_B$-modules $\operatorname{gr}_\bullet^F \mathcal{P}_i$, for $i = -n, \dots, n$. The BGG correspondence associates to each of these graded $\mathcal{S}_B$-modules a complex of graded Ω_B-modules $\mathbf{R}_B(\operatorname{gr}_\bullet^F \mathcal{P}_i)$; recall from §85 Example that

$$\mathbf{R}_B(\operatorname{gr}_\bullet^F \mathcal{P}_i) = \bigoplus_{k=-n}^{n} G_{i,k}[i+k],$$

and that the degree of the summand $G_{i,k}[i+k]$ with respect to the grading is k. Adding all of these objects together, we obtain the first object

$$G = \bigoplus_{i,k=-n}^{n} G_{i,k}[k] = \mathbf{R}_B\left(\bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i]\right).$$

In other words, G is a complex of graded Ω_B-modules, with the summand $G_{i,k}[k]$ in graded degree k; it is the object that corresponds, under the BGG correspondence in §87 Theorem, to the complex of graded $\mathcal{S}_B$-modules

$$\bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i].$$

In the notation that we introduced in Chapter 6, a local section β of Ω_B^1 acts on the complex G via the collection of graded morphisms

$$\pi^*\beta \colon G_{i,k}[k] \to G_{i,k+1}[k+1];$$

of course, this uniquely determines the module structure on G, because Ω_B is generated by Ω_B^1 as an $\mathscr{O}_B$-algebra. We will use this idea several times below.

93. We have a concrete model for G in terms of smooth differential forms. Recall from §34 the definition of the complex $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$. This is the complex of graded $\mathcal{S}_B$-modules with terms

$$\operatorname{gr}_\bullet^F \mathcal{C}_\pi^i = \bigoplus_{p+q=i} \pi_* \mathcal{A}_M^{n+p,n+q} \otimes_{\mathscr{O}_M} \operatorname{Sym}^{\bullet+p}(\mathscr{T}_B),$$

and with differential (in local coordinates)

$$d_\pi\big(\alpha \otimes P\big) = \bar{\partial}\alpha \otimes P + \sum_{j=1}^{n} \pi^*(dt_j) \wedge \alpha \otimes \partial_j P.$$

From Saito's theory, we get an isomorphism

$$\operatorname{gr}_\bullet^F \mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i]$$

in the derived category of graded $\mathcal{S}_B$-modules; here the decomposition is induced by the one in the decomposition theorem. The calculation in §82 Example shows that $\operatorname{gr}_\bullet^F \mathcal{C}_\pi \cong \mathbf{L}_B(M,d)$, where (M,d) is the complex of graded Ω_B-modules with

$$M_k^i = \pi_* \mathcal{A}_M^{n+k,n+i} \quad \text{and} \quad d = (-1)^k \bar{\partial}.$$

The Ω_B-module structure is the obvious one: a holomorphic form $\beta \in \Omega_B^j$ acts via wedge product with the pullback $\pi^*\beta$. Since the BGG correspondence is an equivalence of categories, this means that we have an isomorphism

$$G \cong (M,d)$$

in the derived category of graded Ω_B-modules. This says in particular that the complex (M,d) splits in the derived category (because of the decomposition theorem), which is an extremely deep fact about proper morphisms between Kähler manifolds.

94. Note that G is also related to the direct sum of all the sheaves of holomorphic forms on M, because Saito's formula in (39.1) gives us an isomorphism

$$G \cong \bigoplus_{k=-n}^{n} \mathbf{R}\pi_* \Omega_M^{n+k}[n]$$

in the derived category of graded $\mathscr{O}_B$-modules.

95. Now we introduce the second object. This is the complex

$$G_v = \bigoplus_{i,k} G_{i,k}[k],$$

in which the summand $G_{i,k}[k]$ sits in graded degree $-k$. As a complex of $\mathscr{O}_B$-modules, this is of course isomorphic to G, but the grading and the Ω_B-module structure are different. We turn G_v into a complex of graded Ω_B-modules in the following way. Recall that we have the Weil element w_σ, associated to the $\mathfrak{sl}_2(\mathbb{C})$-representation on $\bigoplus_k \Omega_M^{n+k}$. It determines an automorphism of the complex (M,d) that maps M_k^i isomorphically to M_{-k}^i. According to §64 Lemma, for any local section β of Ω_B^1, it also exchanges the action by the holomorphic form $\pi^*\beta$ agains the action by the holomorphic vector field $v(\beta)$. In other words,

$$\mathsf{w}_\sigma \colon G \to G_v$$

is an isomorphism between G and G_v, as complexes of graded $\mathscr{O}_B$-modules. Note that w_σ respects the grading, but there is no reason why it should preserve the individual summands $G_{i,k}[k]$ in the decompositions of G and G_v.

96. We use this isomorphism to give G_v the structure of a complex of graded Ω_B-modules. §64 Lemma tells us that we have, for every local section $\beta \in \Omega_B^1$, a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathsf{w}_\sigma} & G_v \\ \downarrow \beta & & \downarrow v(\beta) \\ G & \xrightarrow{\mathsf{w}_\sigma} & G_v. \end{array}$$

The Ω_B-module structure on G_v is therefore the unique one for which a locally defined holomorphic form $\beta \in \Omega_B^1$ acts via the collection of graded morphisms

$$v(\beta)\colon G_{i,k}[k] \to G_{i-1,k-1}[k-1].$$

Let me stress that this step of the construction works on the level of smooth forms, meaning on the complex (M, d), because the symplectic form is a holomorphic form of type $(2,0)$. Neither Hodge theory nor the decomposition theorem are needed here.

97. Let us now describe the third object. This is the complex of graded $\mathscr{O}_B$-modules

$$G_f = \bigoplus_{i,k} G_{i,k}[i+k]$$

in which the term $G_{i,k}[k]$ sits in graded degree $i-k$. The reason for this choice of grading is the following. The topmost component of the Kähler form gives us a morphism of complexes $\omega_2\colon G_{i,k} \to G_{i+2,k+1}[2]$; from the resulting representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we get a second Weil element w_{ω_2}. For every $i, k \in \mathbb{Z}$, it induces an isomorphism of complexes

$$\mathsf{w}_{\omega_2}\colon G_{i,k}[k] \to G_{-i,k-i}[k-2i].$$

The resulting isomorphism of complexes of $\mathscr{O}_B$-modules

$$\mathsf{w}_{\omega_2}\colon G_v \to G_f$$

therefore respects the grading exactly when we put the summand $G_{i,k}[i+k]$ in the complex G_f in graded degree $i-k$. Unlike the other isomorphism, this one cannot be defined on the level of smooth forms, because w_{ω_2} does not commute with the operator $\bar{\partial}$; instead, we have to rely on difficult results from Hodge theory (such as the relative Hard Lefschetz theorem) to do the work for us.

98. As in the previous step, we use the isomorphism $\mathsf{w}_{\omega_2}\colon G_v \to G_f$ to turn the object G_f into a complex of graded Ω_B-modules. We can again describe the Ω_B-module structure on G_f very concretely with the help of the results from Chapter 6. Recall from §72 Lemma that the operator $v(\beta)$ is primitive of weight -1 with respect to the representation of $\mathfrak{sl}_2(\mathbb{C})$ determined by ω_2. Consequently, we have

$$\operatorname{Ad} \mathsf{w}_{\omega_2}\big(v(\beta)\big) = \mathsf{w}_{\omega_2} v(\beta)\, \mathsf{w}_{\omega_2}^{-1} = [\omega_2, v(\beta)] = f(\beta)_1.$$

For every local section β of Ω_B^1, we therefore get another commutative diagram

$$\begin{array}{ccc} G_v & \xrightarrow{\mathsf{w}_{\omega_2}} & G_f \\ \downarrow v(\beta) & & \downarrow f(\beta)_1 \\ G_v & \xrightarrow{\mathsf{w}_{\omega_2}} & G_f. \end{array}$$

It follows that the Ω_B-module structure on the complex G_f is the unique one for which a local section β of Ω_B^1 acts via the collection of graded morphisms

$$f(\beta)_1\colon G_{i,k}[i+k] \to G_{i+1,k}[i+k+1].$$

This is compatible with the grading (because the left-hand side sits in graded degree $i-k$ and the right-hand side in graded degree $i-k+1$.)

99. The rest of the proof consists mostly in applying the BGG correspondence for the two graded algebras $\mathcal{S}_B = \operatorname{Sym}(\mathscr{T}_B)$ and $\Omega_B = \bigoplus_j \Omega_B^j$. Recall that we have

$$G = \mathbf{R}_B\left(\bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i]\right).$$

By construction, $\mathsf{w}_{\omega_2}\mathsf{w}_\sigma\colon G \to G_f$ is an isomorphism in the derived category of graded Ω_B-modules; the Ω_B-module structure on G_f has the property that a local section β of Ω_B^1 acts via the operator $f(\beta)_1$. Since $f(\beta)_1\colon G_{i,k}[i] \to G_{i+1,k}[i+1]$ only changes the index i, it is obvious that G_f decomposes, as a complex of graded Ω_B-modules, into a direct sum

$$G_f = \bigoplus_{k=-n}^{n}\left(\bigoplus_{i=-n}^{n} G_{i,k}[i]\right)[k].$$

Let F_{-k} denote the complex of graded $\mathcal{S}_B$-modules that the BGG correspondence associates to the k-th summand in this decomposition; in symbols,

$$F_{-k} = \mathbf{L}_B\left(\bigoplus_{i=-n}^{n} G_{i,k}[i]\right).$$

Since the BGG correspondence is an equivalence of categories (by §87 Theorem), we conclude from the isomorphism $G \cong G_f$ in the derived category of graded Ω_B-modules that

$$\bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i] \cong \bigoplus_{k=-n}^{n} F_k[-k], \tag{99.1}$$

in the derived category of graded $\mathcal{S}_B$-modules.

100. What we need to do now is to prove that $F_k \cong \operatorname{gr}_\bullet^F \mathcal{P}_k$ for all $k = -n, \dots, n$. Since we know next to nothing about the complexes F_k, this may seem impossible – but in fact, we have just enough information to make it work.

101. The left-hand side of (99.1) is a direct sum of graded $\mathcal{S}_B$-modules, each of which is an n-dimensional Cohen-Macaulay module. It follows that each complex F_k also splits into a direct sum of n-dimensional Cohen-Macaulay modules; consequently, we get a decomposition

$$F_k \cong \bigoplus_{\ell\in\mathbb{Z}} F_{k,\ell}[-\ell]$$

in the derived category of graded $\mathcal{S}_B$-modules. Each $F_{k,\ell}$ is a graded $\mathcal{S}_B$-module that is n-dimensional and Cohen-Macaulay. Our task now becomes showing that $F_{k,\ell} = 0$ for $\ell \neq 0$. It turns out that this is a purely formal consequence of what we know about the complexes $G_{i,k}$. On the one hand, the bound on the amplitude of the complex $G_{i,k}$ (in §44 Lemma) implies that F_k is concentrated in nonpositive degrees.

Lemma. *We have $F_{k,\ell} = 0$ for $\ell > 0$.*

Proof. Recall that we defined F_k with the help of the BGG correspondence as

$$F_k = \mathbf{L}_B \left(\bigoplus_{i=-n}^{n} G_{i,-k}[i] \right).$$

According to §44 Lemma, we have $\mathcal{H}^j G_{i,k} = 0$ for $j > k$, and therefore $\mathcal{H}^j\big(G_{i,-k}[i]\big) = \mathcal{H}^{i+j} G_{i,-k} = 0$ for $i + j + k > 0$. Since the summand $G_{i,-k}[i]$ has degree $i + k$ with respect to the grading on G_f (and therefore on the object in parentheses), this is exactly what we need in order to apply §88 Theorem. The conclusion is that $\mathcal{H}^j F_k = 0$ for $j > 0$. ☐

102. We can use duality to prove the vanishing of the graded $\mathcal{S}_B$-modules $F_{k,\ell}$ for $\ell < 0$. For the sake of clarity, let us temporarily write

$$G_k = \bigoplus_{i=-n}^{n} G_{i,-k}[i]$$

for the complex of graded Ω_B-modules on the right-hand side; then $F_k = \mathbf{L}_B(G_k)$. Recall from (43.1) that $G_{-i,-k} \cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\big(G_{i,k}, \omega_B[n]\big)$. Therefore

$$\widehat{G}_k = \mathbf{R}\mathcal{H}om_{\mathscr{O}_B}\Big(G_k, \omega_B[n]\Big) \cong \bigoplus_{i=-n}^{n} G_{i,k}[i],$$

where the notation $\widehat{G}_k$ comes from the proof of §89 Theorem. Note that the summand $G_{i,k}[i]$ again ends up having degree $i - k$ with respect to the induced grading on $\widehat{G}_k$.[7] By the same argument as in §101 Lemma, the complex $\mathbf{L}_B\big(\widehat{G}_k\big)$ is concentrated in degrees ≤ 0. Now §89 Theorem gives

$$\begin{aligned}\mathbf{L}_B(\widehat{G}_k) &\cong \mathbf{R}\mathcal{H}om_{\mathcal{S}_B}\Big(\mathbf{L}_B(G_k), \mathcal{S}_B \otimes \omega_B[n]\Big) \cong \omega_B \otimes \mathbf{R}\mathcal{H}om_{\mathcal{S}_B}\Big(F_k, \mathcal{S}_B[n]\Big) \\ &\cong \bigoplus_{\ell \leq 0} \omega_B \otimes \mathbf{R}\mathcal{H}om_{\mathcal{S}_B}\Big(F_{k,\ell}, \mathcal{S}_B[n]\Big)[\ell].\end{aligned}$$

Since each $F_{k,\ell}$ is an n-dimensional Cohen-Macaulay module, the complex on the right-hand side is concentrated in degrees ≥ 0. But the complex on the left-hand side is concentrated in degrees ≤ 0, and so it must be that $F_{k,\ell} = 0$ for $\ell < 0$.

103. The conclusion is that each complex F_k is actually a single graded $\mathcal{S}_B$-module (in cohomological degree 0). Because of the isomorphism in (99.1), we then get

$$F_k \cong \operatorname{gr}_{\bullet}^F \mathcal{P}_k.$$

If we now use the BGG correspondence in the other direction, we find that

$$\bigoplus_{i=-n}^{n} G_{i,-k}[i] \cong \bigoplus_{i=-n}^{n} G_{k,i}[i+k] \tag{103.1}$$

are isomorphic in the derived category of graded Ω_B-modules. We will now take a short break from proving §22 Theorem and turn to the symplectic relative Hard Lefschetz theorem (in §15 Theorem) and the symmetry conjecture of Shen and Yin (in §12 Conjecture).

[7]In fact, it should be the case that $\widehat{G}_k \cong G_{-k}$, but this would be tedious to check, and fortunately it turns out to be irrelevant for the proof.

104. If we forget the grading and the Ω_B-module structure, (103.1) becomes an isomorphism in the derived category of $\mathscr{O}_B$-modules. After replacing k by $-k$ and using the relative Hard Lefschetz isomorphism in (48.1), we can put it into the form

$$\bigoplus_{i=-n}^{n} G_{i,k}[i+k] \cong \bigoplus_{i=-n}^{n} G_{-k,i}[i] \cong \bigoplus_{i=-n}^{n} G_{k,i+k}[i+2k] \cong \bigoplus_{i=-n}^{n} G_{k,i}[i+k].$$

This is true for every $k = -n, \dots, n$, and so we are in a position where we can apply §57 Theorem. The conclusion is that the morphism

$$\sigma_1^k \colon G_{i,-k} \to G_{i,k}[2k]$$

is an isomorphism for every $k \geq 1$; this establishes the "symplectic relative Hard Lefschetz theorem". It also follows that the complexes $G_{i,k}$ and $G_{k,i}$ are isomorphic in the derived category, as predicted by the symmetry conjecture of Shen and Yin.

105. We can now go back and finish the proof of §22 Theorem. The relative Hard Lefschetz theorem gives us a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the direct sum of all the $G_{i,k}$ (with appropriate shifts). To simplify the notation, let us denote the three operators by the symbols $\mathsf{X}_1, \mathsf{Y}_1, \mathsf{H}_1$, and the Weil element by the symbol w_1. Concretely,

$$\mathsf{X}_1 = \omega_2 \colon G_{i,k} \to G_{i+2,k+1}[2] \quad \text{and} \quad \mathsf{Y}_1 = \mathsf{Y}_{\omega_2} \colon G_{i,k} \to G_{i-2,k-1}[-2],$$

H_1 acts on $G_{i,k}$ as multiplication by the integer i, and the Weil element is the isomorphism

$$\mathsf{w}_1 = \mathsf{w}_{\omega_2} \colon G_{i,k} \to G_{-i,k-i}[-2i].$$

Likewise, the symplectic relative Hard Lefschetz theorem gives us a second representation of $\mathfrak{sl}_2(\mathbb{C})$, for which we use the symbols $\mathsf{X}_2, \mathsf{Y}_2, \mathsf{H}_2$ and w_2. Once again,

$$\mathsf{X}_2 = \sigma_1 \colon G_{i,k} \to G_{i+1,k+2}[2] \quad \text{and} \quad \mathsf{Y}_2 = \mathsf{Y}_{\sigma_1} \colon G_{i,k} \to G_{i-1,k-2}[-2],$$

H_2 acts on $G_{i,k}$ as multiplication by the integer k, and the Weil element is the isomorphism

$$\mathsf{w}_2 = \mathsf{w}_{\sigma_1} \colon G_{i,k} \to G_{i-k,-k}[-2k].$$

We know from §38 Lemma that $[\omega_2, \sigma_1] = 0$, which translates into the relation $[X_1, X_2] = 0$.

106. We can now establish a direct relationship between $\operatorname{gr}_\bullet^F \mathcal{P}_i$ and $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$ with the help of the two $\mathfrak{sl}_2(\mathbb{C})$-representations. Recall from §85 Example that

$$\mathbf{R}_B(\operatorname{gr}_\bullet^F \mathcal{P}_i) = \bigoplus_{k=-n}^{n} G_{i,k}[i+k].$$

Consider the following chain of isomorphisms

$$G_{i,k} \xrightarrow{\mathsf{w}_1} G_{-i,k-i}[-2i] \xrightarrow{\mathsf{w}_2} G_{-k,i-k}[-2k] \xrightarrow{\mathsf{w}_1} G_{k,i}$$

that, very concretely, realizes the symmetry between $G_{i,k}$ and $G_{k,i}$. (In the drawing in §16, the composition $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_1$ is exactly reflection in the third diagonal of the hexagon.)

107. Let $\beta \in \Omega_B^1$ be a locally defined holomorphic 1-form. From the construction in Chapter 6, we get three operators (that are defined on the same open set as β, of course):

$$\pi^*\beta = (\pi^*\beta)_0 \colon G_{i,k} \to G_{i,k+1}[1]$$
$$v(\beta) = v(\beta)_{-1} \colon G_{i,k} \to G_{i-1,k-1}[-1]$$
$$f(\beta)_1 = [X_1, v(\beta)] \colon G_{i,k} \to G_{i,k+1}[1]$$

Let us see how these three operators interact with the two Weil elements w_1 and w_2. We know that $\operatorname{ad} X_1(\pi^*\beta) = 0$ (from §70 Lemma), which implies that

$$\operatorname{Ad}\mathsf{w}_1\big(\pi^*\beta\big) = \mathsf{w}_1\big(\pi^*\beta\big)\mathsf{w}_1^{-1} = \pi^*\beta.$$

We also showed (in §72 Lemma) that $(\operatorname{ad} X_1)^2 v(\beta) = 0$; because $v(\beta)$ has weight -1 with respect to H_1, it follows that

$$\operatorname{Ad}\mathsf{w}_1\big(v(\beta)\big) = [X_1, v(\beta)] = f(\beta)_1.$$

Lastly, we know from §64 Lemma that $[\sigma, v(\beta)] = \pi^*\beta$, and because $v(\beta)$ and $\pi^*\beta$ have weight -1 and 0 with respect to H_1, we get $[X_2, v(\beta)] = [\sigma_1, v(\beta)] = \pi^*\beta$. At the same time, $\pi^*\beta$ has weight 1 with respect to H_2 and commutes with X_2, and therefore

$$\operatorname{Ad}\mathsf{w}_2\big(\pi^*\beta\big) = -[\mathsf{Y}_2, \pi^*\beta] = v(\beta).$$

108. The conclusion of all these computations is that we get a commutative diagram

$$\begin{array}{ccccccc}
G_{i,k} & \xrightarrow{\mathsf{w}_1} & G_{-i,k-i}[-2i] & \xrightarrow{\mathsf{w}_2} & G_{-k,i-k}[-2k] & \xrightarrow{\mathsf{w}_1} & G_{k,i} \\
\downarrow{\scriptstyle \pi^*\beta} & & \downarrow{\scriptstyle \pi^*\beta} & & \downarrow{\scriptstyle v(\beta)} & & \downarrow{\scriptstyle f(\beta)_1} \\
G_{i,k+1}[1] & \xrightarrow{\mathsf{w}_1} & G_{-i,k-i+1}[-2i+1] & \xrightarrow{\mathsf{w}_2} & G_{-k-1,i-k-1}[-2k-1] & \xrightarrow{\mathsf{w}_1} & G_{k+1,i}[1]
\end{array}$$

in which all the horizontal morphisms are isomorphisms. After shifting everything by $i+k$ and taking the direct sum over all $k \in \mathbb{Z}$, this gives us an isomorphism between

$$\mathbf{R}_B(\operatorname{gr}_\bullet^F \mathcal{P}_i) = \bigoplus_{k=-n}^{n} G_{i,k}[i+k]$$

and the complex of graded Ω_B-modules

$$\bigoplus_{k=-n}^{n} G_{k,i}[i+k], \tag{108.1}$$

in the derived category of graded Ω_B-modules. Here the Ω_B-module structure on the second complex is the unique one where $\beta \in \Omega_B^1$ acts through the collection of graded morphisms

$$f(\beta)_1 \colon G_{k,i}[i+k] \to G_{k+1,i}[i+k+1],$$

and the grading has the summand $G_{k,i}[i+k]$ in graded degree k. The argument above shows that this object is isomorphic, via $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_1$, to the object

$$\mathbf{R}_B(\operatorname{gr}_\bullet^F \mathcal{P}_i) = \bigoplus_{k=-n}^{n} G_{i,k}[i+k],$$

which is the image of $\operatorname{gr}_\bullet^F \mathcal{P}_i$ under the BGG correspondence.

109. To conclude the proof, we only need to describe how the object in (108.1) is related to $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$. In the derived category of $\mathscr{O}_B$-modules, we do have an isomorphism

$$\mathbf{R}\pi_* \Omega_M^{n+i}[n] \cong \bigoplus_{k=-n}^{n} G_{k,i}[i]. \tag{109.1}$$

The object on the left-hand side is naturally a module over the algebra

$$\mathbf{R}\pi_* \mathscr{O}_M \cong \bigoplus_{j=0}^{n} \Omega_B^j[-j],$$

where the isomorphism comes from Matsushita's theorem (in §49 Theorem). If we take the associated graded of this action with respect to the perverse filtration, which is just the filtration by increasing values of k in the above decomposition, (109.1) becomes an isomorphism of graded modules

$$\operatorname{gr}_{\bullet}^{P}\left(\mathbf{R}\pi_* \Omega_M^{n+i}[n]\right) \cong \bigoplus_{k=-n}^{n} G_{k,i}[i].$$

Moreover, $\Omega_B^1[-1]$ now acts on the object on the right-hand side exactly through the collection of morphisms $f(\beta)_1 \colon G_{i,k} \to G_{i+1,k}[1]$. We can turn this into an honest action by Ω_B by adding a shift by k to the k-th term in the sum; in this way, we arrive at the object

$$\bigoplus_{k=-n}^{n} G_{k,i}[i+k]$$

with the Ω_B-module structure and the grading that appeared in the proof above.

110. In other words, we need to take the associated graded of $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$ with respect to the perverse filtration, in order to extract from the action by $f(\beta)$ its topmost component $f(\beta)_1$. After adding appropriate shifts (determined by the degree with respect to the grading), we then obtain a complex of graded Ω_B-modules, and under the BGG correspondence, this goes to the graded $\mathcal{S}_B$-module $\operatorname{gr}_{\bullet}^{F} \mathcal{P}_i$. With this, we have proved all the claims that were made in the introduction.

111. The interesting point is that the process we have described does relate $\mathbf{R}\pi_* \Omega_M^{n+i}[n]$ to the filtered $\mathscr{D}$-module $(\mathcal{P}_i, F_{\bullet}\mathcal{P}_i)$, but only after we take the associated graded on both sides: with respect to the perverse filtration on one side, and with respect to the Hodge filtration on the other. I do not know whether one can expect to relate the two sides without going to the associated graded objects.

9 Symmetries and the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$

112. In this chapter, we explain why the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ acts on the direct sum of the complexes $G_{i,k}$ (with suitable shifts). As in Chapter 6, this relies on computations with differential forms on M; the main point is to understand how the Weil element w_σ for the holomorphic symplectic form σ interacts with the Kähler form ω.

113. Recall that the holomorphic symplectic form σ determines an isomorphism $\mathscr{T}_M \cong \Omega_M^1$. If we think of the Kähler form ω as a $\bar{\partial}$-closed $(0,1)$-form with coefficients in Ω_M^1, hence as an element $\omega \in A^{0,1}(M, \Omega_M^1)$, the isomorphism provides us with another element

$$i(\omega) \in A^{0,1}(M, \mathscr{T}_M),$$

which is again $\bar{\partial}$-closed. We can also express the relation between ω and $i(\omega)$ as

$$i(\omega) \lrcorner \sigma = \omega.$$

More generally, contraction with $i(\omega)$ is an operator

$$i(\omega) \lrcorner \colon A^{p,q}(M) \to A^{p-1,q+1}(M),$$

that acts as follows. Let $z_1, \dots, z_{2n}$ be local holomorphic coordinates on M. Then

$$i(\omega) = \sum_{j,k} f_{j,k} \frac{\partial}{\partial z_j} \otimes d\bar{z}_k,$$

and we define the contraction against (p,q)-forms as

$$i(\omega) \lrcorner dz_J \wedge d\bar{z}_K = (-1)^p \sum_{j,k} f_{j,k} \operatorname{sgn}(J,j) dz_{J \setminus \{j\}} \otimes d\bar{z}_k \wedge d\bar{z}_K,$$

where $|J| = p$ and $|K| = q$. The sign $(-1)^p$ is caused by swapping the order of $d\bar{z}_k$ and dz_J.

114. The following lemma is proved in exactly the same way as §64 Lemma.

Lemma. *For every $k, q \in \mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccc} A^{n-k,q}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k,q}(M) \\ \downarrow{\scriptstyle \omega\wedge} & & \downarrow{\scriptstyle i(\omega)\lrcorner} \\ A^{n-k+1,q+1}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k-1,q+1}(M) \end{array}$$

115. Now let us try to understand the commutator of the two operators $\alpha \mapsto \omega \wedge \alpha$ and $\alpha \mapsto i(\omega) \lrcorner \alpha$. We notice that the contraction

$$\Theta = -i(\omega) \lrcorner \omega \in A^{0,2}(M)$$

is a $\bar{\partial}$-closed $(0,2)$-form. The following lemma shows that the commutator in question is nothing but wedge product with Θ.

Lemma. *We have* $\big[\omega\wedge, i(\omega)\lrcorner\big] = \Theta\wedge$.

Proof. This is due to the identity

$$i(\omega) \lrcorner \big(\omega \wedge \alpha\big) = \big(i(\omega) \lrcorner \omega\big) \wedge \alpha + \omega \wedge \big(i(\omega) \lrcorner \alpha\big) = -\Theta \wedge \alpha + \omega \wedge \big(i(\omega) \lrcorner \alpha\big),$$

which is easily proved by a computation in local coordinates. □

116. Because $\Theta = -i(\omega) \lrcorner \omega$ is a $\bar{\partial}$-closed $(0,2)$-form, it acts on the complex $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$. Using the decomposition theorem, it therefore determines a morphism (in the derived category)

$$\Theta\colon \bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[-i] \to \bigoplus_{i=-n}^{n} \operatorname{gr}_\bullet^F \mathcal{P}_i[2-i].$$

As in earlier chapters, this morphism breaks up into a finite sum $\Theta = \Theta_2 + \Theta_1 + \Theta_0 + \cdots$; each component Θ_j is a morphism

$$\Theta_j\colon \operatorname{gr}_\bullet^F \mathcal{P}_i \to \operatorname{gr}_\bullet^F \mathcal{P}_{i+j}[2-j]$$

in the derived category of graded $\operatorname{Sym} \mathscr{T}_B$-modules. We get induced morphisms

$$\Theta_j\colon G_{i,k} \to G_{i+j,k}[2].$$

What matters for our computation is that $\Theta_j = 0$ for $j \geq 3$. This holds because Θ is a $\bar{\partial}$-closed $(0,2)$-form and therefore commutes with the differential in the complex $\operatorname{gr}_\bullet^F \mathcal{C}_\pi$.

117. We now consider how $i(\omega)$ and Θ act on the complex (M, d), where $M_k^i = \pi_* \mathcal{A}_M^{n+k,n+i}$ and $d = (-1)^k \bar{\partial}$. Recall that this complex is isomorphic (in the derived category) to

$$G = \bigoplus_{i,k=-n}^{n} G_{i,k}[k] \cong \bigoplus_{k=-n}^{n} \mathbf{R}\pi_* \Omega_M^{n+k}[n],$$

but now we purposely forget the action by Ω_B, because contraction with $i(\omega)$ does not preserve it. Contraction against $i(\omega)$ and wedge product with Θ define two morphisms

$$i(\omega)\colon G \to G(-1)[1] \quad \text{and} \quad \Theta\colon G \to G[2]$$

in the derived category; the first morphism changes the grading, but the second one preserves it. The analysis in the previous two paragraphs shows that

$$\Theta = -\big[i(\omega), \omega\big] = -\big[\mathsf{w}_\sigma \cdot \omega \cdot \mathsf{w}_\sigma^{-1}, \omega\big],$$

where w_σ is the Weil element for the representation of $\mathfrak{sl}_2(\mathbb{C})$ on

$$\bigoplus_{k=-n}^{n} \Omega_M^{n+k}$$

coming from the action of the symplectic form σ. For the sake of clarity, let us denote the third generator of this representation by the symbol Y_σ.

118. As $[\sigma, \omega] = 0$, the operator ω is primitive with respect to this representation, and so

$$\mathsf{w}_\sigma \cdot \omega \cdot \mathsf{w}_\sigma^{-1} = -[\mathsf{Y}_\sigma, \omega].$$

After combining this with the formula for Θ, we get

$$\Theta = -\big[\mathsf{w}_\sigma \cdot \omega \cdot \mathsf{w}_\sigma^{-1}, \omega\big] = \big[[\mathsf{Y}_\sigma, \omega], \omega\big] = \big[\omega, [\omega, \mathsf{Y}_\sigma]\big].$$

Now we look at the topmost component on each side, with respect to the filtration by the first index i. On the right-hand side, by §122 Lemma, this is the iterated commutator

$$\Big[\omega_2, \big[\omega_2, \mathsf{Y}_{\sigma_1}\big]\Big],$$

which is a morphism $G_{i,k} \to G_{i+3,k}[2]$. The corresponding term on the left-hand side is Θ_3, and we already know that $\Theta_3 = 0$. This observation proves the following lemma.

Lemma. *We have* $\big[\omega_2, [\omega_2, \mathsf{Y}_{\sigma_1}]\big] = 0$*, and therefore* $[\mathsf{Y}_{\omega_2}, \mathsf{Y}_{\sigma_1}] = 0$*.*

Proof. The first assertion is clear. Since $\mathsf{Y}_{\sigma_1} : G_{i,k} \to G_{i-1,k-2}[-2]$, this is saying that Y_{σ_1} is primitive of weight -1 (with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation determined by ω_2), which gives the second assertion. □

119. We can now show that the two $\mathfrak{sl}_2(\mathbb{C})$-representations determined by ω_2 and σ_1 can be combined into a single representation of the larger Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Let us briefly review its structure. The Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is associated to the Dynkin diagram of type A_2, and so it has two simple roots, and the resulting Cartan matrix is

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

According to a theorem by Serre, $\mathfrak{sl}_3(\mathbb{C})$ is therefore generated as a Lie algebra by six elements $\mathsf{e}_1, \mathsf{f}_1, \mathsf{h}_1$, $\mathsf{e}_2, \mathsf{f}_2, \mathsf{h}_2$, subject to the following relations:

1. $[\mathsf{h}_i, \mathsf{h}_j] = 0$, $[\mathsf{h}_i, \mathsf{e}_j] = a_{i,j}\mathsf{e}_j$, and $[\mathsf{h}_i, \mathsf{f}_j] = -a_{i,j}\mathsf{f}_j$
2. $[\mathsf{e}_i, \mathsf{f}_j] = \delta_{i,j}\mathsf{h}_j$, where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise
3. $(\operatorname{ad} \mathsf{e}_i)^{1-a_{i,j}}\mathsf{e}_j = 0$ and $(\operatorname{ad} \mathsf{f}_i)^{1-a_{i,j}}\mathsf{f}_j = 0$.

120. The point is that the operators ω_2 and σ_1 satisfy the Serre relations for $\mathfrak{sl}_3(\mathbb{C})$.

Proposition. *If we define* $\mathsf{e}_1 = \omega_2$*,* $\mathsf{f}_1 = \mathsf{Y}_{\omega_2}$*,* $\mathsf{e}_2 = \mathsf{Y}_{\sigma_1}$*, and* $\mathsf{f}_2 = \sigma_1$*, and let* h_1 *and* h_2 *act on the complex* $G_{i,k}$ *as multiplication by* i *respectively* $-k$*, then these six operators satisfy the Serre relations for the Lie algebra* $\mathfrak{sl}_3(\mathbb{C})$*.*

Proof. The relations on the first line hold because ω_2 and σ_1 each determine a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and because $\omega_2 : G_{i,k} \to G_{i+2,k+1}[2]$ and $\sigma_1 : G_{i,k} \to G_{i+1,k+2}[2]$. The relations on the second line hold because $[\mathsf{e}_1, \mathsf{f}_2] = [\omega_2, \sigma_1] = 0$, and because $[\mathsf{f}_1, \mathsf{e}_2] = [\mathsf{Y}_{\omega_2}, \mathsf{Y}_{\sigma_1}] = 0$ by §118 Lemma. The relations on the third line now follow from the finite-dimensional representation theory of $\mathfrak{sl}_2(\mathbb{C})$: for example, $[\mathsf{h}_1, \mathsf{e}_2] = -\mathsf{e}_2$ and $[\mathsf{f}_1, \mathsf{e}_2] = 0$ are saying that e_2 is primitive of weight -1 with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation $\mathsf{e}_1, \mathsf{f}_1, \mathsf{h}_1$, and consequently $(\operatorname{ad} \mathsf{e}_1)^2 \mathsf{e}_2 = 0$. □

121. Because the operators ω_2 and σ_1 each act with a shift, we need to be a little bit careful if we want to say exactly which object in the derived category the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ acts on. There are two possible answers. One possibility is to take the direct sum

$$\bigoplus_{i,k,\ell \in \mathbb{Z}} G_{i,k}[\ell].$$

All six operators $\mathsf{e}_1, \mathsf{f}_1, \mathsf{h}_1$ and $\mathsf{e}_2, \mathsf{f}_2, \mathsf{h}_2$ clearly act on this object; the disadvantage is that there are infinitely many nonzero terms. The other possibility is to look at the direct sum

$$\bigoplus_{i,k=-n}^{n} G_{i,k}\Big[\lfloor \tfrac{2}{3}(i+k) \rfloor\Big].$$

This only has finitely many nonzero terms, and all six operators act on it; the only disadvantage is that the extra shift – by the integer part of $\frac{2}{3}(i+k)$ – seems somewhat artificial.

122. The proof above depends on knowing the topmost component of the operator Y_σ. The following lemma, whose proof uses the symplectic relative Hard Lefschetz theorem (in §15 Theorem), shows that this is exactly Y_{σ_1}.

Lemma. *The difference* $\mathsf{Y}_\sigma - \mathsf{Y}_{\sigma_1}$ *maps* $G_{i,k}$ *into the sum of the* $G_{i+j,k-2}[-2]$ *with* $j \leq -2$.

Proof. To simplify the notation, let us again denote the operators in the $\mathfrak{sl}_2(\mathbb{C})$-representation determined by σ_1 by the letters $\mathsf{X}_2 = \sigma_1$, Y_2, and H_2. Let us also set $\tau = \mathsf{Y}_\sigma$, and expand this according to its degree in i as $\tau = \sum_j \tau_j$; the individual components are operators

$$\tau_j \colon G_{i,k} \to G_{i+j,k-2}[-2].$$

In these terms, the lemma is asserting that $\tau_j = 0$ for $j \geq 0$, and that $\tau_{-1} = \mathsf{Y}_2$. Let j be the largest integer such that $\tau_j \neq 0$. If $j \geq 0$, then after expanding the relation

$$\mathsf{H}_2 = [\sigma, \tau] = [\mathsf{X}_2 + \sigma_0 + \cdots, \tau_j + \tau_{j-1} + \cdots]$$

by degree, we would get $[\mathsf{X}_2, \tau_j] = 0$. But this is impossible because τ_j has weight -2 with respect to $\operatorname{ad} \mathsf{H}_2$. The same reasoning shows that $\mathsf{H}_2 = [\mathsf{X}_2, \tau_{-1}]$, and as Y_2 is uniquely determined by H_2 and X_2, it follows that $\tau_{-1} = \mathsf{Y}_2$. Consequently, $\tau = \mathsf{Y}_2 + \tau_{-2} + \cdots$. ☐

123. The relations for the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ can also be interpreted nicely in terms of the reflections given by the two Weil elements w_1 and w_2.

Lemma. *We have* $(\operatorname{Ad} \mathsf{w}_1 \circ \operatorname{Ad} \mathsf{w}_2 \circ \operatorname{Ad} \mathsf{w}_1)(\omega_2) = \sigma_1$.

Proof. Recall from §105 that $\mathsf{X}_1 = \omega_2$ and that $\mathsf{X}_2 = \sigma_1$. The definition of the Weil element shows that $\operatorname{Ad} \mathsf{w}_1(\mathsf{X}_1) = \mathsf{w}_1 \mathsf{X}_1 \mathsf{w}_1^{-1} = -\mathsf{Y}_1$. The relations $[\mathsf{Y}_2, \mathsf{Y}_1] = 0$ and $[\mathsf{H}_2, \mathsf{Y}_1] = -\mathsf{Y}_1$ are saying that Y_1 is primitive of weight -1 with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation $\mathsf{X}_2, \mathsf{Y}_2, \mathsf{H}_2$, and therefore $\operatorname{Ad} \mathsf{w}_2(-\mathsf{Y}_1) = [\mathsf{X}_2, -\mathsf{Y}_1] = [\mathsf{Y}_1, \mathsf{X}_2]$. Because $(\operatorname{ad} \mathsf{Y}_1)^2 \mathsf{X}_2 = 0$, this is now primitive of weight -1 with respect to the $\mathfrak{sl}_2(\mathbb{C})$-representation $\mathsf{X}_1, \mathsf{Y}_1, \mathsf{H}_1$, and so finally

$$(\operatorname{Ad} \mathsf{w}_1 \circ \operatorname{Ad} \mathsf{w}_2 \circ \operatorname{Ad} \mathsf{w}_1)(\mathsf{X}_1) = \operatorname{Ad} \mathsf{w}_1\big([\mathsf{Y}_1, \mathsf{X}_2]\big) = \big[\mathsf{X}_1, [\mathsf{Y}_1, \mathsf{X}_2]\big] = \mathsf{X}_2.$$

This proves the assertion. ☐

10 The compact case

124. We end this paper with a short analysis of the compact case. Most notably, we prove that the action by the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ can be upgraded, in the case where M and B are compact, to an action by the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$. Since $\mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{so}_6(\mathbb{C})$, this gives an alternative proof (not relying on the existence of a hyperkähler metric) for a result by Looijenga-Lunts [LL97, §4] and Verbitsky [Ver96]: the cohomology of an irreducible compact hyperkähler manifold with a Lagrangian fibration carries an action by $\mathfrak{so}_6(\mathbb{C})$.

125. We assume from now on that M and B are both compact; in other words, M is a holomorphic symplectic compact Kähler manifold of dimension $2n$, and the base B of our Lagrangian fibration $\pi \colon M \to B$ is a compact Kähler manifold of dimension n. In this case, the holomorphic symplectic form $\sigma \in H^0(M, \Omega_M^2)$ is automatically closed. If M is simply connected, then it has a hyperkähler metric (by Yau's theorem) and B is a product of projective spaces (by Hwang's theorem), but we are not assuming that this is the case.

126. Let $\lambda \in A^{1,1}(B)$ be a Kähler class on B. Recall that we have

$$\mathbf{R}\pi_* \mathbb{Q}_M(n)[2n] \cong \bigoplus_{i=-n}^{n} P_i[-i],$$

and that each P_i is a polarizable Hodge module of weight i on B. Since B is a compact Kähler manifold, the cohomology groups $H^j(B, P_i)$ therefore have Hodge structures of weight $i+j$, and we have an isomorphism of Hodge structures

$$H^{2n+j}(M, \mathbb{Q})(n) \cong \bigoplus_{i=-n}^{n} H^{j-i}(B, P_i).$$

By the Hard Lefschetz theorem (for polarizable Hodge modules on compact Kähler manifolds), cup product with λ determines, for each $j \geq 1$, an isomorphism

$$\lambda^j : H^{-j}(B, P_i) \to H^j(B, P_i)(j)$$

of Hodge structures of weight $i - j$.

127. Since λ is closed, its pullback $\pi^* \lambda \in A^{1,1}(M)$ acts on the complex $\mathcal{C}_\pi$ from §34. Let us check that, with respect to the isomorphism

$$\mathcal{C}_\pi \cong \bigoplus_{i=-n}^{n} (\mathcal{P}_i, F_\bullet \mathcal{P}_i)[-i]$$

from (34.1), the action by $\pi^* \lambda$ is diagonal. With our usual notation, this amounts to saying that $\pi^* \lambda = (\pi^* \lambda)_0$, and that the morphism

$$(\pi^* \lambda)_0 : (\mathcal{P}_i, F_\bullet \mathcal{P}_i) \to (\mathcal{P}_i, F_{\bullet -1} \mathcal{P}_i)[2]$$

is the one induced by the Kähler form $\lambda \in A^{1,1}(B)$. This is a consequence of the projection formula: if we view λ as a morphism $\lambda : \mathbb{C}_B \to \mathbb{C}_B[2]$ in the derived category of constructible sheaves, then the claim is that

$$\mathbf{R}\pi_* \Big(\mathbb{C}_M \xrightarrow{\pi^* \lambda} \mathbb{C}_M[2] \Big) \cong \Big(\mathbb{C}_B \xrightarrow{\lambda} \mathbb{C}_B[2] \Big) \otimes \mathbf{R}\pi_* \mathbb{C}_M.$$

Using the de Rham resolutions of $\mathbb{C}_M$ and $\mathbb{C}_B$ by differential forms, the projection formula in this case amounts to the following statement.

Lemma. *The morphism of complexes*

$$(\mathcal{A}_B^\bullet, d) \otimes (\pi_* \mathcal{A}_M^\bullet, d) \to (\pi_* \mathcal{A}_M^\bullet, d), \quad \alpha \otimes \beta \mapsto \pi^* \alpha \wedge \beta,$$

is a quasi-isomorphism.

Proof. Since we are using Deligne's sign conventions, the tensor product has terms

$$\bigoplus_{i+j=k} \mathcal{A}_B^i \otimes \pi_* \mathcal{A}_M^j$$

and differential $d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^i \alpha \otimes d\beta$. The morphism of complexes is

$$\bigoplus_{i+j=k} \mathcal{A}_B^i \otimes \pi_* \mathcal{A}_M^j \to \pi_* \mathcal{A}_M^k, \quad \sum_{i+j=k} \alpha_i \otimes \beta_j \mapsto \sum_{i+j=k} \pi^* \alpha_i \wedge \beta_j.$$

This is clearly surjective: if $U \subseteq B$ is open, then the element $1 \otimes \beta \in A^0(U) \otimes A^k\bigl(\pi^{-1}(U)\bigr)$ is a preimage for $\beta \in A^k\bigl(\pi^{-1}(U)\bigr)$. To show that the morphism is a quasi-isomorphism, one then argues locally with the help of the Poincaré lemma, which says that $\mathcal{A}_B^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_B$. □

128. The Hard Lefschetz theorem for the Kähler form λ gives us a third action by the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In the rest of this chapter, we shall argue that all three actions can be combined into an action of the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$. Consider the vector spaces

$$H^{i,j,k} = H^{i+j}(B, G_{i,k}) = H^j\bigl(B, \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i)\bigr) \tag{128.1}$$

for $i, j, k \in \mathbb{Z}$. These vector spaces refine the Hodge decomposition on M, taking into account the decomposition theorem for $\pi \colon M \to B$. Indeed, according to the direct image theorem for the cohomology of the polarizable Hodge module P_i on the compact Kähler manifold B, the $\mathbb{Q}$-vector space $H^j(B, P_i)$ carries a Hodge structure of weight $i + j$, and

$$\operatorname{gr}_{-k}^F\bigl(H^j(B, P_i) \otimes_{\mathbb{Q}} \mathbb{C}\bigr) \cong \operatorname{gr}_{-k}^F H^j\bigl(B, \operatorname{DR}(\mathcal{P}_i)\bigr) \cong H^j\bigl(B, \operatorname{gr}_{-k}^F \operatorname{DR}(\mathcal{P}_i)\bigr).$$

This leads to the following isomorphism with the Hodge decomposition on $H^j(B, P_i)$:

$$H^{i,j,k} \cong \bigl(H^j(B, P_i) \otimes_{\mathbb{Q}} \mathbb{C}\bigr)^{k, i+j-k} \tag{128.2}$$

In order to relate this to the Hodge decomposition on M, we use the isomorphism

$$\mathbf{R}\pi_* \Omega_M^{n+k}[n-k] \cong \bigoplus_{i=-n}^{n} G_{i,k}$$

from (39.1). Substituting in the definition of $H^{i,j,k}$, we see immediately that

$$H^{n+k,n+j}(M) \cong H^{n+j}(M, \Omega_M^{n+k}) \cong \bigoplus_{i=-n}^{n} H^{j+k}(B, G_{i,k}) = \bigoplus_{i=-n}^{n} H^{i,j+k-i,k}. \tag{128.3}$$

The vector spaces $H^{i,j,k}$ also interact with duality in the expected way. Indeed, because of (43.1), the Grothendieck dual of $G_{i,k}$ is isomorphic to $G_{-i,-k}$, and so

$$\operatorname{Hom}_{\mathbb{C}}\bigl(H^{i,j,k}, \mathbb{C}\bigr) \cong H^{-i,-j,-k}. \tag{128.4}$$

129. The discussion in the preceding paragraph leads to the following concrete interpretation for the three indices i, j, k, similar to what happens for the complexes $G_{i,k}$:

1. The first index i records the cohomological degree along the fibers of π, in the sense that $H^{i,j,k}$ is associated with the $(n+i)$-th cohomology groups of the fibers.
2. The second index j records the cohomological degree along the base B, in the sense that $H^{i,j,k}$ is associated with the $(n+j)$-th cohomology groups on B.
3. The third index k records the holomorphic degree, in the sense that $H^{i,j,k}$ is associated with the sheaf Ω_M^{n+k} of holomorphic forms of degree $(n+k)$.

130. The Kähler form $\omega \in A^{1,1}(M)$ induces a morphism

$$\omega_2 \colon H^{i,j,k} = H^{i+j}(B, G_{i,k}) \to H^{i+j+2}(B, G_{i+2,k+1}) = H^{i+2,j,k+1},$$

and the relative Hard Lefschetz theorem for ω becomes an isomorphism

$$\omega_2^i \colon H^{-i,j,k} \to H^{i,j,k+i} \quad \text{for } i \geq 1. \tag{130.1}$$

Similarly, the holomorphic symplectic form $\sigma \in H^0(M, \Omega_M^2)$ induces a morphism

$$\sigma_1 \colon H^{i,j,k} = H^{i+j}(B, G_{i,k}) \to H^{i+j+2}(B, G_{i+1,k+2}) = H^{i+1,j+1,k+2},$$

and the symplectic relative Hard Lefschetz theorem (§15 Theorem) becomes an isomorphism

$$\sigma_1^k \colon H^{i,j,-k} \to H^{i+k,j+k,k} \quad \text{for } k \geq 1. \tag{130.2}$$

Finally, by (128.2), the Kähler form $\lambda \in A^{1,1}(B)$ induces a morphism

$$\lambda \colon H^{i,j,k} \cong \bigl(H^j(B, P_i) \otimes_{\mathbb{Q}} \mathbb{C}\bigr)^{k,i+j-k} \to \bigl(H^{j+2}(B, P_i) \otimes_{\mathbb{Q}} \mathbb{C}\bigr)^{k+1,i+j-k+1} \cong H^{i,j+2,k+1},$$

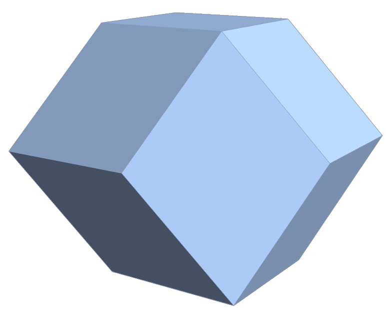
and the Hard Lefschetz theorem for λ becomes an isomorphism

$$\lambda^j \colon H^{i,-j,k} \to H^{i,j,k+j} \quad \text{for } j \geq 1. \tag{130.3}$$

131. From these isomorphisms and §44 Lemma, one can deduce the following result.

Lemma. *We have* $H^{i,j,k} = 0$ *unless* $\max\bigl(|i|, |j|, |k|, |i-k|, |j-k|, |i+j-k|\bigr) \leq n$.

We may arrange the $H^{i,j,k}$ on a three-dimensional grid, by putting the vector space for the multi-index (i,j,k) at the point with coordinates $\frac{1}{2}\bigl(2k-i-j, i\sqrt{3}, j\sqrt{3}\bigr)$. In geometric terms, the conditions in the lemma are describing a *rhombic dodecahedron*:



The lemma is asserting that all the points corresponding to nonzero $H^{i,j,k}$ lie inside this rhombic dodecahedron. The example of a projection $\pi \colon A \times B \to B$ from the product of two n-dimensional abelian varieties shows that this is the best one can expect in general. For irreducible compact hyperkähler manifolds, Nagai's conjecture would imply that the convex hull of the points corresponding to nonzero $H^{i,j,k}$ is actually an octahedron [HM22, §3.8].

132. The three isomorphisms in (130.1), (130.2), and (130.3) give us three reflections

$$(i,j,k) \to (-i, j, k-i), \quad (i,j,k) \to (i, -j, k-j), \quad (i,j,k) \to (i-k, j-k, -k),$$

and a little bit of calculation shows that these three reflections together generate a total of 24 symmetries, with a group structure isomorphic to the symmetric group S_4. This is not surprising, because S_4 is the Weyl group of $\mathfrak{sl}_4(\mathbb{C})$. One of the symmetries is

$$(i,j,k) \to (j, i, i+j-k),$$

and because of (128.3), it induces an isomorphism $H^{p,q}(M) \cong H^{q,p}(M)$. It would be interesting if one could explain this particular symmetry in a geometric way.

133. As in the previous chapter, we need to establish one additional identity in order to have an action by the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$. Once again, the isomorphism $\mathscr{T}_M \cong \Omega_M^1$ associates to the cohomology class of $\pi^*\lambda \in H^1(M, \Omega_M^1)$ a new element

$$i(\pi^*\lambda) \in A^{0,1}(M, \mathscr{T}_M),$$

that is $\bar{\partial}$-closed and satisfies $i(\pi^*\lambda) \lrcorner\, \sigma = \pi^*\lambda$. The following lemma is also proved in the same way as §64 Lemma.

Lemma. *For every $k, q \in \mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccc} A^{n-k,q}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k,q}(M) \\ \Big\downarrow{\scriptstyle (\pi^*\lambda)\wedge} & & \Big\downarrow{\scriptstyle i(\pi^*\lambda)\,\lrcorner} \\ A^{n-k+1,q+1}(M) & \xrightarrow{\mathsf{w}_\sigma} & A^{n+k-1,q+1}(M) \end{array}$$

134. The crucial observation is that the commutator of the two operators $\alpha \mapsto (\pi^*\lambda) \wedge \alpha$ and $\alpha \mapsto i(\omega) \lrcorner\, \alpha$ vanishes.

Lemma. *We have* $\big[(\pi^*\lambda)\wedge, i(\pi^*\lambda)\,\lrcorner\big] = 0$.

Proof. As before, the commutator is the wedge product with the $\bar{\partial}$-closed $(0,2)$-form

$$-i(\pi^*\lambda) \lrcorner (\pi^*\lambda) \in A^{0,2}(M).$$

In a nutshell, this vanishes because the vector fields in $i(\pi^*\lambda)$ are tangent to the fibers of π. It is enough to prove this locally, and so we may assume without loss of generality that

$$\lambda = \sum_{j=1}^n dt_j \wedge \theta_j,$$

where $t_1, \dots, t_n$ are local coordinates on B and $\theta_1, \dots, \theta_n$ are $\bar{\partial}$-closed $(0,1)$-forms. Let η_j be the unique holomorphic vector field such that $\pi^*(dt_j) = \eta_j \lrcorner\, \sigma$. Then

$$i(\pi^*\lambda) = \sum_{j=1}^n \eta_j \otimes \pi^*\theta_j,$$

and consequently

$$-i(\pi^*\lambda) \lrcorner (\pi^*\lambda) = \sum_{j,k=1}^n \big(\eta_j \lrcorner (\pi^* dt_k)\big) \cdot \pi^*(\theta_j \wedge \theta_k) = 0,$$

due to the fact that the vector fields η_j are tangent to the fibers of π. □

135. We now consider how $i(\pi^*\omega)$ acts on the complex (M, d), where $M_k^i = \pi_* \mathcal{A}_M^{n+k,n+i}$ and $d = (-1)^k \bar{\partial}$. Contraction against $i(\pi^*\lambda)$ defines a morphism.

$$i(\pi^*\lambda)\colon G \to G(-1)[1]$$

in the derived category; note that this morphism changes the grading. Since $\pi^*\lambda$ and σ commute (as 2-forms on M), we have $i(\pi^*\lambda) = \mathsf{w}_\sigma \cdot (\pi^*\lambda) \cdot \mathsf{w}_\sigma^{-1} = -[\mathsf{Y}_\sigma, \pi^*\lambda]$, and therefore

$$\big[\pi^*\lambda, [\pi^*\lambda, \mathsf{Y}_\sigma]\big] = -[i(\pi^*\lambda), \pi^*\lambda] = 0.$$

If we now take the topmost component, and remember that the action by $(\pi^*\lambda)_0$ is diagonal and equal to λ, we arrive at the following result.

Lemma. *We have* $\big[\lambda, [\lambda, \mathsf{Y}_{\sigma_1}]\big] = 0$, *and therefore* $[\mathsf{Y}_\lambda, \mathsf{Y}_{\sigma_1}] = 0$.

136. We can finally prove that the three $\mathfrak{sl}_2(\mathbb{C})$-representations determined by ω_2, σ_1, and λ can be combined into a single representation of $\mathfrak{sl}_4(\mathbb{C})$. Recall that the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ is associated to the Dynkin diagram of type A_3, and so it has three simple roots, and the resulting Cartan matrix is

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

By Serre's theorem, $\mathfrak{sl}_4(\mathbb{C})$ is generated as a Lie algebra by nine elements $\mathsf{e}_1, \mathsf{f}_1, \mathsf{h}_1$, $\mathsf{e}_2, \mathsf{f}_2, \mathsf{h}_2$, $\mathsf{e}_3, \mathsf{f}_3, \mathsf{h}_3$, subject to the following relations (which have the same shape as before):

1. $[\mathsf{h}_i, \mathsf{h}_j] = 0$, $[\mathsf{h}_i, \mathsf{e}_j] = a_{i,j}\mathsf{e}_j$, and $[\mathsf{h}_i, \mathsf{f}_j] = -a_{i,j}\mathsf{f}_j$
2. $[\mathsf{e}_i, \mathsf{f}_j] = \delta_{i,j}\mathsf{h}_j$, where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise
3. $(\operatorname{ad}\mathsf{e}_i)^{1-a_{i,j}}\mathsf{e}_j = 0$ and $(\operatorname{ad}\mathsf{f}_i)^{1-a_{i,j}}\mathsf{f}_j = 0$.

137. The proof of the Serre relations is very similar to what we did for $\mathfrak{sl}_3(\mathbb{C})$, and so we will only sketch the argument.

Proposition. *If we define* $\mathsf{e}_1 = \omega_2$, $\mathsf{f}_1 = \mathsf{Y}_{\omega_2}$, $\mathsf{e}_2 = \mathsf{Y}_{\sigma_1}$, $\mathsf{f}_2 = \sigma_1$, $\mathsf{e}_3 = \lambda$, *and* $\mathsf{f}_3 = \mathsf{Y}_\lambda$, *and let* h_1, h_2 *and* h_3 *act on the vector space* $H^{i,j,k}$ *respectively as multiplication by* i, $-k$ *and* j, *then these nine operators satisfy the Serre relations for the Lie algebra* $\mathfrak{sl}_4(\mathbb{C})$.

Proof. Since ω, σ, and $\pi^*\lambda$ commute (as 2-forms on M), it is easy to see that $[\mathsf{e}_1, \mathsf{e}_3] = [\omega_2, \lambda] = 0$ and $[\mathsf{f}_2, \mathsf{e}_3] = [\sigma_1, \lambda] = 0$. Because $[\mathsf{h}_1, \mathsf{e}_3] = 0$, it follows that $[\mathsf{f}_1, \mathsf{e}_3] = 0$, and so the two Lie algebras generated by $\mathsf{e}_1, \mathsf{f}_1, \mathsf{h}_1$ and $\mathsf{e}_3, \mathsf{f}_3, \mathsf{h}_3$ commute; this gives about half of the necessary relations. Among the remaining new relations, the only nontrivial ones are $(\operatorname{ad}\mathsf{e}_3)^2\mathsf{e}_2 = 0$ and $[\mathsf{f}_3, \mathsf{e}_2] = 0$, and these are of course contained in §135 Lemma. □

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