HOLONOMIC D-MODULES ON ABELIAN VARIETIES

CHRISTIAN SCHNELL

Abstract. We study the Fourier-Mukai transform for holonomic $\mathscr{D}$-modules on complex abelian varieties. Among other things, we show that the cohomology support loci of a holonomic $\mathscr{D}$-module are finite unions of linear subvarieties, which go through points of finite order for objects of geometric origin; that the standard t-structure on the derived category of holonomic complexes corresponds, under the Fourier-Mukai transform, to a certain perverse coherent t-structure in the sense of Kashiwara and Arinkin-Bezrukavnikov; and that Fourier-Mukai transforms of simple holonomic $\mathscr{D}$-modules are intersection complexes in this t-structure. This supports the conjecture that Fourier-Mukai transforms of holonomic $\mathscr{D}$-modules are "hyperkähler perverse sheaves".

A. Introduction

1. Agenda. In this paper, we begin a systematic study of holonomic $\mathscr{D}$-modules on complex abelian varieties; recall that a $\mathscr{D}$-module is said to be *holonomic* if its characteristic variety is a Lagrangian subset of the cotangent bundle. Regular holonomic $\mathscr{D}$-modules, which correspond to perverse sheaves under the Riemann-Hilbert correspondence, are familiar objects in complex algebraic geometry. Due to recent breakthroughs by Kedlaya, Mochizuki, and Sabbah (summarized in [Sab13]), we now have an almost equally good understanding of irregular holonomic $\mathscr{D}$-modules, and many important results from the regular case (such as the decomposition theorem or the hard Lefschetz theorem) have been extended to the irregular case. A careful study of an important special case, namely that of complex abelian varieties, may therefore be of some interest.

The original motivation for this project comes from a sequence of papers by Green and Lazarsfeld [GL87, GL91], Arapura [Ara92], and Simpson [Sim93]. In their work on the *generic vanishing theorem*, these authors analyzed the loci

$$S_m^{p,q}(X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^q\bigl(X, \Omega_X^p \otimes L\bigr) \geq m \right\} \subseteq \operatorname{Pic}^0(X),$$

for X a projective (or compact Kähler) complex manifold. Among other things, they showed that each irreducible component of $S_m^{p,q}(X)$ is a translate of a subtorus by a point of finite order; and they obtained bounds on the codimension in the most interesting cases ($p = 0$ and $p = \dim X$). These bounds imply for example that when the Albanese mapping of X is generically finite over its image, all higher cohomology groups of $\omega_X \otimes L$ vanish for a generic line bundle $L \in \operatorname{Pic}^0(X)$.

Hacon pointed out that the codimension bounds can be interpreted as properties of certain coherent sheaves on abelian varieties, and then reproved them using the

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Fourier-Mukai transform [Hac04]. His method applies particularly well to those coherent sheaves that occur in the Hodge filtration of a mixed Hodge module; based on this observation, Popa and I generalized all of the results connected with the generic vanishing theorem (in the projective case) to Hodge modules of geometric origin on abelian varieties [PS13]. As a by-product, we also obtained results about certain regular holonomic $\mathscr{D}$-modules on abelian varieties, namely those that can be realized as direct images of structure sheaves of smooth projective varieties with nontrivial first Betti number. A pretty application of the $\mathscr{D}$-module theory to the geometry of varieties of general type can be found in [PS14].

As we shall see below, all of the results about $\mathscr{D}$-modules in [PS13] remain true for *arbitrary* holonomic $\mathscr{D}$-modules on abelian varieties. In most cases, the proof in the general case turns out to be simpler; we shall also discover that certain statements – such as the codimension bounds – only reveal their true meaning in this broader context.

The principal results about holonomic $\mathscr{D}$-modules are summarized in Section 2 to Section 5; for the convenience of the reader, we also translate everything into the language of perverse sheaves in Section 7. Our main technical tool will be the Fourier-Mukai transform for algebraic $\mathscr{D}$-modules, introduced by Laumon [Lau96] and Rothstein [Rot96], and our results suggest a conjecture about the structure of Fourier-Mukai transforms of (regular or irregular) holonomic $\mathscr{D}$-modules. This conjecture, together with some evidence for it, is described in Section 6.

2. The structure theorem. Let A be a complex abelian variety, and let $\mathscr{D}_A$ be the sheaf of linear differential operators of finite order. The simplest examples of left $\mathscr{D}_A$-modules are line bundles L with integrable connection $\nabla\colon L \to \Omega_A^1 \otimes L$. Because A is an abelian variety, the moduli space $A^\natural$ of such pairs (L,∇) is a quasi-projective algebraic variety of dimension $2\dim A$. The basic idea in the study of $\mathscr{D}_A$-modules is to exploit the fact that $A^\natural$ is so big.

One approach is to consider, for a left $\mathscr{D}_A$-module $\mathcal{M}$, the cohomology groups (in the sense of $\mathscr{D}$-modules) of the various twists $\mathcal{M} \otimes_{\mathscr{O}_A} (L,\nabla)$; we use this symbol to denote the natural $\mathscr{D}_A$-module structure on the tensor product $\mathcal{M} \otimes_{\mathscr{O}_A} L$. That information is contained in the *cohomology support loci* of $\mathcal{M}$, which are the sets

$$S_m^k(A,\mathcal{M}) = \Big\{\, (L,\nabla) \in A^\natural \;\Big|\; \dim \mathbf{H}^k\Big(A, \mathrm{DR}_A\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L,\nabla)\big)\Big) \geq m \,\Big\}. \tag{2.1}$$

The definition works more generally for complexes of $\mathscr{D}$-modules; we are especially interested in the case of a *holonomic complex* $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$, that is to say, a cohomologically bounded complex of $\mathscr{D}_A$-modules with holonomic cohomology sheaves. Our first result is the following structure theorem.

Theorem 2.2. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ be a holonomic complex.*

(a) Each $S_m^k(A,\mathcal{M})$ is a finite union of linear subvarieties of $A^\natural$.
(b) If $\mathcal{M}$ is a semisimple regular holonomic $\mathscr{D}_A$-module of geometric origin, in the sense of [BBD82, 6.2.4]*, then these linear subvarieties are arithmetic.*

Here we are using the new term *(arithmetic) linear subvarieties* for what Simpson called *(torsion) translates of triple tori* in [Sim93, p. 365]; the definition is as follows.

Definition 2.3. A *linear subvariety* of $A^\natural$ is any subset of the form

$$(L,\nabla) \otimes \mathrm{im}\big(f^\natural\colon B^\natural \to A^\natural\big), \tag{2.4}$$

for a surjective morphism of abelian varieties $f\colon A \to B$ with connected fibers, and a line bundle with integrable connection $(L, \nabla) \in A^\natural$. We say that a linear subvariety is *arithmetic* if (L, ∇) can be taken to be a torsion point.[1]

To prove Theorem 2.2, we use the Riemann-Hilbert correspondence. If $\mathcal{M}$ is a holonomic $\mathscr{D}_A$-module, then according to a fundamental theorem by Kashiwara [HTT08, Theorem 4.6.6], its de Rham complex

$$\mathrm{DR}_A(\mathcal{M}) = \Big[\mathcal{M} \to \Omega_A^1 \otimes \mathcal{M} \to \cdots \to \Omega_A^{\dim A} \otimes \mathcal{M}\Big][\dim A],$$

placed in degrees $-\dim A, \dots, 0$, is a perverse sheaf on A. More generally, $\mathrm{DR}_A(\mathcal{M})$ is a constructible complex for any $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ [HTT08, Theorem 4.6.3], and the Riemann-Hilbert correspondence [HTT08, Theorem 7.2.1] asserts that the functor

$$\mathrm{DR}_A\colon \mathrm{D}_{rh}^b(\mathscr{D}_A) \to \mathrm{D}_c^b(\mathbb{C}_A)$$

from regular holonomic complexes to constructible complexes is an equivalence of categories.

Now let $\mathrm{Char}(A)$ be the space of characters of the fundamental group of A; any character $\rho\colon \pi_1(A, 0) \to \mathbb{C}^*$ determines a local system $\mathbb{C}_\rho$ of rank one on A. We define the cohomology support loci of a constructible complex $K \in \mathrm{D}_c^b(\mathbb{C}_A)$ as

$$S_m^k(A, K) = \Big\{ \rho \in \mathrm{Char}(A) \,\Big|\, \dim \mathbf{H}^k\big(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) \geq m \Big\}.$$

The well-known correspondence between vector bundles with integrable connection and representations of the fundamental group gives a biholomorphic mapping

$$\Phi\colon A^\natural \to \mathrm{Char}(A), \quad (L, \nabla) \mapsto \mathrm{Hol}(L, \nabla), \tag{2.5}$$

and it is very easy to show – see Theorem 15.1 below – that the cohomology support loci for $\mathcal{M}$ and $\mathrm{DR}_A(\mathcal{M})$ are related by the formula

$$\Phi\big(A, S_m^k(A, \mathcal{M})\big) = S_m^k\big(A, \mathrm{DR}_A(\mathcal{M})\big). \tag{2.6}$$

The proof of Theorem 2.2 is based on the fact that $\mathrm{Char}(A)$ and $A^\natural$, while isomorphic as complex manifolds, are not isomorphic as complex algebraic varieties. According to a nontrivial theorem by Simpson, a closed algebraic subset $Z \subseteq A^\natural$ is a finite union of linear subvarieties if and only if its image $\Phi(Z) \subseteq \mathrm{Char}(A)$ remains algebraic [Sim93, Theorem 3.1]. We show (in Theorem 14.6 and Proposition 16.2) that cohomology support loci are algebraic subsets of $A^\natural$ and $\mathrm{Char}(A)$; this is enough to prove the first half of Theorem 2.2. To prove the second half, we show (in Section 17) that the cohomology support loci of an object of geometric origin are stable under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$; we can then apply another result by Simpson, namely that every "absolute closed" subset of $A^\natural$ is a finite union of arithmetic linear subvarieties. Another proof is explained in [Sch13].

3. The Fourier-Mukai transform. A second way to present the information about the cohomology of twists of $\mathcal{M}$ is through the *Fourier-Mukai transform* for algebraic $\mathscr{D}_A$-modules, introduced and studied by Laumon [Lau96] and Rothstein [Rot96]. It is an exact functor

$$\mathrm{FM}_A\colon \mathrm{D}_{coh}^b(\mathscr{D}_A) \to \mathrm{D}_{coh}^b(\mathscr{O}_{A^\natural}), \tag{3.1}$$

defined as the integral transform with kernel $(P^\natural, \nabla^\natural)$, the tautological line bundle with relative integrable connection on $A \times A^\natural$. As shown by Laumon and Rothstein, FM_A is an equivalence between the bounded derived category of coherent algebraic $\mathscr{D}_A$-modules, and that of coherent algebraic sheaves on $A^\natural$. In essence, this means that an algebraic $\mathscr{D}$-module on an abelian variety can be recovered from the cohomology of its twists by line bundles with integrable connection.

The support of the complex of coherent sheaves $\mathrm{FM}_A(\mathcal{M})$ is related to the cohomology support loci of $\mathcal{M}$: by the base change theorem, one has

$$\operatorname{Supp} \mathrm{FM}_A(\mathcal{M}) = \bigcup_{k \in \mathbb{Z}} S_1^k(A, \mathcal{M}).$$

In particular, the support is a finite union of linear subvarieties. But the Fourier-Mukai transform of a holonomic complex actually satisfies a much stronger version of Theorem 2.2. We shall say that a subset of $A^\natural$ is *definable in terms of* $\mathrm{FM}_A(\mathcal{M})$ if can be obtained by applying various sheaf-theoretic operations – such as $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_{A^\natural})$, truncation, or restriction to a linear subvariety – to $\mathrm{FM}_A(\mathcal{M})$, and then taking the support of the resulting complex of coherent sheaves.

Theorem 3.2. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ be a holonomic complex on an abelian variety. If a subset of $A^\natural$ is definable in terms of $\mathrm{FM}_A(\mathcal{M})$, then it is a finite union of linear subvarieties. These linear subvarieties are arithmetic whenever $\mathcal{M}$ is a semisimple regular holonomic $\mathscr{D}_A$-module of geometric origin.*

The proof of Theorem 3.2 is based on an analogue of the Fourier-Mukai transform for constructible complexes $K \in \mathrm{D}_c^b(\mathbb{C}_A)$ (explained in Section 14). The main point is that the group ring $R = \mathbb{C}[\pi_1(A, 0)]$ is a representation of the fundamental group, and therefore determines a local system of R-modules $\mathcal{L}_R$ on the abelian variety. Because K is constructible and $p\colon A \to pt$ is proper, the direct image $\mathbf{R}p_*(K \otimes_{\mathbb{C}} \mathcal{L}_R)$ therefore belongs to $\mathrm{D}_{coh}^b(R)$ and gives rise to a complex of coherent algebraic sheaves on the affine algebraic variety $\mathrm{Char}(A) = \operatorname{Spec} R$. When $K = \mathrm{DR}_A(\mathcal{M})$, we show that the resulting complex of coherent analytic sheaves, pulled back along $\Phi\colon A^\natural \to \mathrm{Char}(A)$, is canonically isomorphic to $\mathrm{FM}_A(\mathcal{M})$. Both assertions in Theorem 3.2 then follow as before from Simpson's theorems.

4. Codimension bounds and perverse coherent sheaves. Inequalities for the codimension of cohomology support loci first appeared in the work of Green and Lazarsfeld on the generic vanishing theorem [GL87]. For example, when X is a projective complex manifold whose Albanese mapping is generically finite over its image, Green and Lazarsfeld proved that

$$\operatorname{codim}_{\mathrm{Pic}^0(X)} \left\{ L \in \mathrm{Pic}^0(X) \mid H^k(X, \omega_X \otimes L) \neq 0 \right\} \geq k$$

for every $k \geq 0$. More recently, Popa [Pop12] noticed that such codimension bounds can be expressed in terms of a certain nonstandard t-structure on the derived category, introduced by Kashiwara [Kas04] and Arinkin and Bezrukavnikov [AB10] in their work on "perverse coherent sheaves".

In the context of $\mathscr{D}$-modules on abelian varieties, the relationship between codimension bounds and t-structures is even closer. The first result is that the position of a holonomic complex with respect to the standard t-structure on the category $\mathrm{D}_h^b(\mathscr{D}_A)$ is detected by the codimension of its cohomology support loci.

Theorem 4.1. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ be a holonomic complex. Then one has*

$$\mathcal{M} \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A) \iff \operatorname{codim} S_1^k(A, \mathcal{M}) \geq 2k \text{ for every } k \in \mathbb{Z},$$

$$\mathcal{M} \in \mathrm{D}_h^{\geq 0}(\mathscr{D}_A) \iff \operatorname{codim} S_1^k(A, \mathcal{M}) \geq -2k \text{ for every } k \in \mathbb{Z}.$$

In particular, $\mathcal{M}$ is a single holonomic $\mathscr{D}_A$-module if and only if its cohomology support loci satisfy $\operatorname{codim} S_1^k(A, \mathcal{M}) \geq |2k|$ *for every* $k \in \mathbb{Z}$.

The natural setting for this result is the theory of *perverse coherent sheaves*, developed by Kashiwara and by Arinkin and Bezrukavnikov. As a matter of fact, there is a perverse t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$ with the property that

$$ {}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_{A^\natural}) = \left\{ F \in \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural}) \;\middle|\; \operatorname{codim} \operatorname{Supp} \mathcal{H}^k F \geq 2k \text{ for every } k \in \mathbb{Z} \right\}; $$

it corresponds to the supporting function $m = \left\lfloor \frac{1}{2} \operatorname{codim} \right\rfloor$ on the topological space of the scheme $A^\natural$, in Kashiwara's terminology. Its heart ${}^m\mathrm{Coh}(\mathscr{O}_{A^\natural})$ is the abelian category of *m-perverse coherent sheaves* (see Section 18).

Now Theorem 4.1 is a consequence of the following better result, which says that the Fourier-Mukai transform interchanges the standard t-structure on $\mathrm{D}^b_h(\mathscr{D}_A)$ and the m-perverse t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$.[2]

Theorem 4.2. *Let $\mathcal{M} \in \mathrm{D}^b_h(\mathscr{D}_A)$ be a holonomic complex on A. Then one has*

$$ \begin{aligned} \mathcal{M} \in \mathrm{D}^{\leq k}_h(\mathscr{D}_A) \quad &\Longleftrightarrow \quad \mathrm{FM}_A(\mathcal{M}) \in {}^m\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_{A^\natural}), \\ \mathcal{M} \in \mathrm{D}^{\geq k}_h(\mathscr{D}_A) \quad &\Longleftrightarrow \quad \mathrm{FM}_A(\mathcal{M}) \in {}^m\mathrm{D}^{\geq k}_{coh}(\mathscr{O}_{A^\natural}). \end{aligned} $$

In particular, $\mathcal{M}$ is a single holonomic $\mathscr{D}_A$-module if and only if its Fourier-Mukai transform $\mathrm{FM}_A(\mathcal{M})$ is an m-perverse coherent sheaf on $A^\natural$.

The proofs of both theorems can be found in Section 19. The first part of the argument is to show that when $\mathcal{M}$ is a holonomic $\mathscr{D}_A$-module, the cohomology sheaves $\mathcal{H}^i \mathrm{FM}_A(\mathcal{M})$ are torsion sheaves for $i > 0$. Here the crucial point is that the characteristic variety $\mathrm{Ch}(\mathcal{M})$ inside $T^*A = A \times H^0(A, \Omega^1_A)$ has the same dimension as A itself; this makes the second projection

$$ \mathrm{Ch}(\mathcal{M}) \to H^0(A, \Omega^1_A) $$

finite over a general point of $H^0(A, \Omega^1_A)$. To deduce results about $\mathrm{FM}_A(\mathcal{M})$, we use an extension of the Fourier-Mukai transform to $\mathscr{R}_A$-modules, where $\mathscr{R}_A = R_F \mathscr{D}_A$ is the Rees algebra. Choose a good filtration $F_\bullet\mathcal{M}$, and consider the coherent sheaf $\mathrm{gr}^F\mathcal{M}$ on T^*A determined by the graded $\operatorname{Sym} \mathscr{T}_A$-module $\mathrm{gr}^F_\bullet\mathcal{M}$; its support is precisely $\mathrm{Ch}(\mathcal{M})$. The extended Fourier-Mukai transform of the Rees module $R_F\mathcal{M}$ then interpolates between $\mathrm{FM}_A(\mathcal{M})$ and the complex

$$ \mathbf{R}(p_{23})_* \Big(p_{12}^* P \otimes p_{13}^* (\mathrm{id} \times \iota)^* \, \mathrm{gr}^F\mathcal{M} \Big), $$

and because the higher cohomology sheaves of the latter are torsion, we obtain the result for $\mathrm{FM}_A(\mathcal{M})$. This "generic vanishing theorem" implies also that the cohomology support loci $S_1^k(A, \mathcal{M})$ are proper subvarieties for $k \neq 0$; in the regular case, this result is due to Krämer and Weissauer [KW11, Theorem 1.1].

Once the generic vanishing theorem has been established, Theorem 3.2 implies that $\mathcal{H}^i \mathrm{FM}_A(\mathcal{M})$ is supported in a finite union of linear subvarieties of lower dimension; because of the functoriality of the Fourier-Mukai transform, Theorem 4.2 can then be deduced very easily by induction on the dimension.

From there, the basic properties of the m-perverse t-structure quickly lead to the following result about the Fourier-Mukai transform.

Corollary 4.3. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}_A$-module. The only potentially nonzero cohomology sheaves of the Fourier-Mukai transform* $\mathrm{FM}_A(\mathcal{M})$ *are*

$$\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M}),\ \mathcal{H}^1 \mathrm{FM}_A(\mathcal{M}),\ \dots,\ \mathcal{H}^{\dim A} \mathrm{FM}_A(\mathcal{M}).$$

Their supports satisfy $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i \mathrm{FM}_A(\mathcal{M}) \geq 2i$, *and if* $r \geq 0$ *is the least integer for which* $\mathcal{H}^r \mathrm{FM}_A(\mathcal{M}) \neq 0$, *then* $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r \mathrm{FM}_A(\mathcal{M}) = 2r$.

5. Results about simple holonomic D-modules. According to Theorem 3.2, the Fourier-Mukai transform of a holonomic $\mathscr{D}_A$-module is supported in a finite union of linear subvarieties. For *simple* holonomic $\mathscr{D}_A$-modules, one can say more: the support of the Fourier-Mukai transform is always irreducible, and if it is not equal to $A^\natural$, then the $\mathscr{D}_A$-module is – up to tensoring by a line bundle with integrable connection – pulled back from an abelian variety of lower dimension.

Theorem 5.1. *Let $\mathcal{M}$ be a simple holonomic $\mathscr{D}_A$-module. Then*

$$\operatorname{Supp} \mathrm{FM}_A(\mathcal{M}) = (L, \nabla) \otimes \operatorname{im}\bigl(f^\natural \colon B^\natural \to A^\natural\bigr)$$

is a linear subvariety of $A^\natural$ (in the sense of Definition 2.3), and we have

$$\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla) \simeq f^* \mathcal{N}$$

for a simple holonomic $\mathscr{D}_B$-module $\mathcal{N}$ with $\operatorname{Supp} \mathrm{FM}_B(\mathcal{N}) = B^\natural$.

The idea of the proof is that for some $r \geq 0$, the support of $\mathcal{H}^r \mathrm{FM}_A(\mathcal{M})$ has to contain a linear subvariety $(L, \nabla) \otimes \operatorname{im} f^\natural$ of codimension $2r$. Because of the functoriality of the Fourier-Mukai transform, restricting $\mathrm{FM}_A(\mathcal{M})$ to this subvariety corresponds to taking the direct image $f_+\bigl(\mathcal{M} \otimes (L, \nabla)\bigr)$. We then use adjointness and the fact that $\mathcal{M}$ is simple to conclude that $\mathcal{M} \otimes (L, \nabla)$ is pulled back from B.

One application of Theorem 5.1 is to classify simple holonomic $\mathscr{D}_A$-modules with Euler characteristic zero. Recall that the *Euler characteristic* of a coherent algebraic $\mathscr{D}_A$-module $\mathcal{M}$ is the integer

$$\chi(A, \mathcal{M}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbf{H}^k\bigl(A, \mathrm{DR}_A(\mathcal{M})\bigr).$$

When $\mathcal{M}$ is holonomic, we have $\chi(A, \mathcal{M}) \geq 0$ as a consequence of Theorem 4.2 and the deformation invariance of the Euler characteristic. In the regular case, the following result has been proved in a different way by Weissauer [Wei12, Theorem 2].

Corollary 5.2. *Let $\mathcal{M}$ be a simple holonomic $\mathscr{D}_A$-module. If $\chi(A, \mathcal{M}) = 0$, then there exists an abelian variety B, a surjective morphism $f \colon A \to B$ with connected fibers, and a simple holonomic $\mathscr{D}_B$-module $\mathcal{N}$ with $\chi(B, \mathcal{N}) > 0$, such that*

$$\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla) \simeq f^* \mathcal{N}$$

for a suitable point $(L, \nabla) \in A^\natural$.

Now suppose that $\mathcal{M}$ is a simple holonomic $\mathscr{D}$-module with $\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M}) \neq 0$. In that case, the proof of Theorem 5.1 actually gives the stronger inequalities

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^i \mathrm{FM}_A(\mathcal{M}) \geq 2i + 2 \quad \text{for every } i \geq 1.$$

We deduce from this that $\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ is a reflexive sheaf, locally free on the complement of a finite union of linear subvarieties of codimension ≥ 4. This fact allows us to reconstruct (in Corollary 22.3) the entire complex $\mathrm{FM}_A(\mathcal{M})$ from the locally free sheaf $j^* \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ by applying the functor

$$\tau_{\leq \ell(A)-1} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ \dots \circ \tau_{\leq 2} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ \tau_{\leq 1} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ j_*.$$

Here $\ell(A)$ is the smallest odd integer $\geq \dim A$, and j is the inclusion of the open set where $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M})$ is locally free.

This formula looks a bit like Deligne's formula for the intersection complex of a local system [BBD82, Proposition 2.1.11]. We investigate this analogy in Section 23, where we show that the same formula can be used to define an *intersection complex*

$$\operatorname{IC}_X(\mathscr{E}) \in {}^m\operatorname{Coh}(\mathscr{O}_X),$$

where $j \colon U \hookrightarrow X$ is an open subset of a smooth complex algebraic variety X with $\operatorname{codim}(X \setminus U) \geq 2$, and $\mathscr{E}$ is a locally free coherent sheaf on U. This complex has some of the same properties as its cousin in [BBD82]. In that sense,

$$\operatorname{FM}_A(\mathcal{M}) \simeq \operatorname{IC}_{A^\natural}\bigl(j^*\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M})\bigr)$$

is indeed the intersection complex of a locally free sheaf. When $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M}) = 0$, Theorem 5.1 shows that $\operatorname{FM}_A(\mathcal{M})$ is still the intersection complex of a locally free sheaf, but now on a linear subvariety of $A^\natural$ of lower dimension.

6. A conjecture. By now, it will have become clear that Fourier-Mukai transforms of holonomic $\mathscr{D}_A$-modules are very special complexes of coherent sheaves on the moduli space $A^\natural$. Because the Fourier-Mukai transform

$$\operatorname{FM}_A \colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$$

is an equivalence of categories, this suggests the following general question.

Question. Let $\mathrm{D}^b_h(\mathscr{D}_A)$ denote the full subcategory of $\mathrm{D}^b_{coh}(\mathscr{D}_A)$, consisting of complexes with holonomic cohomology sheaves. What is the image of $\mathrm{D}^b_h(\mathscr{D}_A)$ under the Fourier-Mukai transform? In particular, which complexes of coherent sheaves on $A^\natural$ are Fourier-Mukai transforms of holonomic $\mathscr{D}_A$-modules?

In this section, I would like to propose a conjectural answer to this question. Roughly speaking, the answer seems to be the following:

$$\operatorname{FM}_A\bigl(\mathrm{D}^b_h(\mathscr{D}_A)\bigr) = \text{derived category of } \textit{hyperkähler constructible complexes},$$
$$\operatorname{FM}_A\bigl(\operatorname{Mod}_h(\mathscr{D}_A)\bigr) = \text{abelian category of } \textit{hyperkähler perverse sheaves}.$$

Recall that the space of line bundles with connection is a hyperkähler manifold: as complex manifolds, one has $A^\natural \simeq H^1(A,\mathbb{C})/H^1\bigl(A,\mathbb{Z}(1)\bigr)$, and any polarization of the Hodge structure on $H^1(A,\mathbb{C})$ gives rise to a flat hyperkähler metric on $A^\natural$. Here is some evidence for this point of view:

(1) Finite unions of linear subvarieties of $A^\natural$ are precisely those algebraic subvarieties that are also hyperkähler subvarieties.
(2) Given a holonomic complex $\mathcal{M} \in \mathrm{D}^b_h(\mathscr{D}_A)$, there is a finite stratification of $A^\natural$ by hyperkähler subvarieties such that the restriction of $\operatorname{FM}_A(\mathcal{M})$ to each stratum has locally free cohomology sheaves.
(3) We prove in Section 20 that a complex of coherent sheaves lies in the subcategory $\operatorname{FM}_A\bigl(\mathrm{D}^b_h(\mathscr{D}_A)\bigr)$ if and only if all of its cohomology sheaves do. This gives some justification for using the term "constructible complex".
(4) If we use *quaternionic* dimension, Theorem 4.2 becomes

$$\dim_{\mathbb{H}} \operatorname{Supp} \mathcal{H}^i \operatorname{FM}_A(\mathcal{M}) \leq \dim_{\mathbb{H}} A^\natural - i = \dim A - i$$

for a holonomic $\mathscr{D}_A$-module $\mathcal{M}$; this says that the complex $\operatorname{FM}_A(\mathcal{M})[\dim A]$ is perverse for the usual middle perversity [BBD82, Chapter 2] over $\mathbb{H}$.

(5) For a simple holonomic $\mathscr{D}$-module $\mathcal{M}$, the Fourier-Mukai transform $\mathrm{FM}_A(\mathcal{M})$ is the intersection complex of a locally free sheaf.

Unfortunately, nobody has yet defined a category of hyperkähler perverse sheaves, even in the case of compact hyperkähler manifolds; and our situation presents the additional difficulty that $A^\natural$ is not compact. Nevertheless, I believe that, based on the work of Mochizuki on twistor $\mathscr{D}$-modules [Moc11], it is possible to make an educated guess, at least in the case of semisimple holonomic $\mathscr{D}$-modules.

Conjecture 6.1. *Let $\mathscr{F}$ be a reflexive coherent algebraic sheaf on $A^\natural$. Then there exists a semisimple holonomic $\mathscr{D}_A$-module $\mathcal{M}$ with the property that $\mathscr{F} \simeq \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ if and only if the following conditions are satisfied:*

(a) $\mathscr{F}$ is locally free on the complement of a finite union of linear subvarieties of codimension at least 4.
(b) The resulting locally free sheaf admits a hermitian metric h whose curvature tensor Θ_h is $\mathrm{SU}(2)$-invariant and locally square-integrable on $A^\natural$.
(c) The pointwise norm of Θ_h, taken with respect to h, is in $O\big(d^{-(1+\varepsilon)}\big)$, where d is the distance to the origin in $A^\natural$.

Moreover, $\mathcal{M}$ is regular if and only if the pointwise norm of Θ_h is in $O(d^{-2})$.

There is a certain amount of redundancy in the conditions. In fact, we could start from a holomorphic vector bundle $\mathscr{E}$ on the complement of a finite union of linear subvarieties of codimension ≥ 2, and assume that it admits a hermitian metric h for which (b) and (c) are true. Then h is admissible in the sense of Bando and Siu [BS94], and $\mathscr{E}$ therefore extends uniquely to a reflexive coherent analytic sheaf on $A^\natural$; by virtue of (c), the extension is acceptable in the sense of [Moc11, Chapter 21], and therefore algebraic.[3] In particular, $\mathscr{E}$ itself is algebraic, and the discussion at the end of Section 5 shows that the simple holonomic $\mathscr{D}_A$-module must be

$$\mathrm{FM}_A^{-1}\Big(\mathrm{IC}_{A^\natural}(\mathscr{E})\Big),$$

the inverse Fourier-Mukai transform of the intersection complex of $\mathscr{E}$. The problem is, of course, to show that this is indeed a simple holonomic $\mathscr{D}_A$-module.

The paper [Moc13] establishes a result equivalent to Conjecture 6.1 in the case of elliptic curves. The reason for believing that regularity should correspond to quadratic decay in the curvature is the work of Jardim [Jar02]. In general, the existence of the metric, and the $\mathrm{SU}(2)$-invariance of its curvature, should be consequences of the fact that every simple holonomic $\mathscr{D}$-module lifts to a polarized wild pure twistor $\mathscr{D}$-module. The remaining points will probably require additional methods from analysis. Note that the conjecture is consistent with the result (in Corollary 25.3) that all Chern classes of $\mathrm{FM}_A(\mathcal{M})$ are zero in cohomology.

Another interesting question is whether the existence of the metric in (b) is equivalent to an algebraic condition such as stability. If that was the case, then I would guess that the semistable objects are what corresponds to Fourier-Mukai transforms of not necessarily simple holonomic $\mathscr{D}_A$-modules.

7. Results about perverse sheaves. For the convenience of those readers who are more familiar with constructible complexes and perverse sheaves, we shall now translate our main results into that language. In the sequel, a *constructible complex* on the abelian variety A means a complex K of sheaves of $\mathbb{C}$-vector spaces, whose cohomology sheaves $\mathcal{H}^i K$ are constructible with respect to an algebraic stratification of A, and vanish for i outside some bounded interval. We denote by $\mathrm{D}_c^b(\mathbb{C}_A)$ the bounded derived category of constructible complexes. It is a basic fact [HTT08, Section 4.5] that the hypercohomology groups $\mathbf{H}^i(A, K)$ are finite-dimensional complex vector spaces for any $K \in \mathrm{D}_c^b(\mathbb{C}_A)$.

Now let $\mathrm{Char}(A)$ be the space of characters of the fundamental group; it is also the moduli space for local systems of rank one. For any character $\rho \colon \pi_1(A, 0) \to \mathbb{C}^*$, we denote the corresponding local system on A by the symbol $\mathbb{C}_\rho$. It is easy to see that $K \otimes_{\mathbb{C}} \mathbb{C}_\rho$ is again constructible for any $K \in \mathrm{D}_c^b(\mathbb{C}_A)$; we may therefore define the *cohomology support loci* of $K \in \mathrm{D}_c^b(\mathbb{C}_A)$ to be the subsets

$$S_m^k(A, K) = \left\{ \rho \in \mathrm{Char}(A) \;\middle|\; \dim \mathbf{H}^k\big(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) \geq m \right\}, \tag{7.1}$$

for any pair of integers $k, m \in \mathbb{Z}$. Since the space of characters is very large – its dimension is equal to $2 \dim A$ – these loci should contain a lot of information about the original constructible complex K, and indeed they do.

Our first result is a structure theorem for cohomology support loci.

Definition 7.2. A *linear subvariety* of $\mathrm{Char}(A)$ is any subset of the form

$$\rho \cdot \mathrm{im}\big(\mathrm{Char}(f) \colon \mathrm{Char}(B) \to \mathrm{Char}(A)\big),$$

for a surjective morphism of abelian varieties $f \colon A \to B$ with connected fibers, and a character $\rho \in \mathrm{Char}(A)$. We say that a linear subvariety is *arithmetic* if ρ can be taken to be torsion point of $\mathrm{Char}(A)$.

Theorem 7.3. *Let $K \in \mathrm{D}_c^b(\mathbb{C}_A)$ be a constructible complex.*

(a) Each $S_m^k(A, K)$ is a finite union of linear subvarieties of $\mathrm{Char}(A)$.
(b) If K is a semisimple perverse sheaf of geometric origin [BBD82, 6.2.4]*, then these linear subvarieties are arithmetic.*

Proof. For (a), we use the Riemann-Hilbert correspondence to find a regular holonomic complex $\mathcal{M} \in \mathrm{D}_{rh}^b(\mathscr{D}_A)$ with $\mathrm{DR}_A(\mathcal{M}) \simeq K$. Since $S_m^k(A, K) = \Phi\big(S_m^k(A, \mathcal{M})\big)$ by Theorem 15.1, the assertion follows from Theorem 2.2. The statement in (b) can be deduced from Theorem 17.2 by a similar argument. □

The next result has to do with the codimension of the cohomology support loci. Recall that the category $\mathrm{D}_c^b(\mathbb{C}_A)$ has a nonstandard t-structure

$$\Big({}^\pi\mathrm{D}_c^{\leq 0}(\mathbb{C}_A), {}^\pi\mathrm{D}_c^{\geq 0}(\mathbb{C}_A)\Big),$$

called the *perverse t-structure*, whose heart is the abelian category of perverse sheaves [BBD82]. We show that the position of a constructible complex with respect to this t-structure can be read off from its cohomology support loci.

Theorem 7.4. *Let $K \in \mathrm{D}_c^b(\mathbb{C}_A)$ be a constructible complex. Then one has*

$$\begin{aligned} K \in {}^\pi\mathrm{D}_c^{\leq 0}(\mathbb{C}_A) \quad &\Longleftrightarrow \quad \operatorname{codim} S_1^k(A, K) \geq 2k \text{ for every } k \in \mathbb{Z}, \\ K \in {}^\pi\mathrm{D}_c^{\geq 0}(\mathbb{C}_A) \quad &\Longleftrightarrow \quad \operatorname{codim} S_1^k(A, K) \geq -2k \text{ for every } k \in \mathbb{Z}. \end{aligned}$$

Thus K is a perverse sheaf if and only if $\operatorname{codim} S_1^k(A, K) \geq |2k|$ for every $k \in \mathbb{Z}$.

Proof. Let $\mathcal{M} \in \mathrm{D}^b_{rh}(\mathscr{D}_A)$ be a regular holonomic complex such that $K \simeq \mathrm{DR}_A(\mathcal{M})$. Since $S^k_m(A, K) = \Phi\big(S^k_m(A, \mathcal{M})\big)$, the first assertion is a consequence of Theorem 19.1. Now let $\mathbf{D}_A \colon \mathrm{D}^b_c(\mathbb{C}_A) \to \mathrm{D}^b_c(\mathbb{C}_A)$ be the Verdier duality functor; then

$$S^k_m(A, K) = \langle -1_{\mathrm{Char}(A)} \rangle \, S^{-k}_m(A, \mathbf{D}_A K)$$

by Verdier duality. Since $K \in {}^{\pi}\mathrm{D}^{\geq 0}_c(\mathbb{C}_A)$ if and only if $\mathbf{D}_A K \in {}^{\pi}\mathrm{D}^{\leq 0}_c(\mathbb{C}_A)$, the second assertion follows. The final assertion is clear from the definition of perverse sheaves as the heart of the perverse t-structure on $\mathrm{D}^b_c(\mathbb{C}_A)$. □

A consequence is the following "generic vanishing theorem" for perverse sheaves; a similar – but less precise – statement has been proved some time ago by Krämer and Weissauer [KW11, Theorem 1.1].

Corollary 7.5. *Let $K \in \mathrm{D}^b_c(\mathbb{C}_A)$ be a perverse sheaf on a complex abelian variety. Then the cohomology support loci $S^k_m(A, K)$ are finite unions of linear subvarieties of* $\mathrm{Char}(A)$ *of codimension at least $|2k|$. In particular, one has*

$$\mathbf{H}^k\big(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) = 0$$

for general $\rho \in \mathrm{Char}(A)$ and $k \neq 0$.

The generic vanishing theorem implies that the *Euler characteristic*

$$\chi(A, K) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbf{H}^k\big(A, K\big)$$

of a perverse sheaf on an abelian variety is always nonnegative, a result originally due to Franecki and Kapranov [FK00, Corollary 1.4]. Indeed, from the deformation invariance of the Euler characteristic, we get

$$\chi(A, K) = \chi\big(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) = \dim \mathbf{H}^0\big(A, K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) \geq 0$$

for a general character $\rho \in \mathrm{Char}(A)$. For *simple* perverse sheaves with $\chi(A, K) = 0$, we have the following structure theorem [Wei12, Theorem 2].

Theorem 7.6. *Let $K \in \mathrm{D}^b_c(\mathbb{C}_A)$ be a simple perverse sheaf. If $\chi(A, K) = 0$, then there exists an abelian variety B, a surjective morphism $f \colon A \to B$ with connected fibers, and a simple perverse sheaf $K' \in \mathrm{D}^b_c(\mathbb{C}_B)$ with $\chi(B, K') > 0$, such that*

$$K \simeq f^* K' \otimes_{\mathbb{C}} \mathbb{C}_\rho$$

for some character $\rho \in \mathrm{Char}(A)$.

Proof. This again follows from the Riemann-Hilbert correspondence and the analogous result for simple holonomic $\mathscr{D}_A$-modules in Corollary 5.2. □

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B. The Fourier-Mukai transform

In this chapter, we recall Laumon's construction of the Fourier-Mukai transform for algebraic $\mathscr{D}$-modules on a complex abelian variety [Lau96]. Using a different approach, Rothstein obtained the same results in [Rot96].

9. Operations on D-modules. Let A be a complex abelian variety; we usually put $g = \dim A$. Before introducing the Fourier-Mukai transform, it may be helpful to say a few words about $\mathscr{D}_A$, the sheaf of linear differential operators of finite order. Recall that the tangent bundle of A is trivial; $\mathscr{D}_A$ is therefore generated, as an $\mathscr{O}_A$-algebra, by any basis $\partial_1, \dots, \partial_g \in H^0(A, \mathscr{T}_A)$, subject to the relations

$$[\partial_i, \partial_j] = 0 \text{ and } [\partial_i, f] = \partial_i f, \qquad \text{for } 1 \leq i, j \leq g \text{ and } f \in \Gamma(U, \mathscr{O}_A).$$

By an *algebraic $\mathscr{D}_A$-module*, we mean a sheaf of left $\mathscr{D}_A$-modules that is quasi-coherent as a sheaf of $\mathscr{O}_A$-modules; a $\mathscr{D}_A$-module is *holonomic* if its characteristic variety, as a subset of the cotangent bundle T^*A, has dimension equal to $\dim A$ (and is therefore a finite union of conical Lagrangian subvarieties). Finally, a *holonomic complex* is a complex of $\mathscr{D}_A$-modules $\mathcal{M}$, whose cohomology sheaves $\mathcal{H}^i\mathcal{M}$ are holonomic, and vanish for i outside some bounded interval. We denote by $\mathrm{D}^b_{coh}(\mathscr{D}_A)$ the derived category of cohomologically bounded and coherent $\mathscr{D}_A$-modules, and by $\mathrm{D}^b_h(\mathscr{D}_A)$ the full subcategory of all holonomic complexes. We refer the reader to [HTT08, Chapter 3] for additional details.

Note. Because A is projective, a coherent analytic $\mathscr{D}_A$-module is algebraic if and only if it contains a *lattice*, that is to say, a coherent $\mathscr{O}_A$-submodule that generates it as a $\mathscr{D}_A$-module. By a theorem of Malgrange [Mal04, Theorem 3.1], this is always the case for holonomic $\mathscr{D}_A$-modules; thus there is no difference between holonomic complexes of analytic and algebraic $\mathscr{D}_A$-modules.

Because it will play such an important role below, we briefly discuss the definition of the *de Rham complex*, and especially the conventions about signs. For a single algebraic $\mathscr{D}_A$-module $\mathcal{M}$, we define

$$\mathrm{DR}_A(\mathcal{M}) = \Big[\mathcal{M} \to \Omega_A^1 \otimes \mathcal{M} \to \dots \to \Omega_A^g \otimes \mathcal{M}\Big][g],$$

which we view as a complex of sheaves of $\mathbb{C}$-vector spaces in the analytic topology, placed in degrees $-g, \dots, 0$. The differential is given by $(-1)^g \nabla_{\mathcal{M}}$, where

$$\nabla_{\mathcal{M}} \colon \Omega_A^p \otimes \mathcal{M} \to \Omega_A^{p+1} \otimes \mathcal{M}, \quad \omega \otimes m \mapsto d\omega \otimes m + \sum_{j=1}^g (dz_j \wedge \omega) \otimes \partial_j m;$$

here $dz_1, \dots, dz_g \in H^0(A, \Omega_A^1)$ is the basis dual to $\partial_1, \dots, \partial_g \in H^0(A, \mathscr{T}_A)$. Given a complex of algebraic $\mathscr{D}_A$-modules $(\mathcal{M}^\bullet, d)$, we define

$$\mathrm{DR}_A(\mathcal{M}^\bullet)$$

to be the single complex determined by the double complex $\big(D^{\bullet,\bullet}, d_1, d_2\big)$, whose term in bidegree (i,j) is equal to

$$D^{i,j} = \Omega_A^{g+i} \otimes \mathcal{M}^j,$$

and whose differentials are given by the formulas

$$d_1 = (-1)^g \nabla_{\mathcal{M}^j} \quad \text{and} \quad d_2 = \mathrm{id} \otimes d.$$

Note that, according to the sign rules introduced by Deligne, the differential in the total complex acts as $d_1 + (-1)^i d_2$ on the summand $D^{i,j}$.

The fundamental operations on algebraic $\mathscr{D}$-modules – such as direct and inverse images or duality – are described in [HTT08, Part I]. Here we only recall the notation. Let $f\colon A \to B$ be a morphism of abelian varieties. On the one hand, one has the *direct image functor*

$$f_+ \colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{D}_B);$$

in case f is surjective (and hence smooth), f_+ is given by the formula

$$f_+ \mathcal{M}^\bullet = \mathbf{R} f_* \, \mathrm{DR}_{A/B}(\mathcal{M}^\bullet),$$

where $\mathrm{DR}_{A/B}(\mathcal{M}^\bullet)$ denotes the relative de Rham complex, defined in a similar way as above, but with $g = \dim A$ replaced by the relative dimension $r = \dim A - \dim B$. For holonomic complexes, we have an induced functor

$$f_+ \colon \mathrm{D}^b_h(\mathscr{D}_A) \to \mathrm{D}^b_h(\mathscr{D}_B)$$

since direct images by algebraic morphisms preserve holonomicity [HTT08, Theorem 3.2.3]. We also use the *shifted inverse image functor*

$$f^+ = \mathbf{L} f^* [\dim A - \dim B] \colon \mathrm{D}^b(\mathscr{D}_B) \to \mathrm{D}^b(\mathscr{D}_A);$$

in general, it only preserves coherence when f is surjective (and hence smooth). According to [HTT08, Theorem 3.2.3], we get an induced functor

$$f^+ \colon \mathrm{D}^b_h(\mathscr{D}_B) \to \mathrm{D}^b_h(\mathscr{D}_A).$$

Finally, a very important role will be played by the *duality functor*

$$\mathbf{D}_A \colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{D}_A)^{opp}, \quad \mathbf{D}_A(\mathcal{M}^\bullet) = \mathbf{R}\mathcal{H}om_{\mathscr{D}_A}(\mathcal{M}^\bullet, \mathscr{D}_A) \otimes (\Omega_A^g)^{-1}[g].$$

Note that a $\mathscr{D}_A$-module $\mathcal{M}$ is holonomic exactly when its dual $\mathbf{D}_A(\mathcal{M})$ is again a single $\mathscr{D}_A$-module (viewed as a complex concentrated in degree zero).

10. Definition and basic properties. We now come to the definition of the Fourier-Mukai transform. Following Mazur-Messing [MM74], we let $A^\natural$ denote the moduli space of algebraic line bundles with integrable connection on the abelian variety A. It is naturally a quasi-projective algebraic variety: on the dual abelian variety $\hat{A} = \mathrm{Pic}^0(A)$, there is a canonical extension of vector bundles

$$0 \to \hat{A} \times H^0(A, \Omega_A^1) \to E(A) \to \hat{A} \times \mathbb{C} \to 0, \tag{10.1}$$

whose extension class in

$$\begin{aligned} \mathrm{Ext}^1\bigl(\mathscr{O}_{\hat{A}}, \mathscr{O}_{\hat{A}} \times H^0(A, \Omega_A^1)\bigr) &\simeq H^1(\hat{A}, \mathscr{O}_{\hat{A}}) \otimes H^0(A, \Omega_A^1) \\ &\simeq H^0(A, \mathscr{T}_A) \otimes H^0(A, \Omega_A^1) \end{aligned}$$

is represented by $\sum_j \partial_j \otimes dz_j$. Then $A^\natural$ is isomorphic to the preimage of $\hat{A} \times \{1\}$ inside of $E(A)$, and the projection

$$\pi \colon A^\natural \to \hat{A}, \quad (L, \nabla) \mapsto L,$$

is a torsor for the trivial bundle $\hat{A} \times H^0(A, \Omega_A^1)$. This corresponds to the fact that $\nabla + \omega$ is again an integrable connection for any $\omega \in H^0(A, \Omega_A^1)$. Note that $A^\natural$ is a group under tensor product, with unit element $(\mathscr{O}_A, d)$.

The Fourier-Mukai transform takes bounded complexes of algebraic $\mathscr{D}_A$-modules to bounded complexes of quasi-coherent sheaves on $A^\natural$; we briefly describe it following the presentation in [Lau96, §3]. Let P denote the normalized Poincaré bundle on the product $A \times \hat{A}$. Since $A^\natural$ is the moduli space of line bundles with integrable connection on A, the pullback $P^\natural = (\mathrm{id}_A \times \pi)^* P$ of the Poincaré bundle to the product $A \times A^\natural$ is endowed with a universal integrable connection

$$\nabla^\natural \colon P^\natural \to \Omega^1_{A\times A^\natural/A^\natural} \otimes P^\natural$$

relative to $A^\natural$. The construction of $\nabla^\natural$ can be found in [MM74, Chapter I]. An algebraic left $\mathscr{D}_A$-module $\mathcal{M}$ may be interpreted as a quasi-coherent sheaf of $\mathscr{O}_A$-modules with integrable connection $\nabla \colon \mathcal{M} \to \Omega^1_A \otimes \mathcal{M}$; then

$$p_1^*\nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^\natural$$

defines a relative integrable connection on the tensor product $p_1^*\mathcal{M} \otimes_{\mathscr{O}_{A\times A^\natural}} P^\natural$, and we denote the resulting algebraic $\mathscr{D}_{A\times A^\natural/A^\natural}$-module by the symbol $p_1^*\mathcal{M} \otimes (P^\natural, \nabla^\natural)$. Given a complex of algebraic $\mathscr{D}_A$-modules $(\mathcal{M}^\bullet, d)$, we define

$$\mathrm{DR}_{A\times A^\natural/A^\natural}\Big(p_1^*\mathcal{M}^\bullet \otimes (P^\natural, \nabla^\natural)\Big)$$

as the single complex determined by the double complex $\big(D^{\bullet,\bullet}, d_1, d_2\big)$, whose term in bidegree (i,j) is equal to

$$D^{i,j} = \Omega^{g+i}_{A\times A^\natural/A^\natural} \otimes p_1^*\mathcal{M}^j \otimes P^\natural,$$

and whose differentials are given by the formulas

$$d_1 = (-1)^g\big(p_1^*\nabla_{\mathcal{M}^j} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^\natural\big) \quad \text{and} \quad d_2 = \mathrm{id} \otimes p_1^* d \otimes \mathrm{id}\,.$$

We then define the *Fourier-Mukai transform* of the complex $\mathcal{M}^\bullet$ by the formula

$$\mathrm{FM}_A(\mathcal{M}^\bullet) = \mathbf{R}(p_2)_*\,\mathrm{DR}_{A\times A^\natural/A^\natural}\Big(p_1^*\mathcal{M}^\bullet \otimes (P^\natural, \nabla^\natural)\Big) \tag{10.2}$$

Because every differential in the relative de Rham complex is $\mathscr{O}_{A^\natural}$-linear, $\mathrm{FM}_A(\mathcal{M}^\bullet)$ is naturally a complex of quasi-coherent algebraic sheaves on $A^\natural$. The following fundamental theorem was proved by Laumon [Lau96, Théorème 3.2.1 and Corollaire 3.2.5], and, using a different method, by Rothstein [Rot96, Theorem 6.2].

Theorem 10.3 (Laumon, Rothstein). *The Fourier-Mukai transform gives rise to an equivalence of categories*

$$\mathrm{FM}_A \colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural}) \tag{10.4}$$

between the bounded derived category of coherent algebraic $\mathscr{D}_A$-modules and the bounded derived category of coherent algebraic sheaves on $A^\natural$.

The Fourier-Mukai transform is compatible with various operations on $\mathscr{D}$-modules; here, taken from Laumon's paper, is a list of the basic properties that we will use.

Theorem 10.5 (Laumon). *The Fourier-Mukai transform for algebraic $\mathscr{D}$-modules on abelian varieties enjoys the following properties:*

(a) For $(L,\nabla) \in A^\natural$, denote by $t_{(L,\nabla)} \colon A^\natural \to A^\natural$ the translation morphism. Then one has a canonical and functorial isomorphism

$$\mathrm{FM}_A\big(- \otimes_{\mathscr{O}_A} (L,\nabla)\big) = \mathbf{L}(t_{(L,\nabla)})^* \circ \mathrm{FM}_A\,.$$

(b) *One has a canonical and functorial isomorphism*

$$\mathrm{FM}_A \circ \mathbf{D}_A = \langle -1_{A^\natural} \rangle^* \, \mathbf{R}\mathcal{H}om\big(\mathrm{FM}_A(-), \mathscr{O}_{A^\natural}\big).$$

(c) *For a morphism $f \colon A \to B$ of abelian varieties, denote by $f^\natural \colon B^\natural \to A^\natural$ the induced morphism. Then one has canonical and functorial isomorphisms*

$$\mathbf{L}(f^\natural)^* \circ \mathrm{FM}_A = \mathrm{FM}_B \circ f_+,$$
$$\mathbf{R}f^\natural_* \circ \mathrm{FM}_B = \mathrm{FM}_A \circ f^+.$$

(Note that f^+ only preserves coherence when f is smooth.)

(d) *One has a canonical and functorial isomorphism*

$$\mathrm{FM}_A \circ \big(\mathscr{D}_A \otimes_{\mathscr{O}_A} (-)\big) = \mathbf{L}\pi^* \circ \mathbf{R}\Phi_P,$$

where $\pi \colon A^\natural \to \hat{A}$ denotes the projection, and $\mathbf{R}\Phi_P \colon \mathrm{D}^b_{coh}(\mathscr{O}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$ is the usual Fourier-Mukai transform for coherent sheaves [Muk81].

Proof. (a) is immediate from the properties of the normalized Poincaré bundle on $A \times A^\natural$. (c) is proved in [Lau96, Proposition 3.3.2]; note that "$g - 1 - g_2$" should read "$g_1 - g_2$." The compatibility of the Fourier-Mukai transform with duality in (b) can be found in [Lau96, Proposition 3.3.4]. Lastly, (d) is proved in [Lau96, Proposition 3.1.2]. □

11. The space of generalized connections. During the proof of Theorem 4.2 in Section 19 below, it will be necessary to compare the Fourier-Mukai transform of a $\mathscr{D}$-module to that of the associated graded object $\mathrm{gr}^F_\bullet \mathcal{M}$, for some choice of good filtration $F_\bullet \mathcal{M}$. Here it is convenient to introduce the Rees algebra

$$\mathscr{R}_A = \bigoplus_{k \in \mathbb{Z}} F_k \mathscr{D}_A \otimes z^k \subseteq \mathscr{D}_A[z],$$

and to pass from a filtered $\mathscr{D}$-module to the associated graded $\mathscr{R}$-module

$$R_F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} \otimes z^k \subseteq \mathcal{M} \otimes_{\mathscr{O}_A} \mathscr{O}_A[z, z^{-1}].$$

We shall extend the Fourier-Mukai transform to this setting; the role of $A^\natural$ is played by $E(A)$, the moduli space of line bundles with generalized connection. Recall that $E(A)$ was defined by the extension in (10.1); we begin by explaining another construction, whose idea is originally due to Deligne and Simpson (see [Bon10]).

Definition 11.1. Let X be a complex manifold, and $\lambda \colon X \to \mathbb{C}$ a holomorphic function. A *generalized connection with parameter* λ, or more briefly a λ-*connection*, on a locally free sheaf of $\mathscr{O}_X$-modules $\mathscr{E}$ is a $\mathbb{C}$-linear morphism of sheaves

$$\nabla \colon \mathscr{E} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{E}$$

that satisfies the Leibniz rule with parameter λ, which is to say that

$$\nabla(f \cdot s) = f \cdot \nabla s + df \otimes \lambda s$$

for local sections $f \in \Gamma(U, \mathscr{O}_X)$ and $s \in \Gamma(U, \mathscr{E})$. A λ-connection is called *integrable* if its $\mathscr{O}_X$-linear curvature operator $\nabla \circ \nabla \colon \mathscr{E} \to \Omega^2_X \otimes_{\mathscr{O}_X} \mathscr{E}$ is equal to zero.

Example 11.2. An integrable 1-connection is an integrable connection in the usual sense; an integrable 0-connection is the structure of a Higgs bundle on $\mathscr{E}$.

For an equivalent description of generalized connections, let Δ^1 be the first infinitesimal neighborhood of the diagonal in $X \times X$; it is the closed subscheme defined by the ideal sheaf $\mathcal{I}_\Delta^2$. Consider the short exact sequence

$$0 \longrightarrow \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{E} \longrightarrow (p_1)_* \big(\mathscr{O}_{\Delta^1} \otimes p_2^* \mathscr{E} \big) \xrightarrow{p} \mathscr{E} \longrightarrow 0,$$

where $p_1 \colon X \times X \to X$ and $p_2 \colon X \times X \to X$ are the two projections.

Lemma 11.3. *A λ-connection on $\mathscr{E}$ is the same thing as a morphism*

$$s \colon \mathscr{E} \to (p_1)_* \big(\mathscr{O}_{\Delta^1} \otimes p_2^* \mathscr{E} \big)$$

with the property that $p \circ s = \lambda \operatorname{id}_{\mathscr{E}}$.

Proof. For connections in the usual sense, this is proved in [MM74, Section 3.1]; the argument there carries over to our case. □

On an abelian variety A, the moduli space $E(A)$ of line bundles with integrable λ-connection (for arbitrary $\lambda \in \mathbb{C}$) may be constructed as follows. Observe first that any λ-connection on a line bundle $L \in \operatorname{Pic}^0(A)$ is automatically integrable. To construct the moduli space, let $\mathfrak{m}_A \subseteq \mathscr{O}_A$ denote the ideal sheaf of the unit element $0 \in A$. Restriction of differential forms induces an isomorphism

$$\mathfrak{m}_A / \mathfrak{m}_A^2 \simeq H^0(A, \Omega_A^1) \otimes \mathscr{O}_A / \mathfrak{m}_A,$$

and therefore determines an extension of coherent sheaves

$$0 \to H^0(A, \Omega_A^1) \otimes \mathscr{O}_A / \mathfrak{m}_A \to \mathscr{O}_A / \mathfrak{m}_A^2 \to \mathscr{O}_A / \mathfrak{m}_A \to 0.$$

Let P be the normalized Poincaré bundle on the product $A \times \hat{A}$, and denote by $\mathbf{R}\Phi_P \colon \mathrm{D}^b_{coh}(\mathscr{O}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$ the Fourier-Mukai transform. Then $\mathbf{R}\Phi_P \big(\mathscr{O}_A / \mathfrak{m}_A^2 \big)$ is a locally free sheaf $\mathscr{E}(A)$, and so we obtain an extension of locally free sheaves

$$0 \to H^0(A, \Omega_A^1) \otimes \mathscr{O}_{\hat{A}} \to \mathscr{E}(A) \to \mathscr{O}_{\hat{A}} \to 0 \tag{11.4}$$

on the dual abelian variety $\hat{A}$. The corresponding extension of vector bundles is the one in (10.1). By construction, $E(A)$ comes with two algebraic morphisms $\pi \colon E(A) \to \hat{A}$ and $\lambda \colon E(A) \to \mathbb{C}$. The following lemma shows that there is a universal line bundle with generalized connection on $A \times E(A)$.

Lemma 11.5. *Let $\widetilde{P} = (\operatorname{id} \times \pi)^* P$ denote the pullback of the Poincaré bundle to $A \times E(A)$. Then there is a canonical generalized relative connection*

$$\widetilde{\nabla} \colon \widetilde{P} \to \Omega^1_{A \times E(A)/E(A)} \otimes \widetilde{P}$$

that satisfies the Leibniz rule $\widetilde{\nabla}(f \cdot s) = f \cdot \widetilde{\nabla} s + d_{A \times E(A)/E(A)} f \otimes \lambda s$.

Proof. Let $\mathcal{I}_\Delta$ denote the ideal sheaf of the diagonal in $A \times A$. Let Z be the non-reduced subscheme of $A \times A \times E(A)$ defined by the ideal sheaf $\mathscr{O}_{A \times A \times E(A)} \cdot \mathcal{I}_\Delta^2$. We have a natural exact sequence

$$0 \to \widetilde{P} \otimes H^0(A, \Omega_A^1) \to (p_{13})_* \big(\mathscr{O}_Z \otimes p_{23}^* \widetilde{P} \big) \to \widetilde{P} \to 0, \tag{11.6}$$

and a generalized relative connection is the same thing as a morphism of sheaves

$$\widetilde{P} \to (p_{13})_* \big(\mathscr{O}_Z \otimes p_{23}^* \widetilde{P} \big)$$

whose composition with the morphism to $\widetilde{P}$ acts as multiplication by λ. In fact, there is a canonical choice, which we shall now describe. Consider the morphism

$$f\colon A \times A \to A \times A, \quad f(a,b) = (a, a+b).$$

Since $f \times \mathrm{id}_{E(A)}$ induces an isomorphism between the first infinitesimal neighborhood of $A \times \{0\} \times E(A)$ and the subscheme Z, we have

$$\begin{aligned}(f \times \mathrm{id}_{E(A)})^* \bigl(\mathscr{O}_Z \otimes p_{23}^* \widetilde{P}\bigr) &= p_2^*(\mathscr{O}_A/\mathfrak{m}_A^2) \otimes (m \times \mathrm{id}_{E(A)})^* \widetilde{P} \\ &= p_2^*(\mathscr{O}_A/\mathfrak{m}_A^2) \otimes p_{13}^* \widetilde{P} \otimes p_{23}^* \widetilde{P},\end{aligned}$$

due to the well-known fact that the Poincaré bundle satisfies

$$(m \times \mathrm{id}_{\hat{A}})^* P = p_{13}^* P \otimes p_{23}^* P$$

on $A \times A \times \hat{A}$. Since $p_{13} \circ (f \times \mathrm{id}_{E(A)}) = p_{13}$, we conclude that we have

$$(p_{13})_* \bigl(\mathscr{O}_Z \otimes p_{23}^* \widetilde{P}\bigr) = \widetilde{P} \otimes p_2^* \pi^* \mathbf{R}\Phi_P \bigl(\mathscr{O}_A/\mathfrak{m}_A^2\bigr) = \widetilde{P} \otimes p_2^* \pi^* \mathscr{E}(A)$$

on $A \times E(A)$; more precisely, (11.6) is isomorphic to the tensor product of $\widetilde{P}$ and the pullback of (11.4) by $\pi \circ p_2$.

Now the pullback of the exact sequence (11.4) to $E(A)$ obviously has a splitting of the type we are looking for: indeed, the tautological section of $\pi^*\mathscr{E}(A)$ gives a morphism $\mathscr{O}_{E(A)} \to \pi^*\mathscr{E}(A)$ whose composition with the projection to $\mathscr{O}_{E(A)}$ is multiplication by λ. Thus we obtain a canonical morphism $\widetilde{P} \to \widetilde{P} \otimes p_2^* \pi^* \mathscr{E}(A)$ and hence, by the above, the desired generalized relative connection. □

At any point $e \in E(A)$, we thus obtain a $\lambda(e)$-connection on the line bundle corresponding to $\pi(e) \in \mathrm{Pic}^0(A)$. This shows that $E(A)$ is the moduli space of (topologically trivial) line bundles with integrable generalized connection. Using the properties of the Picard scheme, one can show that $E(A)$ is a fine moduli space in the obvious sense; as we do not need this fact below, we shall not give the proof.

We close this section with two simple lemmas that describe how $E(A)$ and $(\widetilde{P}, \widetilde{\nabla})$ behave under restriction to the fibers of $\lambda\colon E(A) \to \mathbb{C}$.

Lemma 11.7. *We have $\lambda^{-1}(1) = A^\natural$, and the restriction of $(\widetilde{P}, \widetilde{\nabla})$ to $A \times A^\natural$ is equal to $(P^\natural, \nabla^\natural)$.*

Proof. This follows from the construction of $A^\natural$ and $\nabla^\natural$ in [MM74, Chapter I]. □

Recall that the cotangent bundle of A satisfies $T^*A = A \times H^0(A, \Omega_A^1)$, and consider the following diagram:

(11.8)
$$\begin{tikzcd} A \times H^0(A,\Omega_A^1) & A \times \hat{A} \times H^0(A,\Omega_A^1) \arrow[l, "p_{13}"'] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & \hat{A} \times H^0(A,\Omega_A^1) \\ & A \times \hat{A} & \end{tikzcd}$$

Lemma 11.9. *We have $\lambda^{-1}(0) = \hat{A} \times H^0(A, \Omega_A^1)$, and the restriction of $(\widetilde{P}, \widetilde{\nabla})$ to $A \times \hat{A} \times H^0(A, \Omega_A^1)$ is equal to the Higgs bundle*

$$\bigl(p_{12}^* P,\, p_{13}^* \theta_A\bigr),$$

where θ_A denotes the tautological holomorphic one-form on T^*A.

Proof. This follows easily from the proof of Lemma 11.5. □

12. The extended Fourier-Mukai transform. We shall now describe an extension of the Fourier-Mukai transform to the case of algebraic $\mathscr{R}_A$-modules. Since we only need a very special case in this paper, we leave a more careful discussion to a future publication. Recall that $\mathscr{D}_A$ is generated as an $\mathscr{O}_A$-algebra by a basis of vector fields $\partial_1, \dots, \partial_g \in H^0(A, \mathscr{T}_A)$, subject to the relations

$$[\partial_i, \partial_j] = 0 \quad \text{and} \quad [\partial_i, f] = \partial_i f.$$

Likewise, $\mathscr{R}_A$ is generated as an $\mathscr{O}_A[z]$-algebra by $z\partial_1, \dots, z\partial_g$, subject to

$$[z\partial_i, z\partial_j] = 0 \quad \text{and} \quad [z\partial_i, f] = z \cdot \partial_i f.$$

It is not hard to show that $\mathscr{R}_A$ is isomorphic to $(p_1)_* \mathscr{R}_{A\times\mathbb{C}/\mathbb{C}}$, where $\mathscr{R}_{A\times\mathbb{C}/\mathbb{C}}$ denotes the $\mathscr{O}_{A\times\mathbb{C}}$-subalgebra of $\mathscr{D}_{A\times\mathbb{C}/\mathbb{C}}$ generated by $z\mathscr{T}_{A\times\mathbb{C}/\mathbb{C}}$. If M is an algebraic $\mathscr{R}_A$-module, then the associated quasi-coherent sheaf $\widetilde{M}$ on $A \times \mathbb{C}$ is naturally a module over $\mathscr{R}_{A\times\mathbb{C}/\mathbb{C}}$, and vice versa.

Now fix an algebraic $\mathscr{R}_A$-module M. Let $\mathscr{R}_{A\times E(A)/E(A)}$ denote the subalgebra of $\mathscr{D}_{A\times E(A)/E(A)}$ generated by $\lambda \mathscr{T}_{A\times E(A)/E(A)}$. The tensor product

$$(\mathrm{id} \times \lambda)^* \widetilde{M} \otimes_{\mathscr{O}_{A\times E(A)}} \widetilde{P}$$

naturally has the structure of $\mathscr{R}_{A\times E(A)/E(A)}$-module on $A \times E(A)$: concretely, the module structure is given by $\lambda(m \otimes s) = (\lambda m) \otimes s = m \otimes \lambda s$ and $\lambda\partial_i(m \otimes s) = (z\partial_i m) \otimes s + m \otimes \widetilde{\nabla}_{\partial_i}(s)$. We may therefore consider the relative de Rham complex

$$\mathrm{DR}_{A\times E(A)/E(A)}\Big((\mathrm{id} \times \lambda)^* \widetilde{M} \otimes_{\mathscr{O}_{A\times E(A)}} (\widetilde{P}, \widetilde{\nabla})\Big),$$

which is defined just as in the case of $\mathscr{D}$-modules.

Definition 12.1. The *Fourier-Mukai transform* of an algebraic $\mathscr{R}_A$-module M is

$$\widetilde{\mathrm{FM}}_A(M) = \mathbf{R}(p_2)_* \, \mathrm{DR}_{A\times E(A)/E(A)}\Big((\mathrm{id} \times \lambda)^* \widetilde{M} \otimes_{\mathscr{O}_{A\times E(A)}} \widetilde{P}\Big);$$

it is an object of $\mathrm{D}^b\big(\mathscr{O}_{E(A)}\big)$, the bounded derived category of quasi-coherent algebraic sheaves on $E(A)$.

Note. Using the general formalism in [PR01], one can show that the Fourier-Mukai transform induces an equivalence of categories

$$\widetilde{\mathrm{FM}}_A \colon \mathrm{D}^b(\mathscr{R}_A) \to \mathrm{D}^b\big(\mathscr{O}_{E(A)}\big),$$

Since this fact will not be used below, we shall omit the proof.

Lemma 12.2. *If M is a coherent algebraic $\mathscr{R}_A$-module, $\widetilde{\mathrm{FM}}_A(M) \in \mathrm{D}^b_{coh}(\mathscr{O}_{E(A)})$.*

Proof. The proof is the same as in the case of $\mathscr{D}_A$-modules; for more details, refer to [Lau96, Proposition 3.1.2 and Corollaire 3.1.3]. □

13. Compatibility. Just as $\mathscr{R}_A$-modules interpolate between $\mathscr{D}_A$-modules and quasi-coherent sheaves on the cotangent bundle T^*A, the extended Fourier-Mukai transform in Definition 12.1 interpolates between the Fourier-Mukai transform for $\mathscr{D}_A$-modules and the usual Fourier-Mukai transform for quasi-coherent sheaves. The purpose of this section is to make that relationship precise.

Throughout the discussion, let $\mathcal{M}$ be a coherent algebraic $\mathscr{D}_A$-module and $F_\bullet\mathcal{M}$ a good filtration of $\mathcal{M}$ by $\mathscr{O}_A$-coherent subsheaves. The graded $\operatorname{Sym} \mathscr{T}_A$-module

$$\mathrm{gr}^F_\bullet \mathcal{M} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

is then coherent over $\operatorname{Sym} \mathscr{T}_A$, and therefore defines a coherent sheaf on the cotangent bundle T^*A that we shall denote by the symbol $\operatorname{gr}^F \mathcal{M}$. Now consider once more the Rees module

$$R_F\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k\mathcal{M} \cdot z^k \subseteq \mathcal{M} \otimes_{\mathscr{O}_A} \mathscr{O}_A[z, z^{-1}],$$

which is a graded $\mathscr{R}_A$-module, coherent over $\mathscr{R}_A$. The associated quasi-coherent sheaf on $A \times \mathbb{C}$, which we shall denote by the symbol $\widetilde{R_F\mathcal{M}}$, is equivariant for the natural $\mathbb{C}^*$-action on the product. Moreover, it is easy to see that the restriction of $\widetilde{R_F\mathcal{M}}$ to $A \times \{1\}$ is a $\mathscr{D}_A$-module isomorphic to $\mathcal{M}$, while the restriction to $A \times \{0\}$ is a graded $\operatorname{Sym} \mathscr{T}_A$-module isomorphic to $\operatorname{gr}_\bullet^F \mathcal{M}$.

Proposition 13.1. *Let $\mathcal{M}$ be a coherent algebraic $\mathscr{D}_A$-module with good filtration $F_\bullet\mathcal{M}$. Then the extended Fourier-Mukai transform $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M}) \in \mathrm{D}^b_{coh}(\mathscr{O}_{E(A)})$ of the associated graded $\mathscr{R}_A$-module has the following properties:*

(i) It is equivariant for the natural $\mathbb{C}^*$-action on the vector bundle $E(A)$.
(ii) Its restriction to $A^\natural = \lambda^{-1}(1)$ is canonically isomorphic to $\mathrm{FM}_A(\mathcal{M})$.
(iii) Its restriction to $\hat{A} \times H^0(A, \Omega_A^1) = \lambda^{-1}(0)$ is canonically isomorphic to

$$\mathbf{R}(p_{23})_* \Big(p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^* (\mathrm{id} \times \iota)^* \operatorname{gr}^F \mathcal{M} \Big),$$

where the notation is as in the diagram in (11.8) above, and where $\iota = -\mathrm{id}$ is the obvious involution of $H^0(A, \Omega_A^1)$.

Proof. (i) is true because $R_F\mathcal{M}$ is a graded $\mathscr{R}_A$-module, and because $(\widetilde{P}, \widetilde{\nabla})$ and the relative de Rham complex are obviously $\mathbb{C}^*$-equivariant. (ii) follows directly from the definition of the Fourier-Mukai transform, using the base change formula for the morphism $\lambda \colon E(A) \to \mathbb{C}$ and Lemma 11.7.

The proof of (iii) is a little less obvious, and so we give some details. By base change, it suffices to show that the restriction of the relative de Rham complex

$$\mathrm{DR}_{A \times E(A)/E(A)} \Big((\mathrm{id} \times \lambda)^* \widetilde{R_F\mathcal{M}} \otimes_{\mathscr{O}_{A \times E(A)}} (\widetilde{P}, \widetilde{\nabla}) \Big)$$

to $A \times \hat{A} \times H^0(A, \Omega_A^1)$ resolves the coherent sheaf $p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^* (\mathrm{id} \times \iota)^* \operatorname{gr}^F \mathcal{M}$. After a short computation, one finds that this restriction is isomorphic to the tensor product of $p_{12}^* P$ and the pullback, via p_{13}, of the complex

$$\Big[p_1^* \Big(\operatorname{gr}_\bullet^F \mathcal{M} \Big) \to p_1^* \Big(\Omega_A^1 \otimes_{\mathscr{O}_A} \operatorname{gr}_\bullet^F \mathcal{M} \Big) \to \cdots \to p_1^* \Big(\Omega_A^g \otimes_{\mathscr{O}_A} \operatorname{gr}_\bullet^F \mathcal{M} \Big) \Big] [g], \tag{13.2}$$

with differential $p_1^* \Big(\Omega_A^k \otimes \operatorname{gr}_\bullet^F \mathcal{M} \Big) \to p_1^* \Big(\Omega_A^{k+1} \otimes \operatorname{gr}_\bullet^F \mathcal{M} \Big)$ given by the formula

$$\omega \otimes m \mapsto (-1)^g (\theta_A \wedge \omega) \otimes m + (-1)^g \sum_{j=1}^g (dz_j \wedge \omega) \otimes \partial_j m.$$

But since $\operatorname{gr}^F \mathcal{M}$ is the coherent sheaf on $A \times H^0(A, \Omega_A^1)$ corresponding to the graded $\operatorname{Sym} \mathscr{T}_A$-module $\operatorname{gr}_\bullet^F \mathcal{M}$, the complex in (13.2) is a resolution of the coherent sheaf $p_1^* \Omega_A^g \otimes (\mathrm{id} \times \iota)^* \operatorname{gr}^F \mathcal{M}$ by Lemma 13.3 below, and so we get the desired result. ☐

In the proof, we used the following lemma.

Lemma 13.3. *Let $\mathscr{F}$ be a finitely generated* $\operatorname{Sym}\mathscr{T}_A$*-module, and denote by $\tilde{\mathscr{F}}$ the associated coherent sheaf on $T^*A = A \times H^0(A, \Omega_A^1)$. Then the complex*

$$p_1^*\big(\mathscr{O}_A \otimes_{\mathscr{O}_A} \mathscr{F}\big) \to p_1^*\big(\Omega_A^1 \otimes_{\mathscr{O}_A} \mathscr{F}\big) \to \cdots \to p_1^*\big(\Omega_A^g \otimes_{\mathscr{O}_A} \mathscr{F}\big)$$

is a resolution of $p_1^*\Omega_A^g \otimes (\operatorname{id}\times\iota)^*\tilde{\mathscr{F}}$, where ι is the involution of $H^0(A, \Omega_A^1)$.

Proof. By construction, $\mathscr{F} = (p_1)_*\tilde{\mathscr{F}}$; note that $\mathscr{F}' = (p_1)_*(\operatorname{id}\times\iota)^*\tilde{\mathscr{F}}$ is isomorphic to $\mathscr{F}$ as an $\mathscr{O}_A$-module, but sections of $\operatorname{Sym}^k\mathscr{T}_A$ act with an additional factor of $(-1)^k$. The tangent bundle of A is trivial; if we set $V = T_0A$ and $S = \operatorname{Sym} V$, then $\mathscr{T}_A = V \otimes_{\mathbb{C}} \mathscr{O}_A$ and $\operatorname{Sym}\mathscr{T}_A = S \otimes_{\mathbb{C}} \mathscr{O}_A$. Because $p_1 \colon A \times H^0(A, \Omega_A^1) \to A$ is affine, the assertion is equivalent to the exactness of the complex

$$\cdots \longrightarrow \textstyle\bigwedge^2 V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} \mathscr{F} \xrightarrow{d_2} V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} \mathscr{F} \xrightarrow{d_1} S \otimes_{\mathbb{C}} \mathscr{F} \xrightarrow{d_0} \mathscr{F}' \longrightarrow 0,$$

where d_0 is the obvious multiplication map, and where the other differentials

$$d_{k+1} \colon \bigwedge^{k+1} V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} \mathscr{F} \to \bigwedge^{k} V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} \mathscr{F}$$

are given by the formula

$$\begin{aligned}(v_0 \wedge v_1 \wedge \cdots \wedge v_k) \otimes s \otimes f \mapsto \sum_{i=0}^{k} (-1)^i (v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k) \otimes v_i s \otimes f \\ + \sum_{i=0}^{k} (-1)^i (v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k) \otimes s \otimes v_i f.\end{aligned}$$

It is not hard to see that the complex in question is the tensor product (over S) of $\mathscr{F}$ with the analogously defined complex of S-modules

$$\cdots \longrightarrow \textstyle\bigwedge^2 V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} S \xrightarrow{d_2} V \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} S \xrightarrow{d_1} S \otimes_{\mathbb{C}} S \xrightarrow{d_0} S' \longrightarrow 0. \tag{13.4}$$

Note that $d_0 \colon S_k \otimes S_\ell \to S_{k+\ell}$ is $(-1)^k$ times the usual multiplication map. The complex in (13.4) is exact; one way to see this is to observe that $S \otimes_{\mathbb{C}} S$ is isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots, x_g, y_1, \dots, y_g]$, and that the complex is just the usual Koszul complex for the ideal $(x_1 + y_1, \dots, x_g + y_g)$. ☐

C. The structure theorem

The purpose of this chapter is to prove the structure theorems for cohomology support loci of holonomic and constructible complexes.

14. Cohomology of constructible complexes. In this section, we describe an analogue of the Fourier-Mukai transform for constructible complexes on A, and use it to prove that cohomology support loci are algebraic subvarieties of $\operatorname{Char}(A)$. We refer the reader to [HTT08, Section 4.5] and to [Dim04, Chapter 4] for details about constructible complexes and perverse sheaves.

The abelian variety A may be presented as a quotient V/Λ, where V is a complex vector space of dimension g, and $\Lambda \subseteq V$ is a lattice of rank $2g$. Note that V is isomorphic to the tangent space of A at the unit element, while Λ is isomorphic to the fundamental group $\pi_1(A, 0)$. We shall denote by $k[\Lambda]$ the group ring of Λ with coefficients in a subfield $k \subseteq \mathbb{C}$; thus

$$k[\Lambda] = \bigoplus_{\lambda \in \Lambda} k e_\lambda,$$

with $e_\lambda \cdot e_\mu = e_{\lambda+\mu}$. A choice of basis for Λ shows that $k[\Lambda]$ is isomorphic to the ring of Laurent polynomials in $2g$ variables. Any character $\rho \colon \Lambda \to \mathbb{C}^*$ extends uniquely to a homomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\Lambda] \to \mathbb{C}, \quad e_\lambda \mapsto \rho(\lambda),$$

whose kernel is a maximal ideal $\mathfrak{m}_\rho \subseteq \mathbb{C}[\Lambda]$; concretely, $\mathfrak{m}_\rho$ is generated by the elements $e_\lambda - \rho(\lambda)$, for $\lambda \in \Lambda$. It is easy to see that any maximal ideal of $\mathbb{C}[\Lambda]$ is of this form; this means that $\operatorname{Char}(A)$ is the set of $\mathbb{C}$-valued points of the scheme $\operatorname{Spec} \mathbb{C}[\Lambda]$, and therefore naturally an affine complex algebraic variety.

For any $k[\Lambda]$-module M, multiplication by the ring elements e_λ determines a natural action of Λ on the k-vector space M. By the well-known correspondence between representations of the fundamental group and local systems, it thus gives rise to a local system on A.

Definition 14.1. For a $k[\Lambda]$-module M, we denote by the symbol $\mathcal{L}_M$ the corresponding local system of k-vector spaces on A. Concretely,

$$\mathcal{L}_M(U) = \left\{ \ell \colon \pi^{-1}(U) \to M \;\middle|\; \begin{array}{l} \ell \text{ is locally constant, and} \\ \ell(v+\lambda) = e_\lambda^{-1} \ell(v) \text{ for } \lambda \in \Lambda \end{array} \right\}$$

for all open sets $U \subseteq A$, where $\pi \colon V \to A$ is the quotient mapping.

Since $k[\Lambda]$ is commutative, $\mathcal{L}_M$ is actually a *local system of $k[\Lambda]$-modules*. The most important example is $\mathcal{L}_{\mathbb{C}[\Lambda]}$, which is a local system of $\mathbb{C}[\Lambda]$-modules of rank one; one can show that it is isomorphic to the direct image with proper support $\pi_! \mathbb{C}_V$ of the constant local system on the universal covering space $\pi \colon V \to A$. This device allows us to construct $\mathbb{C}[\Lambda]$-modules, and hence quasi-coherent sheaves on $\operatorname{Char}(A)$, by twisting a complex of sheaves of $\mathbb{C}$-vector spaces by a local system of the form $\mathcal{L}_M$, and pushing forward along the morphism $p \colon A \to pt$ to a point.

Proposition 14.2. *Let k be a field, and let $K \in \mathrm{D}_c^b(k_A)$ be a constructible complex of sheaves of k-vector spaces on A. Then for any finitely generated $k[\Lambda]$-module M, the direct image $\mathbf{R}p_*\big(K \otimes_k \mathcal{L}_M\big)$ belongs to $\mathrm{D}_{coh}^b\big(k[\Lambda]\big)$.*

Proof. Since K is a constructible complex of sheaves of k-vector spaces, the tensor product $K \otimes_k \mathcal{L}_M$ is a constructible complex of sheaves of $k[\Lambda]$-modules. By [Dim04, Corollary 4.1.6], its direct image is thus an object of $\mathrm{D}_{coh}^b\big(k[\Lambda]\big)$. □

To understand how $\mathbf{R}p_*\big(K \otimes_k \mathcal{L}_M\big)$ depends on M, we need the following auxiliary result. Recall that a *fine sheaf* on a manifold is a sheaf admitting partitions of unity; such sheaves are acyclic for direct image functors.

Lemma 14.3. *Let $\mathscr{F}$ be a fine sheaf of $\mathbb{C}$-vector spaces on A. Then the space of global sections $H^0\big(A, \mathscr{F} \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$ is a flat $\mathbb{C}[\Lambda]$-module, and for every $\mathbb{C}[\Lambda]$-module M, one has*

$$H^0\big(A, \mathscr{F} \otimes_{\mathbb{C}} \mathcal{L}_M\big) \simeq H^0\big(A, \mathscr{F} \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \otimes_{\mathbb{C}[\Lambda]} M,$$

functorially in M.

Proof. Each sheaf of the form $\mathscr{F} \otimes_{\mathbb{C}} \mathcal{L}_M$ is clearly again a fine sheaf. Consequently, $M \mapsto H^0\big(A, \mathscr{F} \otimes_{\mathbb{C}} \mathcal{L}_M\big)$ is an exact functor from the category of $\mathbb{C}[\Lambda]$-modules to the category of $\mathbb{C}[\Lambda]$-modules. Since the functor also preserves arbitrary direct sums, the result follows from the Eilenberg-Watts theorem in homological algebra [Wat60, Theorem 1]. □

Proposition 14.4. *Let $K \in \mathrm{D}_c^b(\mathbb{C}_A)$. Then for every $\mathbb{C}[\Lambda]$-module M, one has an isomorphism*

$$\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_M\big) \simeq \mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \overset{\mathbf{L}}{\otimes}_{\mathbb{C}[\Lambda]} M,$$

functorial in M.

Proof. We begin by choosing a bounded complex $(\mathscr{F}^\bullet, d)$ of fine sheaves quasi-isomorphic to K. One way to do this is as follows. By the Riemann-Hilbert correspondence, $K \simeq \mathrm{DR}_A(\mathcal{M}^\bullet)$ for some $\mathcal{M}^\bullet \in \mathrm{D}_{rh}^b(\mathscr{D}_A)$; if we now let $\mathcal{A}_A^k$ denote the sheaf of smooth k-forms on the complex manifold A, then by the Poincaré lemma, the single complex determined by the double complex with terms

$$\mathcal{A}_A^{g+i} \otimes_{\mathscr{O}_A} \mathcal{M}^j$$

is a complex of fine sheaves quasi-isomorphic to K. For any such choice, $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_M\big) \in \mathrm{D}_{coh}^b\big(\mathbb{C}[\Lambda]\big)$ is represented by the bounded complex of $\mathbb{C}[\Lambda]$-modules

$$H^0\big(A, \mathscr{F}^\bullet \otimes_{\mathbb{C}} \mathcal{L}_M\big) \simeq H^0\big(A, \mathscr{F}^\bullet \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \otimes_{\mathbb{C}[\Lambda]} M,$$

and so the assertion follows from Lemma 14.3. □

Now let $\rho \in \mathrm{Char}(A)$ be an arbitrary character; recall that $\mathfrak{m}_\rho$ is the maximal ideal of $\mathbb{C}[\Lambda]$ generated by the elements $e_\lambda - \rho(\lambda)$, for $\lambda \in \Lambda$. Using the notation introduced above, we therefore have the alternative description $\mathbb{C}_\rho \simeq \mathcal{L}_{\mathbb{C}[\Lambda]/\mathfrak{m}_\rho}$ for the local system corresponding to ρ.

Corollary 14.5. *For any character $\rho \in \mathrm{Char}(A)$, we have*

$$\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathbb{C}_\rho\big) \simeq \mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \overset{\mathbf{L}}{\otimes}_{\mathbb{C}[\Lambda]} \mathbb{C}[\Lambda]/\mathfrak{m}_\rho$$

as objects of $\mathrm{D}_{coh}^b(\mathbb{C})$.

We may thus consider the complex $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \in \mathrm{D}_{coh}^b\big(\mathbb{C}[\Lambda]\big)$ as being a sort of "Fourier-Mukai transform" of the constructible complex $K \in \mathrm{D}_c^b(\mathbb{C}_A)$. This point of view is justified also by its relationship with the Fourier-Mukai transform for $\mathscr{D}$-modules in Theorem 15.2 below.[4]

The results above are all that is needed to prove that the cohomology support loci of a constructible complex are algebraic subsets of $\mathrm{Char}(A)$.

Theorem 14.6. *If $K \in \mathrm{D}_c^b(\mathbb{C}_A)$, then each cohomology support locus $S_m^k(A, K)$ is an algebraic subset of $\mathrm{Char}(A)$.*

Proof. Recall that $\mathrm{Char}(A)$ is the complex manifold associated with the complex algebraic variety $\operatorname{Spec} \mathbb{C}[\Lambda]$. Thus $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \in \mathrm{D}_{coh}^b\big(\mathbb{C}[\Lambda]\big)$ determines an object in the bounded derived category of coherent algebraic sheaves on $\mathrm{Char}(A)$, whose fiber at any closed point ρ computes the hypercohomology of $K \otimes_{\mathbb{C}} \mathbb{C}_\rho$, according to Corollary 14.5. We conclude that

$$S_m^k(A, K) = \left\{ \rho \in \mathrm{Char}(A) \;\middle|\; \dim \mathbf{H}^k\Big(\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \overset{\mathbf{L}}{\otimes}_{\mathbb{C}[\Lambda]} \mathbb{C}[\Lambda]/\mathfrak{m}_\rho\Big) \geq m \right\},$$

and by Lemma 14.7 below, this description implies that $S^k_m(A,K)$ is an algebraic subset of $\operatorname{Char}(A)$. ☐

For the convenience of the reader, we include the following elementary lemma.

Lemma 14.7. *Let E be a perfect complex on a scheme X. For each $k, m \in \mathbb{Z}$,*

$$\left\{ x \in X \;\middle|\; \dim \mathbf{H}^k\big(E \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}_{X,x}/\mathfrak{m}_x\big) \geq m \right\}$$

is the set of closed points of a closed subscheme of X.

Proof. Since the statement is local, it suffices to consider the case of a complex

$$E^{k-1} \xrightarrow{f} E^k \xrightarrow{g} E^{k+1}$$

of free R-modules of finite rank. For any prime ideal $P \subseteq R$, we shall use the notation

$$E^{k-1} \otimes_R R/P \xrightarrow{f_P} E^k \otimes_R R/P \xrightarrow{g_P} E^{k+1} \otimes_R R/P$$

for the tensor product of the above complex by R/P. Now $\operatorname{rk}(\ker g_P / \operatorname{im} f_P) \geq m$ is clearly equivalent to having $\operatorname{rk}(\ker g_P) \geq m+i$ and $\operatorname{rk}(\operatorname{im} f_P) \leq i$ for some $i \geq 0$. Consequently, the set of prime ideals $P \in \operatorname{Spec} R$ for which $\operatorname{rk}(\ker g_P / \operatorname{im} f_P) \geq m$ is equal to

$$\bigcup_{i \geq 0} \left\{ P \;\middle|\; \operatorname{rk}(\ker g_P) \geq m+i \right\} \cap \left\{ P \;\middle|\; \operatorname{rk}(\ker f_P) \geq \operatorname{rk}(E^{k-1}) - i \right\}$$

But this set can be defined by the vanishing of certain minors of f and g, and is therefore naturally a closed subscheme of $\operatorname{Spec} R$. ☐

For the more arithmetic questions in Section 17, we make the following observation about fields of definition.

Proposition 14.8. *Let k be any subfield of $\mathbb{C}$. If $K \in \mathrm{D}^b_c(k_A)$ is a constructible complex of sheaves of k-vector spaces, then $\mathbf{R}p_*\big(K \otimes_k \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$ is defined over k.*

Proof. Indeed, we have $\mathcal{L}_{\mathbb{C}[\Lambda]} = \mathcal{L}_{k[\Lambda]} \otimes_k \mathbb{C}$, and therefore

$$\mathbf{R}p_*\big(K \otimes_k \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \simeq \mathbf{R}p_*\big(K \otimes_k \mathcal{L}_{k[\Lambda]}\big) \otimes_k \mathbb{C}$$

is obtained by extension of scalars from an object of $\mathrm{D}^b_{coh}\big(k[\Lambda]\big)$. ☐

15. Comparison theorems. Recall that we have a biholomorphic mapping

$$\Phi \colon A^\natural \to \operatorname{Char}(A)$$

that takes a line bundle with integrable connection to the corresponding local system of rank one. In this section, we relate the Fourier-Mukai transform of a holonomic complex $\mathcal{M} \in \mathrm{D}^b_h(\mathscr{D}_A)$ and the transform of the constructible complex $\mathrm{DR}_A(\mathcal{M})$. Our first result is purely set-theoretic, and concerns the relationship between the cohomology support loci of $\mathcal{M}$ and $\mathrm{DR}_A(\mathcal{M})$.

Theorem 15.1. *Let $\mathcal{M} \in \mathrm{D}^b_h(\mathscr{D}_A)$ be a holonomic complex on A. Then*

$$\Phi\big(A, S^k_m(A,\mathcal{M})\big) = S^k_m\big(A, \mathrm{DR}_A(\mathcal{M})\big).$$

for every $k, m \in \mathbb{Z}$.

Proof. Let $\big(\mathcal{M}^\bullet, d\big)$ be the given holonomic complex. For any line bundle with integrable connection (L, ∇), the associated local system $\ker \nabla$ is a subsheaf of L, and we have $(\ker \nabla) \otimes_{\mathbb{C}} \mathscr{O}_A = L$. This means that the natural sheaf morphisms

$$\big(\Omega_A^{g+i} \otimes_{\mathscr{O}_A} \mathcal{M}^j\big) \otimes_{\mathbb{C}} \ker \nabla \to \Omega_A^{g+i} \otimes_{\mathscr{O}_A} \big(\mathcal{M}^j \otimes_{\mathscr{O}_A} L\big)$$

are isomorphisms for every $i, j \in \mathbb{Z}$. Since they are compatible with the differentials $d_1 = (-1)^g \nabla_{\mathcal{M}^j}$ and $d_2 = \mathrm{id} \otimes d$, we obtain an isomorphism of complexes

$$\mathrm{DR}_A(\mathcal{M}^\bullet) \otimes_{\mathbb{C}} \ker \nabla \to \mathrm{DR}_A\big(\mathcal{M}^\bullet \otimes_{\mathscr{O}_A} (L, \nabla)\big),$$

and therefore the desired relation between their hypercohomology groups. □

The second result is much stronger, and directly relates the two complexes $\mathrm{FM}_A(\mathcal{M})$ and $\mathbf{R}p_*\big(\mathrm{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$. This only makes sense on the level of coherent analytic sheaves, because Φ is not algebraic. For the remainder of this section, *every coherent sheaf is a coherent analytic sheaf.*

Theorem 15.2. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ be a holonomic complex on A. Then the complex of coherent analytic sheaves $\mathbf{R}\Phi_* \mathrm{FM}_A(\mathcal{M})$ is quasi-isomorphic to the complex of coherent analytic sheaves determined by $\mathbf{R}p_*\big(\mathrm{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$.*

We shall denote by $R = \mathscr{O}\big(\mathrm{Char}(A)\big)$ the ring of global holomorphic functions on the Stein manifold $\mathrm{Char}(A)$. It is an algebra over $\mathbb{C}[\Lambda]$: the holomorphic function corresponding to e_λ takes a character ρ to the complex number $\rho(\lambda)$. From the induced Λ-action on R, we obtain a local system $\mathcal{L}_R$ on the abelian variety A, and Proposition 14.4 says that

$$\mathbf{R}p_*\big(\mathrm{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \otimes_{\mathbb{C}[\Lambda]} R \simeq \mathbf{R}p_*\big(\mathrm{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_R\big)$$

The idea of the proof of Theorem 15.2 is to relate this complex of finitely generated R-modules to the complex of global sections of $\mathbf{R}\Phi_* \mathrm{FM}_A(\mathcal{M})$.

We begin by recalling the analytic description of the Poincaré bundle on $A \times A^\natural$. Since $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$, the canonical isomorphism

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$$

allows us to identify a group homomorphism $f\colon \Lambda \to \mathbb{C}$ with the induced $\mathbb{R}$-linear functional $f\colon V \to \mathbb{C}$. Now any such $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ gives rise to a translation-invariant complex-valued one-form df on the abelian variety A, and therefore determines a line bundle with integrable connection on A: the underlying smooth line bundle is $A \times \mathbb{C}$; the complex structure is given by $\bar{\partial} + (df)^{0,1}$, and the connection by $\partial + (df)^{1,0}$. Briefly, we say that the line bundle with integrable connection is defined by the operator $d + df$; the local system of its flat sections corresponds to the character $\rho_f\colon \Lambda \to \mathbb{C}^*$, $\rho_f(\lambda) = e^{-f(\lambda)}$. We thus have a commutative diagram:

$$\begin{tikzcd} & A^\natural \arrow[d, "\Phi"] \\ \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \arrow[ur, bend left, "f \mapsto d + df"] \arrow[r, bend right, "f \mapsto e^{-f}"'] & \mathrm{Char}(A) \end{tikzcd}$$

Both horizontal arrows are surjective homomorphisms of complex Lie groups; their kernels are equal to the subspace $\mathrm{Hom}_{\mathbb{Z}}\big(\Lambda, \mathbb{Z}(1)\big)$, where $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z} \subseteq \mathbb{C}$. Note that if $f(\Lambda) \subseteq \mathbb{Z}(1)$, then e^{-f} descends to a nowhere vanishing real analytic function on A. On the product $V \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$, one has the tautological function

$$F \colon V \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathbb{C}, \quad F(v, f) = f(v).$$

Now we can describe the Poincaré bundle $P^\natural$ and the universal relative connection $\nabla^\natural \colon P^\natural \to \Omega^1_{A \times A^\natural / A^\natural} \otimes P^\natural$ on $A \times A^\natural$. The discrete group $\Lambda \times \operatorname{Hom}_{\mathbb{Z}}\bigl(\Lambda, \mathbb{Z}(1)\bigr)$ acts smoothly on the trivial bundle $V \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \times \mathbb{C}$ by the formula

$$(\lambda, \phi) \cdot (v, f, z) = \left(v + \lambda, f + \phi, e^{-\phi(v)} z\right),$$

and the quotient is a smooth line bundle on $A \times A^\natural$. Its space of global sections consists of all smooth functions $\sigma \colon V \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathbb{C}$ that satisfy

$$\sigma(v + \lambda, f + \phi) = e^{-\phi(v)} \sigma(v, f)$$

for every $\lambda \in \Lambda$ and every $\phi \in \operatorname{Hom}_{\mathbb{Z}}\bigl(\Lambda, \mathbb{Z}(1)\bigr)$. The original bundle has both a complex structure and a relative integrable connection, defined by the operator

$$\bigl(d_V + d_V F\bigr) + \bar{\partial}_{\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})};$$

one can show that this operator descends to the quotient, and endows the smooth line bundle from above with a complex structure and a connection relative to $A^\natural$. This gives a useful model for the pair $(P^\natural, \nabla^\natural)$.

We now turn to the proof of Theorem 15.2. Our first concern is to describe the direct image of $(P^\natural, \nabla^\natural)$ under the projection $p_1 \colon A \times A^\natural \to A^\natural$. Because $A^\natural$ is biholomorphic to $\operatorname{Char}(A)$, it is a Stein manifold, and so $R^i (p_1)_* P^\natural = 0$ for $i \geq 1$. (Remember that *every coherent sheaf is a coherent analytic sheaf* in this section.) It is also easy to see from the analytic description of the Poincaré bundle that the sections of $(p_1)_* P^\natural$ over an open set $U \subseteq A$ are

$$\left\{ \sigma \colon \pi^{-1}(U) \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathbb{C} \;\middle|\; \begin{array}{l} \sigma(v, f) \text{ is holomorphic, and} \\ \sigma(v + \lambda, f + \phi) = e^{-\phi(v)} \sigma(v, f) \end{array} \right\}$$

After taking the direct image of $\nabla^\natural \colon P^\natural \to \Omega^1_{A \times A^\natural / A^\natural} \otimes P^\natural$ and using the projection formula, we end up with an operator

$$(p_1)_* \nabla^\natural \colon (p_1)_* P^\natural \to \Omega^1_A \otimes_{\mathscr{O}_A} (p_1)_* P^\natural.$$

The next lemma relates its kernel to the local system defined by $R = \mathscr{O}\bigl(\operatorname{Char}(A)\bigr)$.

Lemma 15.3. *The kernel of $(p_1)_* \nabla^\natural$ is canonically isomorphic to $\mathcal{L}_R$.*

Proof. As with any representation of Λ, the sections of the local system $\mathcal{L}_R$ over an open set $U \subseteq A$ are given by

$$\mathcal{L}_R(U) = \left\{ \ell \colon \pi^{-1}(U) \to R \;\middle|\; \begin{array}{l} \ell \text{ is locally constant, and} \\ \ell(v + \lambda) = e_\lambda^{-1} \ell(v) \text{ for } \lambda \in \Lambda \end{array} \right\}.$$

Now the operator $(p_1)_* \nabla^\natural$ takes a section $\sigma \colon \pi^{-1}(U) \times \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathbb{C}$ of $(p_1)_* P^\natural$ to the section $(d_V + d_V F)(\sigma)$ of $\Omega^1_A \otimes (p_1)_* P^\natural$. This shows that σ lies in the kernel of $(p_1)_* \nabla^\natural$ exactly when $s = e^F \sigma$ is locally constant in its first argument. Remembering that $F(v, f) = f(v)$, we compute that

$$s(v + \lambda, f + \phi) = e^{(f + \phi)(v + \lambda)} e^{-\phi(v)} \sigma(v, f) = e^{f(\lambda)} s(v, f).$$

Consequently, $s(v, -)$ descends to a global holomorphic function on $\operatorname{Char}(A)$, and so we get an element $\ell(v) \in R$. The resulting function $\ell \colon \pi^{-1}(U) \to R$ is locally constant and satisfies $\ell(v+\lambda) = e_\lambda^{-1}\ell(v)$, which means that $\ell \in \mathcal{L}_R(U)$. Because this process is clearly reversible, we obtain the asserted isomorphism. □

We can now complete the proof of Theorem 15.2 by imitating the argument from the set-theoretic result about cohomology support loci.

Proof of Theorem 15.2. Let $(\mathcal{M}^\bullet, d)$ be the given holonomic complex; to shorten the notation, set $K = \mathrm{DR}_A(\mathcal{M}^\bullet)$. Our first task is to construct a morphism from the complex of $\mathbb{C}[\Lambda]$-modules $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$ to the complex of R-modules

$$\mathbf{R}p_* \mathbf{R}\Phi_* \operatorname{FM}_A(\mathcal{M}^\bullet) \simeq \mathbf{R}p_* \operatorname{FM}_A(\mathcal{M}^\bullet)$$

here and elsewhere in the proof, p always denotes the mapping from a complex manifold to a point. From the definition of the Fourier-Mukai transform,

$$\begin{aligned}\mathbf{R}p_* \operatorname{FM}_A(\mathcal{M}^\bullet) &\simeq \mathbf{R}p_*\Big(\mathbf{R}(p_2)_* \operatorname{DR}_{A\times A^\natural/A^\natural}\big(p_1^*\mathcal{M}^\bullet \otimes_{\mathscr{O}_{A\times A^\natural}} (P^\natural, \nabla^\natural)\big)\Big)\\ &\simeq \mathbf{R}p_*\Big(\mathbf{R}(p_1)_* \operatorname{DR}_{A\times A^\natural/A^\natural}\big(p_1^*\mathcal{M}^\bullet \otimes_{\mathscr{O}_{A\times A^\natural}} (P^\natural, \nabla^\natural)\big)\Big).\end{aligned}$$

Because $\Omega^{g+i}_{A\times A^\natural/A^\natural} \simeq p_1^*\Omega_A^{g+i}$, we have a collection of morphisms

$$\Omega_A^{g+i} \otimes_{\mathscr{O}_A} \mathcal{M}^j \otimes_{\mathscr{O}_A} (p_1)_*P^\natural \to (p_1)_*\Big(\Omega^{g+i}_{A\times A^\natural/A^\natural} \otimes_{\mathscr{O}_{A\times A^\natural}} p_1^*\mathcal{M}^j \otimes_{\mathscr{O}_{A\times A^\natural}} P^\natural\Big).$$

According to Lemma 15.3, the subsheaf $\ker (p_1)_*\nabla^\natural$ of $(p_1)_*P^\natural$ is isomorphic to $\mathcal{L}_R$; by composing with the inclusion of $\mathcal{L}_{\mathbb{C}[\Lambda]}$ into $\mathcal{L}_R$, we obtain morphisms

$$\Omega_A^{g+i} \otimes_{\mathscr{O}_A} \mathcal{M}^j \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]} \to (p_1)_*\Big(\Omega^{g+i}_{A\times A^\natural/A^\natural} \otimes_{\mathscr{O}_{A\times A^\natural}} p_1^*\mathcal{M}^j \otimes_{\mathscr{O}_{A\times A^\natural}} P^\natural\Big).$$

These morphisms are clearly compatible with the differentials in both double complexes, and so we obtain a morphism

$$\operatorname{DR}_A(\mathcal{M}^\bullet) \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]} \to (p_1)_* \operatorname{DR}_{A\times A^\natural/A^\natural}\big(p_1^*\mathcal{M}^\bullet \otimes_{\mathscr{O}_{A\times A^\natural}} (P^\natural, \nabla^\natural)\big)$$

between the associated single complexes. Putting everything together, we get a canonical morphism of complexes of $\mathbb{C}[\Lambda]$-modules

$$\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \to \mathbf{R}p_*\mathbf{R}\Phi_* \operatorname{FM}_A(\mathcal{M}^\bullet).$$

In the derived category, it induces a morphism from the complex of coherent analytic sheaves on $\operatorname{Char}(A)$ determined by $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$ to the complex of coherent analytic sheaves $\mathbf{R}\Phi_* \operatorname{FM}_A(\mathcal{M}^\bullet)$; this can be seen, for example, by representing the former by a bounded complex of free $\mathbb{C}[\Lambda]$-modules.

To conclude the proof, we have to show that the morphism is an isomorphism in the derived category. This is equivalent to the cone of the morphism being acyclic, and is therefore a local problem. Because the local ring at every point of the complex manifold $\operatorname{Char}(A)$ is noetherian, and because both complexes are objects of $\mathrm{D}^b_{coh}(\mathscr{O}_{\operatorname{Char}(A)})$, we can apply Nakayama's lemma; it is therefore enough to show that the restriction to every point $\rho \in \operatorname{Char}(A)$ is an isomorphism. But by Corollary 14.5 and base change, this restriction agrees with the morphism in the proof of Theorem 15.1; it is a quasi-isomorphism by the argument given there. □

16. Structure theorem. The goal of this section is to prove Theorem 2.2 and Theorem 7.3, which together describe the structure of cohomology support loci for holonomic complexes of $\mathscr{D}_A$-modules and for constructible complexes of sheaves of $\mathbb{C}$-vector spaces.

Our proof is based on the observation that $\Phi\colon A^\natural \to \mathrm{Char}(A)$ is an isomorphism of complex Lie groups, but not of complex algebraic varieties. The most striking difference between the two sides is that $A^\natural$ does not have any non-constant global algebraic functions [Lau96, Théorème 2.4.1], whereas $\mathrm{Char}(A)$ is affine. Now linear subvarieties are clearly algebraic in both models, because

$$\Phi\Bigl((L,\nabla)\otimes \mathrm{im}\bigl(f^\natural\colon B^\natural \to A^\natural\bigr)\Bigr) = \Phi(L,\nabla)\cdot \mathrm{im}\bigl(\mathrm{Char}(f)\colon \mathrm{Char}(B)\to \mathrm{Char}(A)\bigr).$$

The content of the following result by Simpson [Sim93, Theorem 3.1] is that finite unions of linear subvarieties are the only closed subsets with this property.

Theorem 16.1 (Simpson). *Let Z be a closed algebraic subset of $A^\natural$. If $\Phi(Z)$ is again a closed algebraic subset of $\mathrm{Char}(A)$, then Z is a finite union of linear subvarieties of $A^\natural$, and $\Phi(Z)$ is a finite union of linear subvarieties of $\mathrm{Char}(A)$.*

In Theorem 14.6, we have already seen that cohomology support loci are algebraic subsets of $\mathrm{Char}(A)$. With the help of the Fourier-Mukai transform, it is easy to show that they are also algebraic subsets of $A^\natural$.

Proposition 16.2. *If $\mathcal{M}\in \mathrm{D}^b_{coh}(\mathscr{D}_A)$, then the cohomology support loci $S^k_m(A,\mathcal{M})$ are algebraic subsets of $A^\natural$.*

Proof. Since $A^\natural$ is a quasi-projective algebraic variety, we may represent $\mathrm{FM}_A(\mathcal{M})$ by a bounded complex $(\mathscr{E}^\bullet, d)$ of locally free sheaves on $A^\natural$. Now let (L,∇) be a line bundle with integrable connection, and let $i_{(L,\nabla)}$ denote the inclusion morphism. By the base change theorem,

$$\mathbf{R}i^*_{(L,\nabla)}\,\mathrm{FM}_A(\mathcal{M}) \simeq \mathrm{DR}_A\bigl(\mathcal{M}\otimes_{\mathscr{O}_A}(L,\nabla)\bigr),$$

and so we have

$$S^k_m(A,\mathcal{M}) = \Bigl\{\,(L,\nabla)\in A^\natural \;\Bigm|\; \dim \mathbf{H}^k\bigl(i^*_{(L,\nabla)}(\mathscr{E}^\bullet,d)\bigr)\geq m\,\Bigr\}.$$

Because of Lemma 14.7, this description shows that $S^k_m(A,\mathcal{M})$ is an algebraic subset of $A^\natural$, as claimed. □

We can now prove the two structure theorems from the introduction.

Proof of Theorem 2.2. Let $\mathcal{M}\in \mathrm{D}^b_h(\mathscr{D}_A)$ be a holonomic complex. Then $\mathrm{DR}_A(\mathcal{M})$ is constructible, and we have

$$\Phi\bigl(S^k_m(A,\mathcal{M})\bigr) = S^k_m\bigl(A,\mathrm{DR}_A(\mathcal{M})\bigr)$$

by Theorem 15.1. Proposition 16.2 shows that $S^k_m(A,\mathcal{M})$ is an algebraic subset of $A^\natural$; Theorem 14.6 shows that $S^k_m\bigl(\mathrm{DR}_A(\mathcal{M})\bigr)$ is an algebraic subset of $\mathrm{Char}(A)$. We conclude from Simpson's Theorem 16.1 that both must be finite unions of linear subvarieties of $A^\natural$ and $\mathrm{Char}(A)$, respectively. The assertion about objects of geometric origin is proved in Section 17 below. □

Proof of Theorem 3.2. This follows from Theorem 15.2 and Simpson's results by the same argument as the one just given. □

17. Objects of geometric origin. In this section, we study cohomology support loci for semisimple regular holonomic $\mathscr{D}_A$-modules of geometric origin, as defined in [BBD82, 6.2.4]. To begin with, recall the following definition for mixed Hodge modules, due to Saito [Sai91, Definition 2.6].

Definition 17.1. A mixed Hodge module is said to be *of geometric origin* if it is obtained by applying several of the standard cohomological functors $H^i f_*$, $H^i f_!$, $H^i f^*$, $H^i f^!$, ψ_g, $\phi_{g,1}$, $\mathbf{D}$, $\boxtimes$, $\oplus$, $\otimes$, and $\mathcal{H}om$ to the trivial Hodge structure $\mathbb{Q}^H$ of weight zero, and then taking subquotients in the category of mixed Hodge modules.

One of the results of Saito's theory is that any semisimple perverse sheaf of geometric origin, in the sense of [BBD82, 6.2.4], is a direct summand of a perverse sheaf underlying a mixed Hodge module of geometric origin. Consequently, any semisimple regular holonomic $\mathscr{D}$-module of geometric origin is a direct summand of a $\mathscr{D}$-module that underlies a polarizable Hodge module of geometric origin.

Theorem 17.2. *Let $\mathcal{M}$ be a semisimple regular holonomic $\mathscr{D}_A$-module of geometric origin. Then each cohomology support locus $S^k_m(A, \mathcal{M})$ is a finite union of arithmetic linear subvarieties of $A^\natural$.*

We introduce some notation that will be used during the proof. For any field automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, we obtain from A a new complex abelian variety A^σ. Likewise, an algebraic line bundle (L, ∇) with integrable connection on A gives rise to $(L^\sigma, \nabla^\sigma)$ on A^σ, and so we have a natural isomorphism of abelian groups

$$c_\sigma \colon A^\natural \to (A^\sigma)^\natural.$$

Now recall the following notion, due in a slightly different form to Simpson, who modeled it on Deligne's definition of absolute Hodge classes.

Definition 17.3. A closed subset $Z \subseteq A^\natural$ is said to be *absolute closed* if, for every field automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, the set

$$\Phi\bigl(c_\sigma(Z)\bigr) \subseteq \operatorname{Char}(A^\sigma)$$

is closed and defined over $\bar{\mathbb{Q}}$.

The following theorem about absolute closed subsets is also due to Simpson.

Theorem 17.4 (Simpson)**.** *An absolute closed subset of $A^\natural$ is a finite union of arithmetic linear subvarieties.*

Proof. Simpson's definition [Sim93, p. 376] of absolute closed sets actually contains several additional conditions (related to the space of Higgs bundles); but as he explains, a strengthening of [Sim93, Theorem 3.1], added in proof, makes these conditions unnecessary. In fact, the proof of [Sim93, Theorem 6.1] goes through unchanged with only the assumptions in Definition 17.3. □

With the help of Simpson's result, the proof of Theorem 17.2 is straightforward. We first establish the following lemma.

Lemma 17.5. *Let $M \in \operatorname{MHM}(A)$ be a mixed Hodge module, with underlying filtered $\mathscr{D}_A$-module $(\mathcal{M}, F)$. Then the cohomology support loci of the perverse sheaf $\operatorname{DR}_A(\mathcal{M})$ are algebraic subsets of $\operatorname{Char}(A)$ that are defined over $\bar{\mathbb{Q}}$.*

Proof. By definition, a mixed Hodge module has an underlying perverse sheaf $\operatorname{rat} M$ with coefficients in $\mathbb{Q}$, and $\mathrm{DR}_A(\mathcal{M}) \simeq (\operatorname{rat} M) \otimes_{\mathbb{Q}} \mathbb{C}$. By Proposition 14.8, it follows that $\mathbf{R}p_*\big(\mathrm{DR}_A(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big) \in \mathrm{D}^b_{coh}(\mathbb{C}[\Lambda])$ is obtained by extension of scalars from an object of $\mathrm{D}^b_{coh}\big(\mathbb{Q}[\Lambda]\big)$. The assertion about cohomology support loci now follows easily from Corollary 14.5. ☐

The same result is true for any holonomic $\mathscr{D}_A$-module with $\bar{\mathbb{Q}}$-structure; that is to say, for any holonomic $\mathscr{D}_A$-module whose de Rham complex is the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. This is what Mochizuki calls a "pre-Betti structure" in [Moc10].

Lemma 17.6. *Let $K \in \mathrm{D}^b_c(\bar{\mathbb{Q}}_A)$ be a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. Any simple subquotient of $K \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ is the complexification of a simple subquotient of K.*

Proof. We only have to show that if $K \in \mathrm{D}^b_c(\bar{\mathbb{Q}}_A)$ is a simple perverse sheaf, then $K \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \in \mathrm{D}^b_c(\mathbb{C}_A)$ is also simple. By the classification of simple perverse sheaves, there is an irreducible locally closed subvariety $X \subseteq A$, and an irreducible representation $\rho\colon \pi_1(X) \to \mathrm{GL}_n(\bar{\mathbb{Q}})$, such that K is the intermediate extension of the local system associated with ρ. Since $\bar{\mathbb{Q}}$ is algebraically closed, ρ remains irreducible over $\mathbb{C}$, proving that $K \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ is still simple. ☐

Proof of Theorem 17.2. We first show that this holds when $\mathcal{M}$ underlies a mixed Hodge module M obtained by iterating the standard cohomological functors (but without taking subquotients). Fix two integers k, m, and set $Z = S^k_m(A, \mathcal{M})$. In light of Lemma 17.5, it suffices to prove that each set $c_\sigma(Z)$ is equal to $S^k_m(A^\sigma, \mathcal{M}_\sigma)$ for some polarizable Hodge module $M_\sigma \in \mathrm{MHM}(A^\sigma)$. But since M is of geometric origin, this is obviously the case; indeed, we can obtain M_σ by simply applying σ to the finitely many algebraic varieties and morphisms involved in the construction of M.

Now suppose that $\mathcal{M}$ is an arbitrary semisimple regular holonomic $\mathscr{D}_A$-module of geometric origin. Then $\mathcal{M}$ is a direct sum of simple subquotients of $\mathscr{D}_A$-modules underlying mixed Hodge modules of geometric origin. By the same argument as before, it suffices to show that the perverse sheaf $\mathrm{DR}_A(\mathcal{M}_\sigma)$ is defined over $\bar{\mathbb{Q}}$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Now $\mathrm{DR}_A(\mathcal{M}_\sigma)$ is again a direct sum of simple subquotients of perverse sheaves underlying mixed Hodge modules; by Lemma 17.6, it is therefore the complexification of a perverse sheaf with coefficients in $\bar{\mathbb{Q}}$. We then conclude the proof as above. ☐

D. Codimension bounds and perverse coherent sheaves

After a brief review of perverse coherent sheaves, we investigate how the Fourier-Mukai transform for holonomic complexes interacts with various t-structures.

18. Perverse coherent sheaves. Let X be a smooth complex algebraic variety. In this section, we recall the construction of perverse t-structures on the bounded derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ of coherent algebraic sheaves, following [Kas04]. For a (possibly non-closed) point x of the scheme X, we denote the residue field at the point by $\kappa(x)$, the inclusion morphism by $i_x\colon \operatorname{Spec} \kappa(x) \hookrightarrow X$, and the codimension of the closed subvariety $\overline{\{x\}}$ by $\operatorname{codim}(x) = \dim \mathscr{O}_{X,x}$.

Definition 18.1. A *supporting function* on X is a function $p\colon X \to \mathbb{Z}$ from the underlying topological space of the scheme X to the set of integers, with the property that $p(y) \geq p(x)$ whenever $y \in \overline{\{x\}}$.

Given a supporting function, Kashiwara defines two families of subcategories

$$
\begin{aligned}
{}^p\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_X) &= \Big\{ F \in \mathrm{D}^b_{coh}(\mathscr{O}_X) \;\Big|\; \mathbf{L}i_x^* F \in \mathrm{D}^{\leq k+p(x)}_{coh}\big(\kappa(x)\big) \text{ for all } x \in X \Big\},\\
{}^p\mathrm{D}^{\geq k}_{coh}(\mathscr{O}_X) &= \Big\{ F \in \mathrm{D}^b_{coh}(\mathscr{O}_X) \;\Big|\; \mathbf{R}i_x^! F \in \mathrm{D}^{\geq k+p(x)}_{coh}\big(\kappa(x)\big) \text{ for all } x \in X \Big\}.
\end{aligned}
$$

The following fundamental result is proved in [Kas04, Theorem 5.9] and, based on a suggestion by Deligne, in [AB10, Theorem 3.10].

Theorem 18.2 (Kashiwara, Arinkin-Bezrukavnikov)**.** *The above subcategories define a bounded t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ if and only if the supporting function has the property that*

$$p(y) - p(x) \leq \operatorname{codim}(y) - \operatorname{codim}(x)$$

for every pair of (possibly non-closed) points $x, y \in X$ with $y \in \overline{\{x\}}$.

For example, the function $p = 0$ corresponds to the standard t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. An equivalent way of putting the condition in Theorem 18.2 is that the dual function $\hat{p}(x) = \operatorname{codim}(x) - p(x)$ should again be a supporting function. If that is the case, one has the identities

$$
\begin{aligned}
{}^{\hat{p}}\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_X) &= \mathbf{R}\mathcal{H}om\Big({}^p\mathrm{D}^{\geq -k}_{coh}(\mathscr{O}_X), \mathscr{O}_X\Big)\\
{}^{\hat{p}}\mathrm{D}^{\geq k}_{coh}(\mathscr{O}_X) &= \mathbf{R}\mathcal{H}om\Big({}^p\mathrm{D}^{\leq -k}_{coh}(\mathscr{O}_X), \mathscr{O}_X\Big),
\end{aligned}
$$

which means that the duality functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_X)$ exchanges the two perverse t-structures defined by p and $\hat{p}$.

Definition 18.3. The heart of the t-structure defined by p is denoted

$${}^p\mathrm{Coh}(\mathscr{O}_X) = {}^p\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_X) \cap {}^p\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X),$$

and is called the abelian category of *p-perverse coherent sheaves.*

We are interested in a special case of Theorem 18.2, namely the set of objects $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ with $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(F) \geq 2i$ for all $i \geq 0$. Define a function

$$m\colon X \to \mathbb{Z}, \quad m(x) = \left\lfloor \tfrac{1}{2} \operatorname{codim}(x) \right\rfloor.$$

It is easily verified that both m and the dual function

$$\hat{m}\colon X \to \mathbb{Z}, \quad \hat{m}(x) = \left\lceil \tfrac{1}{2} \operatorname{codim}(x) \right\rceil$$

are supporting functions. As a consequence of Theorem 18.2, m defines a bounded t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; objects in the heart ${}^m\mathrm{Coh}(\mathscr{O}_X)$ will be called *m-perverse coherent sheaves.*[5]

The next lemma follows easily from [Kas04, Lemma 5.5].

Lemma 18.4. *The perverse t-structures defined by m and $\hat{m}$ satisfy*

$$
\begin{aligned}
{}^m\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_X) &= \Big\{ F \in \mathrm{D}^b_{coh}(\mathscr{O}_X) \;\Big|\; \operatorname{codim}\operatorname{Supp}\mathcal{H}^i(F) \geq 2(i-k) \text{ for all } i \in \mathbb{Z} \Big\}\\
{}^{\hat{m}}\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_X) &= \Big\{ F \in \mathrm{D}^b_{coh}(\mathscr{O}_X) \;\Big|\; \operatorname{codim}\operatorname{Supp}\mathcal{H}^i(F) \geq 2(i-k) - 1 \text{ for all } i \in \mathbb{Z} \Big\}.
\end{aligned}
$$

By duality, this also describes the subcategories with $\geq k$.

Consequently, an object $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ is an m-perverse coherent sheaf precisely when $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(F) \geq 2i$ and $\operatorname{codim} \operatorname{Supp} R^i\mathcal{H}om(F, \mathscr{O}_X) \geq 2i-1$ for every integer $i \geq 0$. This shows one more time that the category of m-perverse coherent sheaves is not preserved by the duality functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_X)$.

Lemma 18.5. *If $F \in {}^m\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$ or $F \in {}^{\hat{m}}\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$, then $F \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$.*

Proof. This is obvious from the fact that $\hat{m} \geq m \geq 0$. □

When it happens that both F and $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_X)$ are m-perverse coherent sheaves, F has surprisingly good properties.

Proposition 18.6. *If $F \in {}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_X)$ also satisfies $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_X) \in {}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_X)$, then it has the following properties:*

(i) Both F and $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_X)$ belong to ${}^m\mathrm{Coh}(\mathscr{O}_X)$.
(ii) Let $r \geq 0$ be the least integer with $\mathcal{H}^r(F) \neq 0$; then every irreducible component of $\operatorname{Supp} \mathcal{H}^r(F)$ has codimension $2r$.
(iii) If $\mathcal{H}^0(F)$ is nonzero, then it is a torsion-free sheaf on X.

Proof. The first assertion follows directly from Lemma 18.4. To prove the second assertion, note that we have $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r(F) \geq 2r$. After restricting to an open neighborhood of the generic point of any given irreducible component, it therefore suffices to show that if $\mathcal{H}^i(F) = 0$ for $i < r$, and $\operatorname{codim} \operatorname{Supp} \mathcal{H}^r(F) > 2r$, then $\mathcal{H}^r(F) = 0$. Under these assumptions, we have

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(F) \geq \max(2i, 2r+1) \geq i + r + 1,$$

and therefore $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_X) \in \mathrm{D}^{\geq r+1}_{coh}(\mathscr{O}_X)$ by [Kas04, Proposition 4.3]. The same argument, applied to $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_X)$, now shows that $F \in \mathrm{D}^{\geq r+1}_{coh}(\mathscr{O}_X)$.

Now suppose in addition that $\mathcal{H}^0(F) \neq 0$. Because X is nonsingular, $\mathcal{H}^0(F)$ being torsion-free is equivalent to the inequalities

$$\operatorname{codim} \operatorname{Supp} R^i\mathcal{H}om\big(\mathcal{H}^0(F), \mathscr{O}_X\big) \geq i+1 \quad \text{for every } i \geq 1.$$

To prove these inequalities, let $\tau_{\leq n}$ and $\tau_{\geq n}$ denote the truncation functors with respect to the standard t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; we have a distinguished triangle

$$\big(\tau_{\geq 1}F\big)[-1] \to \mathcal{H}^0(F) \to F \to \tau_{\geq 1}F.$$

After applying the functor $\mathbf{R}\mathcal{H}om(-, \mathscr{O}_X)$, we obtain an exact sequence

$$\cdots \to R^i\mathcal{H}om(F, \mathscr{O}_X) \to R^i\mathcal{H}om\big(\mathcal{H}^0(F), \mathscr{O}_X\big) \to R^{i+1}\mathcal{H}om(\tau_{\geq 1}F, \mathscr{O}_X) \to \cdots$$

By assumption, the support of $R^i\mathcal{H}om(F, \mathscr{O}_X)$ has codimension $\geq 2i$; moreover, the support of $R^{i+1}\mathcal{H}om(\tau_{\geq 1}F, \mathscr{O}_X)$ has codimension $\geq i+2$ by [Kas04, Proposition 4.3]. This is enough to conclude that $\mathcal{H}^0(F)$ is torsion-free. □

19. Description of the t-structure. In this section, we show that the standard t-structure on $\mathrm{D}^b_h(\mathscr{D}_A)$ corresponds, under the Fourier-Mukai transform FM_A, to the m-perverse t-structure.

Theorem 19.1. *Let $\mathcal{M} \in \mathrm{D}^b_h(\mathscr{D}_A)$ be a holonomic complex on A. Then one has*

$$\mathcal{M} \in \mathrm{D}^{\leq k}_h(\mathscr{D}_A) \iff \mathrm{FM}_A(\mathcal{M}) \in {}^m\mathrm{D}^{\leq k}_{coh}(\mathscr{O}_{A^\natural}),$$
$$\mathcal{M} \in \mathrm{D}^{\geq k}_h(\mathscr{D}_A) \iff \mathrm{FM}_A(\mathcal{M}) \in {}^m\mathrm{D}^{\geq k}_{coh}(\mathscr{O}_{A^\natural}).$$

In particular, $\mathcal{M}$ is a single holonomic $\mathscr{D}_A$-module if and only if its cohomology support loci satisfy $\operatorname{codim} S^k_1(A, \mathcal{M}) \geq |2k|$ for every $k \in \mathbb{Z}$.

The first step of the proof consists in the following "generic vanishing theorem" for holonomic $\mathscr{D}_A$-modules. In the regular case, this result is due to Krämer and Weissauer [KW11, Theorem 1.1], whose argument relies on the (difficult) recent proof of Kashiwara's conjecture for semisimple perverse sheaves. By contrast, our argument is completely elementary.

Proposition 19.2. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}_A$-module. Then for every $i > 0$, the support of the coherent sheaf $\mathcal{H}^i \operatorname{FM}_A(\mathcal{M})$ is a proper subset of $A^\natural$.*

Proof. Let $F_\bullet\mathcal{M}$ be a good filtration by $\mathscr{O}_A$-coherent subsheaves; this exists by [HTT08, Theorem 2.3.1]. As in Section 13, we consider the associated coherent $\mathscr{R}_A$-module $R_F\mathcal{M}$ defined by the Rees construction, and its Fourier-Mukai transform

$$\widetilde{\operatorname{FM}}_A(R_F\mathcal{M}) \in \mathrm{D}^b_{coh}\big(\mathscr{O}_{E(A)}\big).$$

By Proposition 13.1, $\widetilde{\operatorname{FM}}_A(R_F\mathcal{M})$ is equivariant for the $\mathbb{C}^*$-action on $E(A)$, and for any $z \neq 0$, its restriction to $\lambda^{-1}(z) \simeq A^\natural$ is isomorphic to $\operatorname{FM}_A(\mathcal{M})$. It is therefore sufficient to prove that the restriction of $\widetilde{\operatorname{FM}}_A(R_F\mathcal{M})$ to $\lambda^{-1}(0) = A \times H^0(A, \Omega_A^1)$ has the asserted property. By Proposition 13.1, this restriction is isomorphic to

$$\mathbf{R}(p_{23})_*\Big(p_{12}^*P \otimes p_{13}^*(\mathrm{id} \times \iota)^* \operatorname{gr}^F\mathcal{M} \otimes p_1^*\Omega_A^g\Big), \tag{19.3}$$

where the notation is as in (11.8). Now $\mathcal{M}$ is holonomic, and so each irreducible component of $\operatorname{Supp}(\operatorname{gr}^F\mathcal{M})$ has dimension g. The support of $p_{13}^*(\mathrm{id} \times \iota)^* \operatorname{gr}^F\mathcal{M}$ therefore has the same dimension as $\hat{A} \times H^0(A, \Omega_A^1)$, which implies that the support of the higher direct image sheaves in (19.3) is a proper subset of $\hat{A} \times H^0(A, \Omega_A^1)$. □

Together with the structure theory for cohomology support loci and basic properties of the Fourier-Mukai transform, this result now allows us to prove the first equivalence asserted in Theorem 19.1.

Lemma 19.4. *For any $\mathcal{M} \in \mathrm{D}_h^{\leq k}(\mathscr{D}_A)$, one has $\operatorname{FM}_A(\mathcal{M}) \in {}^m\mathrm{D}_{coh}^{\leq k}(\mathscr{O}_{A^\natural})$.*

Proof. The proof is by induction on $\dim A$, the statement being obviously true when A is a point. Since FM_A is triangulated, it suffices to prove the statement for $k = 0$. According to Lemma 18.4, what we then need to show is the following: for any holonomic complex $\mathcal{M} \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A)$ concentrated in nonpositive degrees, the Fourier-Mukai transform $\operatorname{FM}_A(\mathcal{M})$ satisfies, for every $\ell \geq 1$, the inequality

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^\ell \operatorname{FM}_A(\mathcal{M}) \geq 2\ell.$$

Let Z be any irreducible component of $\operatorname{Supp} \mathcal{H}^\ell \operatorname{FM}_A(\mathcal{M})$, for some $\ell \geq 1$. By Theorem 3.2, Z is a linear subvariety of $A^\natural$, hence of the form $Z = t_{(L,\nabla)}(\operatorname{im} f^\natural)$ for a surjective morphism $f\colon A \to B$ and a suitable point $(L, \nabla) \in A^\natural$. Furthermore, Proposition 19.2 shows that $\operatorname{codim} Z > 0$, and therefore $\dim B < \dim A$. Setting $r = \dim A - \dim B > 0$, we thus have $\operatorname{codim} Z = 2r$.

Using the properties of the Fourier-Mukai transform listed in Theorem 10.5, we find that the pullback of $\operatorname{FM}_A(\mathcal{M})$ to the subvariety Z is isomorphic to

$$\mathbf{L}(f^\natural)^*\mathbf{L}t_{(L,\nabla)}^* \operatorname{FM}_A(\mathcal{M}) \simeq \operatorname{FM}_B\Big(f_+\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big)\Big) \in \mathrm{D}^b_{coh}(\mathscr{O}_{B^\natural}).$$

From the definition of the direct image functor f_+, it is clear that $f_+\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big)$ belongs to the subcategory $\mathrm{D}_h^{\leq r}(\mathscr{D}_B)$. The inductive assumption now allows us to conclude that the restriction of $\operatorname{FM}_A(\mathcal{M})$ to Z lies in the subcategory ${}^m\mathrm{D}_{coh}^{\leq r}(\mathscr{O}_Z)$.

But Z is an irreducible component of $\operatorname{Supp}\mathcal{H}^\ell \operatorname{FM}_A(\mathcal{M})$; it follows that $\ell \leq r$, and consequently $\operatorname{codim} Z \geq 2\ell$, as asserted. □

Lemma 19.5. *Let $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$ be a holonomic complex. If its Fourier-Mukai transform satisfies $\operatorname{FM}_A(\mathcal{M}) \in \mathrm{D}_{coh}^{\leq k}(\mathscr{O}_{A^\natural})$, then $\mathcal{M} \in \mathrm{D}_h^{\leq k}(\mathscr{D}_A)$.*

Proof. It again suffices to prove this for $k = 0$. By [Lau96, Théorème 3.2.1], we can recover $\mathcal{M}$ – up to canonical isomorphism – from its Fourier-Mukai transform as

$$\mathcal{M} = \langle -1_A \rangle^* \mathbf{R}(p_1)_* \Big(P^\natural \otimes_{\mathscr{O}_{A \times A^\natural}} p_2^* \operatorname{FM}_A(\mathcal{M}) \Big)[g],$$

where $p_1 \colon A \times A^\natural \to A$ and $p_2 \colon A \times A^\natural \to A^\natural$ are the two projections. If we forget about the $\mathscr{D}_A$-module structure and only consider $\mathcal{M}$ as a complex of quasi-coherent sheaves of $\mathscr{O}_A$-modules, we can use the fact that $\pi \colon A^\natural \to A$ is affine to obtain

$$\mathcal{M} = \langle -1_A \rangle^* \mathbf{R}(p_1)_* \Big(P \otimes_{\mathscr{O}_{A \times \hat{A}}} p_2^* \pi_* \operatorname{FM}_A(\mathcal{M}) \Big)[g],$$

where now $p_1 \colon A \times \hat{A} \to A$ and $p_2 \colon A \times \hat{A} \to \hat{A}$. By virtue of Theorem 3.2, every irreducible component of $\operatorname{Supp}\mathcal{H}^\ell \operatorname{FM}_A(\mathcal{M})$ is contained in a linear subvariety of codimension at least 2ℓ; consequently, every irreducible component of $\operatorname{Supp} \pi_* \mathcal{H}^\ell \operatorname{FM}_A(\mathcal{M})$ still has codimension at least ℓ. From this, it is easy to see that $\mathcal{H}^i \mathcal{M} = 0$ for $i > 0$, and hence that $\mathcal{M} \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A)$. □

Proof of Theorem 19.1. The first equivalence is proved in Lemma 19.4 and Lemma 19.5 above. The second equivalence follows from this by duality, using the compatibility of the Fourier-Mukai transform with the duality functors for $\mathscr{D}_A$-modules and $\mathscr{O}_{A^\natural}$-modules (see Theorem 10.5). □

20. Stability under truncation. If the proposal in Section 6 has merit, and Fourier-Mukai transforms of holonomic complexes are indeed "hyperkähler constructible complexes", then a complex of coherent sheaves on $A^\natural$ should belong to the subcategory

$$\operatorname{FM}_A\big(\mathrm{D}_h^b(\mathscr{D}_A)\big) \subseteq \mathrm{D}_{coh}^b(\mathscr{O}_{A^\natural})$$

if and only if all of its cohomology sheaves do. This is because "constructibility" of a complex should be defined in terms of the cohomology sheaves.

In this section, we prove that this is indeed the case. The result is that the subcategory $\operatorname{FM}_A\big(\mathrm{D}_h^b(\mathscr{D}_A)\big)$ is closed under the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ for the *standard* t-structure on $\mathrm{D}_{coh}^b(\mathscr{O}_{A^\natural})$. This is of course equivalent to the statement about cohomology sheaves, but more convenient for doing induction.

Theorem 20.1. *Let $F = \operatorname{FM}_A(\mathcal{M})$ for some $\mathcal{M} \in \mathrm{D}_h^b(\mathscr{D}_A)$. Then for every $n \in \mathbb{Z}$, the truncations $\tau_{\leq n} F$ and $\tau_{\geq n} F$ are again Fourier-Mukai transforms of holonomic complexes.*

Proof. It suffices to show the assertion for $\tau_{\leq 0} F$. Since $\operatorname{FM}_A(\tau_{\geq 1}\mathcal{M}) \in \mathrm{D}_{coh}^{\geq 1}(\mathscr{O}_{A^\natural})$ by Lemma 18.5, we may replace $\mathcal{M}$ by $\tau_{\leq 0}\mathcal{M}$ and assume without loss of generality that $\mathcal{M} \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A)$. Since $\operatorname{codim}\operatorname{Supp}\mathcal{H}^i F \geq 2i$, each $\mathcal{H}^i F$ with $i \geq 1$ is then a torsion sheaf, supported in a finite union of linear subvarieties of $A^\natural$. We shall argue that, by suitably modifying $\mathcal{M}$, it is possible to remove these torsion sheaves from the picture, leaving us with $\tau_{\leq 0} F$.

To measure the difference between F and $\tau_{\leq 0}F$, we introduce the set

$$N(F) = \bigcup_{i \geq 1} \operatorname{Supp} \mathcal{H}^i F,$$

which is again a finite union of linear subvarieties of $A^\natural$. Our goal is to reduce the size of the set $N(F)$, because $N(F)$ is empty if and only if $\tau_{\leq 0}F \simeq F$. Let $i\colon Z \hookrightarrow A^\natural$ be an irreducible component of $N(F)$ of maximal dimension, let $k \geq 1$ be the biggest integer such that Z is an irreducible component of $\operatorname{Supp} \mathcal{H}^k F$, and let $m \geq 1$ be the multiplicity of $\mathcal{H}^k F$ along Z. We shall now describe how to modify $\mathcal{M}$ in a way that reduces the value of m (and eventually also of k) but leaves the set $N(F) \cup \{Z\}$ invariant. After repeating this construction a number of times, we can remove Z from the set $N(F)$, possibly adding linear subvarieties of Z of lower dimension in the process. After finitely many steps, we thus arrive at $N(F) = \emptyset$, which is the desired outcome.

Since Z is a linear subvariety, we have $Z = t_{(L,\nabla)}(\operatorname{im} f^\natural)$, where $f\colon A \to B$ is a morphism of abelian varieties with connected fibers, and $(L, \nabla) \in A^\natural$; to simplify the notation, we shall assume that $(L, \nabla) = (\mathscr{O}_A, d)$. If we set $r = \dim A - \dim B$, then $\operatorname{codim} Z = 2r$; moreover, the fact that Z is an irreducible component of $\operatorname{Supp} \mathcal{H}^k F$ implies that $2r = \operatorname{codim} Z \geq 2k$, and hence that $r \geq k$.

Now consider the pullback $\mathbf{L}i^* F$ to the subvariety Z. By construction, the k-th cohomology sheaf of $\mathbf{L}i^* F$ is supported on all of Z, while all higher cohomology sheaves are torsion. By Theorem 10.5, we have $\mathbf{L}i^* F \simeq \operatorname{FM}_B(f_+\mathcal{M})$, and therefore $\mathcal{H}^k \operatorname{FM}_B(f_+\mathcal{M}) \neq 0$. If we now define

$$\mathcal{N} = \tau_{\geq k} f_+\mathcal{M} \in \mathrm{D}_h^{[k,r]}(\mathscr{D}_B),$$

we obtain a distinguished triangle

$$\operatorname{FM}_B(\tau_{\leq k-1} f_+\mathcal{M}) \to \operatorname{FM}_B(f_+\mathcal{M}) \to \operatorname{FM}_B(\mathcal{N}) \to \operatorname{FM}_B(\tau_{\leq k-1} f_+\mathcal{M})[1];$$

we conclude that $\mathcal{H}^k \operatorname{FM}_B(\mathcal{N})$ and $i^*\mathcal{H}^k F$ are isomorphic at the generic point of Z, and that $\mathcal{H}^i \operatorname{FM}_B(\mathcal{N})$ is a torsion sheaf for $i > k$.

The adjunction morphism $\mathcal{M} \to f^+ f_+\mathcal{M}$ induces a morphism $\mathcal{M} \to f^+\mathcal{N}$. We choose $\mathcal{M}' \in \mathrm{D}_h^b(\mathscr{D}_A)$ so as to have a distinguished triangle

$$\mathcal{M}' \to \mathcal{M} \to f^+\mathcal{N} \to \mathcal{M}'[1].$$

Since f is smooth of relative dimension r, we have $f^+\mathcal{N} \in \mathrm{D}_h^{[k-r,0]}(\mathscr{D}_A)$; consequently, $\mathcal{M}' \in \mathrm{D}_h^{\leq 1}(\mathscr{D}_A)$. Let $F' = \operatorname{FM}_A(\mathcal{M}')$; we claim that $N(F') \subseteq N(F)$, that $\tau_{\leq 0}F' \simeq \tau_{\leq 0}F$, and that the multiplicity of $\mathcal{H}^k F'$ along Z is strictly smaller than that of $\mathcal{H}^k F$.

The first part is obvious: by Theorem 10.5, we have $\operatorname{FM}_A(f^+\mathcal{N}) \simeq \mathbf{R}i_* \operatorname{FM}_B(\mathcal{N})$, and so the support of the Fourier-Mukai transform of $f^+\mathcal{N}$ is entirely contained in Z. Moreover, $\operatorname{FM}_A(f^+\mathcal{N})$ belongs to $\mathrm{D}_{coh}^{\geq k}(\mathscr{O}_{A^\natural})$; the support of $\mathcal{H}^k \operatorname{FM}_A(f^+\mathcal{N})$ is equal to Z, while all higher cohomology sheaves are supported in proper subvarieties of Z. In particular, $\tau_{\leq 0}F' \simeq \tau_{\leq 0}F$.

To prove the assertion about the multiplicity, observe that we have an exact sequence

$$0 \to \mathcal{H}^k F' \to \mathcal{H}^k F \to \mathcal{H}^k \operatorname{FM}_A(f^+\mathcal{N}).$$

Now $\mathcal{H}^k \operatorname{FM}_A(f^+\mathcal{N}) \simeq i_*\mathcal{H}^k \operatorname{FM}_B(\mathcal{N})$ is isomorphic to $i^*\mathcal{H}^k F$ at the generic point of Z. After localizing at the generic point of Z, we can apply Lemma 20.2 below, which implies that the multiplicity of $\mathcal{H}^k F'$ along Z is strictly less than that of $\mathcal{H}^k F$.

To conclude the reduction step, we now set $\mathcal{M}'' = \tau_{\leq 0}\mathcal{M}'$ and $F'' = \mathrm{FM}_A(\mathcal{M}'')$. Then $\mathcal{M}'' \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A)$, we have $\tau_{\leq 0}F'' \simeq \tau_{\leq 0}F$, and while $N(F'') \subseteq N(F)$, the multiplicity of $\mathcal{H}^k F''$ along Z is strictly smaller than that of $\mathcal{H}^k F$. As explained above, this suffices to conclude the proof. □

Lemma 20.2. *Let $(R, \mathfrak{m})$ be a local ring, and let M a nonzero finitely generated R-module with $\mathfrak{m}^n M = 0$ for some $n \geq 1$. If we set $M' = \ker(M \to M/\mathfrak{m}M)$, then*

$$\dim_{R/\mathfrak{m}} M' = \dim_{R/\mathfrak{m}} M - \dim_{R/\mathfrak{m}} M/\mathfrak{m}M < \dim_{R/\mathfrak{m}} M.$$

In particular, the multiplicity of M' is strictly smaller than that of M.

Proof. Nakayama's lemma implies that $M/\mathfrak{m}M \neq 0$. □

Here are several immediate consequences of Theorem 20.1.

Corollary 20.3. *A complex of coherent sheaves on $A^\natural$ is the Fourier-Mukai transform of a holonomic complex if and only if all of its cohomology sheaves are.*

Proof. One direction is obvious because of Theorem 20.1; the other follows from the fact that $\mathrm{D}_h^b(\mathscr{D}_A)$ is a thick subcategory of $\mathrm{D}_{coh}^b(\mathscr{D}_A)$. □

Corollary 20.4. *Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_{A^\natural})$ is the Fourier-Mukai transform of a holonomic complex. Then the same is true for its reflexive hull.*

Proof. The reflexive hull $\mathcal{H}om\big(\mathcal{H}om(\mathscr{F}, \mathscr{O}), \mathscr{O}\big)$ is obtained from $\mathscr{F}$ by dualizing and truncating twice; both operations preserve the property of being the Fourier-Mukai transform of a holonomic complex. □

Corollary 20.5. *Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_{A^\natural})$ is the Fourier-Mukai transform of a holonomic complex. If $\mathscr{F}$ is reflexive, then there is a holonomic $\mathscr{D}_A$-module $\mathcal{M}$ such that $\mathscr{F} \simeq \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$.*

Proof. Let $\mathcal{N} \in \mathrm{D}_h^{\leq 0}(\mathscr{D}_A)$ be such that $\mathscr{F} = \mathrm{FM}_A(\mathcal{N})$. If we now define $\mathcal{M} = \mathcal{H}^0\mathcal{N}$, then the distinguished triangle

$$\mathrm{FM}_A(\tau_{\leq -1}\mathcal{N}) \to \mathscr{F} \to \mathrm{FM}_A(\mathcal{M}) \to \mathrm{FM}_A(\tau_{\leq -1}\mathcal{N})[1]$$

gives us an exact sequence

$$\mathcal{H}^0 \mathrm{FM}_A(\tau_{\leq -1}\mathcal{N}) \to \mathscr{F} \to \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M}) \to \mathcal{H}^1 \mathrm{FM}_A(\tau_{\leq -1}\mathcal{N}). \tag{20.6}$$

By Lemma 19.4, $\mathrm{FM}_A(\tau_{\leq -1}\mathcal{N})$ is an object of ${}^m\mathrm{D}_{coh}^{\leq -1}(\mathscr{O}_{A^\natural})$, and so the supports of the two sheaves on the outside have codimension ≥ 2 and ≥ 4, respectively. Because $\mathscr{F}$ is torsion-free, this forces $\mathscr{F} \to \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ to be injective. But then we know from Proposition 18.6 that $\mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$ is a torsion-free sheaf; putting everything together, it follows that $\mathscr{F} \simeq \mathcal{H}^0 \mathrm{FM}_A(\mathcal{M})$. □

E. Simple holonomic D-modules

This chapter is devoted to a more careful study of Fourier-Mukai transforms of simple holonomic $\mathscr{D}_A$-modules. In particular, we shall discover that they are intersection complexes for the m-perverse coherent t-structure, in a sense made precise below.

21. Classification by support. In this section, we prove a structure theorem for the Fourier-Mukai transform of a simple holonomic $\mathscr{D}_A$-module. The idea is that, in the case of a simple holonomic $\mathscr{D}$-module, the support of the Fourier-Mukai transform must be a single linear subvariety; and if that linear subvariety is not equal to $A^\natural$, then the $\mathscr{D}$-module in question is – up to tensoring by a line bundle – pulled back from a lower-dimensional abelian variety.

Theorem 21.1. *Let $\mathcal{M}$ be a simple holonomic $\mathscr{D}_A$-module, and let $r \geq 0$ be the least integer such that $\mathcal{H}^r\big(\mathrm{FM}_A(\mathcal{M})\big) \neq 0$. Then there is an abelian variety B of dimension $\dim B = \dim A - r$, a surjective morphism $f\colon A \to B$ with connected fibers, and a simple holonomic $\mathscr{D}_B$-module $\mathcal{N}$, such that*

$$\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla) \simeq f^*\mathcal{N}$$

for a suitable point $(L, \nabla) \in A^\natural$. Moreover, we have $\operatorname{Supp} \mathcal{H}^0\big(\mathrm{FM}_B(\mathcal{N})\big) = B^\natural$ and

$$\operatorname{Supp} \mathrm{FM}_A(\mathcal{M}) = (L, \nabla) \otimes \operatorname{im}\big(f^\natural\colon B^\natural \to A^\natural\big).$$

This result clearly implies Theorem 5.1 from the introduction. Here is the proof of the corollary about simple holonomic $\mathscr{D}_A$-modules with Euler characteristic zero.

Proof of Corollary 5.2. Let $(L, \nabla) \in A^\natural$ be a generic point. Because

$$0 = \chi(A, \mathcal{M}) = \chi\big(A, \mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big) = \dim \mathbf{H}^0\Big(A, \mathrm{DR}_A\big(\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla)\big)\Big),$$

we find that the support of $\mathcal{H}^0\, \mathrm{FM}_A(\mathcal{M})$ is a proper subset of $A^\natural$. Both $\mathrm{FM}_A(\mathcal{M})$ and the dual complex belong to ${}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_{A^\natural})$ by Theorem 19.1, and so we conclude from Proposition 18.6 that $\mathcal{H}^0\, \mathrm{FM}_A(\mathcal{M}) = 0$. Now it only remains to apply Theorem 21.1. ☐

For the proof of Theorem 21.1, we need two small lemmas. The first describes the inverse image of a simple holonomic $\mathscr{D}$-module.

Lemma 21.2. *Let $f\colon A \to B$ be a surjective morphism of abelian varieties, with connected fibers. If $\mathcal{N}$ is a simple holonomic $\mathscr{D}_B$-module, then $f^*\mathcal{N}$ is a simple holonomic $\mathscr{D}_A$-module.*

Proof. Since f is smooth, $f^*\mathcal{N} = \mathscr{O}_A \otimes_{f^{-1}\mathscr{O}_B} f^{-1}\mathcal{N}$ is a holonomic $\mathscr{D}_A$-module. According to the classification of simple holonomic $\mathscr{D}$-modules [HTT08, Theorem 3.4.2], there is a locally closed subvariety $X \subseteq B$, and an irreducible representation $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$, such that $\mathcal{N}$ is the minimal extension of the integrable connection on X associated with ρ. Now f has connected fibers, and so the map on fundamental groups

$$f_*\colon \pi_1\big(f^{-1}(X)\big) \to \pi_1(X)$$

is surjective. Clearly, the pullback $f^*\mathcal{N}$ is equal, over $f^{-1}(X)$, to the integrable connection associated with the representation $\rho \circ f_*\colon \pi_1\big(f^{-1}(X)\big) \to \pi_1(X) \to \mathrm{GL}(V)$. This representation is still irreducible because f_* is surjective; to conclude the proof, we shall argue that $f^*\mathcal{N}$ is the minimal extension.

By [HTT08, Theorem 3.4.2], it suffices to show that $f^*\mathcal{N}$ has no submodules or quotient modules that are supported outside of $f^{-1}(X)$. Suppose that $\mathcal{M} \hookrightarrow f^*\mathcal{N}$ is such a submodule. We have $f^+\mathcal{N} = f^*\mathcal{N}[r]$, where $r = \dim A - \dim B$; by adjunction, the morphism $\mathcal{M} \hookrightarrow f^*\mathcal{N}$ corresponds to a morphism $f_+\mathcal{M}[r] \to \mathcal{N}$, which factors uniquely as

$$f_+\mathcal{M}[r] \to \mathcal{H}^r f_+\mathcal{M} \to \mathcal{N}.$$

Since $\mathcal{H}^r f_+\mathcal{M}$ is supported outside of X, this morphism must be zero; consequently, $\mathcal{M} = 0$. A similar result for quotient modules can be derived by applying the duality functor, using [HTT08, Theorem 2.7.1]. This shows that $f^*\mathcal{N}$ is the minimal extension of a simple integrable connection, and hence simple. □

The second lemma deals with restriction to an irreducible component of the support of a complex.

Lemma 21.3. *Let X be a scheme, and let $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$. Suppose that Z is an irreducible component of the support of $\mathcal{H}^r(F)$ that is not contained in $\operatorname{Supp}\mathcal{H}^i(F)$ for any $i > r$. Let $i\colon Z \hookrightarrow X$ be the inclusion. Then the morphism*

$$\mathcal{H}^r(F) \to \mathcal{H}^r\big(\mathbf{R}i_*\mathbf{L}i^*F\big)$$

induced by adjunction is nonzero at the generic point of Z.

Proof. After localizing at the generic point of Z, we may assume that $X = \operatorname{Spec} R$ for a local ring $(R, \mathfrak{m})$, and that $F \in \mathrm{D}^b_{coh}(R)$ is represented by a complex

$$\cdots \longrightarrow F^{r-2} \stackrel{d}{\longrightarrow} F^{r-1} \stackrel{d}{\longrightarrow} F^r$$

of finitely generated free R-modules. Set $M = \mathcal{H}^r(F) = F^r/dF^{r-1}$, which is a finitely generated R-module with $M \neq 0$. Then $\mathcal{H}^r\big(\mathbf{R}i_*\mathbf{L}i^*F\big) \simeq M/\mathfrak{m}M$, and the morphism $M \to M/\mathfrak{m}M$ is nonzero by Nakayama's lemma. □

We can now prove our structure theorem for simple holonomic $\mathscr{D}_A$-modules.[6]

Proof of Theorem 21.1. Let $F = \mathrm{FM}_A(\mathcal{M}) \in \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$. Theorem 19.1 shows that $F \in {}^m\mathrm{Coh}(\mathscr{O}_{A^\natural})$; by duality, it follows that $\mathbf{R}\mathcal{H}om(F, \mathscr{O}_{A^\natural}) \in {}^m\mathrm{Coh}(\mathscr{O}_{A^\natural})$, too. According to Proposition 18.6, the codimension of the support of $\mathcal{H}^r(F)$ is therefore equal to $2r$ for some $r \geq 0$; moreover, each irreducible component of $\operatorname{Supp}\mathcal{H}^r(F)$ is a linear subvariety by Theorem 3.2. After tensoring $\mathcal{M}$ by a suitable line bundle with integrable connection, we may therefore assume that one irreducible component of the support of $\mathcal{H}^r(F)$ is equal to $\operatorname{im} f^\natural$, for a surjective morphism of abelian varieties $f\colon A \to B$ with connected fibers and $\dim B = \dim A - r$.

To produce the required simple $\mathscr{D}_B$-module, consider the direct image $f_+\mathcal{M}$, which belongs to $\mathrm{D}^{\leq r}_h(\mathscr{D}_B)$. We have a distinguished triangle

$$\tau_{\leq r-1}(f_+\mathcal{M}) \to f_+\mathcal{M} \to \mathcal{H}^r(f_+\mathcal{M})[-r] \to \cdots$$

in $\mathrm{D}^b_h(\mathscr{D}_B)$, and hence also a distinguished triangle

$$f^+\tau_{\leq r-1}(f_+\mathcal{M}) \to f^+f_+\mathcal{M} \to f^+\mathcal{H}^r(f_+\mathcal{M})[-r] \to \cdots \tag{21.4}$$

in $\mathrm{D}^b_h(\mathscr{D}_A)$. Since f is smooth, $f^+\mathcal{H}^r(f_+\mathcal{M})[-r] = f^*\mathcal{H}^r(f_+\mathcal{M})$ is a single holonomic $\mathscr{D}_A$-module. Let $\alpha\colon \mathcal{M} \to f^+f_+\mathcal{M}$ be the adjunction morphism.

Now we observe that the induced morphism $\mathcal{M} \to f^*\mathcal{H}^r(f_+\mathcal{M})$ must be nonzero. Indeed, suppose to the contrary that the morphism was zero. Then α factors as

$$\mathcal{M} \to f^+\tau_{\leq r-1}(f_+\mathcal{M}) \to f^+f_+\mathcal{M}.$$

If we apply the Fourier-Mukai transform to this factorization, and use the properties in Theorem 10.5, we obtain

$$F \to \mathbf{R}f^\natural_* \operatorname{FM}_B\big(\tau_{\leq r-1}(f_+\mathcal{M})\big) \to \mathbf{R}f^\natural_*\mathbf{L}(f^\natural)^*F,$$

which is a factorization of the adjunction morphism for the closed embedding $f^\natural$. In particular, we then have

$$\mathcal{H}^r(F) \to \mathcal{H}^r\Big(\mathbf{R}f^\natural_* \operatorname{FM}_B\big(\tau_{\leq r-1}(f_+\mathcal{M})\big)\Big) \to \mathcal{H}^r\big(\mathbf{R}f^\natural_* \mathbf{L}(f^\natural)^* F\big);$$

but because the coherent sheaf in the middle is supported in a subset of $\operatorname{im} f^\natural$ of codimension at least two, this contradicts Lemma 21.3. Therefore, $\mathcal{M} \to f^*\mathcal{H}^r(f_+\mathcal{M})$ is indeed nonzero.

Being a holonomic $\mathscr{D}_B$-module, $\mathcal{H}^r(f_+\mathcal{M})$ admits a finite filtration with simple quotients; consequently, we can find a simple holonomic $\mathscr{D}_B$-module $\mathcal{N}$ and a nonzero morphism $\mathcal{M} \to f^*\mathcal{N}$. Since $\mathcal{M}$ is simple, and $f^*\mathcal{N}$ is also simple by Lemma 21.2, the morphism must be an isomorphism, and so $\mathcal{M} \simeq f^*\mathcal{N}$.

To prove the final assertion, note that $f^*\mathcal{N} = f^+\mathcal{N}[-r]$; on account of Theorem 10.5, we therefore have

$$\operatorname{FM}_A(\mathcal{M}) \simeq \operatorname{FM}_A(f^*\mathcal{N}) \simeq \mathbf{R}f^\natural_* \operatorname{FM}_B(\mathcal{N})[-r].$$

Since $\operatorname{im} f^\natural$ is an irreducible component of the support of $\mathcal{H}^r\big(\operatorname{FM}_A(\mathcal{M})\big)$, it follows that $\operatorname{Supp} \mathcal{H}^0\big(\operatorname{FM}_B(\mathcal{N})\big) = B^\natural$, as claimed. □

The proof also gives the following surprising improvement of Theorem 19.1 for simple holonomic $\mathscr{D}$-modules.

Corollary 21.5. *Let $\mathcal{M}$ be a simple holonomic $\mathscr{D}_A$-module with $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M}) \neq 0$. Then for every $k > 0$, we have* $\operatorname{codim} \operatorname{Supp} \mathcal{H}^k \operatorname{FM}_A(\mathcal{M}) \geq 2k + 2$.

Proof. If we had $\operatorname{codim} \operatorname{Supp} \mathcal{H}^k \operatorname{FM}_A(\mathcal{M}) = 2k$ for some $k > 0$, then the same proof as above would show that $\mathcal{M}$ is pulled back from an abelian variety of dimension $g - k$. This possibility is ruled out by our assumption that $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M}) \neq 0$. □

22. Rigidity of the Fourier-Mukai transform. Corollary 21.5 has a very interesting consequence, namely that the Fourier-Mukai transform of a simple holonomic $\mathscr{D}_A$-module is completely determined by a single locally free sheaf: the restriction of the 0-th cohomology sheaf to the open subset where it is locally free.

Proposition 22.1. *Let $\mathcal{M}$ be a simple holonomic $\mathscr{D}_A$-module, and suppose that the support of $F = \operatorname{FM}_A(\mathcal{M})$ is equal to $A^\natural$. Then $\mathcal{H}^0 F$ is a reflexive coherent sheaf, locally free on the complement of a finite union of linear subvarieties of codimension ≥ 4; and F is uniquely determined by the restriction of $\mathcal{H}^0 F$ to that open subset.*

Proof. We shall use the symbols $\tau_{\leq n}$ and $\tau_{\geq n}$ for the truncation functors with respect to the standard t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$.

Let us first show that $\mathcal{H}^0 F$ is reflexive. We have $F \in {}^m\mathrm{Coh}(\mathscr{O}_{A^\natural})$, and therefore $F \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_{A^\natural})$, according to Theorem 19.1 and Lemma 18.5. This means that we can write down a distinguished triangle

$$\mathcal{H}^0 F \to F \to \tau_{\geq 1} F \to (\mathcal{H}^0 F)[1],$$

and after dualizing, an exact sequence

$$R^i\mathcal{H}om(F, \mathscr{O}) \to R^i\mathcal{H}om\big(\mathcal{H}^0 F, \mathscr{O}\big) \to R^{i+1}\mathcal{H}om\big(\tau_{\geq 1} F, \mathscr{O}\big).$$

Since $\mathbf{R}\mathcal{H}om(F, \mathscr{O})$ is isomorphic to the Fourier-Mukai transform of the simple holonomic $\mathscr{D}_A$-module $\langle -1_A \rangle^* \mathbf{D}_A \mathcal{M}$, we get $\operatorname{codim} \operatorname{Supp} R^i\mathcal{H}om(F, \mathscr{O}) \geq 2i + 2$ for all

$i \geq 1$; on the other hand, we have $\tau_{\geq 1} F \in \mathrm{D}^{\geq 1}_{coh}(\mathscr{O}_{A^\natural})$, and [Kas04, Proposition 4.3] shows that

$$\operatorname{codim} \operatorname{Supp} R^{i+1}\mathcal{H}om\big(\tau_{\geq 1} F, \mathscr{O}\big) \geq i+2.$$

Together, this proves that $\operatorname{codim} \operatorname{Supp} R^i\mathcal{H}om\big(\mathcal{H}^0 F, \mathscr{O}\big) \geq i+2$ for every $i \geq 1$; because $A^\natural$ is nonsingular, these inequalities guarantee that $\mathcal{H}^0 F$ is reflexive.

Now let $j \colon U \hookrightarrow A^\natural$ be the maximal open subset where $\mathcal{H}^0 F$ is locally free. Observe that the complement

$$A^\natural \setminus U = \bigcup_{k \geq 1} \operatorname{Supp} R^k\mathcal{H}om(F, \mathscr{O}_{A^\natural}) = \bigcup_{k \geq 1} \langle -1_{A^\natural} \rangle \operatorname{Supp} \mathcal{H}^k \operatorname{FM}_A\big(\mathbf{D}_A \mathcal{M}\big)$$

is a finite union of linear subvarieties of $A^\natural$, of codimension at least four. Because $\mathcal{H}^0 F$ is reflexive, it is uniquely determined by $j^* \mathcal{H}^0 F$; in fact, we have

$$\mathcal{H}^0 F \simeq j_* j^* \mathcal{H}^0 F.$$

To prove the remaining assertion, it will be enough to show that, for each $n \geq 0$, the truncation $\tau_{\leq n} F$ can be reconstructed starting from $\mathcal{H}^0 F$. We shall argue by induction on $n \geq 0$, using that $\tau_{\leq 0} F = \mathcal{H}^0 F$. Consider the distinguished triangle

$$\tau_{\leq n} F \to F \to \tau_{\geq n+1} F \to (\tau_{\leq n} F)[1].$$

According to Corollary 21.5, we have $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i(F) \geq 2i+2 \geq i+n+3$ for every $i \geq n+1$; in combination with [Kas04, Proposition 4.3], this implies that $\mathbf{R}\mathcal{H}om(\tau_{\geq n+1} F, \mathscr{O})$ belongs to $\mathrm{D}^{\geq n+3}_{coh}(\mathscr{O}_{A^\natural})$. After dualizing, we conclude that

$$\tau_{\leq n+1} \mathbf{R}\mathcal{H}om(F, \mathscr{O}) \simeq \tau_{\leq n+1} \mathbf{R}\mathcal{H}om\big(\tau_{\leq n} F, \mathscr{O}\big).$$

The same argument, applied to the dual complex $\mathbf{R}\mathcal{H}om(F, \mathscr{O})$, shows that

$$\tau_{\leq n+2} F \simeq \tau_{\leq n+2} \mathbf{R}\mathcal{H}om\Big(\tau_{\leq n+1} \mathbf{R}\mathcal{H}om(F, \mathscr{O}), \mathscr{O}\Big).$$

Together, this says that we always have

$$\tau_{\leq n+2} F \simeq \tau_{\leq n+2} \mathbf{R}\mathcal{H}om\Big(\tau_{\leq n+1} \mathbf{R}\mathcal{H}om(\tau_{\leq n} F, \mathscr{O}), \mathscr{O}\Big), \tag{22.2}$$

thereby concluding the proof. □

The procedure used during the proof leads to the following striking result.

Corollary 22.3. *Under the same assumptions as above,* $\operatorname{FM}_A(\mathcal{M})$ *can be reconstructed, up to isomorphism, by applying the functor*

$$\tau_{\leq \ell-1} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ \cdots \circ \tau_{\leq 2} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ \tau_{\leq 1} \circ \mathbf{R}\mathcal{H}om(-, \mathscr{O}) \circ j_*$$

to the locally free sheaf $j^* \mathcal{H}^0 \operatorname{FM}_A(\mathcal{M})$*; here* ℓ *is any odd integer with* $\ell \geq \dim A$.

Proof. This follows (22.2) and the fact that $F \in \mathrm{D}^{\leq \dim A - 1}_{coh}(\mathscr{O}_{A^\natural})$. □

23. Intersection complexes. The formula in Corollary 22.3 for the Fourier-Mukai transform of a simple holonomic $\mathscr{D}_A$-module is similar to Deligne's formula for the intersection complex of a local system. The purpose of this section is to turn that analogy into a rigorous statement.[7]

We shall use the assumptions and notations of Section 18; in particular, X will be a smooth complex algebraic variety. We write $\tau_{\leq n}$ and $\tau_{\geq n}$ for the truncation functors with respect to the standard t-structure on $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. To simplify the notation, let us introduce the symbol

$$\Delta = \mathbf{R}\mathcal{H}om(-, \mathscr{O}_X) \colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_X)^{opp}$$

for the naive duality functor. Define

$$\ell(X) = 2\left\lceil \frac{\dim X + 1}{4} \right\rceil - 1,$$

which is the smallest odd integer such that $2\ell + 1 \geq \dim X$.

Definition 23.1. Let $\mathscr{F}$ be a reflexive coherent sheaf. The complex

$$\mathrm{IC}(\mathscr{F}) = \bigl(\tau_{\leq \ell(X)-1} \circ \Delta \circ \tau_{\leq \ell(X)-2} \circ \Delta \circ \dots \circ \tau_{\leq 2} \circ \Delta \circ \tau_{\leq 1} \circ \Delta\bigr)\mathscr{F}$$

will be called the *intersection complex* of $\mathscr{F}$.

In the definition, $\ell(X)$ may be replaced by any odd integer ℓ with the property that $2\ell+1 \geq \dim X$; will see below that this does not change the resulting complex of coherent sheaves (up to isomorphism).

When the coherent sheaf $\mathscr{F}$ is locally free, we of course have $\mathrm{IC}(\mathscr{F}) \simeq \mathscr{F}$. The first result is that $\mathrm{IC}(\mathscr{F})$ is always an m-perverse coherent sheaf.

Proposition 23.2. *The complex* $\mathrm{IC}(\mathscr{F})$ *is an* m*-perverse coherent sheaf. Its* 0*-th cohomology sheaf is isomorphic to* $\mathscr{F}$*, and*

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^i \, \mathrm{IC}(\mathscr{F}) \geq 2i + 1 \quad \textit{for every } i \geq 1.$$

The dual complex $\Delta\, \mathrm{IC}(\mathscr{F})$ *has the same properties, but* $\mathcal{H}^0 \Delta\, \mathrm{IC}(\mathscr{F}) \simeq \mathcal{H}om(\mathscr{F}, \mathscr{O}_X)$.

Proof. Our main task is to show that both $\mathrm{IC}(\mathscr{F})$ and $\Delta\, \mathrm{IC}(\mathscr{F})$ belong to the subcategory ${}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_X)$. We recursively define a sequence of complexes by setting

$$F_n = \tau_{\leq n-1} \Delta F_{n-1},$$

starting from $F_0 = \mathcal{H}om(\mathscr{F}, \mathscr{O}_X)$. Observe that $F_1 \simeq \mathscr{F}$, because we are assuming that $\mathscr{F}$ is reflexive. We shall first prove by induction on $n \geq 0$ that

$$F_n \in \mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X), \text{ and } \operatorname{codim} \operatorname{Supp} \mathcal{H}^i F_n \geq 2i + 1 \text{ for every } i \geq 1. \tag{23.3}$$

To get started, note that (23.3) is obviously true for $n = 0$ and $n = 1$: indeed, both $F_0 = \mathcal{H}om(\mathscr{F}, \mathscr{O}_X)$ and $F_1 \simeq \mathscr{F}$ are sheaves.

Suppose now that we already know the result for all integers between 0 and n. From the definition of F_n, we obtain a distinguished triangle

$$F_n \to \Delta F_{n-1} \to \tau_{\geq n} \Delta F_{n-1} \to F_n[1];$$

after dualizing again, this becomes

$$\Delta \tau_{\geq n} \Delta F_{n-1} \to F_{n-1} \to \Delta F_n \to \Bigl(\Delta \tau_{\geq n} \Delta F_{n-1}\Bigr)[1]. \tag{23.4}$$

From (23.3), we get $\Delta F_n \in {}^{\hat{m}}\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$; together with Lemma 18.5, this implies that both ΔF_n and $F_{n+1} = \tau_{\leq n} \Delta F_n$ are objects of $\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$.

It remains to show that F_{n+1} also lies in the subcategory ${}^m\mathrm{D}^{\leq 0}_{coh}(\mathscr{O}_X)$. According to [Kas04, Proposition 4.3], which we have already used several times,

$$\operatorname{codim} \operatorname{Supp} \Bigl(\mathcal{H}^i \Delta \tau_{\geq n} \Delta F_{n-1}\Bigr) \geq i + n;$$

from (23.3), we also know that $\operatorname{codim} \operatorname{Supp} \mathcal{H}^i F_{n-1} \geq 2i + 1$ for $i \geq 1$. The distinguished triangle in (23.4) gives an exact sequence

$$\mathcal{H}^i F_{n-1} \to \mathcal{H}^i \Delta F_n \to \mathcal{H}^{i+1} \Delta \tau_{\geq n} \Delta F_{n-1}, \tag{23.5}$$

from which it follows that $\operatorname{codim}\operatorname{Supp}\mathcal{H}^i\Delta F_n \geq \min(2i+1, i+n+1) \geq 2i+1$ as long as $1 \leq i \leq n$. This means that $F_{n+1} = \tau_{\leq n}\Delta F_n$ also satisfies (23.3). We have thus proved that (23.3) is true for all $n \geq 0$. A useful consequence is that

$$\mathcal{H}^i F_n = 0 \quad \text{once } i > b(X) = \left\lceil \frac{\dim X}{2} \right\rceil - 1,$$

which means that $F_n \in \mathrm{D}^{\leq b(X)}_{coh}(\mathscr{O}_X)$ for every $n \geq 0$.

Next, let us show that the sequence of complexes F_n eventually settles into the pattern $F, \Delta F, F, \Delta F, \dots$. Here we need a bound on the amplitude of ΔF_n:

$$\mathcal{H}^i \Delta F_n = 0 \quad \text{if } i > b(X) \text{ and } i > \dim X - n - 1 \tag{23.6}$$

The proof is again by induction on $n \geq 0$: the statement is obviously true for $n = 0$ and $n = 1$; for the inductive step, one uses (23.5). Looking back at (23.4), we can then say that $\tau_{\geq n}\Delta F_{n-1} = 0$ as soon as $n > b(X)$ and $n > \dim X - (n-1) - 1$; this translates into the condition that

$$2n - 1 \geq \dim X.$$

If that is the case, we get

$$F_{n-1} \simeq \Delta F_n \quad \text{and} \quad F_{n+1} = \tau_{\leq n}\Delta F_n \simeq F_{n-1},$$

and so from that point on, the sequence of complexes alternates between F_{n-1} and ΔF_{n-1}. Consequently, the intersection complex of $\mathscr{F}$ satisfies

$$\mathrm{IC}(\mathscr{F}) \simeq F_\ell \quad \text{and} \quad \Delta\,\mathrm{IC}(\mathscr{F}) \simeq F_{\ell+1}$$

where ℓ is any odd integer such that $2\ell + 1 \geq \dim X$. Since it is obvious from the definition that $\mathcal{H}^0\,\mathrm{IC}(\mathscr{F})$ is isomorphic to $\mathscr{F}$, we have now proved everything that was asserted. □

As in the case of Fourier-Mukai transforms, the fact that $\mathscr{F}$ is reflexive means that $\mathrm{IC}(\mathscr{F})$ is already determined by the restriction of $\mathscr{F}$ to an open subset.

Corollary 23.7. *Let $j\colon U \hookrightarrow X$ denote the maximal open subset where $\mathscr{F}$ is locally free. Then the intersection complex of $\mathscr{F}$ satisfies*

$$\mathrm{IC}(\mathscr{F}) \simeq \mathrm{IC}(j_*j^*\mathscr{F}),$$

and is therefore uniquely determined by the locally free sheaf $j^*\mathscr{F}$.

Proof. The codimension of $X \setminus U$ is ≥ 2, and so $j_*j^*\mathscr{F}$ is isomorphic to $\mathscr{F}$. □

We can therefore extend the definition of the intersection complex to locally free sheaves that are defined on the complement of a subset of codimension at least two.

Definition 23.8. Let $j\colon U \hookrightarrow X$ be an open subset with $\operatorname{codim}(X \setminus U) \geq 2$, and let $\mathscr{E}$ be a locally free coherent sheaf on U. Then the complex

$$\mathrm{IC}_X(\mathscr{E}) = \mathrm{IC}(j_*\mathscr{E})$$

will be called the *intersection complex* of $\mathscr{E}$ (with respect to X).

Our next result is that $\mathrm{IC}_X(\mathscr{E})$ actually behaves like an intersection complex: it does not have nontrivial subobjects or quotient objects whose support is properly contained in X. For technical reasons, the statement is not symmetric.

Proposition 23.9. *Let $\mathscr{F}$ be a torsion-free coherent sheaf on X.*

(a) If a subobject of $\mathrm{IC}_X(\mathscr{E})$ *in the abelian category* ${}^m\mathrm{Coh}(\mathscr{O}_X)$ *is supported on a proper subset of* X*, then that subobject must be zero.*
(b) If a quotient object of $\mathrm{IC}_X(\mathscr{E})$ *in the abelian category* ${}^{\hat{m}}\mathrm{Coh}(\mathscr{O}_X)$ *is supported on a proper subset of* X*, then that quotient object must be zero.*

Proof. Since Δ interchanges the two abelian categories ${}^m\mathrm{Coh}(\mathscr{O}_X)$ and ${}^{\hat{m}}\mathrm{Coh}(\mathscr{O}_X)$, it suffices to prove (a) for both $\mathrm{IC}_X(\mathscr{E})$ and $\Delta\,\mathrm{IC}_X(\mathscr{E})$. Set $\mathscr{F} = j_*\mathscr{E}$. We shall only deal with $\mathrm{IC}(\mathscr{F})$; the argument in the other case is exactly the same.

Suppose then that we have a subobject E of $\mathrm{IC}(\mathscr{F})$, with $\operatorname{Supp} E \neq X$; we need to show that $E = 0$. Because ${}^m\mathrm{Coh}(\mathscr{O}_X)$ is defined as the heart of a t-structure, E being a subobject means that we have a distinguished triangle

$$E \to \mathrm{IC}(\mathscr{F}) \to F \to E[1]$$

in which E and F are objects of ${}^m\mathrm{Coh}(\mathscr{O}_X)$. After dualizing, this becomes

$$\Delta F \to \Delta\,\mathrm{IC}(\mathscr{F}) \to \Delta E \to \Delta F[1].$$

Now $\Delta F \in {}^{\hat{m}}\mathrm{Coh}(\mathscr{O}_X)$ has the property that $\operatorname{codim}\operatorname{Supp}\mathcal{H}^i\Delta F \geq 2i-1$; we also know from Proposition 23.2 that $\operatorname{codim}\operatorname{Supp}\mathcal{H}^i\Delta\,\mathrm{IC}(\mathscr{F}) \geq 2i+1$ for $i \geq 1$. Looking at the second distinguished triangle, we find that

$$\operatorname{codim}\operatorname{Supp}\mathcal{H}^i\Delta E \geq 2i+1$$

for $i \geq 1$; in fact, this also holds for $i = 0$ because $\operatorname{Supp} E \neq X$. This clearly means that $\Delta E \in {}^m\mathrm{D}^{\geq 0}_{coh}(\mathscr{O}_X)$, too. Now apply Proposition 18.6 to get $\Delta E = 0$. □

Note that the distinction between ${}^m\mathrm{Coh}(\mathscr{O}_X)$ and ${}^{\hat{m}}\mathrm{Coh}(\mathscr{O}_X)$ does not matter if we only consider objects for which the support of every cohomology sheaf has even dimension. This is the case for Fourier-Mukai transforms of holonomic complexes.

Proposition 23.10. *Suppose that a reflexive coherent sheaf* $\mathscr{F}$ *on* $A^\natural$ *is the Fourier-Mukai transform of a holonomic complex. Then the same is true for* $\mathrm{IC}(\mathscr{F})$*.*

Proof. A look at the formula for the intersection complex in Definition 23.1 shows that it is obtained by repeatedly dualizing and truncating. By Theorem 10.5 and Theorem 20.1, both operations preserve the subcategory $\mathrm{FM}_A\big(\mathrm{D}^b_h(\mathscr{D}_A)\big)$. □

As an object of the abelian category $\mathrm{FM}_A\big(\mathrm{D}^b_h(\mathscr{D}_A)\big)$, the intersection complex of $\mathscr{F}$ now has neither subobjects nor quotient objects that are supported in a proper subset of $A^\natural$ (except for the zero object). This shows that the analogy with the intersection complex in [BBD82] is meaningful.

F. Miscellaneous results

This concluding chapter contains a small number of additional results about Fourier-Mukai transforms of holonomic complexes.

24. A criterion for holonomicity. In this section, we give a necessary and sufficient condition for a complex of coherent sheaves on $A^\natural$ to be of the form $\mathrm{FM}_A(\mathcal{M})$ for a holonomic complex $\mathcal{M}$. Unfortunately, this condition is not very useful in practice: on the one hand, it is hard to verify; on the other hand, it does not directly imply any of the properties of Fourier-Mukai transforms of holonomic complexes that we already know.[8]

Proposition 24.1. *The following conditions on* $F \in \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$ *are equivalent:*

(a) *F is the Fourier-Mukai transform of a holonomic complex.*
(b) *For every $L \in \operatorname{Pic}^0(\hat{A})$, the cohomology groups*

$$\mathbf{H}^k\big(A^\natural, F \otimes \pi^* L\big)$$

are finite-dimensional for every $k \in \mathbb{Z}$.

Proof. Since $\mathrm{FM}_A \colon \mathrm{D}^b_{coh}(\mathscr{D}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{A^\natural})$ is an equivalence of categories, there is a (essentially unique) complex of $\mathscr{D}$-modules $\mathcal{M} \in \mathrm{D}^b_{coh}(\mathscr{D}_A)$ with $F \simeq \mathrm{FM}_A(\mathcal{M})$. According to [HTT08, Theorem 3.3.1], $\mathcal{M}$ is a holonomic complex if and only if

$$i_a^+ \mathcal{M} \in \mathrm{D}^b_{coh}(\mathbb{C})$$

for every point $a \in A$. Let $P_a \in \operatorname{Pic}^0(\hat{A})$ denote the line bundle on $\hat{A}$ corresponding to the point $a \in A$. If $p \colon A^\natural \to pt$ is the morphism to a point, we have

$$i_a^+ \mathcal{M} = \mathbf{R}p_*\big(\mathrm{FM}_A(\mathcal{M}) \otimes \pi^* P_a^{-1}\big)$$

by Theorem 10.5; this immediately implies the asserted equivalence. $\square$

25. Chern characters. The purpose of this section is to compute the algebraic Chern character of $\mathrm{FM}_A(\mathcal{M})$, for $\mathcal{M}$ a holonomic $\mathscr{D}_A$-module.

For a smooth algebraic variety X, we denote by $CH(X)$ the algebraic Chow ring of X. To begin with, observe that since $\pi \colon A^\natural \to \hat{A}$ is an affine bundle in the Zariski topology, the pullback map $\pi^* \colon CH(\hat{A}) \to CH(A^\natural)$ is an isomorphism.

Proposition 25.1. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}_A$-module. Then the algebraic Chern character of the Fourier-Mukai transform $\mathrm{FM}_A(\mathcal{M})$ lies in the subring of $CH(\hat{A})$ generated by $CH^1_1(\hat{A}) = \operatorname{Pic}^0(\hat{A})$.*

Proof. Since $\pi \colon E(A) \to \hat{A}$ is an algebraic vector bundle containing $A^\natural = \lambda^{-1}(1)$, pullback of cycles induces isomorphisms

$$CH(\hat{A}) \simeq CH\big(E(A)\big) \simeq CH(A^\natural).$$

As in the proof of Proposition 19.2, choose a good filtration $F_\bullet \mathcal{M}$ and consider the Fourier-Mukai transform $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ of the associated Rees module. Its restriction to $A^\natural$ is isomorphic to $\mathrm{FM}_A(\mathcal{M})$, and so it suffices to show that the Chern character of $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ is contained in the subring generated by $\operatorname{Pic}^0(\hat{A})$. Since $\lambda^{-1}(0) = \hat{A} \times H^0(A, \Omega_A^1)$, we only need to prove this after restricting to $\hat{A} \times \{\omega\} \subseteq \lambda^{-1}(0)$, for any choice of $\omega \in H^0(A, \Omega_A^1)$.

By Proposition 13.1, the restriction of $\widetilde{\mathrm{FM}}_A(R_F\mathcal{M})$ to $\lambda^{-1}(0)$ is isomorphic to

$$\mathbf{R}(p_{23})_*\Big(p_{12}^* P \otimes p_1^* \Omega_A^g \otimes p_{13}^*(\mathrm{id} \times \iota)^* \operatorname{gr}^F \mathcal{M}\Big). \tag{25.2}$$

Since $\mathcal{M}$ is holonomic, the support of $\operatorname{gr}^F\mathcal{M}$ is of pure dimension g. Now choose $\omega \in H^0(A, \Omega_A^1)$ general enough that the restriction of $\operatorname{gr}^F\mathcal{M}$ to $A \times \{\omega\}$ is a coherent sheaf with zero-dimensional support. The restriction of (25.2) to $\hat{A} \times \{\omega\}$ is then the Fourier-Mukai transform of a coherent sheaf on A with zero-dimensional support; its algebraic Chern character must therefore be contained in the subring of $CH(\hat{A})$ generated by $\operatorname{Pic}^0(\hat{A})$. $\square$

Corollary 25.3. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}_A$-module. Then all the Chern classes of $\mathrm{FM}_A(\mathcal{M})$ are zero in the singular cohomology ring of $A^\natural$.*

26. Ampleness results. The following result says that, in most cases, the singular locus of a holonomic $\mathscr{D}$-module is an ample divisor.

Theorem 26.1. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}$-module with support equal to A. Let $D(\mathcal{M})$ be the image in A of the projectivized characteristic variety of $\mathcal{M}$. If $\chi(A, \mathcal{M}) \neq 0$, then $D(\mathcal{M})$ is an ample divisor.*

Proof. Since the Euler characteristic of any holonomic $\mathscr{D}_A$-module is nonnegative, it suffices to prove the assertion when $\mathcal{M}$ is simple. If $D(\mathcal{M})$ is not an ample divisor, then all of its codimension one components are fibered in a fixed abelian subvariety, and so there is a morphism $f \colon A \to B$, with $\dim B = \dim A - r$, such that no codimension one component of $D(\mathcal{M})$ dominates B. On $A \setminus D(\mathcal{M})$, we have a vector bundle with integrable connection; its restriction to a general fiber of f extends to a vector bundle with integrable connection on the entire fiber. We can then argue as in the proof of Theorem 21.1 to show that $\mathcal{M} \otimes_{\mathscr{O}_A} (L, \nabla) \simeq f^* \mathcal{N}$ for a simple holonomic $\mathscr{D}_B$-module $\mathcal{N}$. But this implies that $\chi(A, \mathcal{M}) = 0$. $\square$

The following result is inspired by the application of Hodge modules and their Fourier-Mukai transforms to varieties of general type in [PS14].

Theorem 26.2. *Let $\mathcal{M}$ be a holonomic $\mathscr{D}$-module on A, and suppose that for some ample line bundle L, there is a nonzero morphism of $\mathscr{O}_A$-modules $L \to \mathcal{M}$.*

(a) If $\operatorname{Supp} \mathcal{M} = A$, then the singular locus $D(\mathcal{M})$ is an ample divisor.
(b) The projection $\operatorname{Ch}(\mathcal{M}) \to H^0(A, \Omega_A^1)$ is surjective.

Proof. Both assertions will be proved if we manage to show that $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M}) \neq 0$. The given morphism induces a nonzero morphism of $\mathscr{D}_A$-modules

$$\mathscr{D}_A \otimes_{\mathscr{O}_A} L \to \mathcal{M},$$

and therefore a nonzero morphism

$$\operatorname{FM}_A\bigl(\mathscr{D}_A \otimes_{\mathscr{O}_A} L\bigr) \to \operatorname{FM}_A(\mathcal{M})$$

between their Fourier-Mukai transforms. By Theorem 10.5, we have

$$\operatorname{FM}_A\bigl(\mathscr{D}_A \otimes L\bigr) = \mathbf{L}\pi^* \mathbf{R}\Phi_P(L) = \pi^* \mathscr{E}_L,$$

where $\mathscr{E}_L = (p_2)_*(P \otimes p_1^* L)$ is a locally free sheaf of rank $H^0(A, L)$ on $\hat{A}$. Because $\operatorname{FM}_A(\mathcal{M}) \in \mathrm{D}_{coh}^{\geq 0}(\mathscr{O}_{A^\natural})$, the morphism factors through $\mathcal{H}^0 \operatorname{FM}_A(\mathcal{M})$, which is only possible if the latter is not zero. $\square$

Notes

[1] We use the word *linear* because the linear subvarieties of $A^\natural$ are precisely those whose preimage in the universal cover are linear subspaces. The reason for the term *arithmetic* is as follows. Let $\operatorname{Char}(A)$ be the space of characters of the fundamental group of A; it is also a complex algebraic variety, biholomorphic to $A^\natural$, but with a different algebraic structure. When A is defined over a number field, the torsion points are precisely those points on the algebraic varieties $A^\natural$ and $\operatorname{Char}(A)$ that are simultaneously defined over a number field in both [Sim93, Proposition 3.4].

[2] This is very surprising at first glance, because one expects the Fourier-Mukai transform to interchange "local" and "global" data. But both t-structures in this result are in fact defined by local conditions.

[3] This is just to give an idea; in reality, the extension is typically not locally free, and so it is more difficult to prove that it is algebraic.

[4] In this setting, the transform does *not* determine the original constructible complex: for example, if K is any constructible sheaf whose support is a finite union of points of A, then $\mathbf{R}p_*\big(K \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}[\Lambda]}\big)$ is a free $\mathbb{C}[\Lambda]$-module of finite rank.

[5] We use this letter because m and $\hat{m}$ are as close as one can get to "middle perversity". There is of course no actual middle perversity for coherent sheaves, because the equality $p = \hat{p}$ cannot hold unless X is a point.

[6] The proof can be simplified by using the fact that semi-simplicity is preserved under projective direct images. This is a very hard theorem by Sabbah and Mochizuki – see [Sab13] for the history of this result – and so it seemed better to give an elementary proof.

[7] In [AB10, Section 4], Arinkin and Bezrukavnikov also define a notion of "coherent IC-sheaves". Unfortunately, I do not know how their definition relates to Definition 23.1.

[8] In a sense, this answers the question raised in Section 6 – but not in a way that is particularly useful. The problem is that the condition is Proposition 24.1 is a *global* one, whereas we are really asking for a *local* characterization of Fourier-Mukai transforms of holonomic complexes.

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA
E-mail address: cschnell@math.sunysb.edu