HIGHER MULTIPLIER IDEALS

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Dedicated to Rob Lazarsfeld on the occasion of his 70th birthday

Abstract. We associate a family of ideal sheaves to any $\mathbb{Q}$-effective divisor on a complex manifold, called the higher multiplier ideals, using the theory of mixed Hodge modules and V-filtrations. This family is indexed by two parameters, an integer indicating the Hodge level and a rational number, and these ideals admit a weight filtration. When the Hodge level is zero, they recover the usual multiplier ideals.

We study the local and global properties of higher multiplier ideals systematically. In particular, we prove vanishing theorems and restriction theorems, and provide criteria for the nontriviality of the new ideals. The main idea is to exploit the global structure of the V-filtration along an effective divisor using the notion of twisted Hodge modules. In the local theory, we introduce the notion of the center of minimal exponent, which generalizes the notion of minimal log canonical center. As applications, we prove some cases of conjectures by Debarre, Casalaina-Martin and Grushevsky on singularities of theta divisors on principally polarized abelian varieties and the geometric Riemann-Schottky problem.

Contents

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1. Introduction

Let X be a complex manifold of dimension n, and let D be an effective divisor on X. We construct a family of ideal sheaves $\mathcal{I}_{k,\alpha}(D)$ from (X, D), using the Hodge theory of the Kashiwara-Malgrange filtration along D; these are indexed by $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$. We defer the somewhat technical construction to the end of the introduction. One purpose of this work is to show that these ideal sheaves can serve as a new measure of singularities of D and lead to a number of applications for singularities of divisors. The theory is most interesting in those cases where there is no useful information left in the usual multiplier ideals; this happens for example for theta divisors on principally polarized abelian varieties (more generally when (X, D) is log canonical).

We call the new ideals *higher multiplier ideals*, because $\mathcal{I}_{0,<-\alpha}(D) = \mathcal{J}(X, \alpha D)$ is the usual multiplier ideal of the effective $\mathbb{Q}$-divisor αD. (Here the subscript $< \alpha$ is an abbreviation for $\alpha - \varepsilon$, where $\varepsilon > 0$ is a small positive real number.) When D is reduced,

$\mathcal{I}_{k,\alpha}(D)$ recover Saito's "microlocal multiplier ideal sheaves" [65]; in this work we study them from a more global point of view and prove many new results about them. In the introduction, we outline the most important result: a version of the Nadel vanishing theorem; restriction and semicontinuity theorems; a numerical jumping criterion; and a host of smaller results. This makes the theory almost as good as that of usual multiplier ideals.

1.1. **Basic properties of higher multiplier ideals.** To begin with, our higher multiplier ideals $\mathcal{I}_{k,\alpha}(D)$ have similar formal properties as usual multiplier ideals. One has the increasing property

$$\mathcal{I}_{k,\alpha}(D) \subseteq \mathcal{I}_{k,\beta}(D), \quad \text{whenever } \alpha \leq \beta,$$

and as with usual multiplier ideals, we call $\alpha \in \mathbb{Q}$ a *jumping number* if $\mathcal{I}_{k,<\alpha}(D) \neq \mathcal{I}_{k,\alpha}(D)$. The graded pieces

$$\mathcal{G}_{k,\alpha}(D) = \mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,<\alpha}(D) \tag{1.1}$$

are supported inside the singular locus of D if $\alpha \in (-1, 0]$. They have an additional "weight filtration", indexed by $\mathbb{Z}$, whose subquotients we denote by the symbol $\mathrm{gr}_\ell^W \mathcal{G}_{k,\alpha}(D)$. This induces a weight filtration $W_\bullet \mathcal{I}_{k,\alpha}(D)$ via (1.1). Here are some basic properties of these ideals.

(I) One has $\mathcal{I}_{k,<k}(D) = \mathcal{O}_X$, i.e. all jumping numbers of $\mathcal{I}_{k,\alpha}(D)$ are less than k (Proposition 5.10).

(II) The sequence of ideal sheaves $\{\mathcal{I}_{k,\alpha}(D)\}_{\alpha\in\mathbb{Q}}$ is discrete and right continuous. For any $\alpha \in \mathbb{Q}$, there exist isomorphisms

$$\mathcal{I}_{k,\alpha-1}(D) \xrightarrow{\sim} \mathcal{I}_{k,\alpha}(D) \otimes \mathcal{O}_X(-D), \quad \text{for } \alpha < 0,$$
$$\mathcal{I}_{k+1,\alpha+1}(D) \xrightarrow{\sim} \mathcal{I}_{k,\alpha}(D), \quad \text{for } \alpha \geq -1.$$

Therefore all $\mathcal{I}_{k,\alpha}(D)$ are controlled by those with $\alpha \in [-1, 0]$; see Proposition 5.10.

(III) If D is smooth, then $\mathcal{I}_{k,\alpha}(D) = \mathcal{O}_X$ for all $\alpha \in [-1, 0]$ and all k; Theorem 1.2 provides an effective converse statement.

(IV) One has $\mathcal{I}_{k+1,\alpha}(D) \subseteq \mathcal{I}_{k,\alpha}(D)$ for $\alpha \in \mathbb{Q}, k \in \mathbb{N}$ (Corollary 5.17).

(V) The minimal exponent $\tilde{\alpha}_D$ of any effective divisor D, defined in [63], can be characterized using the first jumping number of $\{\mathcal{I}_{k,\alpha}(D)\}_{k\in\mathbb{N},\alpha\in(-1,0]}$:

$$\begin{aligned} \tilde{\alpha}_D &= \min_{k\in\mathbb{N},\alpha\in(-1,0]} \{k - \alpha \mid \mathcal{I}_{k,<\alpha}(D) \subsetneq \mathcal{O}_X\} \\ &= \min_{k\in\mathbb{N},\alpha\in(-1,0]} \{k - \alpha \mid \mathcal{G}_{k,\alpha}(D) \neq 0\}, \end{aligned} \tag{1.2}$$

see Lemma 5.26. This generalizes the fact that the log canonical threshold of D is the first jumping number of the multiplier ideals $\{\mathcal{J}(\beta D)\}_{\beta\geq 0}$. It also means that the minimal exponent is completely determined by the vanishing cycle mixed Hodge modules along D; see (5.6) and §2.1. Moreover, this provides a close connection between the notion of k-du Bois and k-rational singularities, introduced recently in [21] and [26], and higher multiplier ideals. Because these notions for hypersurfaces can be completely controlled by minimal exponents.

(VI) The minimal exponent $\tilde{\alpha}_D$ is a root of the Bernstein-Sato polynomial of D. Generalizing the relation between minimal exponent and jumping numbers in the previous statement, locally for any $x \in X$, choose f_x to be the local defining equation of D near x. Then the set of roots of the Bernstein-Sato polynomial of f_x is the set of all jumping numbers of $\{\mathcal{I}_{k,\bullet}(D)_x\}$ for all k, modulo $\mathbb{Z}$. This is essentially due to Malgrange [39], see Proposition 5.10.

(VII) The definition of higher multiplier ideals naturally extends to effective $\mathbb{Q}$-divisors for $\alpha \leq 0$. In §5.3, we show that for an effective divisor D, one has

$$\mathcal{I}_{k,m\alpha}(D) \otimes \mathcal{O}_X(kD) \cong \mathcal{I}_{k,\alpha}(D) \otimes \mathcal{O}_X(kmD).$$

Then if E is an effective $\mathbb{Q}$-divisor, we choose $m \geq 1$ such that mE has integer coefficients, and set

$$S_{k,\alpha}(E) := \mathcal{I}_{k,\alpha/m}(mE) \otimes \mathcal{O}_X(kmE).$$

This is a well-defined rank-one torsion-free sheaf, due to the isomorphism above. Therefore the reflexive hall of $S_{k,\alpha}(E)$ is a line bundle and we can extract an ideal sheaf $\mathcal{I}_{k,\alpha}(E)$ out of $S_{k,\alpha}(E)$; for details, see §5.3. Consequently, we will focus on the case of effective divisors in the introduction; many local properties easily carry over to $\mathbb{Q}$-divisors.

Let us discuss some local properties of higher multiplier ideals. First, they behave well under restriction; for more details, see Theorem 7.6.

Theorem 1.1. *Let $i : H \hookrightarrow X$ be the closed embedding of a smooth hypersurface that is not entirely contained in the support of D so that the pullback $D_H = i^*D$ is defined. Then one has an inclusion*

$$\mathcal{I}_{k,\alpha}(D_H) \subseteq \mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_H,$$

where the latter is defined as the image of $\{i^*\mathcal{I}_{k,\alpha}(D) \to i^*\mathcal{O}_X = \mathcal{O}_H\}$. If H is sufficiently general, then

$$\mathcal{I}_{k,\alpha}(D_H) = \mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_H.$$

There are variants of the restriction theorem for the weight filtrations $W_\ell\mathcal{I}_{k,\alpha}(D)$ and $W_\ell\mathcal{G}_{k,\alpha}(D)$, see Theorem 7.9. By a standard argument, the restriction theorem implies the semicontinuity of higher multiplier ideals in families (Theorem 7.12). With the help of these theorems, we can show that the presence of very singular points forces certain higher multiplier ideals to jump. Using multiplicity as a coarse measure for the singularities, we define

$$\mathrm{Sing}_m(D) := \{x \in X \mid \mathrm{mult}_x D \geq m\}.$$

Theorem 1.2. *Let D be an effective divisor on a complex manifold X of dimension n. Suppose that $Z \subseteq \mathrm{Sing}_m(D)$ is an irreducible component of dimension d. Write $n - d = km + r$ with $k \in \mathbb{N}$ and $0 \leq r \leq m - 1$. Then*

$$\mathcal{I}_{k,<\alpha}(D) \neq \mathcal{O}_X, \quad \textit{for some } \alpha \geq -r/m.$$

For the more precise statement about the containment of $\mathcal{I}_{k,\alpha}(D)$ in certain symbolic powers of $\mathcal{I}_Z$, see Theorem 7.17. The proof works by using the restriction and semicontinuity theorems to reduce the problem to the case of an isolated singularity of multiplicity m and codimension $n-d$ whose projectivized tangent cone $X_m \subseteq \mathbf{P}^{n-d-1}$ is smooth. In this case, the sheaf $\mathcal{G}_{k,\alpha}(D)$ is then computed in Proposition 6.5 and Theorem 6.6 using the primitive cohomology of the cyclic covering of $\mathbf{P}^{n-d-1}$ branched along X_m; this is one of the most difficult computations in this paper and uses the work of Qianyu Chen [13] and Saito's bistrict direct images. The numbers above can be remembered using Griffiths formulas for the Hodge numbers of hypersurfaces (because the cyclic covering mentioned above is a hypersurface in $\mathbf{P}^{n-d}$).

We close the discussion of local properties with the generalization of minimal log canonical centers. Let D be an effective divisor on a complex manifold X. In birational geometry, the log canonical center is another basic invariant of the pair (X, D). For example, it is a foundational result of Kawamata [28] that any minimal log canonical center is normal and has at worst rational singularities. By the work of Lichtin [38] and Kollár [31], one has

$$\operatorname{lct}(D) = \min\{1, \tilde{\alpha}_D\}, \tag{1.3}$$

where $\tilde{\alpha}_D$ is again the so-called minimal exponent of the divisor D. In particular, $\tilde{\alpha}_D$ is a more refined invariant than $\operatorname{lct}(D)$. By (1.2), we see that the minimal exponent $\tilde{\alpha}_D$ can be characterized using the higher multiplier ideals. Therefore it is natural to study the notion of "*the center of minimal exponent*", which is defined as the subscheme $Y \subseteq X$ such that

$$\operatorname{gr}_\ell^W \mathcal{G}_{k,\alpha}(D) = \mathcal{O}_Y, \tag{1.4}$$

where $\tilde{\alpha}_D = k - \alpha$ for $k \in \mathbb{N}$, $\alpha \in (-1, 0]$ and ℓ is the largest integer such that $\operatorname{gr}_\ell^W \mathcal{G}_{k,\alpha}(D) \neq 0$; for more details, see §7.4. In Theorem 7.36, we prove

Theorem 1.3. *Every connected component of the center of minimal exponent of* (X, D) *is irreducible, reduced, normal and has at worst rational singularities.*

This theorem relies on a Hodge module theoretic statement (Proposition 7.37). Using the same statement, we can recover the criterion due to Ein-Lazarsfeld [19] (for detecting normality and rational singularities of a hypersurface D in terms of the adjoint ideal $\operatorname{adj}(D)$) as well as the aforementioned result by Kawamata under additional assumptions, see Corollary 7.41 and Corollary 7.42. The key idea is that the adjoint ideal and minimal log canonical centers can be related to weight filtrations on higher multiplier ideals. We also compute the center of minimal exponent for theta divisors of hyperelliptic curves and Brill-Noether general curves, see Example 7.39.

Regarding the global properties of higher multiplier ideals, we prove several vanishing theorems, similar in spirit to Nadel's vanishing theorem for usual multiplier ideals, but involving the de Rham-type complex ($n = \dim X$)

$$\begin{aligned}\operatorname{gr}_\ell^W K_{k,\alpha}(D) = [\Omega_X^{n-k} \otimes \operatorname{gr}_\ell^W \mathcal{G}_{0,\alpha}(D) \to \cdots \\ \cdots \to \Omega_X^{n-1} \otimes L^{k-1} \otimes \operatorname{gr}_\ell^W \mathcal{G}_{k-1,\alpha}(D) \to \Omega_X^n \otimes L^k \otimes \operatorname{gr}_\ell^W \mathcal{G}_{k,\alpha}(D)][k].\end{aligned}$$

Recall that once we go beyond the first step in the Hodge filtration, vanishing theorems for mixed Hodge modules always involve complexes of sheaves (see [68], for example). The following result is proved in Theorem 8.1.

Theorem 1.4. *Let D be an effective divisor on a projective complex manifold X. Let $k \in \mathbb{N}, \ell \in \mathbb{Z}$ and $\alpha \in [-1, 0]$. Furthermore, let B be an effective divisor such that the $\mathbb{Q}$-divisor $B + \alpha D$ is ample. Then*

$$H^i\left(X, \operatorname{gr}_\ell^W K_{k,\alpha}(D) \otimes \mathcal{O}_X(B)\right) = 0, \quad \textit{for every } i > 0.$$

This is proved by relating higher multiplier ideals with the notion of *twisted polarizable Hodge modules*, which give us a handle on their global properties. They will be discussed later in the introduction. Assuming the triviality of higher multiplier ideals of lower orders, one can obtain a more precise vanishing theorem for $\mathcal{I}_{k,\alpha}(D), \mathcal{G}_{k,\alpha}(D)$ and $W_\ell \mathcal{I}_{k,\alpha}(D)$, see Corollary 8.2, Corollary 8.3 and Corollary 8.5. The vanishing theorem improves dramatically when X is an abelian variety or a projective space and D is itself an ample divisor, see §8.2 and §8.3.

Theorem 1.5. *Let D be an effective divisor on an abelian variety A such that the line bundle $L = \mathcal{O}_A(D)$ is ample. For any line bundle $\rho \in \operatorname{Pic}^0(A)$ and $i \geq 1$, we have*

(1) $H^i\left(A, L^{k+1} \otimes W_\ell \mathcal{G}_{k,\alpha}(D) \otimes \rho\right) = 0$ for $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$ and $\alpha \in (-1, 0]$.
(2) $H^i(A, L^k \otimes \mathcal{I}_{k,0}(D) \otimes \rho) = 0$ for $k \geq 1$.

(3) $H^i(A, L^{k+1} \otimes \mathcal{I}_{k,\alpha}(D) \otimes \rho) = 0$ *for* $k \in \mathbb{N}$ *and* $\alpha \in [-1, 0)$.

Theorem 1.6. *Let D be a reduced hypersurface of degree d in $\mathbf{P}^n$. For $k \in \mathbb{N}$ and $i \geq 1$, we have*

(1) $H^i\left(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes W_\ell \mathcal{G}_{k,\alpha}(D)\right) = 0$ *for* $\alpha \in (-1, 0]$, $\ell \in \mathbb{Z}$ *and* $m > d(k-\alpha)-n-1$.
(2) $H^i\left(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{I}_{k,0}(D)\right) = 0$ *for* $k \geq 1$ *and* $m \geq kd - n - 1$.
(3) $H^i\left(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{I}_{k,\alpha}(D)\right) = 0$ *for* $\alpha \in [-1, 0)$ *and* $m \geq (k+1)d - n - 1$.

1.2. **Twisted Hodge modules.** An important feature of the present work is that we can relate higher multiplier ideals to twisted $\mathscr{D}$-modules, which give us a handle on their global properties. More precisely, we introduce a notion of *twisted polarizable Hodge modules*, which provides a convenient framework for discussing global structures of nearby and vanishing cycles along effective divisors.

First, let us recall the notion of twisted $\mathscr{D}$-modules, developed by Bernstein and Beilinson in [5, §2]. Roughly speaking, these are objects that are locally $\mathscr{D}$-modules, but with a different transition rule from one coordinate chart to another. The twisting depends on two parameters: a holomorphic line bundle L on the complex manifold X and a complex number $\alpha \in \mathbb{C}$. Like the sheaf of differential operators $\mathscr{D}_X$ itself, the sheaf of *αL-twisted differential operators* $\mathscr{D}_{X,\alpha L}$ is a noncommutative $\mathcal{O}_X$-algebra; it still has an order filtration $F_\bullet$ such that

$$\operatorname{gr}_k^F \mathscr{D}_{X,\alpha L} \cong \operatorname{Sym}^k \mathscr{T}_X,$$

where $\mathscr{T}_X$ is the tangent bundle of X. In particular, we have $\operatorname{gr}_\bullet^F \mathscr{D}_{X,\alpha L} \cong \operatorname{gr}_\bullet^F \mathscr{D}_X$. But unlike in the case of $\mathscr{D}_X$ where $F_1\mathscr{D}_X = \mathcal{O}_X \oplus \mathscr{T}_X$, the sequence

$$0 \to \mathcal{O}_X \to F_1\mathscr{D}_{X,\alpha L} \to \mathscr{T}_X \to 0$$

does not split; instead, its extension class is equal to $\alpha \cdot c_1(L)$ in $\operatorname{Ext}^1_X(\mathscr{T}_X, \mathcal{O}_X) \cong H^1(X, \Omega^1_X)$. We give a construction of $\mathscr{D}_{X,\alpha L}$ using differential operators on the total space of the line bundle L in §3.2. An *αL-twisted $\mathscr{D}$-module* is a *right* module over $\mathscr{D}_{X,\alpha L}$. One crucial difference with usual $\mathscr{D}$-modules is that there is no de Rham complex for twisted $\mathscr{D}$-modules, because there is no longer an action by $\mathscr{T}_X$. But we do have $\operatorname{gr}_1^F \mathscr{D}_{X,\alpha L} \cong \mathscr{T}_X$, and so the "graded pieces of the de Rham complex", by which we mean the complex

$$\operatorname{gr}_p^F \operatorname{DR}(\mathcal{M}) = \Big[\operatorname{gr}_{p-n}^F \mathcal{M} \otimes \bigwedge^n \mathscr{T}_X \to \cdots \\ \cdots \to \operatorname{gr}_{p-2}^F \mathcal{M} \otimes \bigwedge^2 \mathscr{T}_X \to \operatorname{gr}_{p-1}^F \mathcal{M} \otimes \mathscr{T}_X \to \operatorname{gr}_p^F \mathcal{M}\Big][n]$$

still make sense for a twisted $\mathscr{D}$-module $\mathcal{M}$ with a good filtration $F_\bullet\mathcal{M}$.

Together with Claude Sabbah, the first author is developing a theory of *complex mixed Hodge modules*, where one removes perverse sheaves from the picture and describes polarizations as certain distribution-valued pairings on the underlying $\mathscr{D}$-modules. In §3.8 below, this formalism is extended to *αL-twisted polarizable Hodge modules*: for $\alpha \in \mathbb{R}$, these are filtered $\mathscr{D}_{X,\alpha L}$-modules with a pairing valued in the sheaf of αL-twisted currents (that transform by $|g|^{-2\alpha}$ under a change in trivialization of L); see also [66].

We prove a general vanishing theorem of twisted polarizable Hodge modules; for more details, see Theorem 4.7. A version of this theorem has been independently proved by Dougal Davis and Kari Vilonen [14, Theorem 1.4].

Theorem 1.7. *Let D be an effective divisor on a projective complex manifold X and denote $L = \mathcal{O}_X(D)$. For any $\alpha \in \mathbb{Q}$, let M be an αL-twisted Hodge module with strict* *support X and let B be an effective divisor on X such that the $\mathbb{Q}$-divisor $B+\alpha D$ is ample. Then for any $k \in \mathbb{Z}$, we have*

$$H^i\left(X, \operatorname{gr}_k^F \mathrm{DR}(\mathcal{M}) \otimes \mathcal{O}_X(B)\right)=0, \quad \textit{for every } i>0,$$

where $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered twisted $\mathscr{D}$-module underlying M.

The vanishing Theorem 1.4 will be reduced to this theorem. As another application, in §4.1 we give a quick proof of a log Kawamata-Viehweg type Akizuki-Nakano vanishing theorem in [1].

1.3. **Higher multiplier ideals and twisted Hodge modules.** We now give the construction of higher multiplier ideals and discuss how they are related to twisted Hodge modules.

Let D be an effective divisor on a complex manifold X. Let $L=\mathcal{O}_X(D)$ be the corresponding holomorphic line bundle and let $s \in H^0(X, L)$ be a section with $\operatorname{div}(s)=D$. We view s as a closed embedding $s: X \to L$ into the $(n+1)$-dimensional total space of the line bundle, which is also denoted by L. Let

$$M=s_*\mathbb{Q}_X^H[n] \in \mathrm{MHM}(L)$$

be the direct image of the constant Hodge module on X, where $\mathrm{MHM}(L)$ is the category of graded-polarizable mixed Hodge modules on L. There are several interesting filtrations on the underlying $\mathscr{D}$-module of M. First, the filtered *right* $\mathscr{D}$-module underlying M is

$$(\mathcal{M}, F_\bullet\mathcal{M})=s_+(\omega_X, F_\bullet\omega_X), \tag{1.5}$$

where ω_X is viewed as a right $\mathscr{D}$-module with $F_{-n}\omega_X=\omega_X$, $F_{-n-1}\omega_X=0$ and s_+ is the direct image functor for filtered $\mathscr{D}$-modules. One can show that

$$\operatorname{gr}_{-n+k}^F \mathcal{M} \cong s_*(\omega_X \otimes L^k)$$

for $k \in \mathbb{N}$. On the other hand, let $V_\bullet\mathcal{M}$ be the V-filtration relative to the zero section of L; locally it is the V-filtration of Kashiwara and Malgrange relative to any local equation for D (see §2). For every $\alpha \in \mathbb{Q}$, the sheaf $\operatorname{gr}_{-n+k}^F V_\alpha\mathcal{M}$ is a coherent sub-$\mathcal{O}$-module of $\operatorname{gr}_{-n+k}^F \mathcal{M}$. Then the higher multiplier ideal $\mathcal{I}_{k,\alpha}(D)$ is defined as a unique coherent ideal sheaf satisfying

$$\operatorname{gr}_{-n+k}^F V_\alpha\mathcal{M}=s_*\left(\omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D)\right). \tag{1.6}$$

Similarly, we define $\mathcal{I}_{k,<\alpha}(D)$ using $V_{<\alpha}\mathcal{M}$. The discreteness of V-filtration implies that $\mathcal{I}_{k,<\alpha}(D)=\mathcal{I}_{k,\alpha-\epsilon}(D)$ for $0<\epsilon \ll 1$.

Now we relate the higher multiplier ideals with twisted Hodge modules. Let

$$\operatorname{gr}_\alpha^V \mathcal{M}=V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$$

be the associated graded of the V-filtration. There is a weight filtration $W_\bullet(N)\operatorname{gr}_\alpha^V \mathcal{M}$ on $\operatorname{gr}_\alpha^V \mathcal{M}$ induced by the nilpotent monodromy operator N, see §3.11. By (1.1) and (1.6), one has an isomorphism $\omega_X \otimes L^k \otimes \mathcal{G}_{k,\alpha}(D) \cong \operatorname{gr}_{-n+k}^F \operatorname{gr}_\alpha^V \mathcal{M}$, which induces a weight filtration $W_\bullet\mathcal{G}_{k,\alpha}(D)$ on $\mathcal{G}_{k,\alpha}(D)$, and we denote by

$$\operatorname{gr}_\ell^W \mathcal{G}_{k,\alpha}(D):=W_\ell\mathcal{G}_{k,\alpha}(D)/W_{\ell-1}\mathcal{G}_{k,\alpha}(D)$$

the graded pieces. The following result is proved in Proposition 3.12.

Proposition 1.8. *For $\ell \in \mathbb{Z}$ and $\alpha \in [-1,0]$, the pair*

$$(\operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}, F_{\bullet+\lfloor\alpha\rfloor} \operatorname{gr}_\ell^W \operatorname{gr}_\alpha^V \mathcal{M})$$

is a filtered αL-twisted $\mathscr{D}$-module that underlies an αL-twisted polarizable Hodge module on X. Moreover, for any $k \in \mathbb{N}$, we have an isomorphism of coherent $\mathcal{O}_X$-modules

$$\omega_X \otimes L^k \otimes \mathrm{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) \cong \mathrm{gr}^F_{-n+k} \mathrm{gr}^{W(N)}_\ell \mathrm{gr}^V_\alpha \mathcal{M}.$$

Theorem 1.4 then follows from the combination of Proposition 1.8 and the general vanishing theorem 1.7. Note that locally the direct sum $\oplus_{-1\leq\alpha<0} \mathrm{gr}^V_\alpha \mathcal{M}$ is a mixed Hodge module in Saito's sense, but the individual summands $\mathrm{gr}^V_\alpha \mathcal{M}$ are (locally) complex mixed Hodge modules without any rational structure. This is one of the reason why we need the theory of complex mixed Hodge modules.

1.4. **Comparison with other ideals.** The higher multiplier ideals are closely related to other generalization of multiplier ideals. Let D be an effective divisor on a complex manifold X.

First, the higher multiplier ideals $\mathcal{I}_{k,\alpha}(D)$ can recover the microlocal V-filtration $\tilde{V}^\bullet\mathcal{O}_X$, which is an decreasing filtration induced on $\mathcal{O}_X$ by D, and vice versa. It is first defined in the case of global hypersurfaces by Saito [64, 65] and later generalized to arbitrary effective divisors in [41]. The minimal exponent $\tilde{\alpha}_D$ can be characterized as the first jumping number of microlocal V-filtration. The following result is proved in Corollary 5.16.

Proposition 1.9. *Let D be an effective divisor on X. Then*

$$\tilde{V}^\beta\mathcal{O}_X = \begin{cases} \mathcal{I}_{\lfloor\beta\rfloor,-\{\beta\}}(D) & \text{if } \beta \notin \mathbb{N}, \\ \mathcal{I}_{\beta-1,-1}(D) & \text{if } \beta \in \mathbb{N}_{\geq 1}, \end{cases}$$

where $\lfloor\beta\rfloor$ and $\{\beta\}$ are the integer and fractional parts of β. Conversely,

$$\mathcal{I}_{k,\alpha}(D) = \begin{cases} \tilde{V}^{k-\alpha}\mathcal{O}_X, & \text{if } \alpha \geq -1, \\ \tilde{V}^{k-(\alpha+t)}\mathcal{O}_X \otimes \mathcal{O}_X(-tD), & \text{if } \alpha < -1 \text{ and } -1 \leq t+\alpha < 0 \text{ for a unique } t \in \mathbb{N}. \end{cases}$$

This proposition is very useful for computational purposes. For example, it induces a Thom-Sebastiani formula for higher multiplier ideals and one can use it to compute $\mathcal{I}_{k,\alpha}(D)$ for hypersurfaces with weighted homogeneous isolated singularities, see §6.3 and Remark 6.10.

There is another well-known generalization of multiplier ideals of $\mathbb{Q}$-divisors, called *Hodge ideals*, proposed by the first author [67] and then developed by Mustaţă and Popa in a series of work [45, 46, 47]. Later, Olano further studied the weight filtration on Hodge ideals, called *weighted Hodge ideals*, for reduced divisors in [51, 52]. We show the following comparison result; for details, see §5.5.

Proposition 1.10. *Let D be a reduced effective divisor on a smooth algebraic variety. For any $\alpha \in [-1,0)$ and $k \in \mathbb{N}$, we have*

$$\mathcal{I}_{k,\alpha}(D) \equiv I_k(-\alpha D) \quad \text{mod } \mathcal{I}_D, \tag{1.7}$$

where $I_k(-\alpha D)$ is the k^{th} Hodge ideal of the $\mathbb{Q}$-divisor $-\alpha D$. Furthermore, for any $\ell \in \mathbb{Z}$, one has

$$W_\ell\mathcal{I}_{k,-1}(D) = \begin{cases} I_0^{W_{\ell+1}}(D), & \text{if } k = 0, \\ I_k^{W_{\ell+1}}(D) \quad \text{mod } \mathcal{I}_D, & \text{if } k \geq 1, \end{cases} \tag{1.8}$$

where $W_\bullet\mathcal{I}_{k,-1}(D)$ is induced by the weight filtration $W_\bullet\mathcal{G}_{k,-1}(D)$ (see Definition 5.8) and $I_k^{W_\bullet}(D)$ is the weight filtration on the Hodge ideal $I_k(D)$.

The first equality (1.7) follows from Proposition 1.9 and a result of Mustaţă and Popa [47]; for the weighted version (1.8) we follow a different path: the idea is that Hodge ideals depend on the $\mathscr{D}$-module $\mathcal{O}_X(*D)$, which is determined by $X \setminus D$, and higher multiplier ideals are controlled by $\mathscr{D}$-modules $\mathrm{gr}^V_\alpha \mathcal{M}$ related to the embedding $D \hookrightarrow X$. These $\mathscr{D}$-modules are related by the two distinguished triangles associated to a closed embedding. The additional shift by 1 for the weighted case comes from the weight convention on $\mathrm{gr}^V_{-1} \mathcal{M}$, see (2.3). It happens already for ordinary singularities that the equality (1.7) can hold without $\mathcal{I}_D$ and can also fail, see Example 5.19 and Example 6.20.

1.5. **Applications.** The first sets of application of higher multiplier ideals is a better understanding of minimal exponents. Using fundamental theorems of higher multiplier ideals and the characterization (1.2), we deduce some properties of minimal exponents of hypersurfaces, see Corollary 7.21.

Proposition 1.11. *Let D be an effective divisor on a complex manifold X.*

(1) Let $m \geq 2$ and suppose that $Z \subseteq \mathrm{Sing}_m(D) = \{x \in D \mid \mathrm{mult}_x(D) \geq m\}$ is an irreducible component of dimension d. Then

$$\tilde{\alpha}_D \leq \frac{\mathrm{codim}_X(Z)}{m}. \tag{1.9}$$

(2) Let $H \subseteq X$ be a smooth hypersurface that is not entirely contained in the support of D. Denote by $D_H := D|_H$. Then

$$\tilde{\alpha}_{D_H} \leq \tilde{\alpha}_D.$$

Equality holds if H is sufficiently general.

(3) Consider a smooth morphism $\pi : X \to T$, together with a section $s : T \to X$ such that $s(T) \subseteq D$. If D does not contain any fiber of π, so that for every $t \in T$ the divisor $D_t = D|_{\pi^{-1}(t)}$ is defined, then the function

$$T \to \mathbb{Q}, \quad t \mapsto \tilde{\alpha}_{D_t, s(t)}$$

is lower semicontinuous.

The second and the third statement have been proved in [47, Theorem E] using Hodge ideals. Combining (1.9) with a lower bound from [47, Corollary D] in terms of log resolutions of (X, D), we can compute the minimal exponent in certain circumstances as follows (see Proposition 7.24).

Proposition 1.12. *Let D be an effective divisor on a complex manifold X. Assume there exists a log resolution $\pi : \tilde{X} \to X$ of (X, D) satisfying the following conditions:*

- *the proper transform $\tilde{D}$ is smooth and $\pi^* D$ has simple normal crossing support,*
- *the morphism π is the iterated blow up of X along (the proper transform of) all irreducible components of $\mathrm{Sing}_m(D)$ for all $m \geq 2$, and all the blow-up centers are smooth.*

Then

$$\tilde{\alpha}_D = \min \frac{\mathrm{codim}_X(Z)}{m},$$

where the minimum runs through all $m \geq 2$ and all irreducible components Z of $\mathrm{Sing}_m(D)$.

We use this to compute several geometrically interesting examples, including the theta divisor on the Jacobian of a smooth hyperelliptic curve (see Theorem 9.6) and the secant hypersurface of a curve embedded by a positive enough line bundle (see Example 7.25), which seem quite hard to compute directly.

Higher multiplier ideals also give new applications to singularities of divisors on projective spaces and theta divisors.

Proposition 1.13. *Let D be a reduced hypersurface of degree d in $\mathbf{P}^n$ with $n \geq 3$. The the set of isolated singular points on D of multiplicity $m \geq 2$ imposes independent conditions on hypersurfaces of degree at least*

$$\left\lceil \frac{n+1-\lceil n/m \rceil}{m-1} \right\rceil \cdot d - n - 1.$$

If $n = 3$ and D has only nodal singularities, then the bound above is $2d-4$, which is still one worse than Severi's $2d-5$ bound. In general, our bound is better than the ones obtained by [45, Corollary H] using Hodge ideals, and thus is better than what is known for most other n and m in the literature, see the further discussion in §8.4.

For abelian varieties, we have the following application to theta divisors and the geometric Riemann-Schottky problem. Let us recall the following conjecture from [11]; for more detailed discussion, see §9. Let (A, Θ) be an indecomposable principally polarized abelian variety, i.e. Θ is irreducible.

Conjecture 1.14. If (A, Θ) is not a hyperelliptic Jacobian or the intermediate Jacobian of a smooth cubic threefold, then

$$\dim \mathrm{Sing}_m(\Theta) \leq g - 2m$$

for every $m \geq 2$.

When $m = 2$, the conjecture is due to Debarre [15], which says that an irreducible ppav is the Jacobian of a hyperelliptic curve if and only if the singular locus of Θ has codimension 3 in A. Our first application is a partial solution of this conjecture.

Theorem 1.15. *Assume the center of minimal exponent Y of (A, Θ), defined using* (1.4), *is a one dimensional scheme, then*

$$\dim \mathrm{Sing}_m(\Theta) \leq g - 2m + 1, \textit{for all } m \geq 2,$$

and Y must be a smooth hyperelliptic curve. Moreover, if there exists $m \geq 2$ such that

$$\dim \mathrm{Sing}_m(\Theta) = g - 2m + 1,$$

then one of the following holds

(1) either $(A, \Theta) = (\mathrm{Jac}(Y), \Theta_{\mathrm{Jac}(Y)})$,
(2) or $g(Y) = 2m$, $\dim A = 2m-1$, the minimal exponent of Θ is $\frac{2m-1}{m}$ and Θ has a singular point of multiplicity m.

We also have the following general statement, due to Popa [55].

Proposition 1.16. *A modified Conjecture A of Pareschi and Popa* [53] *implies Conjecture 1.14.*

Here we need to modify this conjecture slightly, see §9.4. The idea is that, assume $\dim \mathrm{Sing}_m(\Theta) \geq g - 2m + 1$, then we can show that the center of minimal exponent Y of (A, Θ) generates A and the twisted ideal sheaf $\mathcal{I}_Y(2\Theta)$ has the IT_0 property. Then a modified version of [53, Conjecture A] perdicts that (A, Θ) must be a Jacobian or the intermediate Jacobian of a smooth cubic threefold. By a result of Martens, if (A, Θ) is a Jacobian and $\dim \mathrm{Sing}_m(\Theta) \geq g - 2m + 1$, it must be hyperelliptic. Therefore Conjecture 1.14 holds.

To have a better understanding of the picture, we also compute certain higher multiplier ideals for theta divisors in the boundary case of Conjecture 1.14, see Theorem 9.6 and Theorem 9.8.

Finally, assuming Θ has only isolated singularities, it is proved by Mustaţă and Popa [45, Theorem I] that $\mathrm{mult}_x(\Theta) \leq (g+1)/2$ for every $x \in \Theta$. We give an alternative proof of their result using higher multiplier ideals in §9.5.

1.6. **Statement in terms of left $\mathscr{D}$-modules.** In this work, we work exclusively with *right* $\mathscr{D}$-modules (because this is more natural where spaces with singularities are involved), but one can also use *left* $\mathscr{D}$-modules to define higher multiplier ideals, where the notation is more aligned with the classical theory. To illustrate this, let us state some results.

Let us repeat the set-up in the beginning of introduction: assume D is an effective divisor on X. Let $L = \mathcal{O}_X(D)$ be the corresponding holomorphic line bundle, and let $s \in H^0(X, L)$ be a section with $\mathrm{div}(s) = D$, which is also viewed as a closed embedding $s : X \to L$, where L is the total space of line bundle on L. Set

$$M = s_* \mathbb{Q}_X^H[n] \in \mathrm{MHM}(L)$$

be the direct image of the constant Hodge module on X. Consider the underlying *left* filtered $\mathscr{D}_X$-module

$$(\mathcal{M}, F_\bullet \mathcal{M}) = s_+(\mathcal{O}_X, F_\bullet \mathcal{O}_X),$$

with $F_0\mathcal{O}_X = \mathcal{O}_X$ and $F_{-1}\mathcal{O}_X = 0$. Let $V^\bullet \mathcal{M}$ be the V-filtration of M relative to the zero section of L.

For each $k \in \mathbb{N}$ and $\beta \in \mathbb{Q}$, we define $\mathcal{J}_k(\beta D)$ to be the unique ideal sheaf on X satisfying

$$s_*(\mathcal{J}_k(\beta D) \otimes \mathcal{O}_X(kD)) = \mathrm{gr}_k^F V^{>\beta} M.$$

Using the translation rule between left and right $\mathscr{D}$-modules, we have

$$\mathcal{J}_k(\beta D) = \mathcal{I}_{k,<-\beta}(D), \quad \mathcal{I}_{k,\beta}(D) = \mathcal{J}_k((-\beta - \epsilon)D).$$

On the associated graded level we have

$$\frac{\mathcal{J}_k((\beta - \epsilon)D)}{\mathcal{J}_k(\beta D)} = \mathcal{G}_{k,-\beta}(D).$$

Using this, all results discussed in the earlier part of Introduction can be translated. For example, the Budur-Saito result translates into

$$\mathcal{J}_0(\alpha D) = \mathcal{J}(\alpha D), \quad \forall \alpha \geq 0.$$

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2. Kashiwara-Malgrange V-filtrations

The work of Budur and Saito [9] reinterprets multiplier ideals using the V-filtration (or Kashiwara-Malgrange filtration). The theory of higher multiplier ideals is a further development of this circle of ideas. As a preparation, in this section we recall the definition of the rational Kashiwara-Malgrange filtration and how it interacts with the Hodge filtration in the theory of mixed Hodge modules. For normal crossing divisors, there are explicit formulas for V-filtrations by Saito [61] and Qianyu Chen [13]. Together with bifiltered direct images, these results enable the computation of higher multiplier ideals in terms of log resolutions. Towards the end of this section, we prove some new technical results about V-filtrations, which are essential for the theory of higher multiplier ideals of $\mathbb{Q}$-divisors.

2.1. **Nearby and vanishing cycles, and V-filtrations.** The most important operations in Saito's theory are the nearby and vanishing cycle functors. Let M be a mixed Hodge module on X, with underlying perverse sheaf K and filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. Suppose that $f : X \to \mathbb{C}$ is a nonconstant holomorphic function. The *nearby cycles* respectively *unipotent vanishing cycles*

$$\psi_f M = \bigoplus_{|\lambda|=1} \psi_{f,\lambda} M \quad \text{and} \quad \phi_{f,1} M$$

are again mixed Hodge modules on X; they are supported on the hypersurface $X_0 = f^{-1}(0)$, and contain information about the behaviour of M near X_0. The underlying perverse sheaf $\psi_f K$ was constructed by Deligne: he sets $\psi_f K = i^{-1}\mathbf{R}k_*(k^{-1}K)[-1]$, where k and i are as follows:

$$\begin{tikzcd} \tilde{X} \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] & X_0 \arrow[l, "i"'] \arrow[d] \\ \mathbb{C} \arrow[r, "e^{2\pi i z}"] & \mathbb{C} & \{0\} \arrow[l] \end{tikzcd}$$

Roughly speaking, Deligne's definition pulls K back to the "generic fiber" of f, and then retracts onto the "special fiber", with a shift to keep the complex perverse. Accordingly, $\psi_f K$ contains more information about the behaviour of K near the hypersurface X_0 than the naive restriction $i^{-1}K$. The deck transformation $z \mapsto z + 1$ induces a monodromy transformation $T \in \operatorname{Aut}(\psi_f K)$, and

$$\psi_f K = \bigoplus_{|\lambda|=1} \psi_{f,\lambda} K$$

is the decomposition into generalized eigenspaces; the eigenvalues are actually roots of unity. Somewhat indirectly, $\phi_{f,1}K$ is then constructed as a part of the following distinguished triangle

$$i^{-1}K[-1] \to \psi_{f,1}K \to \phi_{f,1}(K) \to i^{-1}K.$$

The analogous construction for $\mathscr{D}$-modules needs the V-filtration of Kashiwara and Malgrange. Let $\mathcal{M}$ be a right $\mathscr{D}$-module on X and let $f : X \to \mathbb{C}$ be a non-constant holomorphic function. The V-filtration only makes sense for smooth hypersurfaces and so we use the graph embedding

$$i_f : X \to X \times \mathbb{C}, \quad i_f(x) = (x, f(x))$$

and work with the filtered right $\mathscr{D}$-module

$$(\mathcal{M}_f, F_\bullet\mathcal{M}_f) = (i_f)_+(\mathcal{M}, F_\bullet\mathcal{M}) \tag{2.1}$$

on $X \times \mathbb{C}$. Let t be the coordinate on $\mathbb{C}$, and $\partial_t = \partial/\partial_t$ the corresponding vector field. Inside $\mathscr{D}_{X\times\mathbb{C}}$, we have the subsheaf $V_0\mathscr{D}_{X\times\mathbb{C}}$ of those differential operators that preserve the ideal sheaf of $X \times \{0\}$; it is generated by $\mathscr{D}_X$ and the additional two operators t and $t\partial_t$.

Definition 2.1. The *Kashiwara-Malgrange V-filtration* on $\mathcal{M}_f$ is an increasing filtration $V_\bullet\mathcal{M}_f$, indexed by $\mathbb{Q}$, with the following properties:

(1) Each $V_\alpha\mathcal{M}_f$ is coherent over $V_0\mathscr{D}_{X\times\mathbb{C}}$.

(2) $V_\alpha \mathcal{M}_f$ is indexed right-continuously and discretely, i.e.

$$V_\alpha \mathcal{M}_f = \bigcap_{\beta > \alpha} V_\beta \mathcal{M}_f,$$

so that

$$V_{<\alpha} \mathcal{M}_f := \cup_{\beta < \alpha} V_\beta \mathcal{M}_f = V_{\alpha - \epsilon} \mathcal{M}_f$$

for some $0 < \epsilon \ll 1$. Moreover, the set of $\alpha \in \mathbb{Q}$ such that

$$\operatorname{gr}_\alpha^V \mathcal{M}_f := V_\alpha \mathcal{M}_f / V_{<\alpha} M_f \neq 0$$

is discrete.

(3) One has $V_\alpha \mathcal{M}_f \cdot t \subseteq V_{\alpha-1} \mathcal{M}_f$, with equality for $\alpha < 0$.
(4) One has $V_\alpha \mathcal{M}_f \cdot \partial_t \subseteq V_{\alpha+1} \mathcal{M}_f$.
(5) The operator $t\partial_t - \alpha$ acts nilpotently on $\operatorname{gr}_\alpha^V \mathcal{M}_f$.

Remark 2.2. If $\mathcal{M}$ is holonomic, Kashiwara [27] proved that the V-filtration exists and is unique.

Remark 2.3. The notion of V-filtration can be easily extended to any effective divisor in X. This is because $V_\alpha \mathcal{M}_f$ only depends on the ideal sheaf generated by f, see [7, Proposition 1.5], hence only on the ideal sheaf of the divisor.

Remark 2.4. If $f^{-1}(0)$ is smooth, then there exists a unique filtration $V_\alpha \mathcal{M}$ with the same properties, where one sets $t = f$ and replaces $V_0 \mathscr{D}_{X \times \mathbb{C}}$ by $V_0 \mathscr{D}_X$.

If $\mathcal{M}$ is regular holonomic, Kashiwara [27] showed that each $\operatorname{gr}_\alpha^V \mathcal{M}_f$ is again a regular holonomic $\mathscr{D}_X$-module, with a nilpotent endomorphism $N = t\partial_t - \alpha$. Furthermore, let $K = \mathrm{DR}(\mathcal{M})$, he proved that for $-1 \leq \alpha < 0$, one has

$$\mathrm{DR}(\operatorname{gr}_\alpha^V \mathcal{M}_f) \cong \psi_{f, e^{2\pi i \alpha}} K, \quad \text{and} \quad \mathrm{DR}(\operatorname{gr}_0^V \mathcal{M}_f) \cong \phi_{f,1} K; \tag{2.2}$$

under these isomorphisms, the monodromy operator T becomes equal to $e^{2\pi i \alpha} \cdot e^{2\pi i N}$. To summarize, the $\mathscr{D}$-module theoretic nearby and vanishing cycles can be described in the following way. Let $F_\bullet \mathcal{M}$ be a good filtration on $\mathcal{M}$. For $-1 \leq \alpha < 0$, the nearby cycles for the eigenvalue $\lambda = e^{2\pi i \alpha}$ are described by the following data:

$$\psi_{f,\lambda} M = \left(\operatorname{gr}_\alpha^V \mathcal{M}_f, F_{\bullet - 1} \operatorname{gr}_\alpha^V \mathcal{M}_f, \psi_{f,\lambda} K \right), \quad W_\bullet \operatorname{gr}_\alpha^V \mathcal{M}_f = W_{\bullet + n - 1}(N) \operatorname{gr}_\alpha^V \mathcal{M}_f, \tag{2.3}$$

where $W_\bullet(N)$ means the weight filtration of N and

$$F_\bullet \operatorname{gr}_\alpha^V \mathcal{M}_f := \frac{F_\bullet \mathcal{M}_f \cap V_\alpha \mathcal{M}_f}{F_\bullet \mathcal{M}_f \cap V_{<\alpha} \mathcal{M}_f}.$$

For the unipotent vanishing cycles, this changes to

$$\phi_{f,1} M = \left(\operatorname{gr}_0^V \mathcal{M}_f, F_\bullet \operatorname{gr}_0^V \mathcal{M}_f, \phi_{f,1} K \right), \quad W_\bullet \operatorname{gr}_0^V \mathcal{M}_f = W_{\bullet + n}(N) \operatorname{gr}_0^V \mathcal{M}_f. \tag{2.4}$$

These shifts are needed to make the definition work out properly in all cases.

In the definition of mixed Hodge modules, Saito imposes the following two additional conditions on how the rational V-filtration interacts with the Hodge filtration $F_\bullet \mathcal{M}_f$.

Definition 2.5. We say that the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet \mathcal{M})$ is *quasi-unipotent* along $f = 0$ if all eigenvalues of the monodromy operator on $\psi_f K$ are roots of unity, and if the V-filtration $V_\bullet \mathcal{M}_f$ satisfies the following two additional conditions:

(1) $t : F_p V_\alpha \mathcal{M}_f \to F_p V_{\alpha-1} \mathcal{M}_f$ is an isomorphism for $\alpha < 0$ and $p \in \mathbb{Z}$.
(2) $\partial_t : F_p \operatorname{gr}_\alpha^V \mathcal{M}_f \to F_{p+1} \operatorname{gr}_{\alpha+1}^V \mathcal{M}_f$ is an isomorphism for $\alpha > -1$ and $p \in \mathbb{Z}$.

Remark 2.6. If $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered $\mathscr{D}$-module underlying a mixed Hodge module on X, then $(\mathcal{M}, F_\bullet\mathcal{M})$ is quasi-unipotent along any local holomorphic function. If $(\mathcal{M}, F_\bullet\mathcal{M})$ has strict support X, then $\partial_t : F_p \operatorname{gr}^V_\alpha \mathcal{M}_f \to F_{p+1} \operatorname{gr}^V_{\alpha+1} \mathcal{M}_f$ is surjective for $\alpha = -1$ and $p \in \mathbb{Z}$.

Let us end this section with a result about how the V-filtration behaves under non-characteristic restriction to a hypersurface. This is proved in [16, Thm. 1.1].

Lemma 2.7. *Let t be a smooth function on X, let $i : H \hookrightarrow X$ be a hypersurface that is not contained in $X_0 = t^{-1}(0)$, and set $\mathcal{M}_H = i^*\mathcal{M} = \omega_{H/X} \otimes_{\mathcal{O}_X} \mathcal{M}$. Suppose that H is non-characteristic with respect to both $\mathcal{M}$ and $\mathcal{M}(*X_0)$, then there is a natural isomorphism induced by i:*

$$V_\alpha(\mathcal{M}_H) = \omega_{H/X} \otimes_{\mathcal{O}_X} V_\alpha\mathcal{M}, \quad \textit{for all } \alpha,$$

where $V_\alpha(\mathcal{M}_H)$ is the V-filtration with respect to $t|_H$.

2.2. **Birational formula for V-filtrations.** We use the work of Qianyu Chen [13] and Saito's bifiltered direct images to give a formula for the associated graded of the V-filtration in terms of log resolutions, using sheaves of log forms. It is a crucial tool for the computation of higher multiplier ideals of ordinary singularities in §6.

The set up of this section is the same as the one in §1.3 of the Introduction. Let D be an effective divisor on a complex manifold X of dimension n and set $L = \mathcal{O}_X(D)$. Let $s : X \to L$ be the graph embedding associated to a section $s \in H^0(X, L)$ so that $D = \operatorname{div}(s)$. Denote by

$$M_X = s_*\mathbb{Q}^H_X[n] \in \operatorname{MHM}(L),$$

the direct image of the constant Hodge module on X. Denote by $V_\bullet\mathcal{M}_X$ the V-filtration of $\mathcal{M}_X$ with respect to the zero section of L. Let $\pi : Y \to X$ be a log resolution of (X, D) and write

$$\pi^*D = \sum_{i\in I} e_iY_i, \quad e_i \in \mathbb{N}, \tag{2.5}$$

which is a normal crossing divisor on Y. Analogous to the construction of M, we define

$$M_Y = s_{Y,*}\mathbb{Q}^H_Y[n] \in \operatorname{MHM}(L_Y),$$

where $s_Y : Y \to L_Y$ is the graph embedding of the divisor π^*D and L_Y is the total space of $\mathcal{O}_Y(\pi^*D)$.

Notation 2.8. Here we use M_X and M_Y to stress the dependence on spaces. In the below, we follow the notation in [13] closely with minor differences: Chen's Y is our divisor π^*D, and Chen's X is our ambient space Y.

For any $\alpha \in [-1, 0)$, define the index set

$$I_\alpha := \{i \in I \mid e_i \cdot \alpha \in \mathbb{Z}\}.$$

For any subset $J \subseteq I_\alpha$, we define a pair (Y^J, E) as follows. First set

$$Y^J := \bigcap_{j\in J} Y_j,$$

with the following commutative diagram

$$\begin{tikzcd} Y^J \arrow[r, "\tau^J"] \arrow[dr, "\pi^J"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Then consider a normal crossing divisor on Y^J,

$$E := \bigcup_{i\in I\setminus I_\alpha} \left(Y_i \cap Y^J\right),$$

the union of divisors in the set $I \setminus I_\alpha$ restricting to Y^J. It is direct to check that

$$E = \operatorname{supp}\{\alpha\pi^* D\}|_{Y^J}, \tag{2.6}$$

where $\{\sum a_i E_i\} = \sum\{a_i\}E_i$ and $\operatorname{supp}(\sum b_i E_i) = \sum E_i$. For a normal crossing divisor $E \subset X$, denote by $\Omega^\bullet_X(\log E)$ the sheaves of log forms on X with poles along E.

Proposition 2.9. *Fix $\alpha \in [-1, 0)$, $k \in \mathbb{N}$ and $r, p \in \mathbb{Z}$, then the following results hold.*

(1) One has

$$(\operatorname{gr}^V_\alpha \mathcal{M}_X, F_\bullet) = R^0\pi_+(\operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet). \tag{2.7}$$

Moreover, $(\operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_X, F_\bullet)$ is the cohomology of

$$R^{-1}\pi_+(\operatorname{gr}^W_{r+1} \operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet) \to R^0\pi_+(\operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet) \to R^1\pi_+(\operatorname{gr}^W_{r-1} \operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet).$$

(2) There is a nilpotent operator N on $\operatorname{gr}^V_\alpha \mathcal{M}_Y$ so that $N^{|I_\alpha|} = 0$, where $|I_\alpha|$ is the cardinality of I_α. For any $r \geq 0$, there is a filtered isomorphism

$$(\operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet) \cong \bigoplus_{\ell\geq 0} N^\ell \left(\bigoplus_{\substack{J\subseteq I_\alpha,\\ |J|=r+1+2\ell}} \tau^J_+(\mathcal{V}_{\alpha,J}, F_{\bullet-\ell})(-\operatorname{codim}_Y(Y^J)) \right), \tag{2.8}$$

where $(\mathcal{V}_{\alpha,J}, F_\bullet)$ is a filtered $\mathscr{D}$-module on Y^J and the Tate twist is $(\mathcal{N}, F_\bullet\mathcal{N})(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$. Moreover, if $\alpha \in (-1, 0)$, then

$$\operatorname{gr}^F_{-\dim Y^J+k} \operatorname{DR}_{Y^J}(\mathcal{V}_{\alpha,J}) \cong \Omega^{\dim Y^J-k}_{Y^J}(\log E) \otimes_{\mathcal{O}_{Y^J}} \mathcal{O}_Y(\lfloor\alpha\pi^* D\rfloor)[k]. \tag{2.9}$$

For $\alpha = -1$, one has

$$(\mathcal{V}_{-1,J}, F_\bullet) \cong (\omega_{Y^J}, F_\bullet). \tag{2.10}$$

(3) We have

$$\operatorname{gr}^V_\alpha \mathcal{M}_X = 0, \quad \textit{whenever } I_\alpha = \varnothing. \tag{2.11}$$

Remark 2.10. When apply the proposition in practice, one needs to compute $\operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y$ using (2.8). There one needs to be careful to note that $\operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y$ is the direct sum of direct images of $\mathscr{D}$-modules on Y^J. For the direct image induced by $\tau^J : Y^J \to Y$, one has

$$\operatorname{gr}^F_p(\tau^J_+\mathcal{N}) \neq \tau_* \operatorname{gr}^F_p \mathcal{N}!$$

See Lemma 6.4. See Proposition 6.5 for how to apply this birational formula in practice.

Proof. Since π is proper and $\alpha \in [-1, 0)$, the bistrictness property in Saito's bifiltered direct images for Hodge and V-filtrations [61, (3.3.3)-(3.3.5)] (see also [9, (3.2.2)]) implies that

$$(\operatorname{gr}^V_\alpha \mathcal{M}_X, F_\bullet) = \pi_+(\operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet), \tag{2.12}$$

thus (2.7) holds. Moreover, by [61, Proposition 5.3.5], there is a weight spectral sequence

$$E_1^{r,s} = R^{r+s}\pi_+(\operatorname{gr}^W_{-r}(\operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet)) \Longrightarrow R^{r+s}\pi_+(\operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet),$$

which degenerates at E_2-page.

Concerning the Lefschetz decomposition, we can assume $D = \operatorname{div}(f)$. Then $\mathcal{O}_Y(\pi^*D) \cong \mathcal{O}_Y$ and by shrinking the target slightly there is a proper holomorphic morphism

$$f \circ \pi : Y \to X \to \Delta,$$

such that f is smooth away from the origin and the central fiber $Y_0 = (f \circ \pi)^{-1}(0) = \pi^*D$ is a normal crossing divisor. Therefore we are in the setting of Chen [13] and we can use his work to compute the primitive pieces of $(\operatorname{gr}_r^W \operatorname{gr}_\alpha^V \mathcal{M}_Y, F_\bullet)$ with respect to the nilpotent operator N. But this needs a bit translation as follows. In [13, Theorem C], Chen constructed a filtered *right* $\mathscr{D}_Y$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ equipped with an operator $R : (\mathcal{M}, F_\bullet\mathcal{M}) \to (\mathcal{M}, F_{\bullet+1}\mathcal{M})$ with eigenvalues in $[0,1) \cap \mathbb{Q}$ so that there is a filtered quasi-isomorphism

$$(\operatorname{DR}_Y \mathcal{M}, F_\bullet) \xrightarrow{\sim} (\Omega_{Y/\Delta}^{\bullet+n-1}(\log Y_0)|_{Y_0}, F_\bullet),$$

where $n = \dim Y$ and the right hand side is Steenbrink's relative log de Rham complex with the "stupid" Hodge filtration, see [13, Page 2]. Moreover, the operator $\operatorname{DR}_Y(R)$ can be identified with Steenbrink's operator $[\nabla] \in \operatorname{End}_{D_c^b(Y,\mathbb{C})}(\Omega_{Y/\Delta}^{\bullet+n-1}(\log Y_0)|_{Y_0})$, where $D_c^b(Y,\mathbb{C})$ stands for the derived category of $\mathbb{C}$-constructible sheaves on Y. On the other hand, Steenbrink [71] proved that there is a quasi-isomorphism

$$\Omega_{Y/\Delta}^{\bullet+n-1}(\log Y_0)|_{Y_0} \xrightarrow{\sim} \psi_{f\circ\pi}(\mathbb{C}_Y[n]),$$

where the latter is the nearby cycle complex. Moreover, he proved that the operator $[\nabla]$ corresponds to the monodromy operator on the nearby cycle (because $[\nabla]$ is the Gauss-Manin connection). For $\beta \in [0,1)$, denote by $\mathcal{M}_\beta$ the β-generalized eigenspace of $\mathcal{M}$ with respect to R, i.e. $R_\beta := R - \beta$ is nilpotent on $\mathcal{M}_\beta$. Combining these two quasi-isomophisms, one concludes that

$$\operatorname{DR}_Y(\mathcal{M}_\beta) \xrightarrow{\sim} \psi_{f\circ\pi, e^{2\pi i\beta}}(\mathbb{C}_Y)[n].$$

For $\alpha \in [-1,0)$, using (2.2) one can show that there is a nilpotent operator N on $\operatorname{gr}_\alpha^V \mathcal{M}_Y$ (locally induced by $t\partial_t - \alpha$) and

$$\operatorname{DR}_Y(\operatorname{gr}_\alpha^V \mathcal{M}_Y) \cong \psi_{f\circ\pi, e^{2\pi i\alpha}}(\mathbb{C}_Y)[n].$$

For a more detailed discussion, see §2.1. This leads to the following translation rule between $\operatorname{gr}_\alpha^V \mathcal{M}_Y$ in our paper and $\mathcal{M}_\beta$ in [13].

Lemma 2.11. *One has*

$$(\operatorname{gr}_\alpha^V \mathcal{M}_Y, F_\bullet) = \begin{cases} (\mathcal{M}_{-\alpha}, F_{\bullet+1}\mathcal{M}_{-\alpha}) & \text{if } \alpha \in (-1,0), \\ (\mathcal{M}_0, F_{\bullet+1}\mathcal{M}_0) & \text{if } \alpha = -1. \end{cases}$$

The monodromy operator N on $\operatorname{gr}_\alpha^V \mathcal{M}_Y$ corresponds to the operator R_β on $\mathcal{M}_\beta$, where $\beta = -\alpha$ if $\alpha \in (-1,0)$ and $\beta = 0$ if $\alpha = -1$.

With the translation above, Chen's result can be stated as follows. For the ease of notation, we only deal with the case $\alpha \in (-1,0)$; the case $\alpha = -1$ is similar. Set $|I_\alpha| = \ell + 1$ for $\ell \in \mathbb{N}$, then $N^{\ell+1} = 0$ on $\operatorname{gr}_\alpha^V \mathcal{M}_Y$ with the weight filtration

$$0 \subseteq W_{-\ell} \operatorname{gr}_\alpha^V \mathcal{M}_Y \subseteq \cdots \subseteq W_\ell \operatorname{gr}_\alpha^V \mathcal{M}_Y = \operatorname{gr}_\alpha^V \mathcal{M}_Y.$$

In [13, Theorem 7.5], it is shown that N induces a filtered morphism

$$N : (\operatorname{gr}_r^W \operatorname{gr}_\alpha^V \mathcal{M}_Y, F_\bullet) \to (\operatorname{gr}_{r-2}^W \operatorname{gr}_\alpha^V \mathcal{M}_Y, F_{\bullet+1}), \quad \forall r \in \mathbb{Z},$$

so that N^r is an isomorphism if $r \geq 0$. Then for $r \in \mathbb{N}$, the primitive part $\mathcal{P}_{\alpha,r}$ with respect to N is defined by

$$\mathcal{P}_{\alpha,r} := \ker\{N^{r+1} : \operatorname{gr}_r^W \operatorname{gr}_\alpha^V \mathcal{M}_Y \to \operatorname{gr}_{-r-2}^W \operatorname{gr}_\alpha^V \mathcal{M}_Y\},$$

and there is a Hodge filtration $F_\bullet\mathcal{P}_{\alpha,r}$ induced by $F_\bullet \operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y$ such that

$$F_\bullet \operatorname{gr}^W_r \operatorname{gr}^V_\alpha \mathcal{M}_Y = \bigoplus_{\ell\geq 0} N^\ell F_{\bullet-\ell}\mathcal{P}_{\alpha,r+2\ell}. \tag{2.13}$$

Furthermore, [13, Theorem 7.13] shows that there is an isomorphism of filtered $\mathscr{D}$-modules:

$$(\mathcal{P}_{\alpha,r}, F_\bullet) \cong \bigoplus_{J\subseteq I_\alpha, |J|=r+1} \tau^J_+(\mathcal{V}_{\alpha,J}, F_\bullet)(-r), \tag{2.14}$$

where $\tau^J : Y^J \to Y$ is the closed embedding, $(\mathcal{V}_{\alpha,J}, F_\bullet)$ is a filtered $\mathscr{D}$-module on Y^J, $r = \operatorname{codim}_Y(Y^J) - 1$, and the Tate twist is $(\mathcal{N}, F_\bullet\mathcal{N})(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$.

The filtered $\mathscr{D}$-module $(\mathcal{V}_{\alpha,J}, F_\bullet)$ is induced by an integrable log connection on the line bundle $\mathcal{O}_Y(-\lceil -\alpha\pi^*D\rceil) = \mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor)$ (see [13, §7.4]). In a short word, this log connection is defined by

$$\nabla : \mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor) \to \Omega^1_Y(\log \sum_{i\in I\setminus I_\alpha} Y_i) \otimes \mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor)$$

$$\nabla s = \sum_{i\in I\setminus I_\alpha} \{\alpha e_i\} \frac{dz_i}{z_i} \otimes s, \tag{2.15}$$

where z_i is the local equation of Y_i. This connection has poles along Y_i for $i \in I \setminus I_\alpha$, with eigenvalues $\{\alpha e_i\}$. The filtered $\mathscr{D}$-module $(\mathcal{V}_{\alpha,J}, F_\bullet)$ is defined by

$$\mathcal{V}_{\alpha,J} := \big(\omega_{Y^J}(\log E) \otimes_{\mathcal{O}_{Y^J}} (\tau^J)^*(\mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor), \nabla)\big) \otimes_{\mathscr{D}_{Y^J(\log E)}} \mathscr{D}_{Y^J}, \tag{2.16}$$

where

- $\mathscr{D}_{Y^J(\log E)} \subseteq \mathscr{D}_{Y^J}$ is the subsheaf of differential operators preserving the ideal $\mathcal{I}_E$,
- $\omega_{Y^J}(\log E)$ is the sheaf of top holomorphic forms on Y^J with log poles along E, which is a right $\mathscr{D}_{Y^J(\log E)}$-module,
- the Hodge filtration $F_\bullet\mathcal{V}_{\alpha,J}$ is induced by the order filtration on $\mathscr{D}_{Y^J}$ (see [13, Page 51 before Lemma 7.9]).

By [13, Lemma 7.9], there is a filtered quasi-isomorphim in $\mathcal{D}^b_{\mathrm{coh}}(Y^J)$:

$$(\mathrm{DR}_{Y^J}(\mathcal{V}_{\alpha,J}), F_\bullet) \xrightarrow{\sim} (\Omega^{\dim Y^J+\bullet}_{Y^J}(\log E) \otimes_{\mathcal{O}_{Y^J}} \mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor), F_\bullet), \tag{2.17}$$

where $\dim Y^J = n - r - 1$ and the filtration on the right hand side is induced by the "stupid" filtration. In particular, we have a quasi-isomorphism:

$$\operatorname{gr}^F_{-\dim Y^J+k} \mathrm{DR}_{Y^J}(\mathcal{V}_{\alpha,J}) \xrightarrow{\sim} \Omega^{\dim Y^J-k}_{Y^J}(\log E) \otimes_{\mathcal{O}_{Y^J}} \mathcal{O}_Y(\lfloor \alpha\pi^*D\rfloor)[k].$$

Therefor, (2.8) and (2.9) hold using Lemma 2.11 (note that the Hodge filtration is shifted by 1). The case of $\alpha = -1$ proceeds in the same fashion: the only difference is that the corresponding log connection comes from the line bundle $\mathcal{O}_Y$. Hence we need to replace all the terms $\mathcal{O}(\lfloor \alpha\pi^*D\rfloor)$ above by $\mathcal{O}$.

The statement (2.11) is a direct corollary of the previous statements. □

2.3. **V-filtrations for powers of functions.** In this section, we prove a technical result about V-filtrations that is needed to define a version of higher multiplier ideals for $\mathbb{Q}$-divisors (see §5.3). It includes a formula for the nearby cycles of a $\mathscr{D}$-module with respect to a power of a function; in the case of twistor $\mathscr{D}$-modules, this is due to Sabbah [59, 3.3.13]. Let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function on a complex manifold X. We want to relate the nearby cycles and the V-filtration with respect to the two functions

f and f^m. Let $M \in \mathrm{MHM}(X)$, and denote by $(\mathcal{M}, F_\bullet \mathcal{M})$ the underlying filtered right $\mathscr{D}$-module. For $m \geq 1$, consider the graph embedding

$$i_m : X \to X \times \mathbb{C}, \quad i_m(x) = \big(x, f(x)^m\big).$$

In order to keep the notation consistent, we set

$$M_m := (i_m)_* M \in \mathrm{MHM}(X \times \mathbb{C}),$$

and denote the underlying filtered $\mathscr{D}$-module by $(\mathcal{M}_m, F_\bullet \mathcal{M}_m) = (i_m)_+(\mathcal{M}, F_\bullet \mathcal{M})$. Let $V_\bullet \mathcal{M}_m$ denote the V-filtration with respect to the function $t : X \times \mathbb{C} \to \mathbb{C}$.

Proposition 2.12 (V-filtrations for f^m)**.** *For any real number $\alpha < 0$, there is a natural isomorphism of filtered $\mathscr{O}_X$-modules*

$$\phi_\alpha \colon \big(V_{m\alpha} \mathcal{M}_1, F_\bullet V_{m\alpha} \mathcal{M}_1\big) \to \big(V_\alpha \mathcal{M}_m, F_\bullet V_\alpha \mathcal{M}_m\big)$$

such that $\phi_\alpha(v \cdot t^m) = \phi_\alpha(v) \cdot t$ and $\phi_\alpha(v \cdot \frac{1}{m} t\partial_t) = \phi_\alpha(v) \cdot t\partial_t$ for all local sections $v \in V_{m\alpha}\mathcal{M}_1$. If multiplication by f is injective on $\mathcal{M}$, then the same is true for $\alpha = 0$.

Corollary 2.13. *On the level of mixed Hodge modules, this gives*

$$\psi_{f^m,\lambda} M \cong \psi_{f,\lambda^m} M \quad \text{and} \quad \phi_{f^m,1} M \cong \phi_{f,1} M,$$

under the assumption that multiplication by f is injective on $\mathcal{M}$.

The proof takes up the remainder of this section. We observe that the graph embeddings fit into a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, bend right, "i_m"'] & X \times \mathbb{C}_t \arrow[r, "j"] \arrow[d, "p"] & X \times \mathbb{C}_t \times \mathbb{C}_s \arrow[dl, bend left, "q"] \\
& X \times \mathbb{C}_s &
\end{tikzcd}$$

in which $j(x,t) = (x,t,t^m), p(x,t) = (x,t^m)$, and $q(x,t,s) = (x,s)$. To distinguish the two copies of $X \times \mathbb{C}$, let us denote the two coordinate functions by t and s, as indicated in the diagram above; then $s \circ p = t^m$. Since the diagram is commutative, we have

$$M_m \cong p_* M_1 \cong q_*\big(j_* M_1\big). \tag{2.18}$$

Our strategy is to compute these direct images, and then use the bistrictness of direct images with respect to the Hodge and V-filtration.

We begin by describing the $\mathscr{D}$-modules involved in the computation. Since i_1 is a closed embedding, the $\mathscr{D}$-module underlying the direct image $M_1 = (i_1)_* M$ is

$$\mathcal{M}_1 = \mathcal{M} \otimes_\mathbb{C} \mathbb{C}[\partial_t],$$

with the right $\mathscr{D}_{X \times \mathbb{C}_t}$-module structure determined by the following formulas:

$$\begin{aligned}
(u \otimes \partial_t^k) \cdot t &= uf \otimes \partial_t^k + ku \otimes \partial_t^{k-1} \\
(u \otimes \partial_t^k) \cdot \partial_t &= u \otimes \partial_t^{k+1} \\
(u \otimes \partial_t^k) \cdot x_j &= ux_j \otimes \partial_t^k \\
(u \otimes \partial_t^k) \cdot \partial_j &= u\partial_j \otimes \partial_t^k - u\frac{\partial f}{\partial x_j} \otimes \partial_t^{k+1}
\end{aligned}$$

Here $x_1, \dots, x_n$ are local coordinates on X, and $\partial_j = \partial/\partial x_j$ are the corresponding vector fields. The Hodge filtration $F_\bullet \mathcal{M}_1$ is given by

$$F_p \mathcal{M}_1 = \sum_{k \in \mathbb{Z}} F_{p-k} \mathcal{M} \otimes \partial_t^k.$$

Similarly, the $\mathscr{D}$-module underlying the direct image $\widetilde{M}_1 = j_* M_1$ is

$$\widetilde{\mathcal{M}}_1 = \mathcal{M}_1 \otimes_{\mathbb{C}} \mathbb{C}[\partial_s],$$

with the right $\mathscr{D}_{X\times\mathbb{C}_t\times\mathbb{C}_s}$-module structure determined by the formulas

$$\begin{aligned}(v\otimes\partial_s^\ell)\cdot s &= vt^m\otimes\partial_s^\ell + \ell v\otimes\partial_s^{\ell-1}\\ (v\otimes\partial_s^\ell)\cdot\partial_s &= v\otimes\partial_s^{\ell+1}\\ (v\otimes\partial_s^\ell)\cdot P &= vP\otimes\partial_s^\ell \quad \text{for any } P\in\mathscr{D}_X\\ (v\otimes\partial_s^\ell)\cdot\partial_t &= v\partial_t\otimes\partial_s^\ell - mvt^{m-1}\otimes\partial_s^{\ell+1}\end{aligned}$$

and the Hodge filtration

$$F_p\widetilde{\mathcal{M}}_1 = \sum_{\ell\in\mathbb{Z}} F_{p-\ell}\mathcal{M}_1\otimes\partial_s^\ell.$$

Lastly, the filtered $\mathscr{D}$-module underlying $M_m \cong q_*\widetilde{M}_1$ can be computed by the relative de Rham complex, and this gives us a short exact sequence

$$0 \longrightarrow (\widetilde{\mathcal{M}}_1, F_{\bullet-1}\widetilde{\mathcal{M}}_1) \xrightarrow{\partial_t} (\widetilde{\mathcal{M}}_1, F_\bullet\widetilde{\mathcal{M}}_1) \xrightarrow{\pi} (\mathcal{M}_m, F_\bullet\mathcal{M}_m) \longrightarrow 0. \tag{2.19}$$

The next step is to compute the V-filtrations. Since M is a mixed Hodge module, the V-filtration $V_\bullet\mathcal{M}_1$ with respect to the function t exists, and with the convenient shorthand $F_pV_\alpha\mathcal{M}_1 = F_p\mathcal{M}_1\cap V_\alpha\mathcal{M}_1$, the two morphisms

$$t\colon F_pV_\alpha\mathcal{M}_1 \to F_pV_{\alpha-1} \quad\text{and}\quad \partial_t\colon F_p\operatorname{gr}_\alpha^V\mathcal{M}_1 \to F_p\operatorname{gr}_{\alpha+1}^V\mathcal{M}_1$$

are isomorphisms for $\alpha < 0$ respectively $\alpha > 0$ (see Definition 2.5). The V-filtration $V_\bullet\mathcal{M}_m$ with respect to the function s also exists and has the same properties. The following lemma describes the V-filtration of $\widetilde{\mathcal{M}}_1$ with respect to the function $s\colon X\times\mathbb{C}_t\times\mathbb{C}_s\to\mathbb{C}_s$, which is inspired by [62, Theorem 3.4].

Lemma 2.14. *The V-filtration on $\widetilde{\mathcal{M}}_1 = \mathcal{M}_1\otimes_{\mathbb{C}}\mathbb{C}[\partial_s]$ is given by the formula*

$$V_\alpha\widetilde{\mathcal{M}}_1 = \sum_{k,\ell\in\mathbb{N}} \Bigl(V_{m(\alpha-\ell)}\mathcal{M}_1\otimes\partial_s^\ell\Bigr)\cdot\partial_t^k.$$

Proof. It is easy to see that $V_\alpha\widetilde{\mathcal{M}}_1$ is preserved by the action of $\mathscr{D}_{X\times\mathbb{C}_t}$, and that one has $V_\alpha\widetilde{\mathcal{M}}_1\cdot\partial_s\subseteq V_{\alpha+1}\widetilde{\mathcal{M}}_1$ and $V_\alpha\widetilde{\mathcal{M}}_1\cdot s\subseteq V_{\alpha-1}\widetilde{\mathcal{M}}_1$ for every $\alpha\in\mathbb{R}$, with the second inclusion being an equality for $\alpha<0$. It remains to show that each $V_\alpha\widetilde{\mathcal{M}}_1$ is coherent over $V_0\mathscr{D}_{X\times\mathbb{C}_t\times\mathbb{C}_s}$, and that the operator $s\partial_s-\alpha$ acts nilpotently on the quotient $\operatorname{gr}_\alpha^V\widetilde{\mathcal{M}}_1$. Both of these facts rely on the following important identity, which is readily proved using the formulas for the $\mathscr{D}$-module structure on $\widetilde{\mathcal{M}}_1$:

$$(v\otimes\partial_s^\ell)\cdot(s\partial_s-\alpha) = \frac{1}{m}\Bigl(v\bigl(t\partial_t - m(\alpha-\ell)\bigr)\otimes\partial_s^\ell - (v\otimes\partial_s^\ell)\cdot t\partial_t\Bigr) \tag{2.20}$$

Let us first prove the coherence. Set $a = \max(0,\lfloor\alpha\rfloor)$. If $\alpha\notin\mathbb{N}$, then $\alpha - a < 0$, and so for every integer $\ell\geq a+1$, we have

$$V_{m(\alpha-\ell)}\mathcal{M}_1\otimes\partial_s^\ell = V_{m(\alpha-a)}\mathcal{M}_1\cdot t^{m(\ell-a)}\otimes\partial_s^\ell = \Bigl(V_{m(\alpha-a)}\mathcal{M}_1\otimes 1\Bigr)\cdot s^{\ell-a}\partial_s^\ell$$

by the properties of the V-filtration on $\mathcal{M}_1$. Together with the relation in (2.20), this allows us to eliminate all the terms with $\ell\geq a+1$ from the formula for $V_\alpha\widetilde{\mathcal{M}}_1$. If $\alpha\in\mathbb{N}$, then $\alpha=a$, and for every integer $\ell\geq a+1$, we still have

$$V_{m(\alpha-\ell)}\mathcal{M}_1\otimes\partial_s^\ell = V_{-1}\mathcal{M}_1\cdot t^{m-1+m(\ell-a-1)}\otimes\partial_s^\ell = \Bigl(V_{-1}\mathcal{M}_1\cdot t^{m-1}\otimes 1\Bigr)s^{\ell-a-1}\partial_s^\ell.$$

With the help of the identity

$$vt^{m-1} \otimes \partial_s^{\ell+1} = \frac{1}{m}\Big(v\partial_t \otimes \partial_s^\ell - (v \otimes \partial_s^\ell) \cdot \partial_t\Big)$$

we can then again eliminate all terms with $\ell \geq a+1$ from the formula for $V_\alpha \widetilde{\mathcal{M}}_1$. In both cases, the conclusion is that

$$V_\alpha \widetilde{\mathcal{M}}_1 = \sum_{k \in \mathbb{N}} \sum_{\ell=0}^{\max(0,\lfloor \alpha \rfloor)} \Big(V_{m(\alpha-\ell)}\mathcal{M}_1 \otimes \partial_s^\ell\Big) \cdot \partial_t^k. \tag{2.21}$$

Since each $V_{m(\alpha-\ell)}\mathcal{M}_1$ is finitely generated over $V_0\mathscr{D}_{X\times\mathbb{C}_t}$, this shows that $V_\alpha\widetilde{\mathcal{M}}_1$ is finitely generated over $V_0\mathscr{D}_{X\times\mathbb{C}_t\times\mathbb{C}_s}$.

Finally, we argue that $s\partial_s - \alpha$ acts nilpotently on $\operatorname{gr}_\alpha^V \widetilde{\mathcal{M}}_1$. For this, it suffices to show that if $v \in V_{m(\alpha-\ell)}\mathcal{M}_1$ is any local section, then

$$(v \otimes \partial_s^\ell) \cdot (s\partial_s - \alpha)^N \in V_{<\alpha}\widetilde{\mathcal{M}}_1$$

for $N \gg 0$. But since $t\partial_t - m(\alpha-\ell)$ acts nilpotently on $\operatorname{gr}_{m(\alpha-\ell)}^V \mathcal{M}_1$, this is an easy consequence of the identity in (2.20). ☐

The next task is to compute the intersection

$$F_pV_\alpha\widetilde{\mathcal{M}}_1 = F_p\widetilde{\mathcal{M}}_1 \cap V_\alpha\widetilde{\mathcal{M}}_1.$$

This is the content of the following lemma.

Lemma 2.15. *For every $\alpha < 0$ and every $p \in \mathbb{Z}$, one has*

$$F_pV_\alpha\widetilde{\mathcal{M}}_1 = \sum_{k\in\mathbb{N}} \Big(F_{p-k}V_{m\alpha}\mathcal{M}_1 \otimes 1\Big) \cdot \partial_t^k.$$

This also holds for $\alpha = 0$, provided that multiplication by f is injective on $\mathcal{M}$.

Proof. Since the right-hand side is obviously contained in the left-hand side, it suffices to prove the reverse inclusion. According to (2.21), any local section $w \in V_\alpha\widetilde{\mathcal{M}}_1$ can be written, for some $d \in \mathbb{N}$, in the form

$$w = \sum_{k=0}^d \big(v_k \otimes 1\big) \cdot \partial_t^k,$$

with local sections $v_0, \dots, v_d \in V_{m\alpha}\mathcal{M}_1$. Using the formulas for the $\mathscr{D}$-module structure on $\widetilde{\mathcal{M}}_1$, this expression for w can of course be rewritten as

$$\sum_{\ell=0}^d u_\ell \otimes \partial_s^\ell,$$

which is a local section of $F_p\widetilde{\mathcal{M}}_1$ exactly when $u_\ell \in F_{p-\ell}\mathcal{M}_1$ for every $0 \leq \ell \leq d$. We now analyze these expressions from the top down. A short computation shows that

$$u_d = (-1)^d m^d \cdot v_d t^{d(m-1)} \in F_{p-d}V_{m\alpha-d(m-1)}\mathcal{M}_1.$$

Now there are two cases. One case is $\alpha < 0$. Here multiplication by $t^{d(m-1)}$ is a filtered isomorphism, and therefore $v_d \in F_{p-d}V_{m\alpha}$. This means that $v_d \otimes \partial_s^d$ is contained in the right-hand side, and so we can subtract it from the expression for w, and finish the proof by induction on $d \geq 0$. The other case is $\alpha = 0$. Here we can only conclude that $v_d t \in F_{p-d}V_{-1}\mathcal{M}_1$. But since M is a mixed Hodge module, the variation morphism $t\colon \operatorname{gr}_0^V \mathcal{M}_1 \to \operatorname{gr}_{-1}^V \mathcal{M}_1$ is strict with respect to the Hodge filtration. It follows that there is some $v_d' \in F_{p-d}V_0\mathcal{M}_1$ such that $v_d' t = v_d t$. Because multiplication by f is injective on

$\mathcal{M}$, multiplication by t is injective on $\mathcal{M}_1$, and therefore $v_d \in F_{p-d}V_0\mathcal{M}_1$. We can then finish the proof exactly as in the case $\alpha < 0$. □

Now let us go back to the short exact sequence in (2.19). Since M is a mixed Hodge module, the relative de Rham complex computing the direct image $M_m \cong q_*\widetilde{M}_1$ is bistrict with respect to the Hodge filtration and the V-filtration [61, (3.3.3)-(3.3.5)]. For $\alpha \leq 0$ and $p \in \mathbb{Z}$, we therefore get an induced short exact sequence

$$0 \longrightarrow F_{p-1}V_\alpha\widetilde{\mathcal{M}}_1 \xrightarrow{\partial_t} F_pV_\alpha\widetilde{\mathcal{M}}_1 \xrightarrow{\pi} F_pV_\alpha\mathcal{M}_m \longrightarrow 0. \tag{2.22}$$

For the remainder of the argument, we shall assume that either $\alpha < 0$, or $\alpha = 0$ and multiplication by f is injective on $\mathcal{M}$. Under this assumption, we have

$$F_pV_\alpha\widetilde{\mathcal{M}}_1 = \sum_{k\in\mathbb{N}} \Big(F_{p-k}V_{m\alpha}\mathcal{M}_1 \otimes 1\Big) \cdot \partial_t^k = F_pV_{m\alpha}\mathcal{M}_1 \otimes 1 + F_{p-1}V_\alpha\widetilde{\mathcal{M}}_1 \cdot \partial_t.$$

We can now conclude from the exactness of (2.22) that the morphism

$$\phi_\alpha \colon F_pV_{m\alpha}\mathcal{M}_1 \to F_pV_\alpha\mathcal{M}_m, \quad \phi_\alpha(v) = \pi(v \otimes 1),$$

is an isomorphism of $\mathscr{O}_X$-modules. From the formulas for the $\mathscr{D}$-module structure on $\widetilde{\mathcal{M}}_1$, we obtain the identities

$$\phi_\alpha(v) \cdot s = \phi_\alpha(v \cdot t^m) \quad \text{and} \quad \phi_\alpha(v) \cdot s\partial_s = \phi_\alpha\left(v \cdot \frac{1}{m}t\partial_t\right).$$

This proves Proposition 2.12, up to the change in notation caused by using s for the coordinate function on the graph embedding $i_m \colon X \to X \times \mathbb{C}_s$ by the function f^m. The asserted identities for the nearby and vanishing cycles of the mixed Hodge module M follow by passing to the graded quotients gr_α^V for $\alpha \in [-1, 0]$.

3. Twisted $\mathscr{D}$-modules

In this section, we give a brief summary of the theory of twisted $\mathscr{D}$-modules by Beilinson and Bernstein [5], both from a local and global point of view. We then introduce *twisted Hodge modules*, by generalizing the theory of polarizable complex Hodge modules developed by Sabbah and the first author [60] to the setting of twisted $\mathscr{D}$-modules. We show that twisted Hodge modules appear naturally as nearby and vanishing cycles with respect to divisors. In §4, we prove a general vanishing theorem for twisted Hodge modules, extending Saito's vanishing theorem for Hodge modules. These global results are crucial for the properties of higher multiplier ideals in later chapters.

3.1. **Local trivializations.** We first explain a convenient way to deal with local trivializations. Let X be a complex manifold, and L a holomorphic line bundle on X. A local trivialization of L is a pair (U, ϕ), where $U \subseteq X$ is an open subset and

$$\phi \colon L|_U \to U \times \mathbb{C}$$

is an isomorphism of holomorphic line bundles. We can restrict a local trivialization to any open subset of U in the obvious way. In order to compare different local trivializations, it is therefore enough to consider local trivializations over the same open subset. Suppose now that we have two local trivializations (U, ϕ) and (U, ϕ'). The composition

$$\phi' \circ \phi^{-1} \colon U \times \mathbb{C} \to L|_U \to U \times \mathbb{C}$$

then has the form $(x, t) \mapsto \big(x, g(x)t\big)$ for a unique holomorphic function $g \in \Gamma(U, \mathscr{O}_U^\times)$ that is nonzero everywhere on U. This way of thinking about local trivializations eliminates all the unnecessary subscripts.

We can also describe the change of trivialization in terms of sections. Let $s \in \Gamma(U, L)$ be the nowhere vanishing section of L determined by (U, ϕ): the composition

$$\phi \circ s \colon U \to L|_U \to U \times \mathbb{C}$$

satisfies $(\phi \circ s)(x) = (x, 1)$. Similarly, define $s' \in \Gamma(U, L)$ using the local trivialization (U, ϕ'). Then $(\phi' \circ s)(x) = (x, g(x))$ and $(\phi' \circ s')(x) = (x, 1)$, and therefore $s = g \cdot s'$.

3.2. **The sheaf of twisted differential operators.** Let X be a complex manifold of dimension n, and let L be a holomorphic line bundle on X. We view L as a complex manifold of dimension $n+1$, and denote the bundle projection by $p\colon L \to X$. On L, we have the usual sheaf of differential operators $\mathscr{D}_L$. Let $\mathcal{I}_X \subseteq \mathscr{O}_L$ be the ideal sheaf of the zero section. This gives us an increasing filtration

$$V_j \mathscr{D}_L = \left\{ P \in \mathscr{D}_L \;\middle|\; P \cdot \mathcal{I}_X^i \subseteq \mathcal{I}_X^{i-j} \text{ for all } i \geq 0 \right\}.$$

We are only going to use $V_0 \mathscr{D}_L$, which consists of those differential operators that preserve the ideal of the zero section, and the quotient

$$\operatorname{gr}_0^V \mathscr{D}_L = V_0 \mathscr{D}_L / V_{-1} \mathscr{D}_L = V_0 \mathscr{D}_L / \mathcal{I}_X V_0 \mathscr{D}_L.$$

This quotient is naturally a sheaf of $\mathscr{O}_X$-bimodules. A local trivialization (U, ϕ) for L determines an isomorphism of sheaves of algebras (and $\mathscr{O}_U$-bimodules)

$$(\phi^{-1})_* \colon \mathscr{D}_U[t\partial_t] \to \operatorname{gr}_0^V \mathscr{D}_L|_U.$$

We denote by $\theta \in \Gamma(U, \operatorname{gr}_0^V \mathscr{D}_L)$ the image of $t\partial_t = t \cdot \partial/\partial t$. This is actually a well-defined global section of $\operatorname{gr}_0^V \mathscr{D}_L$; the invariant description is as the vector field tangent to the natural $\mathbb{C}^\times$-action on the line bundle L. This tells us what $\operatorname{gr}_0^V \mathscr{D}_L$ looks like locally.

Let us now try to understand what happens when we change the trivialization. Suppose that (U, ϕ) and (U, ϕ') are two local trivializations of L, and that we have local holomorphic coordinates $x_1, \dots, x_n$ on the open set U. Set $\partial_j = \partial/\partial x_j$. A short computation using the chain rule shows that the composition

$$\phi'_* \circ (\phi^{-1})_* \colon \mathscr{D}_U[t\partial_t] \to \operatorname{gr}_0^V \mathscr{D}_L|_U \to \mathscr{D}_U[t\partial_t]$$

acts on the vector fields $\partial_1, \dots, \partial_n, t\partial_t$ in the following way:

$$t\partial_t \mapsto t\partial_t, \quad \partial_j \mapsto \partial_j + g^{-1} \frac{\partial g}{\partial x_j} \cdot t\partial_t$$

We use this to define the sheaf of twisted differential operators.

Definition 3.1. Let X be a complex manifold, L a holomorphic line bundle on X, $\alpha \in \mathbb{R}$ a real number. The sheaf of αL-twisted differential operators on X is the quotient

$$\mathscr{D}_{X, \alpha L} = \operatorname{gr}_0^V \mathscr{D}_L / (\theta - \alpha) \operatorname{gr}_0^V \mathscr{D}_L.$$

In a local trivialization (U, ϕ), the sheaf of αL-twisted differential operators is just

$$\mathscr{D}_{X, \alpha L}|_U \cong \mathscr{D}_U[t\partial_t] / (t\partial_t - \alpha) \mathscr{D}_U[t\partial_t] \cong \mathscr{D}_U.$$

We only see the difference with usual differential operators when we change the trivialization: if (U, ϕ) and (U, ϕ') are two local trivializations, then

$$\phi'_* \circ (\phi^{-1})_* \colon \mathscr{D}_U \to \mathscr{D}_{X, \alpha L}|_U \to \mathscr{D}_U$$

acts on the coordinate vector fields $\partial_1, \dots, \partial_n$ as

$$\partial_j \mapsto \partial_j + \alpha \cdot g^{-1} \frac{\partial g}{\partial x_j}. \tag{3.1}$$

This formula is the reason for the name "twisted" differential operators.

3.3. **Twisted $\mathscr{D}$-modules.** We keep the notation from above. An αL-twisted $\mathscr{D}$-module on X is simply a module over the sheaf $\mathscr{D}_{X,\alpha L}$ of αL-twisted differential operators. We generally work with *right* modules (because this is more appropriate when dealing with singularities). Suppose that $\mathcal{M}$ is a sheaf of right $\mathscr{D}_{X,\alpha L}$-modules. If we have a local trivialization (U, ϕ) for L, then the isomorphism

$$(\phi^{-1})_* \colon \mathscr{D}_U \to \mathscr{D}_{X,\alpha L}|_U \tag{3.2}$$

gives $\mathcal{M}|_U$ the structure of a usual $\mathscr{D}_U$-module. A twisted $\mathscr{D}$-module therefore looks like a usual $\mathscr{D}$-module in any local trivialization of L, but the action by vector fields changes according to the formula in (3.1) from one local trivialization to another. Since it is easy to get confused about the signs, we give the local formulas. Let $\mathcal{M}$ be a right module over $\mathscr{D}_{X,\alpha L}$. For every local trivialization (U, ϕ), we get a right $\mathscr{D}_U$-module $\mathcal{M}_{(U,\phi)}$, whose $\mathscr{D}$-module structure is defined by the rule

$$(m \cdot P)_{(U,\phi)} = m \cdot (\phi^{-1})_*(P), \quad \text{for every } P \in \Gamma(U, \mathscr{D}_U).$$

If (U, ϕ') is a second local trivialization, then we obtain

$$\left(m \cdot \left(\partial_j + \alpha g^{-1} \frac{\partial g}{\partial x_j} \right) \right)_{(U,\phi')} = m \cdot (\phi'^{-1})_* \big(\phi'_* \circ (\phi^{-1})_*(\partial_j) \big) = \big(m \cdot \partial_j \big)_{(U,\phi)}.$$

The $\mathscr{D}$-module structure therefore changes from one local trivialization to another in accordance with the identity in (3.1).

Let us also convince ourselves that tensoring by the line bundle L changes the twisting parameter in the expected way. Suppose that $\mathcal{M}$ is a right $\operatorname{gr}_0^V \mathscr{D}_L$-module. The tensor product $\mathcal{M} \otimes L$ is naturally a sheaf of right modules over $L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L$. If we view θ as a morphism $\theta \colon \mathscr{O}_X \to \operatorname{gr}_0^V \mathscr{D}_L$, we see that $L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L$ also has a global section that we denote by the same letter θ.

Lemma 3.2. *There is a canonical isomorphism*

$$\operatorname{gr}_0^V \mathscr{D}_L \cong L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L$$

that takes the global section θ on the left-hand side to $\theta + 1$.

Proof. We give a local proof to show how the formulas work. Let us first work out the local description of $L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L$. Let (U, ϕ) be a local trivialization of L, and let $s \in \Gamma(U, L)$ be the resulting nowhere vanishing section. Denote by $s^{-1} \in \Gamma(U, L^{-1})$ the induced section of the dual line bundle. We get an isomorphism

$$\mathscr{D}_U[t\partial_t] \to L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L|_U, \quad P \mapsto s^{-1} \otimes (\phi^{-1})_*(P) \otimes s.$$

Let $s' \in \Gamma(U, L)$ be the section determined by a second local trivialization (U, ϕ'). This gives us a second isomorphism

$$\mathscr{D}_U[t\partial_t] \to L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L|_U, \quad Q \mapsto s'^{-1} \otimes (\phi'^{-1})_*(Q) \otimes s'.$$

If we compose the first isomorphism with the inverse of the second one, and remember that $s = gs'$, we find that the change of trivialization is

$$\mathscr{D}_U[t\partial_t] \to \mathscr{D}_U[t\partial_t], \quad P \mapsto g^{-1} \cdot \phi'_* \circ (\phi^{-1})_*(P) \cdot g.$$

In local coordinates $x_1, \dots, x_n$ as above, this acts on the vector fields $\partial_1, \dots, \partial_n, t\partial_t$ by

$$t\partial_t \mapsto t\partial_t, \quad \partial_j \mapsto \partial_j + g^{-1} \frac{\partial g}{\partial x_j} \cdot (t\partial_t + 1).$$

This shows that the collection of isomorphisms

$$\mathscr{D}_U[t\partial_t] \to \mathscr{D}_U[t\partial_t], \quad t\partial_t \mapsto t\partial_t + 1, \quad \partial_j \mapsto \partial_j,$$

give us the desired isomorphism between $\operatorname{gr}_0^V \mathscr{D}_L$ and $L^{-1} \otimes \operatorname{gr}_0^V \mathscr{D}_L \otimes L$. ∎

Remark 3.3. In particular, Lemma 3.2 says that if $\mathcal{M}$ is an αL-twisted right $\mathscr{D}$-module, then the tensor product $\mathcal{M} \otimes L$ is an $(\alpha+1)L$-twisted right $\mathscr{D}$-module. More generally, if $\mathcal{M}$ is a right $\mathrm{gr}_0^V \mathscr{D}_L$-module on which the operator $\theta - \alpha$ acts nilpotently, then $\mathcal{M} \otimes L$ is again a right $\mathrm{gr}_0^V \mathscr{D}_L$-module on which $\theta - (\alpha+1)$ acts nilpotently. Note that this only works nicely for right modules: if $\mathcal{M}$ is an αL-twisted left $\mathscr{D}$-module, then $L \otimes \mathcal{M}$ is $(\alpha-1)L$-twisted. This is one of many reasons why it is better to use right $\mathscr{D}_{X,\alpha L}$-modules.

3.4. **Twisted currents.** We also need a notion of twisted currents, in order to define hermitian pairings on twisted $\mathscr{D}$-modules. We first introduce the space of twisted test functions. These are compactly supported sections of a certain smooth line bundle that we now describe. Fix a real number $\alpha \in \mathbb{R}$. The principal $\mathbb{C}^\times$-bundle corresponding to the holomorphic line bundle L is obtained by removing the zero section from L. Let L_α be the smooth line bundle associated to the representation

$$\mathbb{C}^\times \to \mathrm{GL}_1(\mathbb{C}), \quad z \mapsto |z|^{2\alpha}.$$

We can also describe L_α in terms of local trivializations. A local trivialization (U, ϕ) of L determines an isomorphism

$$\phi_\alpha \colon L_\alpha|_U \to U \times \mathbb{C}.$$

Let (U, ϕ') be a second local trivialization, and let $g \in \Gamma(U, \mathscr{O}_U^\times)$ be the unique nowhere vanishing holomorphic function such that $(\phi' \circ \phi^{-1})(x) = (x, g(x)t)$. Then we have

$$\phi'_\alpha \circ \phi_\alpha^{-1} \colon U \times \mathbb{C} \to L_\alpha|_U \to U \times \mathbb{C}, \quad (x,t) \mapsto \Bigl(x, |g(x)|^{2\alpha} t\Bigr).$$

An αL-twisted test function is a compactly supported smooth section of the smooth line bundle L_α. We give this space the topology that agrees with the usual topology on the space of compactly supported smooth functions in any local trivialization of L.

Definition 3.4. An *αL-twisted current* is a continuous linear functional on the space of $(-\alpha L)$-twisted test functions. We denote by $\mathfrak{C}_{X,\alpha L}$ the sheaf of αL-twisted currents.

Here is a more concrete description. An αL-twisted test function $\varphi \in \Gamma_c(X, L_\alpha)$ is the same thing as a collection of smooth functions $\varphi_{(U,\phi)} \colon U \to \mathbb{C}$, one for each local trivialization (U, ϕ) of the line bundle L, that are compatible with restriction to open subsets, and are related to each other by the formula

$$\varphi_{(U,\phi')} = |g|^{-2\alpha} \cdot \varphi_{(U,\phi)},$$

where $(\phi' \circ \phi^{-1})(x,t) = (x, g(x)t)$ is the transition from one local trivialization to the other. Of course, the union of the supports of all the functions $\varphi_{(U,\phi)}$ must be a compact subset of X. Dually, an αL-twisted current $C \in \Gamma(X, \mathfrak{C}_{X,\alpha L})$ is the same thing as a collection of currents $C_{(U,\phi)} \in \Gamma(U, \mathfrak{C}_X)$ that are compatible with restriction, and are related to each other by the formula

$$C_{(U,\phi')} = C_{(U,\phi)} \cdot |g|^{2\alpha}.$$

Let us return to the general properties of twisted currents. We denote the natural pairing between twisted currents and twisted test functions by the symbol

$$\langle C, \varphi \rangle \in \mathbb{C},$$

for $C \in \Gamma(U, \mathfrak{C}_{X,\alpha L})$ a twisted current and $\varphi \in \Gamma_c(U, L_{-\alpha})$ a twisted test function. As usual, operations on twisted currents are defined in terms of the corresponding operations on twisted test functions. For example, the complex conjugate of a twisted current is defined by the formula

$$\bigl\langle \overline{C}, \varphi \bigr\rangle = \overline{\langle C, \overline{\varphi} \rangle}.$$

The sheaf of αL-twisted currents has the structure of a right $\mathscr{D}_{X,\alpha L}$-module. This can be seen as follows. Each local trivialization (U, ϕ) for L determines an isomorphism

$$\mathfrak{C}_U \cong \mathfrak{C}_{X,\alpha L}|_U,$$

where $\mathfrak{C}_U$ is the sheaf of currents on U and is of course a right $\mathscr{D}_U$-module. Moreover, the transition from one local trivialization to another works correctly. Indeed, if we have a twisted current C, represented by a collection of currents $C_{(U,\phi)} \in \Gamma(U, \mathfrak{C}_X)$ such that

$$C_{(U,\phi')} = C_{(U,\phi)}|g|^{2\alpha},$$

then a brief computation shows that

$$\big(C_{(U,\phi)}\partial_j\big)|g|^{2\alpha} = C_{(U,\phi)}|g|^{2\alpha}\left(\partial_j + \alpha g^{-1}\frac{\partial g}{\partial x_j}\right) = C_{(U,\phi')}\left(\partial_j + \alpha g^{-1}\frac{\partial g}{\partial x_j}\right).$$

This proves that $\mathfrak{C}_{X,\alpha L}$ is an αL-twisted right $\mathscr{D}$-module.

3.5. **Flat hermitian pairings on twisted $\mathscr{D}$-modules.** Fix a real number $\alpha \in \mathbb{R}$. A flat hermitian pairing on an αL-twisted right $\mathscr{D}$-module $\mathcal{M}$ is a morphism of sheaves

$$S\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_{X,\alpha L}$$

with the following properties. First, S is hermitian symmetric, in the sense that for any two local sections $m', m'' \in \Gamma(U, \mathcal{M})$, one has

$$S(m'', m') = \overline{S(m', m'')}$$

as twisted currents on U. Second, S is $\mathscr{D}_{X,\alpha L}$-linear in its first argument, meaning that

$$S(m'P, m'') = S(m', m'')P$$

for every twisted differential operator $P \in \Gamma(U, \mathscr{D}_{X,\alpha L})$. It follows that S is conjugate $\mathscr{D}_{X,\alpha L}$-linear in its second argument. In any local trivialization of L, the twisted $\mathscr{D}$-module $\mathcal{M}$ becomes a usual $\mathscr{D}$-module, and the flat hermitian pairing S becomes a sesquilinear pairing as in [60, Ch. 12].

3.6. **Good filtrations.** The sheaf $\mathrm{gr}_0^V \mathscr{D}_L$ inherits an increasing filtration $F_\bullet \mathrm{gr}_0^V \mathscr{D}_L$ from the order filtration on $\mathscr{D}_L$. Locally, this is just the usual order filtration on differential operators. Indeed, if (U, ϕ) is a local trivialization for L, then under the isomorphism

$$(\phi^{-1})_*\colon \mathscr{D}_U[t\partial_t] \to \mathrm{gr}_0^V \mathscr{D}_L|_U,$$

the filtration $F_\bullet \mathrm{gr}_0^V \mathscr{D}_L|_U$ is just the order filtration on $\mathscr{D}_U[t\partial_t]$, with $t\partial_t$ being considered as a differential operator of order 1. Globally, the first nonzero piece of our filtration is $F_0 \mathrm{gr}_0^V \mathscr{D}_L \cong \mathscr{O}_X$; the next graded piece $\mathrm{gr}_1^F \mathrm{gr}_0^V \mathscr{D}_L$ sits in a short exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\theta} \mathrm{gr}_1^F \mathrm{gr}_0^V \mathscr{D}_L \longrightarrow \mathscr{T}_X \longrightarrow 0,$$

whose extension class in $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{T}_X, \mathscr{O}_X) \cong H^1(X, \Omega_X^1)$ is the first Chern class $c_1(L)$. The order filtration on $\mathrm{gr}_0^V \mathscr{D}_L$ induces a filtration on $\mathscr{D}_{X,\alpha L} = \mathrm{gr}_0^V \mathscr{D}_L/(\theta - \alpha)\,\mathrm{gr}_0^V \mathscr{D}_L$, and from the above, we get a short exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow F_1\mathscr{D}_{X,\alpha L} \longrightarrow \mathscr{T}_X \longrightarrow 0, \tag{3.3}$$

whose extension class is now $\alpha \cdot c_1(L)$. Note that

$$\mathrm{gr}_\bullet^F \mathscr{D}_{X,\alpha L} \cong \mathrm{gr}_\bullet^F \mathscr{D}_X \cong \mathrm{Sym}^\bullet \mathscr{T}_X$$

is isomorphic to the symmetric algebra on the tangent sheaf $\mathscr{T}_X$, just as in the untwisted case. This is a consequence of (3.1).

Good filtrations on twisted $\mathscr{D}$-modules are defined just as in the untwisted case. Let $\mathcal{M}$ be a right $\mathscr{D}_{X,\alpha L}$-module, and let $F_\bullet\mathcal{M}$ be an exhaustive increasing filtration by coherent $\mathscr{O}_X$-submodules such that, locally on X, one has $F_k\mathcal{M} = 0$ for $k \ll 0$. We say that such a filtration $F_\bullet\mathcal{M}$ is a good filtration if

$$F_k\mathcal{M} \cdot F_\ell\mathscr{D}_{X,\alpha L} \subseteq F_{k+\ell}\mathcal{M},$$

with equality for $k \gg 0$. As usual, this is equivalent to the condition that

$$\operatorname{gr}^F_\bullet \mathcal{M} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

is coherent over $\operatorname{gr}^F_\bullet \mathscr{D}_{X,\alpha L} \cong \operatorname{Sym}^\bullet \mathscr{T}_X$.

3.7. **Graded pieces of the de Rham complex.** One important difference between twisted $\mathscr{D}$-modules and usual $\mathscr{D}$-modules is that there is no de Rham complex for twisted $\mathscr{D}$-modules (unless $\alpha = 0$), because there is no longer an action by $\mathscr{T}_X$. This is due to the fact that the short exact sequence in (3.3) does not split (unless $\alpha = 0$). But we do have $\operatorname{gr}^F_1 \mathscr{D}_{X,\alpha L} \cong \mathscr{T}_X$, so the notion of "graded pieces of the de Rham complex" still makes sense for a twisted $\mathscr{D}$-module $\mathcal{M}$ with a good filtration $F_\bullet\mathcal{M}$.

Definition 3.5. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a twisted $\mathscr{D}$-module on X with a good filtration. For every $k \in \mathbb{Z}$, the *graded piece of the de Rham complex* is defined by

$$\operatorname{gr}^F_k \operatorname{DR}(\mathcal{M}) := \left[\operatorname{gr}^F_{k-n} \mathcal{M} \otimes \bigwedge^n \mathscr{T}_X \to \cdots \to \operatorname{gr}^F_{k-1} \mathcal{M} \otimes \mathscr{T}_X \to \operatorname{gr}^F_k \mathcal{M}\right][n], \tag{3.4}$$

where $n = \dim X$ and where the differential is induced by the multiplication morphism $\operatorname{gr}^F_k \mathcal{M} \otimes \operatorname{gr}^F_1 \mathscr{D}_{X,\alpha L} \to \operatorname{gr}^F_{k+1} \mathcal{M}$ and the isomorphism $\operatorname{gr}^F_1 \mathscr{D}_{X,\alpha L} \cong \mathscr{T}_X$.

3.8. **Twisted Hodge modules.** Before we can define twisted Hodge modules, which are the main objects in this chapter, we briefly review the theory of complex Hodge modules from [60, §14]. We will give a simplified version of the definition (without weights) that is sufficient for our purposes.

Let X be a complex manifold of dimension n. A complex Hodge module on X consists of the following three pieces of data:

1. A regular holonomic right $\mathscr{D}_X$-module $\mathcal{M}$.
2. An increasing filtration $F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_X$-submodules. This filtration needs to be good, which means that it is exhaustive; that $F_k\mathcal{M} = 0$ for $k \ll 0$ locally on X; and that one has $F_k\mathcal{M} \cdot F_\ell\mathscr{D}_X \subseteq F_{k+\ell}\mathcal{M}$, with equality for $k \gg 0$.
3. A flat hermitian pairing $S\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$, valued in the sheaf of currents of bidegree (n, n) on X. Again, S needs to be hermitian symmetric and $\mathscr{D}_X$-linear in its first argument (and therefore conjugate linear in its second argument).

The object $M = (\mathcal{M}, F_\bullet\mathcal{M}, S)$ is a polarized Hodge module if it satisfies several additional conditions that are imposed on the nearby and vanishing cycle functors with respect to holomorphic functions on open subsets of X. An important point is that the definition is local: if the restriction of M to every subset in an open covering of X is a polarized Hodge module on that open subset, then M is a polarized Hodge module on X.

Remark 3.6. Note that there are no weights in the simplified definition above. In order to have an intrinsic notion of weights, one needs to work with triples of the form

$$\big((\mathcal{M}', F_\bullet\mathcal{M}'), (\mathcal{M}'', F_\bullet\mathcal{M}''), S\big),$$

where $\mathcal{M}'$ and $\mathcal{M}''$ are regular holonomic right $\mathscr{D}_X$-modules with good filtrations $F_\bullet\mathcal{M}'$ and $F_\bullet\mathcal{M}''$, and where $S\colon \mathcal{M}' \otimes_{\mathbb{C}} \mathcal{M}'' \to \mathfrak{C}_X$ is a flat sesquilinear pairing. In this formulation, a polarization is then a certain kind of isomorphism between $\mathcal{M}'$ and $\mathcal{M}''$ that is compatible with the filtrations and the pairing [60, Ch. 14].

We now come to the main definition of this chapter. It is modelled on the definition of polarized complex Hodge modules [60, §14.2], but with twisted $\mathscr{D}$-modules in place of usual ones. This works because being a polarized complex Hodge module is a local condition.

Let X be a complex manifold, L a holomorphic line bundle on X, and $\alpha \in \mathbb{R}$ a real number. We again consider objects of the type $(\mathcal{M}, F_\bullet\mathcal{M}, S)$, where $\mathcal{M}$ is a right $\mathscr{D}_{X,\alpha L}$-module with a good filtration $F_\bullet\mathcal{M}$, and where

$$S\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_{X,\alpha L}$$

is a flat hermitian pairing on $\mathcal{M}$. If (U, ϕ) is a local trivialization of the line bundle L, the restriction $(\mathcal{M}, F_\bullet\mathcal{M}, S)|_U$ to the open subset U becomes, via the isomorphism in (3.2), a usual filtered $\mathscr{D}_U$-module with a flat hermitian pairing.

Definition 3.7 (Twisted polarized Hodge modules)**.** We say that an object $(\mathcal{M}, F_\bullet\mathcal{M}, S)$ is an αL-twisted polarized Hodge module if, for any local trivialization (U, ϕ), the restriction $(\mathcal{M}, F_\bullet\mathcal{M}, S)|_U$ is a polarized complex Hodge module in the usual sense.

Because of the local nature of the definition, all local properties of polarized complex Hodge modules (such as existence of a decomposition by strict support or the strictness of morphisms) immediately carry over to the twisted setting [60, §14.2].

3.9. **Direct images and the decomposition theorem.** Another important difference between twisted $\mathscr{D}$-modules and usual $\mathscr{D}$-modules is that one cannot take the direct image of a twisted $\mathscr{D}$-module (or twisted Hodge module) along a proper morphism $f\colon X \to Y$ unless the line bundle L is pulled back from Y.

Let us start with a few remarks about the direct image functor for twisted $\mathscr{D}$-modules. Let $f\colon X \to Y$ be a proper holomorphic mapping between complex manifolds, and let L be a holomorphic line bundle on Y, viewed as a complex manifold of dimension $\dim Y + 1$ via the bundle projection $p\colon L \to Y$. Let $L_X = X \times_Y L$ be the pullback of the line bundle to X, as in the commutative diagram below.

$$\begin{tikzcd} L_X \arrow[r] \arrow[d] & L \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $\alpha \in \mathbb{R}$. As in the untwisted case [60, 8.6.4], we introduce the transfer module

$$\mathscr{D}_{X\to Y,\alpha L} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_{Y,\alpha L}.$$

This is an $(\mathscr{D}_{X,\alpha L_X}, f^{-1}\mathscr{D}_{Y,\alpha L})$-bimodule: the right $f^{-1}\mathscr{D}_{Y,\alpha L}$-module structure is the obvious one, and the left $\mathscr{D}_{X,\alpha L_X}$-module structure is induced by the morphism

$$F_1\mathscr{D}_{X,\alpha L_X} \to f^*F_1\mathscr{D}_{Y,\alpha L} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}F_1\mathscr{D}_{Y,\alpha L},$$

which is part of the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, equal] & F_1\mathscr{D}_{X,\alpha L_X} \arrow[r] \arrow[d] & \mathscr{T}_X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_X \arrow[r] & f^*F_1\mathscr{D}_{Y,\alpha L} \arrow[r] & f^*\mathscr{T}_Y \arrow[r] & 0 \end{tikzcd}$$

We can then define the direct image functor

$$f_+\colon D^b_{coh}(\mathscr{D}_{X,\alpha L_X}) \to D^b_{coh}(\mathscr{D}_{Y,\alpha L})$$

from the derived category of right $\mathscr{D}_{X,\alpha L_X}$-modules to that of right $\mathscr{D}_{Y,\alpha L}$-modules as

$$f_+(-) = \mathbf{R}f_*\big(- \otimes^{\mathbf{L}}_{\mathscr{D}_{X,\alpha L_X}} \mathscr{D}_{X\to Y,\alpha L}\big),$$

which is essentially the same formula as in the untwisted case [60, §8.7]. If $\phi\colon L|_U \to U\times\mathbb{C}$ is a local trivialization of L, then we get an induced trivialization of L_X over the open subset $f^{-1}(U)$, and so twisted $\mathscr{D}$-modules on U and $f^{-1}(U)$ are the same thing as usual $\mathscr{D}$-modules. It is then easy to see that the diagram

$$\begin{tikzcd} D^b_{coh}(\mathscr{D}_{X,\alpha L_X}) \arrow[r] \arrow[d, "f_+"] & D^b_{coh}(\mathscr{D}_{f^{-1}(U)}) \arrow[d, "f_+"] \\ D^b_{coh}(\mathscr{D}_{Y,\alpha L}) \arrow[r] & D^b_{coh}(\mathscr{D}_U) \end{tikzcd}$$

is commutative, where the horizontal arrows are restriction to the open subsets U and $f^{-1}(U)$ and $f_+\colon D^b_{coh}(\mathscr{D}_{f^{-1}(U)}) \to D^b_{coh}(\mathscr{D}_U)$ is the usual direct image functor for right $\mathscr{D}$-modules. By the same method as in [60, §8.7], the definition of the direct image functor can be extended to filtered $\mathscr{D}$-modules (using the natural filtration on the transfer module), and as in [60, §12], a flat hermitian pairing $S\colon \mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}} \to \mathfrak{C}_{X,\alpha L_X}$ on an αL_X-twisted $\mathscr{D}$-module induces flat sesquilinear pairings

$$S_i\colon \mathscr{H}^i f_+\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathscr{H}^{-i} f_+\mathcal{M}} \to \mathfrak{C}_{Y,\alpha L}.$$

We can now state the decomposition theorem for twisted Hodge modules.

Theorem 3.8. *Let $f\colon X\to Y$ be a projective morphism between complex manifolds, let L be a holomorphic line bundle on Y, and set $L_X = f^*L$. If $(\mathcal{M}, F_\bullet\mathcal{M}, S)$ is an αL_X-twisted polarized Hodge module on X, then each*

$$\mathscr{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$$

with the induced polarization, is an αL-twisted polarized Hodge module on Y. Moreover, the decomposition theorem

$$f_+(\mathcal{M}, F_\bullet\mathcal{M}) \cong \bigoplus_{i\in\mathbb{Z}} \mathscr{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$$

holds in the derived category of filtered twisted $\mathscr{D}_Y$-modules.

Proof. Locally on Y, the direct image functor for twisted $\mathscr{D}$-modules agrees with the usual direct image functor for $\mathscr{D}$-modules. All the local assertions in the theorem therefore follow from [60, §14.3], and in particular, each $\mathscr{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict. Let ω be the first Chern class of a relatively ample line bundle. From the relative Hard Lefschetz theorem for complex Hodge modules, applied locally, it follows that

$$\omega^i\colon \mathscr{H}^{-i} f_+(\mathcal{M}, F_\bullet\mathcal{M}) \to \mathscr{H}^i f_+(\mathcal{M}, F_{\bullet-i}\mathcal{M})$$

is an isomorphism for every $i\geq 0$. Just as in the untwisted case, this implies the decomposition theorem. The relative Hard Lefschetz theorem also gives us a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the direct sum of all the $\mathscr{H}^i f_+\mathcal{M}$, and we again let w denote the corresponding Weil element. Then [60, §14.3], applied locally, shows that the flat hermitian pairing

$$(-1)^{i(i-1)/2} S_i \circ (\mathrm{id}\otimes\mathsf{w})\colon \mathscr{H}^i f_+\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathscr{H}^i f_+\mathcal{M}} \to \mathfrak{C}_{Y,\alpha L}$$

polarizes $\mathscr{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$, which is therefore a twisted polarized Hodge module. $\square$

3.10. **Kashiwara's equivalence.** Let $i\colon Y \hookrightarrow X$ be the inclusion of a closed submanifold. Let L be a holomorphic line bundle on X and set $L_Y = i^*L$. For any $\alpha \in \mathbb{R}$, there is a direct image functor [5, §2.2]

$$i_+ : \{\alpha L_Y\text{-twisted } \mathscr{D}\text{-modules}\} \to \{\alpha L\text{-twisted } \mathscr{D}\text{-modules}\},$$

which gives a twisted version of Kashiwara's equivalence [5, §2.5.5(iv)].

Theorem 3.9 (Beilinson-Bernstein). *The functor i_+ induces an equivalence between the category of coherent αL_Y-twisted right $\mathscr{D}$-modules and the category of coherent αL-twisted right $\mathscr{D}$-modules whose support is contained in the submanifold Y.*

Since the definition of twisted Hodge modules is local, the functor i_+ extends to

$$i_+ : \{\alpha L_Y\text{-twisted Hodge modules on } Y\} \to \{\alpha L\text{-twisted Hodge modules on } X\},$$

using the direct image functor for polarized complex Hodge modules.

Theorem 3.10. *The functor i_+ induces an equivalence between the category of αL_Y-twisted polarized Hodge modules on Y and the category of αL-twisted polarized Hodge modules on X whose support is contained in the submanifold Y.*

Proof. This follows from the twisted Kashiwara's equivalence (Theorem 3.9) and the strict Kashiwara's equivalence for complex polarized Hodge modules [60, Proposition 9.6.2]. □

3.11. **Nearby and vanishing cycle for divisors.** Now let us explain why the nearby and vanishing cycles of a complex Hodge module with respect to a divisor are naturally twisted Hodge modules.

Let X be a complex manifold and let $\mathcal{M}$ be a regular holonomic right $\mathscr{D}_X$-module. Let D be an effective divisor on X, set $L = \mathcal{O}_X(D)$, and let $s \in H^0(X, L)$ be a global section such that $\operatorname{div}(s) = D$. We view L as a complex manifold of dimension $\dim X + 1$, and the section s as a closed embedding $s\colon X \hookrightarrow L$. Let

$$\mathcal{M}_L = s_+\mathcal{M}$$

be the direct image $\mathscr{D}$-module on L.

The zero section of L induces a filtration $V_\bullet\mathcal{M}_L$; this is defined locally as the V-filtration in Definition 2.1, but the resulting filtration on $\mathcal{M}_L$ is actually globally well-defined [7, Proposition 1.5]. Each $V_\alpha\mathcal{M}_L$ is a sheaf of $V_0\mathscr{D}_L$-modules and each $\operatorname{gr}_\alpha^V \mathcal{M}_L$ is therefore a well-defined $\operatorname{gr}_0^V \mathscr{D}_L$-module, but unlike in the local setting, it is not a $\mathscr{D}_X$-module. We get closer to $\mathscr{D}$-modules once we take the associate graded of the weight filtration. Recall from §3.2 that the operator $\theta \in F_1 \operatorname{gr}_0^V \mathscr{D}_L$ is globally well-defined and central, then the properties of V-filtration implies that the operator

$$N := \theta - \alpha \tag{3.5}$$

acts nilpotently on $\operatorname{gr}_\alpha^V \mathcal{M}_L$. Let $W_\bullet(N)$ denote the weight filtration of this nilpotent operator. Since $\mathscr{D}_{X,\alpha L} = \operatorname{gr}_0^V \mathscr{D}_L/(\theta - \alpha)\operatorname{gr}_0^V \mathscr{D}_L$, it follows that each subquotient

$$\operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L$$

has the structure of an αL-twisted right $\mathscr{D}$-module.

Now let us suppose that $\mathcal{M}$ comes with a flat hermitian pairing

$$S\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$$

into the sheaf of currents of bidegree (n, n). In that case, we get an induced flat hermitian pairing on $\operatorname{gr}_\alpha^V \mathcal{M}$, but now valued in the sheaf of twisted currents.

Lemma 3.11. *For each $\alpha \in [-1, 0]$, we have an induced flat hermitian pairing*

$$S_\alpha \colon \operatorname{gr}_\alpha^V \mathcal{M}_L \otimes_{\mathbb{C}} \overline{\operatorname{gr}_\alpha^V \mathcal{M}_L} \to \mathfrak{C}_{X,\alpha L}.$$

The operator $\theta \in \operatorname{End}(\operatorname{gr}_\alpha^V \mathcal{M}_L)$ is self-adjoint with respect to S_α.

Proof. The point is that the construction of the induced pairing on nearby and vanishing cycles in [60] transforms correctly from one local trivialization of L to another. We will give the proof for $-1 \leq \alpha < 0$, the case $\alpha = 0$ being similar.

Let $\phi \colon L|_U \to U \times \mathbb{C}$ be a local trivialization of the line bundle L, and denote by t the coordinate function on $\mathbb{C}$. The restriction of $V_\alpha \mathcal{M}_L$ to the open subset $U \times \mathbb{C}$ is a module over $V_0 \mathscr{D}_L$, and the restriction of $\operatorname{gr}_\alpha^V \mathcal{M}_L$ to the open set U is a right $\mathscr{D}_U$-module. Let us quickly review the construction of the induced pairing S_α, using the Mellin transform. Let $\varphi \in A_c(U)$ be a test function with compact support in U, and let $\eta \colon \mathbb{C} \to [0, 1]$ be a compactly support smooth function that is identically equal to 1 near the origin. Let $m_1, m_2 \in \Gamma(U \times \mathbb{C}, V_\alpha \mathcal{M}_L)$ The expression

$$\Big\langle S(m_1, m_2), |t|^{2s} \eta(t) \varphi \Big\rangle$$

is a holomorphic function of $s \in \mathbb{C}$ as long as $\operatorname{Re} s \gg 0$, and extends to a meromorphic function on all of $\mathbb{C}$ (using the properties of the V-filtration). One can show that the residue at $s = \alpha$ depends continuously on the test function φ, and that the formula

$$\Big\langle S_\alpha([m_1], [m_2]), \varphi \Big\rangle = \operatorname{Res}_{s=\alpha} \Big\langle S(m_1, m_2), |t|^{2s} \eta(t) \varphi \Big\rangle$$

defines a flat hermitian pairing on the right $\mathscr{D}_U$-module $\operatorname{gr}_\alpha^V \mathcal{M}_L|_U$. Moreover, the endomorphism $\theta = t\partial_t$ is self-adjoint with respect to this pairing.

To prove the lemma, it is enough to check that these pairings transform correctly from one local trivialization to another. Let us denote by $C_{(U,\phi)} = S_\alpha([m_1], [m_2])$ the current constructed above. Let (U, ϕ') be a second trivialization such that

$$(\phi' \circ \phi^{-1})(x, t) = \big(x, g(x)t\big) \quad \text{and } g \in \Gamma(U, \mathscr{O}_X^\times).$$

If we let t' be the resulting holomorphic coordinate on $\mathbb{C}$, we have $t' = gt$. The same formula as above then defines a second current

$$\big\langle C_{(U,\phi')}, \varphi \big\rangle = \operatorname{Res}_{s=\alpha} \Big\langle S(m_1, m_2), |t'|^{2s} \eta(t') \varphi \Big\rangle$$

Since g is holomorphic, the right-hand side evaluates to

$$\begin{aligned} \operatorname{Res}_{s=\alpha} \Big\langle S(m_1, m_2), |g|^{2s} |t|^{2s} \eta(gt) \varphi \Big\rangle &= \operatorname{Res}_{s=\alpha} \Big\langle S(m_1, m_2), |g|^{2\alpha} |t|^{2s} \eta(gt) \varphi \Big\rangle \\ &= \big\langle C_{(U,\phi)}, |g|^{2\alpha} \varphi \big\rangle, \end{aligned}$$

and so we arrive at the identity $C_{(U,\phi')} = C_{(U,\phi)} \cdot |g|^{2\alpha}$. This is enough to conclude that the currents on each local trivialization glue together into an αL-twisted current on X. □

As long as $\alpha \in \mathbb{R}$, the nilpotent endomorphism $N = \theta - \alpha$ is self-adjoint with respect to the pairing S_α, and so the pairing descends to the graded quotients of the weight filtration. This gives us flat sesquilinear pairings

$$S_\alpha \colon \operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L \otimes_{\mathbb{C}} \overline{\operatorname{gr}_{-\ell}^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L} \to \mathfrak{C}_{X,\alpha L}.$$

We need one extra piece of data to describe the induced polarization. The direct sum

$$\bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L$$

carries a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. If we denote the three generators by

$$\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then H acts as multiplication by ℓ on $\operatorname{gr}_\ell^{W(N)}$, and Y acts as $N = \theta - \alpha$. The Weil element

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

has the property that $\mathsf{w}\mathsf{H}\mathsf{w}^{-1} = -\mathsf{H}$, $\mathsf{w}\mathsf{X}\mathsf{w}^{-1} = -\mathsf{Y}$, and $\mathsf{w}\mathsf{Y}\mathsf{w}^{-1} = -\mathsf{X}$, and therefore determines an isomorphism between $\operatorname{gr}_{-\ell}^{W(N)}$ and $\operatorname{gr}_\ell^{W(N)}$. It follows that

$$S_\alpha \circ (\mathrm{id} \otimes \mathsf{w}) \colon \operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L \otimes_{\mathbb{C}} \overline{\operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L} \to \mathfrak{C}_{X,\alpha L}$$

is a flat hermitian pairing on the indicated αL-twisted $\mathscr{D}$-module. We can use this to show that the nearby and vanishing cycles of a complex Hodge module are naturally twisted Hodge modules.

Proposition 3.12. *Let $(\mathcal{M}, F_\bullet\mathcal{M}, S)$ be a polarized complex Hodge module on X. For any real number $\alpha \in [-1, 0)$ and any integer $\ell \in \mathbb{Z}$, the object*

$$\Big(\operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L, F_{\bullet-1} \operatorname{gr}_\ell^{W(N)} \operatorname{gr}_\alpha^V \mathcal{M}_L, S_\alpha \circ (\mathrm{id} \otimes \mathsf{w})\Big)$$

is a polarized αL-twisted Hodge module on X. The object

$$\Big(\operatorname{gr}_\ell^{W(N)} \operatorname{gr}_0^V \mathcal{M}_L, F_\bullet \operatorname{gr}_\ell^{W(N)} \operatorname{gr}_0^V \mathcal{M}_L, S_0 \circ (\mathrm{id} \otimes \mathsf{w})\Big)$$

is a polarized complex Hodge module (without any twisting).

Proof. The claim is that both objects are polarized complex Hodge modules in any local trivialization of L. This follows from the definition of polarized complex Hodge modules, because in any local trivialization, the two objects are exactly the nearby cycles and unipotent vanishing cycles of $(\mathcal{M}, F_\bullet\mathcal{M}, S)$. ☐

Remark 3.13. From (2.9) and Lemma (4.11), we also see that $\operatorname{gr}_r^W \operatorname{gr}_\alpha^V \mathcal{M}_L$ is naturally twisted by αL for $\alpha \in (-1, 0)$.

3.12. **Untwists.** The following lemma provides a procedure to "untwist" a twisted Hodge module when the twisting is an integer.

Lemma 3.14. *If $M = (\mathcal{M}, F_\bullet\mathcal{M}, S)$ is an αL-twisted Hodge module, then*

$$M \otimes L := (\mathcal{M} \otimes L, F_\bullet\mathcal{M} \otimes L, S \otimes L)$$

is an $(\alpha+1)L$-twisted Hodge module. Here $S \otimes L$ is a current defined by multiplication of S by $|g|^2$ when one changes the trivialization of L by g.

In particular, if $\alpha \in \mathbb{Z}$ and $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies an αL-twisted Hodge module, then $(\mathcal{M}, F_\bullet\mathcal{M}) \otimes L^{-\alpha}$ underlies a polarized complex Hodge module.

Proof. It suffices to check that if $\mathcal{M}$ is a right αL-twisted $\mathscr{D}$-module, then $\mathcal{M} \otimes L$ is a right $(\alpha+1)L$-twisted $\mathscr{D}$-module, which follows from Remark 3.3. ☐

4. Vanishing theorems for twisted Hodge modules

In this section, we prove a general vanishing theorem for twisted Hodge modules, extending Saito's vanishing theorem [62, §2.g] (see also [68]). We work with algebraic setting in this section. Before the proof, we need the notion of non-characteristic for twisted $\mathscr{D}$-modules and twisted Hodge modules. First, let us recall the situation of filtered $\mathscr{D}$-modules.

Definition 4.1. [61, §3.5.1] Let $f : Y \to X$ be a morphism of smooth algebraic varieties. We say that f is *non-characteristic* for a filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ if the following conditions are satisfied:

(1) $\mathcal{H}^i(f^{-1}\operatorname{gr}^F_\bullet \mathcal{M} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y) = 0$ for all $i \neq 0$, where
$$\operatorname{gr}^F_\bullet \mathcal{M} = \bigoplus_k F_k\mathcal{M}/F_{k-1}\mathcal{M}.$$

(2) The morphism $df : (pr_2)^{-1}\mathrm{Char}(\mathcal{M}) \to T^*Y$ is finite, where
$$\begin{array}{ccc} Y \times_X T^*X & \xrightarrow{df} & T^*Y \\ \big\downarrow{\scriptstyle pr_2} & & \\ T^*X, & & \end{array}$$
and pr_2 is the natural projection and $df(y,\xi) := (df)^*\xi$ for $y \in Y, \xi \in T^*_{f(y)}X$.

Remark 4.2. To check condition (1), it suffices to prove that $\mathcal{O}_Y$ is a flat module over $f^{-1}\mathcal{O}_X$. The condition (2) means that f is non-characteristic for the $\mathscr{D}$-module $\mathcal{M}$, and can be checked as follows (see [61, 3.5.1.3] and discussion after [68, Definition 9.1]). Consider a Whitney stratification $\{S_\beta\}$ of X such that $\mathrm{Char}(\mathcal{M}) \subseteq \bigcup_\beta T^*_{S_\beta}X$, then the condition (2) is satisfied if the fiber product $S_\beta \times_X Y$ is smooth for every β.

Let L be a line bundle on X, $\alpha \in \mathbb{Q}$ and let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered αL-twisted $\mathscr{D}$-module with a good filtration.

Definition 4.3. We say that f is *non-characteristic* for the filtered twisted $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ if for every open subset $U \subseteq X$ trivializing L, the induced morphism $f^{-1}(U) \to U$ is non-characteristic for the filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})|_U$.

Lemma 4.4. *Let M be a αL-twisted polarized Hodge module on X with strict support Z. If $f : Y \to X$ is non-characteristic for the underlying twisted $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$. Then we have isomorphisms*
$$f^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}_Y, \quad f^*F_\bullet\mathcal{M} \xrightarrow{\sim} F_\bullet\mathcal{M}_Y,$$
where $(\mathcal{M}_Y, F_\bullet\mathcal{M}_Y)$ underlies an αL_Y-twisted polarized Hodge module with strict support $f^{-1}(Z)$ and $L_Y = f^*L$.

Proof. This can be argued in the same way as in [68, Theorem 9.3] by using the equivalence between a polarized complex Hodge module of weight w with strict support Z and a polarized complex variation of Hodge structures of weight $w - \dim Z$ on a Zariski-open subset of the smooth locus of Z. □

Now we extend Saito's vanishing theorem to complex Hodge modules.

Theorem 4.5. *Let X be a smooth projective variety and let M be a polarized complex Hodge module on X with strict support Z, where Z is reduced and irreducible. Let L be an ample line bundle on Z, then one has*
$$H^i(Z, \operatorname{gr}^F_p \mathrm{DR}(\mathcal{M}) \otimes L) = 0, \quad \text{for } i > 0, \text{ and } p \in \mathbb{Z}.$$

If Ω^1_X *is trivial, then*

$$H^i(Z, \operatorname{gr}^F_p \mathcal{M} \otimes L) = 0, \quad \text{for } i > 0, \text{ and } p \in \mathbb{Z},$$

where L *is any ample line bundle on* Z.

Proof. We sketch a proof following the method of [68], which is based on the Esnault-Viehweg method. In this proof, we use left $\mathscr{D}$-modules and explain how ingredients for complex Hodge modules from [60] fit together to give such a statement.

First we need to show that each

$$\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) = \left[\operatorname{gr}^F_p \mathcal{M} \to \Omega^1_X \otimes \operatorname{gr}^F_{p+1} \mathcal{M} \to \cdots \to \Omega^n_X \otimes \operatorname{gr}^F_{p+n} \mathcal{M}\right][n]$$

is a well-defined complex of coherent $\mathcal{O}_Z$-modules, where $n = \dim X$. One needs to show that if f is an arbitrary local section of the ideal sheaf $\mathcal{I}_Z$, then $f \cdot \operatorname{gr}^F_p \mathcal{M} = 0$, which is the consequence of $(\mathcal{M}, F_\bullet\mathcal{M})$ being strictly $\mathbb{R}$-specializable along $f = 0$. In fact one only needs the condition (for right Hodge modules)

$$(F_p \operatorname{gr}^V_\alpha \mathcal{M}) \cdot \partial_t = F_{p+1} \operatorname{gr}^V_{\alpha+1} \mathcal{M}, \quad \text{for all } p \in \mathbb{Z} \text{ and } \alpha > -1,$$

see [69, Exercise 11.3]. This is satisfied by [60, Definition 10.6.1, Condition (b)].

Then one needs to show the compatibility of the de Rham complex with the duality functor, i.e. if M is a polarizable complex Hodge module on an n-dimensional complex manifold X of weight w, then any polarization on M induces an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}), \omega_X[n]) \xrightarrow{\sim} \operatorname{gr}^F_{-p-w} \operatorname{DR}(\mathcal{M}).$$

This is proved in [60, Corollary 8.8.22 (6)] (the underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ of a complex Hodge module is a holonomic $R_F\mathscr{D}_X$-module and $\mathcal{E}xt^i_{R_F\mathscr{D}_X}(R_F\mathcal{M}, R_F\mathscr{D}_X)$ is a strict $R_F\mathscr{D}_X$-module for every i). It can be compared with [61, Lemme 5.1.13]. This compatibility is needed because then the desired vanishing statement is equivalent to

$$H^i(Z, \operatorname{gr}^F_p \operatorname{DR}(\mathcal{M}) \otimes L^{-1}) = 0, \quad \forall i < 0, \text{ and } p \in \mathbb{Z}.$$

Lastly, we need the non-characteristic pull back of complex Hodge modules (c.f. Lemma 4.4) and the E_1-degeneration of Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \operatorname{gr}^F_{-p} \operatorname{DR}(\mathcal{M})) \Longrightarrow H^{p+q}(X, \operatorname{DR}(\mathcal{M}))$$

for a complex Hodge module M on a smooth projective variety X. This is proved in [60, Theorem 14.3.1]. □

Remark 4.6. There are two more strategies for the proof of Theorem 4.5: either one follows Saito's original proof [62, §2.g] and uses necessary ingredients from [60], or one generalizes Hyunsuk Kim's analytic proof for [29, Theorem 1.4] to complex Hodge modules.

Now we can prove the vanishing theorem for twisted Hodge modules. If $\alpha = 0$, this recovers Theorem 4.5.

Theorem 4.7. *Let* D *be an effective divisor on a projective complex manifold* X *and denote* $L = \mathcal{O}_X(D)$. *For any* $\alpha \in \mathbb{Q}$, *let* M *be an* αL*-twisted Hodge module with strict support* Z *and let* B *be an effective divisor on* Z *such that the* $\mathbb{Q}$*-divisor* $B + \alpha D|_Z$ *is ample. Then we have*

$$H^i\left(Z, \operatorname{gr}^F_k \operatorname{DR}(\mathcal{M}) \otimes \mathcal{O}_Z(B)\right) = 0, \quad \text{for every } i > 0 \text{ and } k \in \mathbb{Z}.$$

If Ω^1_X *is trivial, then*

$$H^i\left(Z, \operatorname{gr}^F_k \mathcal{M} \otimes \mathcal{O}_Z(B)\right) = 0, \quad \text{when } i > 0 \text{ and } k \in \mathbb{Z}.$$

Proof. The general idea is to go to a finite branched covering on which αD becomes integral, and then to apply the untwisting Lemma 3.14 as well as the vanishing theorem 4.5 for complex Hodge modules.

Write $\alpha = p/m$ with $\gcd(p,m) = 1$. By a result of Bloch and Gieseker (see [6] and [33, Proposition 2.67]), there is

$(*)$ a finite flat morphism $f : Y \to X$, with Y smooth projective, $f^*L = mL_Y$

for a line bundle L_Y on Y.

We claim that one can choose f to be non-characteristic for the filtered twisted $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ in the sense of Definition 4.3. To prove this, let us recall the construction of Bloch-Gieseker covering following [35, Theorem 4.1.10]. By writing L as the difference of two very ample divisors and applying the following construction to each of them, we can assume that there is a finite-to-one mapping $\phi : X \to \mathbf{P}^r$ such that $L = \phi^*\mathscr{O}_{\mathbf{P}^r}(1)$. Fix a branched covering $\mu : \mathbf{P}^r \to \mathbf{P}^r$ such that $\mu^*\mathscr{O}_{\mathbf{P}^r}(1) = \mathscr{O}_{\mathbf{P}^r}(m)$. Given $g \in G := \mathrm{GL}(r+1)$ acting on $\mathbf{P}^r$ in the natural way, denote by $\mu_g : \mathbf{P}^r \to \mathbf{P}^r$ the composition $\mu_g = g \circ \mu$. Define $Y_g := X \times_{\mathbf{P}^r} \mathbf{P}^r$, with a map f_g as in the following Cartesian square

$$\begin{tikzcd} Y_g \arrow[r] \arrow[d, "f_g"] & \mathbf{P}^r \arrow[d, "\mu_g"] \\ X \arrow[r, "\phi"] & \mathbf{P}^r. \end{tikzcd}$$

It is shown in [35, Theorem 4.1.10] that if $g \in G$ is sufficiently general, then $f_g : Y_g \to X$ satisfies the condition in $(*)$. We will show further that one can choose g such that $f_g : Y_g \to X$ is non-characteristic for $(\mathcal{M}, F_\bullet\mathcal{M})$. Let us choose an open cover $\{U_i\}$ of X trivializing L so that $M|_{U_i}$ is a Hodge module. By Definition 4.3, it suffices to check conditions (1),(2) in Definition 4.1 for the morphism $f^{-1}(U_i) \to U_i$. Consider the mapping

$$m : G \times \mathbf{P}^r \to \mathbf{P}^r, \quad (g, z) \mapsto g \cdot \mu(z),$$

and form the fiber product $W = X \times_{\mathbf{P}^r} (G \times \mathbf{P}^r)$ with the following commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "n"] \arrow[rr, bend left, "b"] & G \times \mathbf{P}^r \arrow[r, "pr_1"] \arrow[d, "m"] & G \\ X \arrow[r, "\phi"] & \mathbf{P}^r. & \end{tikzcd}$$

The fiber of b over $g \in G$ is just Y_g. The Generic Flatness Theorem implies that for a generic $g \in G$, we have $\mathscr{O}_{Y_g}$ is a flat $f^{-1}\mathscr{O}_X$-module, which verifies the condition (1) by Remark 4.2. For the condition (2), choose a Whitney stratification $\{S_\beta\}$ of X such that

$$\mathrm{Char}(\mathcal{M}|_{U_i}) \subseteq \bigcup_\beta T^*_{S_\beta \cap U_i} U_i, \quad \text{for all } i, \beta.$$

By the proof of [24, III.10.8], we see that m is a smooth morphism, hence the base change n and $S_\beta \times_X W \to S_\beta$ are smooth, therefore the fiber product $S_\beta \times_X W$ is smooth. Applying the Generic Smooth Theorem applied to the map $S_\beta \times_X W \to G$, we see that $S_\beta \times_X Y_g$ is smooth, then the condition (2) is satisfied by Remark 4.2.

With such a choice of f, by Lemma 4.4 and note that $\alpha f^*Y = mL_Y$, one knows that the pullback

$$(\mathcal{M}_Y, F_\bullet\mathcal{M}_Y) := \omega_{Y/X} \otimes (\mathcal{M}, F_\bullet\mathcal{M})$$

still underlies a pL_Y-twisted polarized Hodge module with strict support $f^{-1}(Z)$ and

$$\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}_Y) \otimes f^*\mathcal{O}_Z(B) = \omega_{Y/X} \otimes f^*(\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}) \otimes \mathcal{O}_Z(B)).$$

Then by Lemma 4.8, it is enough to prove the vanishing of

$$H^i\left(Y, \mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}_Y) \otimes f^*\mathcal{O}_Z(B)\right) = H^i\left(Y, \mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}_Y \otimes L_Y^{-p})) \otimes f^*\mathcal{O}_Z(B) \otimes L_Y^p\right).$$

Since p is an integer, Lemma 3.14 implies that $(\mathcal{M}_Y, F_\bullet\mathcal{M}_Y) \otimes L_Y^{-p}$ underlies a polarized complex Hodge module with strict support $f^{-1}(Z)$. Note that the line bundle

$$f^*\mathcal{O}_Z(B) \otimes L_Y^p|_{f^{-1}(Z)} = f^*\mathcal{O}_Z(B + \alpha D|_Z)$$

is ample by assumption and finiteness of f, therefore the desired vanishing follows from Theorem 4.5.

If Ω_X^1 is trivial, one can argue as in the proof of [56, Lemma 2.5]. □

Lemma 4.8. *Let $f : Y \to X$ be a finite surjective morphism of smooth complex projective varieties and let $E^\bullet$ be a bounded complex of coherent sheaves on X. Then the natural homomorphism*

$$H^j(X, E^\bullet) \to H^j(Y, \omega_{Y/X} \otimes f^*E^\bullet)$$

induced by f is injective. In particular, if $H^j(Y, \omega_{Y/X} \otimes f^*E^\bullet) = 0$ for some $j \geq 0$, then $H^j(X, E^\bullet) = 0$.

Proof. The proof is similar to [35, Lemma 4.1.14]. By the projection formula and the finiteness of f, we have

$$\begin{aligned} H^j(Y, \omega_{Y/X} \otimes f^*E^\bullet) &= H^j(X, \mathbf{R}f_*(\omega_{Y/X} \otimes f^*E^\bullet)) \\ &= H^j(X, \mathbf{R}f_*\omega_{Y/X} \otimes E^\bullet) = H^j(X, f_*\omega_{Y/X} \otimes E^\bullet). \end{aligned}$$

Since the natural inclusion $\mathcal{O}_X \to f_*\omega_{Y/X}$ is splitted via the trace map $f_*\omega_{Y/X} \to \mathcal{O}_X$, we conclude that $E^\bullet \to f_*\omega_{Y/X} \otimes E^\bullet$ also splits. The stated injectivity follows. □

4.1. **A Kawamata-Viehweg type log Akizuki-Nakano vanishing theorem.** We discuss some consequences of the vanishing Theorem 4.7 for twisted Hodge modules. As an illustration, we give a quick proof of a Kawamata-Viehweg type log Akizuki-Nakano vanishing, which is a special case of [1, Theorem 2.1.1]. The key idea is to construct an αL-twisted Hodge module associated to $\mathcal{O}_X(\lfloor \alpha D \rfloor)$ where D is a normal crossing divisor and $L = \mathcal{O}_X(D)$. We would like to thank Jakub Witaszek for pointing this out.

Let $A = \sum_i e_i A_i$ be a $\mathbb{Q}$-divisor. There are several related divisors

$$\lceil A \rceil, \quad \lfloor A \rfloor, \text{ and } \{A\}$$

defined by applying the same operation to coefficients e_i. For example,

$$\lceil A \rceil := \sum_i \lceil e_i \rceil A_i, \quad \{A\} := \sum_i \{e_i\} A_i.$$

We also denote the support of A to be

$$\mathrm{supp} A := \sum_i A_i.$$

It is immediate to see that

$$\mathrm{supp}\{A\} + \lfloor A \rfloor = \lceil A \rceil.$$

We prove the following Kawamata-Viehweg type log Akizuki-Nakano vanishing theorem.

Theorem 4.9. *Let X be a smooth projective variety of dimension n and let A be an ample $\mathbb{Q}$-divisor on X with normal crossing support. Then*

$$H^i(X, K_X + \lceil A \rceil) = 0, \quad \forall i > 0.$$

More generally, we have

$$H^p(X, \Omega_X^q(\log \operatorname{supp}\{A\}) \otimes \mathscr{O}_X(\lfloor A \rfloor)), \quad \textit{whenever } p + q > n.$$

Here $\Omega_X^q(\log \operatorname{supp}\{A\})$ *is the sheaf of log forms along the normal crossing divisor* $\operatorname{supp}\{A\}$.

Remark 4.10. This recovers [1, Theorem 2.1.1] for the case of

$$A = A, \quad D = G = \operatorname{supp}\{A\}, \quad F = \lceil A \rceil - A.$$

The idea of the proof is to produce a (positively) twisted polarized Hodge module M such that the underlying filtered twisted $\mathscr{D}$-module satisfies

$$\operatorname{gr}^F_{-n+p} \operatorname{DR}_X(\mathcal{M}) = \Omega_X^{n-p}(\log \operatorname{supp}\{A\}) \otimes \mathscr{O}_X(\lfloor A \rfloor)[p].$$

Then the desired vanishing statement will follow from Theorem 4.7. To do this, we need twisted version of some constructions in §2.2 and so we use similar notations. Let X be a complex manifold of dimension n and let

$$D = \sum_{i \in I} e_i Y_i, \quad e_i \in \mathbb{N}$$

be a normal crossing divisor on X. For $\alpha \in (0, 1]$, let

$$I_\alpha = \{i \in I \mid e_i \cdot \alpha \in \mathbb{Z}\}, \quad E = \sum_{i \in I \setminus I_\alpha} Y_i = \operatorname{supp}\{\alpha D\}, \quad L = \mathscr{O}_X(D). \tag{4.1}$$

We consider a log version of twisted differential operators in §3.2. Let

$$\mathscr{D}_{X, \alpha L(\log E)} \subseteq \mathscr{D}_{X, \alpha L}$$

be the sub-algebra of $\mathscr{D}_{X,\alpha L}$ locally generated by elements preserving the ideal sheaf of E. Over an open subset U trivializing L, one has $\mathscr{D}_{X,\alpha L(\log E)}|_U = \mathscr{D}_U(\log E|_U) \subseteq \mathscr{D}_U$, the differential operators over U preserving the ideal sheaf of $E|_U$.

Lemma 4.11. *The line bundle $\mathscr{O}_X(\lfloor \alpha D \rfloor)$ is equipped with a left $\mathscr{D}_{X,\alpha L(\log E)}$-module structure.*

Proof. The construction is similar to the one in [13, §7.4]. To make ideas more clear, let us first give a less rigorous discussion. If we allow the expression $\mathscr{O}_X(\alpha D)$, we can write

$$\begin{aligned} \mathscr{O}_X(\lfloor \alpha D \rfloor) &= \mathscr{O}_X(\alpha D - \{\alpha D\}) \\ &= \mathscr{O}_X(\alpha D) \otimes \mathscr{O}_X\left(\sum_{i \in I \setminus I_\alpha} -\{\alpha e_i\} Y_i\right). \end{aligned}$$

Thus on any open subset U trivializing $\mathscr{O}_X(D)$, one has

$$\mathscr{O}_X(\lfloor \alpha D \rfloor) \cong \mathscr{O}_X\left(\sum_{i \in I \setminus I_\alpha} -\{\alpha e_i\} Y_i\right).$$

Then one can define a $\mathscr{D}_U(\log E|_U)$-action as in (2.15) and would expect globally this gives a *left* $\mathscr{D}_{X,\alpha L(\log E)}$-module in the presence of $\mathscr{O}_X(\alpha D)$.

To make it more rigorously, let N be the greatest common divisor of $\{e_i\}_{i\in I_\alpha}$ and consider the line bundle

$$L_\alpha := \mathcal{O}_Y\left(-\sum_{i\in I_\alpha}\frac{e_i}{N}Y_i\right).$$

Note that αN is an integer. Then we have

$$\mathcal{O}_X(\lfloor\alpha D\rfloor)) = (L_\alpha)^{-\alpha N}\otimes\mathcal{O}_X\left(\sum_{i\in I\setminus I_\alpha}\lfloor\alpha e_i\rfloor Y_i\right)$$

On the other hand, one has

$$(L_\alpha)^{-N} = \mathcal{O}_X(D)\otimes\mathcal{O}_X\left(-\sum_{i\in I\setminus I_\alpha}e_iY_i\right).$$

Let $\phi : L|_U \to U\times\mathbb{C}$ be the local trivialization of the line bundle $L = \mathcal{O}_X(D)$. Let $\ell\in\Gamma^0(U, L_\alpha)$ be a local section. Then $\phi_*(\ell^{-N}) = \prod_{i\in I\setminus I_\alpha} z_i^{e_i}$, where z_i is the local equation of Y_i. Let

$$s = \ell^{-\alpha N}\prod_{i\in I\setminus I_\alpha} z_i^{-\lfloor\alpha e_i\rfloor}$$

be a local frame in $\Gamma^0(U,\mathcal{O}_X(\lfloor\alpha D\rfloor))$, then

$$\phi_*(s) = \prod_{i\in I\setminus I_\alpha} z_i^{\alpha e_i-\lfloor\alpha e_i\rfloor} = \prod_{i\in I\setminus I_\alpha} z_i^{\{\alpha e_i\}},$$

Now we define a *left* $\mathscr{D}_U(\log E|_U)$-action by

$$\partial_i\cdot\phi_*(s) = \begin{cases}\frac{\{\alpha e_i\}}{z_i}\cdot\phi_*(s), & \text{if } i\in I\setminus I_\alpha,\\ 0, & \text{if } i\in I_\alpha.\end{cases}$$

It is immediate to verify that this gives a *left* $\mathscr{D}_{X,\alpha L(\log E)}$-module.

□

Now we define a *right* filtered $\mathscr{D}_{X,\alpha L}$-module similar to (2.16). Since the sheaf $\omega_X(\log E)$ is a natural right $\mathscr{D}_X(\log E)$-module, thus $\omega_X(\log E)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\lfloor\alpha D\rfloor)$ has a *right* $\mathscr{D}_{X,\alpha L(\log E)}$-module structure and

$$\mathcal{M} := (\omega_X(\log E)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\lfloor\alpha D\rfloor))\otimes_{\mathscr{D}_{X,\alpha L(\log E)}}\mathscr{D}_{X,\alpha L}$$

is a *right* $\mathscr{D}_{X,\alpha L}$-module. There is a good filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$ induced by the order filtration on $\mathscr{D}_{X,\alpha L}$, similar to $F_\bullet\mathcal{V}_{\alpha,J}$ in (2.16). Similar to (2.17), one can show that

$$\operatorname{gr}^F_{-n+p}\mathrm{DR}_X(\mathcal{M}) = \Omega_X^{n-p}(\log E)\otimes\mathcal{O}_X(\lfloor\alpha D\rfloor)[p],\quad\forall p. \tag{4.2}$$

We claim that the filtered twisted $\mathscr{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies an αL-twisted polarized Hodge module. Since locally $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered $\mathscr{D}$-module underlying the constant Hodge module, it suffices to write down a flat Hermitian pairing

$$S : \mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\to\mathfrak{C}_{X,\alpha L}.$$

The construction is similar to [13, (8.46)]. As before, on an open subset U trivializing $\mathcal{O}_X(D)$, we have

$$\begin{aligned}\mathcal{M}|_U &\cong (\omega_U\otimes\mathcal{O}_U(\lfloor\alpha D\rfloor + E))\otimes\mathscr{D}_U\\ &\cong\left(\omega_U\otimes\mathcal{O}_U(\sum_{i\in I\setminus I_\alpha}(-\{\alpha e_i\}+1)Y_i)\right)\otimes\mathscr{D}_U,\end{aligned}$$

where $E = \sum_{i \in I \setminus I_\alpha} Y_i$. The Hermitian pairing $S|_U$ is induced by

$$S([s_1 \otimes P_1], [s_2 \otimes P_2]) = \frac{(-1)^{\frac{n(n+1)}{2}}}{(2\pi\sqrt{-1})^n} \int_X (P_1\overline{P_2}-)(s_1 \wedge \overline{s_2})_h,$$

where $P_i \in \Gamma^0(U, \mathscr{D}_U)$ and $s_i \in \Gamma^0(U, \omega_U \otimes \mathcal{O}_U(\sum_{i\in I\setminus I_\alpha}(-\{\alpha e_i\}+1)Y_i))$. Here $(s_1 \wedge \overline{s_2})_h$ is the top form induced by the canonical singular Hermitian metric $|-|_h$ on the line bundle

$$\mathcal{O}_U(\sum_{i\in I\setminus I_\alpha}(-\{\alpha e_i\}+1)Y_i)$$

with weight function $\prod_{i\in I\setminus I_\alpha} z_i^{\{\alpha e_i\}-1}$. One can check that it glues to a pairing into $\mathfrak{C}_{X,\alpha L}$ and is $\mathscr{D}_{X,\alpha L}$-linear. Then Theorem 4.7 together with (4.2) imply the following

Lemma 4.12. *If X is smooth projective, D is an ample normal crossing divisor on X and $\alpha \in (0,1]$, then*

$$H^{i+p}(X, \Omega_X^{n-p}(\log E) \otimes \mathcal{O}_X(\lfloor \alpha D \rfloor)) = 0, \quad \forall i > 0. \tag{4.3}$$

Proof of Theorem 4.9. Let us choose the smallest integer N such that $N \cdot A$ has integer coefficients. Then $D := N \cdot A$ is an ample normal crossing divisor on X and we choose $\alpha = \frac{1}{N} \in (0,1]$. In the notation of (4.1) one has

$$E = \operatorname{supp}\{\alpha D\} = \operatorname{supp}\{A\}, \quad \lfloor \alpha D \rfloor = \lfloor A \rfloor.$$

Now (4.3) gives the desired vanishing

$$H^{i+p}(X, \Omega_X^{n-p}(\log E) \otimes \mathcal{O}_X(\lfloor A \rfloor)) = 0, \quad \forall i > 0.$$

For $p = 0$, we have $\Omega_X^n(\log E) = \mathcal{O}_X(K_X + E)$, hence

$$H^i(X, K_X + \lceil A \rceil) = H^i(X, K_X + \operatorname{supp}\{A\} + \lfloor A \rfloor) = 0, \quad \forall i > 0.$$

This proves the Kawamata-Viehweg vanishing. $\square$

5. Higher multiplier ideals: definition and first properties

In this section, we define higher multiplier ideals for effective $\mathbb{Q}$-divisors and establish some first properties. The more detailed local and global studies are carried out in the later sections. Here is the main set-up of the following sections.

Set-up 5.1. Let X be a complex manifold of dimension n and denote by $\mathbb{Q}_X^H[n]$ the constant Hodge module on X, with the underlying filtered $\mathscr{D}$-module $(\omega_X, F_\bullet\omega_X)$ where

$$F_{-n+k}\omega_X = \begin{cases} \omega_X & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Let D be an effective divisor on X. Let $L = \mathcal{O}_X(D)$ be the associated holomorphic line bundle and also denote by L the total space of the line bundle, with the natural projection $p : L \to X$. Let $s \in H^0(X, L)$ be a section with $\operatorname{div}(s) = D$, which is viewed as a closed embedding

$$s : X \to L,$$

such that $p \circ s = \mathrm{id}_X$. Consider the direct image Hodge module

$$M := s_*\mathbb{Q}_X^H[n],$$

with the underlying filtered $\mathscr{D}$-module

$$(\mathcal{M}, F_\bullet\mathcal{M}) = s_+(\omega_X, F_\bullet\omega_X).$$

5.1. **Definition.** First we need a lemma on the associated graded of the Hodge filtration on $\mathcal{M}$.

Lemma 5.2. *With Set-up 5.1, we have*

$$\mathrm{gr}^F_{-n+k}\mathcal{M} \cong \begin{cases} s_*(\omega_X \otimes L^k), & \text{if } k \geq 0. \\ 0, & \text{if } k < 0. \end{cases}$$

Proof. For any open subset U such that $L|_U \cong \mathcal{O}_U$, let $f : U \to \mathbb{C}$ be a local function such that $D|_U = \mathrm{div}(f)$. Then the map s is given by the graph embedding of f:

$$s|_U = i_f : U \to U \times \mathbb{C}, \quad x \mapsto (x, f(x)).$$

The underlying filtered $\mathscr{D}$-module of $s_*(\mathbb{Q}^H_X[n]|_U) = s_*\mathbb{Q}^H_U[n]$ can be computed as follows. Let t be the holomorphic coordinate on the second component of $U \times \mathbb{C}$. Using the formula for the direct image by a closed embedding and a change of coordinates $t \mapsto t - f(x), x \mapsto x$, one has

$$s_+(\omega_X|_U) \cong \sum_{\ell \in \mathbb{N}} \omega_U \otimes \partial_t^\ell, \quad F_k s_+(\omega_X|_U) \cong \sum_{0 \leq \ell \leq k} \omega_U \otimes \partial_t^\ell.$$

Thus for $k \geq 0$, we have

$$\mathrm{gr}^F_{-n+k}\, s_+(\omega_X|_U) \cong \omega_U \otimes \partial_t^k, \tag{5.1}$$

and $\mathrm{gr}^F_{-n+k}\, s_+(\omega_X|_U) = 0$ if $k < 0$. Let us examine how the isomorphism (5.1) depends on the choice of trivialization. Note that we can identify $t : U \times \mathbb{C} \to \mathbb{C}$ as the local $\mathcal{O}_U$-generator e of $\Gamma(U, L)$ so that the trivialization is given by

$$\mathcal{O}_X(U) \xrightarrow{\sim} \Gamma(U, L), \quad f \mapsto f \cdot e. \tag{5.2}$$

Now we change the trivialization (5.2) by $g \in \mathcal{O}_X^\times(U)$ so that

$$\mathcal{O}_X(U) \to \Gamma(U, L) \to \mathcal{O}_X(U), \quad f \mapsto g \cdot f.$$

Since the local generator e changes to $g^{-1} \cdot e$, the local coordinate t also changes to $t' = g^{-1} \cdot t$. By the chain rule, we have $\partial_{t'} = g \cdot \partial_t$. This gives the following commutative diagram

$$\begin{tikzcd} \mathrm{gr}^F_{-n+k}\, s_+(\omega_X|_U) \arrow[r, "\sim"] \arrow[d, "="] & \omega_X \otimes \partial_t^k \arrow[d, "\cdot g^k"] \\ \mathrm{gr}^F_{-n+k}\, s_+(\omega_X|_U) \arrow[r, "\sim"] & \omega_X \otimes \partial_{t'}^k \end{tikzcd}$$

Therefore, we conclude that there is a global isomorphism

$$\mathrm{gr}^F_{-n+k}\, s_+\mathcal{M} = \mathrm{gr}^F_{-n+k}\, s_+(\omega_X) \cong s_*(\omega_X \otimes L^k).$$

□

Let $V_\bullet\mathcal{M}$ be the V-filtration relative to the zero section of L; locally it is the Kashiwara-Malgrange V-filtration relative to the local defining equation defined in §2 (see Remark 2.3). By property (1) in Definition 2.1, each $V_\alpha\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_L$-modules. Therefore there is an inclusion of coherent sheaves

$$\mathrm{gr}^F_{-n+k}\, V_\alpha\mathcal{M} := \frac{F_{-n+k}\mathcal{M} \cap V_\alpha\mathcal{M}}{F_{-n+k-1}\mathcal{M} \cap V_\alpha\mathcal{M}} \hookrightarrow \mathrm{gr}^F_{-n+k}\mathcal{M} \cong s_*(\omega_X \otimes L^k),$$

by Lemma 5.2. It naturally leads to the following.

Definition 5.3. Let D be an effective divisor on X. For any $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$, the *higher multiplier ideal* $\mathcal{I}_{k,\alpha}(D)$ is defined to be the unique coherent ideal sheaf on X such that

$$\operatorname{gr}^F_{-n+k} V_\alpha \mathcal{M} \cong s_*(\omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D)). \tag{5.3}$$

Similarly, we define $\mathcal{I}_{k,<\alpha}(D)$ to be the unique ideal sheaf such that

$$\operatorname{gr}^F_{-n+k} V_{<\alpha} \mathcal{M} \cong s_*(\omega_X \otimes L^k \otimes \mathcal{I}_{k,<\alpha}(D)).$$

Remark 5.4. The sheaf $\mathcal{I}_{k,\alpha}(D)$ can be locally described as follows. Assume $D = \operatorname{div}(f)$ for some holomorphic function $f : X \to \mathbb{C}$. Let $i_f : X \to X \times \mathbb{C}$ be the graph embedding and t be the coordinate on $\mathbb{C}$. Under the isomorphism $\operatorname{gr}^F_{-n+k} i_{f,+}(\omega_X) \cong \omega_X \otimes \partial_t^k$ (see (5.1)), we have

$$(\omega_X \otimes \mathcal{I}_{k,\alpha}(D)) \otimes \partial_t^k = \operatorname{gr}^F_{-n+k} V_\alpha i_{f,+}(\omega_X). \tag{5.4}$$

Remark 5.5. If $k = 0$, Budur and Saito's result [9, Theorem 0.1] can be restated as: for any $\alpha \in \mathbb{Q}$ one has

$$\mathcal{I}_{0,\alpha}(D) = \mathcal{J}(X, (-\alpha - \epsilon)D), \quad \text{for some } 0 < \epsilon \ll 1, \tag{5.5}$$

the right hand side is the usual multiplier ideal. Equivalently, for any $\alpha \in \mathbb{Q}$ one has

$$\mathcal{I}_{0,<\alpha}(D) = \mathcal{J}(X, -\alpha D).$$

By the properties of V-filtrations in Definition 2.1, one has $\mathcal{I}_{k,\alpha}(D) \subseteq \mathcal{I}_{k,\beta}(D)$ whenever $\alpha \leq \beta$. Therefore one can talk about the associated graded pieces.

Definition 5.6. The graded pieces are defined by

$$\mathcal{G}_{k,\alpha}(D) := \mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,<\alpha}(D).$$

We call $\alpha \in \mathbb{Q}$ a *jumping number* if $\mathcal{I}_{k,<\alpha}(D) \neq \mathcal{I}_{k,\alpha}(D)$, or equivalently $\mathcal{G}_{k,\alpha}(D) \neq 0$. There is an induced isomorphism

$$\operatorname{gr}^F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M} \cong \omega_X \otimes L^k \otimes \mathcal{G}_{k,\alpha}(D). \tag{5.6}$$

In §3.11, we construct a nilpotent operator N (see (3.5)) on $\operatorname{gr}^V_\alpha \mathcal{M}$ and let $W_\bullet(N)$ denote the weight filtration of this nilpotent operator.

Definition 5.7. The *weight filtration* $W_\bullet \mathcal{G}_{k,\alpha}(D)$ is induced by the weight filtration $W_\bullet(N) \operatorname{gr}^V_\alpha \mathcal{M}$ and the isomorphism (5.6). For each $\ell \in \mathbb{Z}$, the graded piece is denoted by

$$\operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) := W_\ell \mathcal{G}_{k,\alpha}(D)/W_{\ell-1}\mathcal{G}_{k,\alpha}(D).$$

One has

$$W_\ell(N) \operatorname{gr}^F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M} \cong \omega_X \otimes L^k \otimes W_\ell \mathcal{G}_{k,\alpha}(D), \tag{5.7}$$

$$\operatorname{gr}^F_{-n+k} \operatorname{gr}^{W(N)}_\ell \operatorname{gr}^V_\alpha \mathcal{M} \cong \omega_X \otimes L^k \otimes \operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D). \tag{5.8}$$

Definition 5.8. The weight filtration $W_\bullet \mathcal{I}_{k,\alpha}(D)$ is defined as the preimage of $W_\bullet \mathcal{G}_{k,\alpha}(D)$ under the quotient map $\mathcal{I}_{k,\alpha}(D) \to \mathcal{G}_{k,\alpha}(D)$. In particular, there is a short exact sequence for any $\ell \in \mathbb{Z}$

$$0 \to \mathcal{I}_{k,<\alpha}(D) \to W_\ell \mathcal{I}_{k,\alpha}(D) \to W_\ell \mathcal{G}_{k,\alpha}(D) \to 0. \tag{5.9}$$

5.2. **First properties.** Using the properties of V-filtrations and Hodge filtrations in the theory of Hodge modules, we establish some first properties of higher multiplier ideals. Before doing so, we need a global version of action by t and ∂_t.

Lemma 5.9. *For any α and $p \in \mathbb{Z}$, there exist morphisms*

$$F_pV_\alpha\mathcal{M} \to F_pV_{\alpha-1}\mathcal{M} \otimes_{\mathcal{O}_X} L, \tag{5.10}$$

$$F_pV_\alpha\mathcal{M} \to F_{p+1}V_{\alpha+1}\mathcal{M} \otimes_{\mathcal{O}_X} L^{-1}, \tag{5.11}$$

$$F_p \operatorname{gr}^V_\alpha \mathcal{M} \to F_{p+1} \operatorname{gr}^V_{\alpha+1} \mathcal{M} \otimes_{\mathcal{O}_X} L^{-1}, \tag{5.12}$$

where (5.10) *is locally induced by t and is an isomorphism for $\alpha < 0$;* (5.11), (5.12) *are both locally induced by ∂_t and* (5.12) *is an isomorphism for $\alpha > -1$ and is surjective for $\alpha = -1$.*

Proof. The proof is similar to the proof of Lemma 5.2. Let $U \subseteq X$ be an open such that $L|_U \cong \mathcal{O}_U$. If the trivialization of L is changed by $g \in \mathcal{O}_X^\times(U)$, then the coordinate t on $\mathbb{C}$ is changed to $g \cdot t$. Therefore the multiplication by t globalizes to

$$\operatorname{gr}^V_\alpha \mathcal{M} \to \operatorname{gr}^V_{\alpha-1} \mathcal{M} \otimes_{\mathcal{O}_X} L.$$

Similarly, one can prove the action of ∂_t globalizes to (5.11) and (5.12). Since ω_X underlies the Hodge module $\mathbb{Q}^H_X[n]$, the desired properties for (5.10) and (5.12) follows from the properties in Definition 2.5 and Remark 2.6 ($\mathbb{Q}^H_X[n]$ has strict support X). $\square$

Proposition 5.10. *Let D be an effective divisor on a complex manifold X. Fix $k \in \mathbb{N}$.*

- *(I) If $\alpha \leq \beta$, then $\mathcal{I}_{k,\alpha}(D) \subseteq \mathcal{I}_{k,\beta}(D)$. The sequence of ideal sheaves $\{\mathcal{I}_{k,\alpha}(D)\}_{\alpha\in\mathbb{Q}}$ is discrete and right continuous, the set of jumping numbers is discrete.*
- *(II) One has $\mathcal{I}_{k,<k}(D) = \mathcal{O}_X$.*
- *(III) For any α, there exist morphisms*

$$\mathcal{I}_{k,\alpha}(D) \to \mathcal{I}_{k,\alpha-1}(D) \otimes \mathcal{O}_X(D), \tag{5.13}$$

$$\mathcal{I}_{k,\alpha}(D) \to \mathcal{I}_{k+1,\alpha+1}(D), \tag{5.14}$$

$$\mathcal{G}_{k,\alpha}(D) \to \mathcal{G}_{k+1,\alpha+1}(D), \tag{5.15}$$

so that (5.13) *is an isomorphism for $\alpha < 0$,* (5.14) *is an isomorphism for $\alpha \geq -1$ and* (5.15) *is isomorphic for $\alpha > -1$, surjective for $\alpha = -1$.*

- *(IV) For $k \geq 1$, there are two short exact sequences:*

$$0 \to L^k \otimes \mathcal{I}_{k,0}(D) \otimes \omega_X \to L^{k+1} \otimes \mathcal{I}_{k,-1}(D) \otimes \omega_X \to \operatorname{gr}^F_{-n+k} \omega_X(*D) \to 0, \tag{5.16}$$

$$0 \to \operatorname{gr}^F_{-n+k} \omega_X(!D) \otimes L \to L^k \otimes \mathcal{G}_{k-1,-1}(D) \otimes \omega_X \to L^k \otimes \mathcal{G}_{k,0}(D) \otimes \omega_X \to 0, \tag{5.17}$$

where $(\omega_X(*D), F)$ and $(\omega_X(!D), F)$ are filtered $\mathscr{D}$-modules underlying the mixed Hodge modules $j_*\mathbb{Q}^H_{X\setminus D}[n]$ and $j_!\mathbb{Q}^H_{X\setminus D}[n]$, and $j : X \setminus D \hookrightarrow X$ is the open embedding.

- *(V) Let $x \in D$ and f_x be the local function so that $f_x(x) = 0$ and $\operatorname{div}(f_x) = D$ locally. Then the set of roots of the Bernstein-Sato polynomial of f_x is the set of jumping numbers of $\{\mathcal{I}_{k,\bullet}(D)_x\}$ for all k, modulo $\mathbb{Z}$.*

Remark 5.11. For $k = 0$, we have $\mathcal{I}_{0,<0}(D) = \mathcal{O}_X$, which agrees with the fact that one only consider $\mathcal{J}(\beta D)$ for $\beta > 0$, using $\mathcal{I}_{0,<\alpha}(D) = \mathcal{J}(-\alpha D)$. But for $k \geq 1$, we can have $\alpha \geq 0$, which provides new information and will be useful in applications. For $k = 0$, the isomorphism (5.13) for $\alpha < 0$ recovers the well-known periodicity of jumping numbers for multiplier ideals [36, Example 9.3.24]. One can compare (V) with [47, Proposition 6.14] for Hodge ideals.

Proof. The statement (I) follows from properties of V-filtration in Definition 2.1.

For (II), if $\alpha \geq k$, the surjectivity of (5.12) for $\alpha \geq -1$ induces a surjection

$$F_{-n-1}\operatorname{gr}^V_{\alpha-k-1}\mathcal{M} \twoheadrightarrow F_{-n+k}\operatorname{gr}^V_\alpha \mathcal{M}\otimes L^{-k-1}.$$

Since $F_{-n-1}\mathcal{M}=0$ by Lemma 5.2, we have $F_{-n+k}\operatorname{gr}^V_\alpha\mathcal{M}=0$ for $\alpha\geq k$. This means that $F_{-n+k}\mathcal{M}\subseteq V_{<k}\mathcal{M}$ and therefore

$$\operatorname{gr}^F_{-n+k} V_{<k}\mathcal{M} = \operatorname{gr}^F_{-n+k}\mathcal{M},$$

i.e. $\mathcal{I}_{k,<k}(D)=\mathcal{O}_X$. This proves (II).

For (III), the existence and properties of (5.13), (5.14) and (5.15) follow from Lemma 5.9 and Definition 5.3, except for the property of (5.14). To prove the property of (5.14), i.e.

$$\mathcal{I}_{k,\alpha}(D)\xrightarrow{\sim}\mathcal{I}_{k+1,\alpha+1}(D),\quad \text{whenever } \alpha\geq -1,$$

let us consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{k,\alpha}(D) \arrow[r]\arrow[d] & \mathcal{I}_{k,k}(D) \arrow[r]\arrow[d] & \mathcal{I}_{k,k}(D)/\mathcal{I}_{k,\alpha}(D) \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & \mathcal{I}_{k+1,\alpha+1}(D) \arrow[r] & \mathcal{I}_{k+1,k+1}(D) \arrow[r] & \mathcal{I}_{k+1,k+1}(D)/\mathcal{I}_{k+1,\alpha+1}(D) \arrow[r] & 0,
\end{tikzcd}$$

where vertical arrows come from (5.14). It is clear by (II) that the second vertical map is an isomorphism. The third vertical map is also an isomorphism by (5.15), since $\mathcal{I}_{k,k}(D)/\mathcal{I}_{k,\alpha}(D)$ is a finite extension of $\mathcal{G}_{k,\beta}(D)$ for $\beta\in(\alpha,k]$ and $\beta>\alpha\geq -1$. We conclude by snake lemma that the first vertical map is an isomorphism as well.

To prove (IV), let $j: X\setminus D\hookrightarrow X$ and $i: D\hookrightarrow X$ be the open and closed embeddings. We use the functorial triangles from [62, (4.4.1)]. Since $i_*(H^0(i^!\mathbb{Q}^H_X[n]))=0$, we have a short exact sequence of mixed Hodge modules

$$0\to\mathbb{Q}^H_X[n]\to j_*\mathbb{Q}^H_{X\setminus D}[n]\to i_*(H^1 i^!\mathbb{Q}^H_X[n])(-1)\to 0, \tag{5.18}$$

with the underlying filtered $\mathscr{D}$-modules

$$0\to(\omega_X,F)\to(\omega_X(*D),F)\to i_*(H^1 i^!(\omega_X,F))\to 0. \tag{5.19}$$

Since $\mathbb{Q}^H_X[n]$ has strict support X, the underlying filtered $\mathscr{D}$-module of $i_*(H^1i^!\mathbb{Q}^H_X[n])$ can be computed as the cokernel of the injective morphism

$$\mathrm{var}:\left(\operatorname{gr}^V_0\mathcal{M},F_\bullet\operatorname{gr}^V_0\mathcal{M}\right)\to\left(\operatorname{gr}^V_{-1}\mathcal{M}\otimes L,F_\bullet\operatorname{gr}^V_{-1}\mathcal{M}\otimes L\right),$$

where locally $\mathrm{var}=t$ by Lemma 5.9. Since $\operatorname{gr}^F_{-n}\omega_X=\omega_X$ and $\operatorname{gr}^F_{-n+k}\omega_X=0$ for $k\geq 1$, combined with (5.6) and (5.19), one has a short exact sequence for $k\geq 1$:

$$0\to\omega_X\otimes L^k\otimes\mathcal{G}_{k,0}(D)\to\omega_X\otimes L^{k+1}\otimes\mathcal{G}_{k,-1}(D)\to\operatorname{gr}^F_{-n+k}\omega_X(*D)\to 0. \tag{5.20}$$

By (III), one has $\mathcal{I}_{k,<0}(D)\cong L\otimes\mathcal{I}_{k,<-1}(D)$ and thus it gives (5.16). In addition, we also obtain

$$\omega_X\otimes L\otimes\mathcal{G}_{0,-1}(D)\cong\frac{\operatorname{gr}^F_{-n}\omega_X(*D)}{\omega_X}, \tag{5.21}$$

which will be used in later sections.

Now, let us consider the exact sequence dual to (5.18):

$$0\to i_*(H^{-1}i^*\mathbb{Q}^H_X[n])\to j_!\mathbb{Q}^H_{X\setminus D}[n]\to\mathbb{Q}^H_X[n]\to 0.$$

Dually, because $\mathbb{Q}^H_X[n]$ has strict support X, the underlying filtered $\mathscr{D}$-module of $i_*(H^{-1}i^*\mathbb{Q}^H_A[g])$ is computed by the kernel of the surjective morphism

$$\mathrm{can}:\left(\operatorname{gr}^V_{-1}\mathcal{M},F_{\bullet-1}\operatorname{gr}^V_{-1}\mathcal{M}\right)\to\left(\operatorname{gr}^V_0\mathcal{M}\otimes L^{-1},F_\bullet\operatorname{gr}^V_0\mathcal{M}\otimes L^{-1}\right),$$

where locally can $= \partial_t$ by Lemma 5.9. This leads to (5.17).

The statement (V) follows from a result of Malgrange [39]: every root of the Bernstein-Sato polynomial of a function f must be the eigenvalue of the monodromy operator on the nearby cycle $\psi_f\mathbb{C}$, modulo $\mathbb{Z}$, and vice versa. On the other hand, by (III) we know that modulo $\mathbb{Z}$ every jumping number is equal to a jumping number in $[-1,0)$, which is the range of α in nearby cycles, see (2.2). $\square$

Remark 5.12. The property $F_{-n+k}\operatorname{gr}_\alpha^V\mathcal{M} = 0$ for $\alpha \geq k$ is the global version of [9, 2.1.4].

Remark 5.13. It is impossible to upgrade (5.17) to a short exact sequence

$$0 \to \operatorname{gr}^F_{-n+k}\omega_X(!D) \otimes L \to L^k \otimes \mathcal{I}_{k-1,-1}(D) \otimes \omega_X \to L^k \otimes \mathcal{I}_{k,0}(D) \otimes \omega_X \to 0.$$

Because in general it is not true that the morphism $\mathcal{I}_{k-1,<-1}(D) \to \mathcal{I}_{k,<0}(D)$ locally induced by ∂_t is an isomorphism. For example, by (5.5) and Lemma 9.11 we will see that $\mathcal{I}_{0,<-1}(\Theta) = \mathcal{O}_A(-\Theta)$ and $\mathcal{I}_{1,<0}(\Theta) = \mathcal{O}_A$, where (A,Θ) is an indecomposable principally polarized abelian variety.

5.3. **Higher multiplier ideals of $\mathbb{Q}$-divisors.** The definition of higher multiplier ideals can be extended to $\mathbb{Q}$-divisors, although the twisting causes some complications. Suppose that D is an effective divisor on X, defined by a global section $s \in H^0(X,L)$. For any integer $m \geq 1$, we denote by M_m the Hodge module on the total space of the line bundle L^m, obtained by the graph embedding along the section s^m defining the divisor mD. Let $(\mathcal{M}_m, F_\bullet\mathcal{M}_m)$ be the underlying filtered $\mathscr{D}$-module. It follows from Proposition 2.12 (applied locally) that we have an isomorphism of $\mathscr{O}_X$-modules

$$F_{-n+k}V_{m\alpha}\mathcal{M}_1 \cong F_{-n+k}V_\alpha\mathcal{M}_m,$$

for $\alpha \leq 0$ and $k \in \mathbb{Z}$. In light of Definition 5.3, this is saying that

$$\mathcal{I}_{k,m\alpha}(D) \otimes \mathscr{O}_X(kD) \cong \mathcal{I}_{k,\alpha}(mD) \otimes \mathscr{O}_X(kmD). \tag{5.22}$$

Both sides are torsion-free coherent $\mathscr{O}_X$-modules of rank 1. We can use this formula in order to extend the definition of higher multiplier ideals to effective $\mathbb{Q}$-divisors.

Let E be an effective $\mathbb{Q}$-divisor on a complex manifold X. Let $m \geq 1$ be a positive integer with the property that mE has integer coefficients. For $\alpha \leq 0$ and $k \in \mathbb{N}$, we then define the torsion-free coherent $\mathscr{O}_X$-module

$$\mathcal{S}_{k,\alpha}(E) = \mathcal{I}_{k,\alpha}(E) \otimes \mathscr{O}_X(kE) \underset{\text{def}}{=} \mathcal{I}_{k,\alpha/m}(mE) \otimes \mathscr{O}_X(kmE).$$

Of course, the notation on the left-hand side is purely symbolic, since $\mathscr{O}_X(kE)$ does not make sense as a line bundle. As a consequence of Proposition 2.12, the resulting $\mathscr{O}_X$-module is (up to isomorphism) independent of the choice of m.

For $k = 0$, we get a well-defined ideal sheaf $\mathcal{I}_{0,\alpha}(E)$ in this way. As expected, it agrees with the usual multiplier ideal sheaf of the $\mathbb{Q}$-divisor $-(\alpha+\varepsilon)E$ for small $\varepsilon > 0$.

Lemma 5.14. *Let E be an effective $\mathbb{Q}$-divisor on a complex manifold. Then*

$$\mathcal{I}_{0,<\alpha}(E) = \mathcal{J}(X, -\alpha E)$$

for every $\alpha \leq 0$.

Proof. Let $m \geq 1$ be such that mE has integer coefficients. Then by definition and (5.5),

$$\mathcal{I}_{0,<\alpha}(D) = \mathcal{I}_{0,<\alpha/m}(mE) = \mathcal{J}\left(X, -\frac{\alpha}{m}\cdot mE\right) = \mathcal{J}(X, -\alpha E),$$

the last equality being of course a basic property of multiplier ideals. $\square$

For $k \geq 1$, it is only the rank-one torsion-free sheaf $\mathcal{S}_{k,\alpha}(E) = \mathcal{I}_{k,\alpha}(E) \otimes \mathscr{O}_X(kE)$ that is globally well-defined. In order to get an actual sheaf of ideals, we observe that the reflexive hull of $\mathcal{S}_{k,\alpha}(E)$ is a line bundle. Consequently, we have

$$\mathcal{S}_{k,\alpha}(E) = \mathcal{I}'_{k,\alpha}(E) \otimes \mathcal{S}_{k,\alpha}(E)^{**}$$

for a unique coherent sheaf of ideals in $\mathscr{O}_X$. By construction, the cosupport of this ideal sheaf has codimension ≥ 2 in X. In the case of a $\mathbb{Z}$-divisor D, this ideal $\mathcal{I}'_{k,\alpha}(D)$ is the result of removing from $\mathcal{I}_{k,\alpha}(D)$ its divisorial part. All the local properties of higher multiplier ideals therefore carry over to the setting of $\mathbb{Q}$-divisors. We leave the details to the interested readers.

5.4. **Comparison with the microlocal V-filtration.** We compare the higher multiplier ideals with Saito's microlocal V-filtration [64, 65]. To recall the definition from [65], we need to temporarily work with *left* $\mathscr{D}$-modules and *decreasing* V-filtrations. First let us consider the local situation: let X be a complex manifold of dimension n and let f be a holomorphic function on X. Consider the graph embedding

$$i_f : X \to X \times \mathbb{C}_t, \quad x \mapsto (x, f(x)),$$

where t is the coordinate on $\mathbb{C}$ and the following two filtered left $\mathscr{D}$-modules

$$(\mathcal{B}_f, F) := (i_f)_+(\mathcal{O}_X, F), \quad (\tilde{\mathcal{B}}_f, F) := \mathcal{O}_X \otimes_{\mathbb{C}} (\mathbb{C}[\partial_t, \partial_t^{-1}], F). \tag{5.23}$$

Here $(i_f)_+$ is the direct image functor for filtered $\mathscr{D}$-modules and $\mathcal{B}_f \cong \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$. The Hodge filtration on $\mathcal{B}_f$ and $\tilde{\mathcal{B}}_f$ are defined by

$$F_k\mathcal{B}_f := \sum_{0 \leq \ell \leq k} \mathcal{O}_X \otimes \partial_t^\ell, \quad F_k\tilde{\mathcal{B}}_f := \sum_{\ell \leq k} \mathcal{O}_X \otimes \partial_t^\ell,$$

so that

$$\operatorname{gr}_k^F \mathcal{B}_f = \mathcal{O}_X \otimes \partial_t^k, \quad \forall k \in \mathbb{N}, \quad \text{and } \operatorname{gr}_k^F \tilde{\mathcal{B}}_f = \mathcal{O}_X \otimes \partial_t^k, \quad \forall k \in \mathbb{Z}.$$

Therefore, there is a natural isomorphism:

$$\mathcal{O}_X \otimes 1 \cong \operatorname{gr}_0^F \tilde{\mathcal{B}}_f. \tag{5.24}$$

The microlocal V-filtration on $\tilde{\mathcal{B}}_f$ along $t = 0$ is defined by

$$V^\beta\tilde{\mathcal{B}}_f = \begin{cases} V^\beta\mathcal{B}_f \oplus (\mathcal{O}_X[\partial_t^{-1}]\partial_t^{-1}) & \text{if } \beta \leq 1, \\ \partial_t^{-j} \cdot V^{\beta-j}\tilde{\mathcal{B}}_f & \text{if } \beta > 1, \beta - j \in (0,1]. \end{cases} \tag{5.25}$$

Here $\partial_t t - \alpha$ acts nilpotently on $\operatorname{gr}_V^\alpha \mathcal{B}_f$. The microlocal V-filtration on $\mathcal{O}_X$ is then induced by (5.24)

$$\tilde{V}^\beta\mathcal{O}_X \otimes 1 := \operatorname{gr}_0^F V^\beta\tilde{\mathcal{B}}_f.$$

Then we have the following statement.

Lemma 5.15. *With the notation above. If $0 < \beta \leq 1$ and $k \in \mathbb{N}$, then*

$$\tilde{V}^{k+\beta}\mathcal{O}_X \otimes \partial_t^k = \operatorname{gr}_k^F V^\beta\mathcal{B}_f.$$

under the identification $\mathcal{O}_X \otimes \partial_t^k = \operatorname{gr}_k^F \mathcal{B}_f$.

Proof. Let $u \in \tilde{V}^{k+\beta}\mathcal{O}_X$. Since $k \in \mathbb{N}$ and $\beta \in (0,1]$, by (5.25) there exist an integer $\ell \geq 0$ and elements $u_{-\ell}, u_{-\ell+1}, \ldots u_0 \in \mathcal{O}_X$, such that

- $u = u_0$,
- $\sum_{-\ell \leq i \leq 0} u_i \otimes \partial_t^i \in V^{k+\beta}\tilde{\mathcal{B}}_f \cap F_0\tilde{\mathcal{B}}_f$, where

$$\partial_t^k \cdot V^{k+\beta}\tilde{\mathcal{B}}_f = V^\beta\mathcal{B}_f \oplus \mathcal{O}_X[\partial_t^{-1}]\partial_t^{-1}.$$

Therefore

$$\sum_{-\ell\leq i\leq 0} u_i\otimes\partial_t^{k+i}=\partial_t^k\cdot(\sum_{-\ell\leq i\leq 0} u_i\otimes\partial_t^i)\in V^\beta\mathcal{B}_f\oplus\mathcal{O}_X[\partial_t^{-1}]\partial_t^{-1}.$$

Since $k\geq 0$ and $u\otimes\partial_t^k=u_0\otimes\partial_t^k$, we know that

$$\sum_{k+i\geq 0} u_i\otimes\partial_t^{k+i}\in V^\beta\mathcal{B}_f\cap F_k\mathcal{B}_f.$$

Hence the class of $u\otimes\partial_t^k$ is an element in $\mathrm{gr}_k^F V^\beta\mathcal{B}_f$ via the isomorphism $\mathrm{gr}_k^F\mathcal{B}_f\cong\mathcal{O}_X\otimes\partial_t^k$. This induces a map

$$\begin{aligned}\tilde{V}^{k+\beta}\mathcal{O}_X\otimes\partial_t^k&\to\mathrm{gr}_k^F V^\beta\mathcal{B}_f,\\ u\otimes\partial_t^k&\mapsto[u\otimes\partial_t^k].\end{aligned}\tag{5.26}$$

Now, let us show (5.26) is an isomorphism. First, suppose that $[u\otimes\partial_t^k]=0\in\mathrm{gr}_k^F V^\beta\mathcal{B}_f$, then $u\otimes\partial_t^k\in F_{k-1}V^\beta\mathcal{B}_f$. We must have $u=0$ and hence (5.26) is injective. For the surjectivity, assume $[v\otimes\partial_t^k]\in\mathrm{gr}_k^F V^\beta\mathcal{B}_f$ for some $v\in\mathcal{O}_X$, then there exist $v_0,\ldots,v_k\in\mathcal{O}_X$ such that

$$v=v_k,\quad\text{and}\quad\sum_{0\leq i\leq k} v_i\otimes\partial_t^i\in V^\beta\mathcal{B}_f.$$

Then

$$\sum_{0\leq i\leq k} v_i\otimes\partial_t^{i-k}\in\partial_t^{-k}\cdot V^\beta\mathcal{B}_f\cap F_0\tilde{\mathcal{B}}_f\subseteq V^{k+\beta}\tilde{\mathcal{B}}_f\cap F_0\tilde{\mathcal{B}}_f.$$

Therefore $v\otimes 1\in\tilde{V}^{k+\beta}\mathcal{O}_X\otimes 1$, this means that $v\in\tilde{V}^{k+\beta}\mathcal{O}_X$. Therefore (5.26) is also surjective.

□

Let D be an effective divisor on a complex manifold X. There is a microlocal V-filtration $V^\bullet\mathcal{O}_X$ along D, see [41, §1]. The point is that the filtration $\tilde{V}^\bullet\mathcal{O}_X$ does not depend on the choice of a local defining equation, see [41, Remark 1.3].

Corollary 5.16. *With the notation above. For any $k\in\mathbb{N}$, one has*

$$\mathcal{I}_{k,\alpha}(D)=\begin{cases}\tilde{V}^{k-\alpha}\mathcal{O}_X, & \text{if } \alpha\geq -1,\\ \tilde{V}^{k-(\alpha+t)}\mathcal{O}_X\otimes\mathcal{O}_X(-tD), & \text{if } \alpha<-1 \text{ and } t\in\mathbb{N} \text{ so that } -1\leq\alpha+t<0.\end{cases}$$

Conversely, for any rational number $\beta>0$, we have

$$\tilde{V}^\beta\mathcal{O}_X=\mathcal{I}_{\lfloor\beta-\epsilon\rfloor,\lfloor\beta-\epsilon\rfloor-\beta}(D)=\begin{cases}\mathcal{I}_{\lfloor\beta\rfloor,-\{\beta\}}(D) & \text{if } \beta\notin\mathbb{N},\\ \mathcal{I}_{\beta-1,-1}(D) & \text{if } \beta\in\mathbb{N}_{\geq 1}.\end{cases}\tag{5.27}$$

Proof. We can check the statements locally and assume $D=\mathrm{div}(f)$ for some $f:X\to\mathbb{C}$. Then $\mathcal{M}$ in Set-up 5.1 becomes $\widetilde{\omega}_X:=(i_f)_+\omega_X$. We have the following transition rules between left and right $\mathscr{D}$-modules:

$$\omega_X\otimes_{\mathcal{O}_X}V^\beta\mathcal{B}_f=V_{-\beta}\widetilde{\omega}_X,\quad \omega_X\otimes_{\mathcal{O}_X}\mathrm{gr}_k^F\mathcal{B}_F\cong\mathrm{gr}^F_{-\dim X+k}\widetilde{\omega}_X.$$

Then for any $k\in\mathbb{N}$ and $\alpha\in[-1,0)$, by Lemma 5.15 and (5.4) we have

$$\tilde{V}^{k-\alpha}\mathcal{O}_X\otimes\partial_t^k=\mathrm{gr}_k^F V^{-\alpha}\mathcal{B}_f=\mathrm{gr}^F_{-\dim X+k}V_\alpha\tilde{\omega}_X\otimes_{\mathcal{O}_X}\omega_X^{-1}=\mathcal{I}_{k,\alpha}(D)\otimes\partial_t^k,$$

which gives

$$\mathcal{I}_{k,\alpha}(D)=\tilde{V}^{k-\alpha}(D),\quad\text{when } -1\leq\alpha<0.$$

The case $\alpha\geq 0$ follows from (5.14) being an isomorphism for $\alpha\geq -1$. If $\alpha<-1$, we use that (5.13) is an isomorphism when $\alpha<0$. □

Corollary 5.17. *One has*

$$\mathcal{I}_{k+1,\alpha}(D) \subseteq \mathcal{I}_{k,\alpha}(D), \quad \text{for all } \alpha \in \mathbb{Q}, k \in \mathbb{N}.$$

More generally, if $\alpha_1, \alpha_2 \geq -1$, *then*

$$\mathcal{I}_{k_1,\alpha_1}(D) \subseteq \mathcal{I}_{k_2,\alpha_2}(D), \quad \text{whenever } k_1 - \alpha_1 \geq k_2 - \alpha_2.$$

Proof. It follows from Corollary 5.16 and that $\tilde{V}^{\bullet}\mathcal{O}_X$ is an decreasing filtration. □

5.5. **Comparison with (weighted) Hodge ideals.** We compare higher multiplier ideals with the (weighted) Hodge ideals from the work of Mustaţă-Popa and Olano [45, 46, 52].

Let D be a reduced effective divisor on a smooth algebraic variety X. Denote by $I_k(\beta D)$ the k-th Hodge ideal associated to the $\mathbb{Q}$-divisor βD. By [45, Proposition 10.1] and (5.5), one has

$$I_0(D) = \mathcal{J}(X, (1-\epsilon)D) = \mathcal{I}_{0,-1}(D).$$

for some $0 < \epsilon \ll 1$. For general k, we have the following comparison.

Lemma 5.18. *For any rational number* $-1 \leq \alpha < 0$, *we have*

$$\mathcal{I}_{k,\alpha}(D) \equiv I_k(-\alpha D) \quad \mathit{mod}\, \mathcal{I}_D.$$

Proof. By [47, Theorem A'], for any $\beta > 0$, one has

$$I_k(\beta D) \equiv \tilde{V}^{k+\beta}\mathcal{O}_X \mod \mathcal{I}_D, \tag{5.28}$$

where $\tilde{V}^{\bullet}\mathcal{O}_X$ is the microlocal V-filtration along D. Since $-\alpha > 0$, together with Corollary 5.16 this gives

$$\mathcal{I}_{k,\alpha}(D) \equiv \tilde{V}^{k-\alpha}\mathcal{O}_X \equiv I_k(-\alpha D) \mod \mathcal{I}_D.$$

□

Example 5.19. Assume D has an ordinary singularity at x with $\mathrm{mult}_x(D) = m \geq 2$. In Theorem 6.6 we compute $\mathcal{I}_{k,\alpha}(D)$ and it implies the following: write $\dim X = n = km + r$ with $k \in \mathbb{N}$ and $0 \leq r \leq m-1$, then one has

$$\mathcal{I}_{k,\alpha}(D)_x = I_k\left(-\alpha D\right)_x, \quad \forall \max(-1, -1 - \frac{r-1}{m}) \leq \alpha < 0,$$

i.e. (5.28) holds without $\mathcal{I}_D$. This can be checked as follows: set $\alpha = -p/m$ such that $p \in [1, m]$ and $p \leq m + r - 1$. Then $(k-1)m + \lceil -\alpha m \rceil < n$ and $k \leq n-2$, so that we can apply (6.17) in Theorem 6.6 and [46, Example 11.7] to get

$$I_k(-\alpha D)_x = \mathfrak{m}_x^{p-r} = \mathfrak{m}_x^{km + \lceil -\alpha m \rceil - n} = \mathcal{I}_{k,-\alpha}(D)_x.$$

The first interesting case is $n = km$, $r = 0$ and $\alpha = -1$. In this case, $I_k(D)_x$ is not known; on the other hand, (6.17) implies that $\mathcal{I}_{k,-1}(D)_x = (J_F, \mathfrak{m}_x^m)$, where F is the equation of $\mathbf{P}(C_xD)$ inside $\mathbf{P}(T_xX)$. If $k > n/m$, there are examples where the equality (5.28) fails without $\mathcal{I}_D$: $D = \mathrm{div}(f)$, $f = x^3 + y^3 + z^3$ with $n = m = 3$, $k = 2$ and $\alpha = -1$. This is a consequence of [65, §2.4, Remarks (ii)] and Corollary 5.16.

Example 5.20. Weighted homogeneous polynomials with isolated singularities provide another example where (5.28) holds without $\mathcal{I}_D$ for small k, see Example 6.20.

Next, we give a comparison with weighted Hodge ideals [52], generalizing Lemma 5.18. To do this, let us start by giving a more precise comparison than Lemma 5.18 for $\alpha = -1$. For $k = 0$, we have

$$\frac{\mathcal{I}_{0,-1}(D)}{\mathcal{O}_X(-D)} = \mathcal{G}_{0,-1}(D) = \frac{\mathrm{gr}^F_{-n}\,\mathcal{O}_X(*D) \otimes \mathcal{O}_X(-D)}{\mathcal{O}_X(-D)} = \frac{I_0(D)}{\mathcal{O}_X(-D)}, \tag{5.29}$$

where the second equality uses (5.21) and $\mathrm{gr}^F_{-n}\,\omega_X(*D) \cong \omega_X \otimes \mathrm{gr}^F_0\,\mathcal{O}_X(*D)$.

Lemma 5.21. *For $k \geq 1$, we have an isomorphism*

$$\frac{\mathcal{I}_{k,-1}(D)}{\mathcal{I}_{k,0}(D) \otimes \mathcal{O}_X(-D)} \cong \frac{I_k(D)}{I_{k-1}(D) \otimes \mathcal{O}_X(-D)}.$$

Proof. By definition, one has $F_k\mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k(D)$. This gives a natural inclusion $I_{k-1}(D) \otimes \mathcal{O}_X(-D) \hookrightarrow I_k(D)$ and thus for $k \geq 1$ one has

$$\mathrm{gr}^F_k\,\mathcal{O}_X(*D) \otimes \mathcal{O}_X((-k-1)D) = \frac{I_k(D)}{I_{k-1}(D) \otimes \mathcal{O}_X(-D)}.$$

On the other hand, recall from (5.16) that there is a short exact sequence for $k \geq 1$:

$$0 \to \mathcal{I}_{k,0}(D) \otimes \mathcal{O}_X(-D) \to \mathcal{I}_{k,-1}(D) \to \mathrm{gr}^F_k\,\mathcal{O}_X(*D) \otimes \mathcal{O}_X((-k-1)D) \to 0, \tag{5.30}$$

where $\mathrm{gr}^F_k\,\mathcal{O}_X(*D) \otimes \omega_X \cong \mathrm{gr}^F_{-n+k}\,\omega_X(*D)$. This finishes the proof. ☐

Remark 5.22. Lemma 5.21 implies Lemma 5.18 when $\alpha = -1$. It remains interesting to see if Lemma 5.21 extends to arbitrary α.

Recall from [52] the *weighted Hodge ideal* is defined by

$$F_kW_{n+\ell}\mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k^{W_\ell}(D),$$

so that the weight starts with $\ell = 0$. If $k = 0$, it is studied as the *weighted multiplier ideal* in [51]. Now we generalize Lemma 5.21 to the following.

Lemma 5.23. *Fix $\ell \geq 0$. For $k \geq 1$, there is an isomorphism*

$$\frac{W_\ell\mathcal{I}_{k,-1}(D)}{W_\ell\mathcal{I}_{k,0}(D) \otimes \mathcal{O}_X(-D)} \cong \frac{I_k^{W_{\ell+1}}(D)}{I_{k-1}^{W_{\ell+1}}(D) \otimes \mathcal{O}_X(-D)}. \tag{5.31}$$

For $k = 0$, we have

$$W_\ell\mathcal{I}_{0,-1}(D) = I_0^{W_{\ell+1}}(D), \quad W_\ell\mathcal{G}_{0,-1}(D) = I_0^{W_{\ell+1}}(D)/\mathcal{O}_X(-D). \tag{5.32}$$

Proof. For $k \geq 1$, the short exact sequence (5.20) implies that

$$0 \to \mathcal{G}_{k,0}(D) \otimes \mathcal{O}_X(-D) \to \mathcal{G}_{k,-1}(D) \to \mathrm{gr}^F_k\,\mathcal{O}_X(*D) \otimes \mathcal{O}_X((-k-1)D) \to 0.$$

Since the weight filtration is strict, one has

$$\frac{W_\ell\mathcal{G}_{k,-1}(D)}{W_\ell\mathcal{G}_{k,0}(D) \otimes \mathcal{O}_X(-D)} \cong \frac{I_k^{W_{\ell+1}}(D)}{I_{k-1}^{W_{\ell+1}}(D) \otimes \mathcal{O}_X(-D)},$$

where the shift by 1 comes from the weight convention on $\mathrm{gr}^V_{-1}\mathcal{M}$ in (2.3) and the Tate twist in (5.18). Because the weight filtration $W_\bullet\mathcal{I}_{k,\alpha}(D)$ is induced by $W_\bullet\mathcal{G}_{k,\alpha}(D)$ and we have an isomorphism $\mathcal{I}_{k,<0}(D) \otimes \mathcal{O}_X(-D) \cong \mathcal{I}_{k,<-1}(D)$ by (5.13), then (5.31) follows.

Similarly, (5.29) implies that $W_\ell\mathcal{G}_{0,-1}(D) = I_0^{W_{\ell+1}}(D)/\mathcal{O}_X(-D)$. Since $W_\ell\mathcal{I}_{0,-1}(D)$ is the preimage of $W_\ell\mathcal{G}_{0,-1}(D)$ under the natural map $\mathcal{I}_{0,-1}(D) \to \mathcal{G}_{0,-1}(D)$, we must have

$$I_0^{W_{\ell+1}}(D) \subseteq W_\ell\mathcal{I}_{0,-1}(D).$$

Because both ideals $I_0^{W_{\ell+1}}(D)$ and $W_\ell\mathcal{I}_{0,-1}(D)$ have the same quotient after $\mathcal{O}_X(-D)$ (which is $W_\ell\mathcal{G}_{0,-1}(D)$), they must equal to each other.

☐

Corollary 5.24. *For any $k \geq 1$, one has*

$$W_\ell \mathcal{I}_{k,-1}(D) \equiv I_k^{W_{\ell+1}}(D) \mod \mathcal{I}_D, \quad \textit{for any } \ell.$$

For $k = 0$, then

$$W_{-1}\mathcal{I}_{0,-1}(D) = \mathcal{O}_X(-D), \quad W_0\mathcal{I}_{0,-1}(D) = \mathrm{adj}(D), \quad W_{n-1}\mathcal{I}_{0,-1}(D) = \mathcal{I}_{0,-1}(D),$$

and

$$W_\ell \mathcal{G}_{0,-1}(D) = \begin{cases} 0 & \textit{if } \ell \leq -1, \\ \mathrm{adj}(D)/\mathcal{O}_X(-D) & \textit{if } \ell = 0, \end{cases}$$

where $\mathrm{adj}(D)$ *is the adjoint ideal of D.*

Proof. Besides Lemma 5.23, we also use the fact that $I_0^{W_0}(D) = \mathcal{O}_X(-D)$, $I_0^{W_n}(D) = I_0(D)$ and $I_0^{W_1}(D) = \mathrm{adj}(D)$ from [51, Theorem A]. □

Remark 5.25. Using the comparison result above, many results of weighted multiplier ideals and weighted Hodge ideals from [51, 52] can be translated to $W_\bullet \mathcal{I}_{k,\alpha}(D)$. We leave the details to interested readers.

5.6. **Minimal exponents as jumping numbers.** Let f be a holomorphic function on X. The minimal exponent of f, denoted by $\tilde{\alpha}_f$ in [64], is defined to be the negative of the largest root of the reduced Bernstein-Sato polynomial $\tilde{b}_f(s) := b_f(s)/(s+1)$, where $b_f(s)$ is the Bernstein-Sato polynomial of f. If D is an effective divisor, the minimal exponent is defined by

$$\tilde{\alpha}_D = \min_{x \in D} \tilde{\alpha}_{f_x}, \tag{5.33}$$

where f_x is the local function of D such that $f_x(x) = 0$. We can interpret $\tilde{\alpha}_D$ as the first jumping number of higher multiplier ideals, up to a shift.

Lemma 5.26. *Let D be an effective divisor on X, then*

$$\begin{aligned} \tilde{\alpha}_D &= \min\{k - \alpha, k \in \mathbb{N}, \alpha \in (-1, 0] \mid \mathcal{G}_{k,\alpha}(D) \neq 0\} \\ &= \min\{k - \alpha, k \in \mathbb{N}, \alpha \in (-1, 0] \mid \mathcal{I}_{k,<\alpha}(D) \subsetneq \mathcal{O}_X\}. \end{aligned}$$

Proof. For the first equality, because $\mathcal{G}_{k,\alpha}(D) \neq 0$ if and only if $\mathcal{G}_{k,\alpha}(f_x) \neq 0$ for some local function f_x of D, we can assume $D = \mathrm{div}(f)$ with $f : X \to \mathbb{C}$. By [42, Proposition 2.14] and [65, (1.3.8)], one has

$$\tilde{\alpha}_D = \min\{k + \beta, k \in \mathbb{N}, \beta \in [0, 1) \mid \mathrm{gr}_k^F \mathrm{gr}_V^\beta \mathcal{B}_f \neq 0\},$$

where $\mathcal{B}_f$ is the direct image of $\mathcal{O}_X$ from (5.23). Then the first equality follows from the right-to-left $\mathscr{D}$-module transformation

$$\omega_X \otimes L^k \otimes \mathcal{G}_{k,\alpha}(D) \cong \mathrm{gr}_{-n+k}^F \mathrm{gr}_\alpha^V \mathcal{M} = \omega_X \otimes \mathrm{gr}_k^F \mathrm{gr}_V^{-\alpha} \mathcal{B}_f.$$

To prove the second equality, write $\tilde{\alpha}_D = k - \alpha$ with $k \in \mathbb{N}, \alpha \in (-1, 0]$, such that $\mathcal{G}_{k,\alpha}(D) \neq 0$ and

$$\mathcal{G}_{j,\beta}(D) = 0, \quad \forall j \in \mathbb{N}, \beta \in (-1, 0], \text{ and } j - \beta < k - \alpha. \tag{5.34}$$

Now it suffices to show

$$\mathcal{G}_{k,\beta}(D) = 0, \quad \forall \beta \in (\alpha, k),$$

since Proposition 5.10 gives $\mathcal{I}_{k,<k}(D) = \mathcal{O}_X$. This can be proved as follows: if $\beta \in (\alpha, 0]$, then $\mathcal{G}_{k,\beta}(D) = 0$ by (5.34); if $\beta \in (0, k)$, then (5.15) being isomorphism for $\alpha > -1$ implies that

$$\mathcal{G}_{k,\beta}(D) \cong \mathcal{G}_{k-1,\beta-1}(D) \cong \ldots \cong \mathcal{G}_{k-\lfloor\beta\rfloor-1,\beta-\lfloor\beta\rfloor-1}(D) = 0,$$

because $\beta - \lfloor \beta \rfloor - 1 \in (-1, 0]$. This finishes the proof.

□

6. Examples

In this section, we compute higher multiplier ideals for several important classes of divisors, including ordinary singularities, normal crossing divisors and diagonal hypersurfaces (for example cusps defined by $x^2 + y^3$). They are crucial for the further investigation of local properties of higher multiplier ideals in §7.

6.1. **Ordinary singularities.** Let D be an effective divisor on a complex manifold X of dimension $n \geq 3$. Following [45], we say a point $x \in D$ is an *ordinary* singular point if the projectivized tangent cone of D at x, denoted by $\mathbf{P}(C_xD)$, is smooth. We give a detailed analysis of $\mathcal{I}_{k,\alpha}(D)_x$ and $\mathcal{G}_{k,\alpha}(D)_x$, see Theorem 6.6.

Since this is a local question, let us first analyze the local model, which is the cone over a smooth hypersurface in a projective space. The set-up is as follows.

Set-up 6.1. -

- Let $n \geq 3$ and let $(Z_0, \dots, Z_{n-1})$ be a homogeneous coordinate of $\mathbf{P}^{n-1}$. Let X_m be a smooth hypersurface in $\mathbf{P}^{n-1}$ of degree $m \geq 2$, cut out by a homogeneous polynomial F. Let $D = \{F = 0\} \subseteq \mathbb{C}^n$ be the affine cone over X_m. Denote by
$$S = \mathrm{Sym}^\bullet(Z_0, \dots, Z_{n-1}), \quad J_F = S \cdot (\frac{\partial F}{\partial Z_0}, \dots, \frac{\partial F}{\partial Z_{n-1}})$$
the homogeneous coordinate ring of $\mathbf{P}^{n-1}$ and the Jacobian ideal of F.
- Let $W \to \mathbf{P}^{n-1}$ be the degree m cyclic covering of $\mathbf{P}^{n-1}$ branched along X_m, associated to the section $F \in H^0(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\otimes m})$. Let $(Z_0, \dots, Z_{n-1}, t)$ be a homogeneous coordinate on $\mathbf{P}^n$ so that W can be realized as a smooth hypersurface of degree m:
$$W = \{F(Z_0, \dots, Z_{n-1}) - t^m = 0\} \subseteq \mathbf{P}^n.$$
- There is an $e^{2\pi i/m}$-action on $\mathbf{P}^n$ induced by
$$t \mapsto e^{2\pi i/m} t, \quad Z_i \mapsto Z_i,$$
which induces an action on W. Denote by $H^{p,q}_{\mathrm{prim}}(W, \mathbb{C})$ the primitive (p,q)-cohomology (same for $H^{p,q}_{\mathrm{prim}}(X_m)$) and $H^{p,q}_{\mathrm{prim}}(W, \mathbb{C})_\lambda$ the λ-eigenspace of the action.

To explain the computation on $\mathcal{G}_{k,\alpha}(D)$, let us first relate the (p,q)-cohomology of the cyclic cover with the Jacobian ring S/J_F.

Lemma 6.2. *With Set-up 6.1. Fix integers $p \in [1, m]$ and $k \in [0, n-1]$, then*
$$H^{n-1-k,k}_{\mathrm{prim}}(W, \mathbb{C})_{e^{2\pi i p/m}} \cong \begin{cases} 0, & \text{if } p = m, \\ (S/J_F)^{m(k+1)-p-n}, & \text{if } 1 \leq p \leq m-1. \end{cases}$$
$$H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}) \cong (S/J_F)^{mk-n},$$
where $(S/J_F)^\ell$ is the degree ℓ component, with the convention that it is zero if $\ell \notin \mathbb{N}$.

Proof. Denote by
$$J_f := \mathrm{Sym}^\bullet(Z_0, \dots, Z_n) \cdot \left(t^{m-1}, \frac{\partial F}{\partial Z_0}, \cdots, \frac{\partial F}{\partial Z_{n-1}}\right),$$

the Jacobian ideal of $f = F - t^m$, where $Z_n = t$. Since $W = \{f = 0\} \subseteq \mathbf{P}^n$ is a smooth hypersurface of degree m, by [73, Corollary 6.12], there is a natural isomorphism

$$(\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f)^{m(k+1)-n-1} \xrightarrow{\sim} H^{n-1-k,k}_{\mathrm{prim}}(W), \tag{6.1}$$

Recall that there is an action of $e^{2\pi i/m}$ on $\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)$, which induces an action on W and $\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f$.

Claim 6.3. *The isomorphism* (6.1) *induces an isomorphism of eigenspaces:*

$$\big((\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f)^{m(k+1)-n-1}\big)_{e^{2\pi i(p-1)/m}} \xrightarrow{\sim} H^{n-1-k,k}_{\mathrm{prim}}(W)_{e^{2\pi ip/m}}, \quad \forall p \in [1, m]. \tag{6.2}$$

Proof of claim. Let us recall the construction of (6.1) using Griffiths' residue. Let Ω be a generator of $H^0(\mathbf{P}^n, \omega_{\mathbf{P}^n}(n+1))$ given by

$$\Omega := \sum_i (-1)^i Z_i dZ_0 \wedge \ldots \wedge d\hat{Z}_i \wedge \ldots \wedge dZ_n.$$

where $Z_n = t$. Denote $U = \mathbf{P}^n \setminus W$ and consider the following composition of maps

$$\begin{aligned} H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m(k+1)-n-1)) &\to F^{n-k}H^n(U, \mathbb{C}) \cong F^{n-k-1}H^{n-1}_{\mathrm{prim}}(W) \\ &\twoheadrightarrow \mathrm{gr}_F^{n-k-1} H^{n-1}_{\mathrm{prim}}(W) = H^{n-1-k,k}_{\mathrm{prim}}(W), \end{aligned}$$

where the first map associates a polynomial P to the residue of the class of the meromorphic form $\frac{P\Omega}{f^{k+1}}$. Griffiths [22] showed that this composition map is surjective and the kernel is generated by $(J_f)^{m(k+1)-n-1}$, so this induces (6.1). Observe that

$$(e^{2\pi i/m} \cdot \Omega) = e^{2\pi i/m}\Omega, \quad e^{2\pi i/m} \cdot f = f,$$

hence

$$e^{2\pi i/m} \cdot \frac{P\Omega}{f^{k+1}} = \frac{(e^{2\pi i/m} \cdot P)(e^{2\pi i/m} \cdot \Omega)}{(e^{2\pi i/m} \cdot f)^{k+1}} = e^{2\pi i/m}\frac{(e^{2\pi i/m} \cdot P)\Omega}{f^{k+1}}.$$

Therefore

$$e^{2\pi i/m} \cdot P = e^{2\pi i(p-1)/m}P \Longleftrightarrow e^{2\pi i/m} \cdot \frac{\Omega P}{f^{k+1}} = e^{2\pi ip/m}\frac{\Omega P}{f^{k+1}},$$

and (6.2) follows. ☐

To finish the proof, let us fix an integer $\ell \geq 0$ and analyze $(\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f)^\ell_{e^{2\pi i(p-1)/m}}$, which is generated by the class of t^{p-1}. There are two cases.

- If $p = m$, then $(\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f)^\ell_{e^{2\pi i(p-1)/m}} = 0$ for all ℓ, because the generating class t^{m-1} is zero in $\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f$.
- If $1 \leq p \leq m-1$, then

$$(\mathrm{Sym}^\bullet(Z_0, \ldots, Z_n)/J_f)^\ell_{e^{2\pi i(p-1)/m}} \cong (\mathrm{Sym}^\bullet(Z_0, \ldots, Z_{n-1})/J_F)^{\ell-(p-1)} \cdot t^{p-1}.$$

We then obtain the statement for W by setting $\ell = m(k+1) - n - 1$ and using (6.2). The statement for X_m directly follows from [73, Corollary 6.12].

☐

Let us also recall a formula regarding the direct image of filtered $\mathscr{D}$-modules.

Lemma 6.4. *Let X be a complex manifold and let $i : X \hookrightarrow X \times \mathbb{C}^n$ be the closed embedding induced by $x \mapsto (x, 0)$. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered $\mathscr{D}$-module on X, then*

$$\mathrm{gr}_p^F(i_+\mathcal{M}) \cong \bigoplus_{\ell \geq 0} (\mathrm{gr}^F_{p-\ell}\,\mathcal{M})^{\oplus\binom{n+\ell-1}{\ell}}.$$

Proof. Let $t_1, \ldots, t_n$ be the holomorphic coordinates on $\mathbb{C}^n$ and set $\partial_i = \partial/\partial t_i$. It follows immediately from the formula

$$i_+\mathcal{M} \cong \sum_{a_1,\ldots,a_n\in\mathbb{N}} \mathcal{M} \otimes \partial_1^{a_1} \cdots \partial_n^{a_n}$$

with filtration given by

$$F_p(i_+\mathcal{M}) \cong \sum_{a_1,\ldots,a_n\in\mathbb{N}} F_{p-(a_1+\cdots+a_n)}\mathcal{M} \otimes \partial_1^{a_1} \cdots \partial_n^{a_n}.$$

□

Proposition 6.5. *With Set-up 6.1. Denote by $x \in D$ the unique cone point. Fix $k \in \mathbb{N}$ and $\alpha \in (-1, 0]$. Then*

$$\operatorname{supp} \mathcal{G}_{k,\alpha}(D) \subseteq \{x\} \text{ and } \mathcal{G}_{k,\alpha}(D) = 0, \text{ whenever } \alpha m \notin \mathbb{Z}.$$

Furthermore, assume $\alpha m \in \mathbb{Z}$, then the weight filtration on $\mathcal{G}_{k,\alpha}(D)$ is trivial and the following hold.

- *If $m(k-\alpha) - n \in [0, m-1]$ and $k \in [0, n-1]$, one has*

$$\mathcal{G}_{k,\alpha}(D)_x \cong (S/J_F)^{m(k-\alpha)-n}, \tag{6.3}$$

  where $S = \mathrm{Sym}^\bullet(Z_0, \ldots, Z_{n-1})$. More precisely,

$$\mathcal{G}_{k,\alpha}(D)_x \cong \begin{cases} H^{n-1-k,k}_{\mathrm{prim}}(W, \mathbb{C})_{e^{2\pi i\alpha}} & \text{if } \alpha \in (-1,0), \\ H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}) & \text{if } \alpha = 0. \end{cases} \tag{6.4}$$

- *In general, we have*

$$\mathcal{G}_{k,\alpha}(D)_x \cong \bigoplus_{\substack{0\le \ell \le \frac{m(k-\alpha)-n}{m} \\ k-\ell\in[0,n-1]}} \left((S/J_F)^{m(k-\alpha-\ell)-n}\right)^{\oplus\binom{n+\ell-1}{\ell}}. \tag{6.5}$$

Proof. Denote by $X = \mathbb{C}^n$. Since $\alpha \in (-1, 0]$, it is clear that the support of $\mathcal{G}_{k,\alpha}(D)$ is contained in the support of the vanishing cycle of $\mathbb{C}_X$ along the divisor D by (2.2), which is $D_{\mathrm{Sing}} = \{x\}$.

To compute $\mathcal{G}_{k,\alpha}(D)$, the plan is to use the birational formula in Proposition 2.9 in terms of a log resolution. Let $\pi : Y \to X$ be the blow up of X along x, which is a log resolution of (X, D). We have $\pi^*D = \tilde{D} + mE$, where E is the exceptional divisor and $\tilde{D}$ is the proper transform of D. Moreover

$$E \cong \mathbf{P}^{n-1}, \quad E \cap \tilde{D} \cong X_m, \quad \mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^{n-1}}(-1). \tag{6.6}$$

The geometry is summarized in the following diagram

$$\begin{tikzcd} X_m \cong E \cap \tilde{D} \arrow[r, "i_{X_m}"] \arrow[d, "\pi_{X_m}"] & Y \arrow[d, "\pi"] & E \arrow[l, "i_E"'] \arrow[d, "\pi_E"] \\ \{x\} \arrow[r, "i"] & X & \{x\} \arrow[l, "i"'] \end{tikzcd}$$

Let $\mathcal{M}_X, \mathcal{M}_Y$ be the $\mathscr{D}$-module associated to the total embedding of the divisor D and π^*D, respectively, as in the set up of Proposition 2.9. Then (2.11) implies that

$$\mathrm{gr}^V_\alpha \mathcal{M}_X = 0, \text{ whenever } \alpha m \notin \mathbb{Z}.$$

From now on, we assume $\alpha m \in \mathbb{Z}$ and write

$$\alpha = -q/m, \quad \text{for some } 0 \le q \le m-1.$$

There are two cases. Assume $\alpha \in (-1,0)$, then E is the only divisor on Y whose multiplicity multiplies α is equal to an integer, so Proposition 2.9 implies that the weight filtration on $\operatorname{gr}^V_\alpha \mathcal{M}_Y$ is trivial (so are $\operatorname{gr}^V_\alpha \mathcal{M}_X$ and $\mathcal{G}_{k,\alpha}(D)_x$). Then (2.7) and (2.8) gives

$$\begin{aligned}(\operatorname{gr}^V_\alpha \mathcal{M}_X, F_\bullet) =&R^0\pi_+(\operatorname{gr}^V_\alpha \mathcal{M}_Y, F_\bullet)\\ =&R^0(\pi\circ i_E)_+(\mathcal{V}_{\alpha,J}, F_{\bullet+1})\\ =&i_+\circ R^0\pi_{E,+}(\mathcal{V}_{\alpha,J}, F_{\bullet+1}),\end{aligned}$$

where $J=\{E\}$. Denote by $(\mathcal{N}_\alpha, F_\bullet) = R^0\pi_{E,+}(\mathcal{V}_{\alpha,J}, F_{\bullet+1})$, which is a filtered $\mathscr{D}$-module over $\pi(E)=\{x\}$. Then by (2.9) and the formula for direct images of filtered $\mathscr{D}$-modules under projection (π_E is a projection), one has

$$\begin{aligned}\operatorname{gr}^F_{-n+k}\mathcal{N}_\alpha &= R^0\pi_{E,*}(\operatorname{gr}^F_{-n+k+1}\operatorname{DR}_E(\mathcal{V}_{\alpha,J}))\\ &= R^0\pi_{E,*}(\operatorname{gr}^F_{-\dim E+k}\operatorname{DR}_E(\mathcal{V}_{\alpha,J}))\\ &= H^k(E, \Omega_E^{\dim E-k}(\log E\cap\tilde{D})\otimes\mathcal{O}_E(\lfloor\alpha\pi^*D\rfloor|_E))\\ &\cong H^k(\mathbf{P}^{n-1}, \Omega^{(n-1)-k}_{\mathbf{P}^{n-1}}(\log X_m)\otimes\mathcal{O}_{\mathbf{P}^{n-1}}(-(m-q))).\end{aligned}$$

The last equality uses (6.6) to obtain

$$\mathcal{O}_E(\lfloor\alpha\pi^*D\rfloor|_E) = \mathcal{O}_E((-\tilde{D}-qE)|_E)\cong\mathcal{O}_{\mathbf{P}^{n-1}}(-(m-q)).$$

Now let us identify the cohomology group above with (p,q)-cohomology of W. Denote by $\mu: W\to\mathbf{P}^{n-1}$ the cyclic covering map. The $e^{2\pi i/m}$-action on W induces an action on $\mu_*\Omega^p_W$, whose eigenspace decomposition is (see [68, §10] or [20, §3])

$$\mu_*\Omega^p_W\cong\Omega^p_{\mathbf{P}^{n-1}}\oplus\bigoplus_{i=1}^{m-1}\Omega^p_{\mathbf{P}^{n-1}}(\log X_m)\otimes\mathcal{O}_{\mathbf{P}^{n-1}}(-i)$$

for $0\le p\le\dim W=n-1$; here $\Omega^p_{\mathbf{P}^{n-1}}(\log X_m)\otimes\mathcal{O}_{\mathbf{P}^{n-1}}(-\ell)$ is the $e^{2\pi i\ell/m}$-eigenspace. Since μ is a finite map, it follows that

$$\begin{aligned}H^{p,q}(W,\mathbb{C}) &= H^q(W,\Omega^p_W) = H^q(\mathbf{P}^{n-1},\mu_*\Omega^p_W)\\ \cong& H^q(\mathbf{P}^{n-1},\Omega^p_{\mathbf{P}^{n-1}})\oplus\bigoplus_{1\le i\le m-1}H^q\left(\mathbf{P}^{n-1},\Omega^p_{\mathbf{P}^{n-1}}(\log X_m)\otimes\mathcal{O}(-i)\right).\end{aligned}$$

Moreover, we can also identify the eigenspaces:

$$\begin{aligned}H^{n-1-k,k}_{\text{prim}}(W,\mathbb{C})_{e^{-2\pi iq/m}} &= H^{n-1-k,k}_{\text{prim}}(W,\mathbb{C})_{e^{2\pi i(m-q)/m}}\\ &\cong H^k(\mathbf{P}^{n-1},\Omega^{(n-1)-k}_{\mathbf{P}^{n-1}}(\log X_m)\otimes\mathcal{O}(-(m-q))),\end{aligned}\tag{6.7}$$

for $1\le q\le m-1$. Hence we conclude that for $\alpha\in(-1,0)$

$$(\operatorname{gr}^V_\alpha\mathcal{M}_X, F_\bullet) = i_+(\mathcal{N}_\alpha, F_\bullet),\ \text{and}\ \operatorname{gr}^F_{-n+k}\mathcal{N}_\alpha\cong H^{n-1-k,k}_{\text{prim}}(W,\mathbb{C})_{e^{2\pi i\alpha}},\tag{6.8}$$

where $(\mathcal{N}_\alpha, F_\bullet)$ is a filtered $\mathscr{D}$-module over $\{x\}$.

Assume $\alpha=0$. Since $\pi: Y\to X$ is the blow up of $\mathbb{C}^n$ along the origin, it is a direct computation to see that

$$R^0\pi_+\omega_Y[n]=\omega_X[n]\oplus i_+\mathcal{P},\ \text{where}\ \mathcal{P}=\begin{cases}0, & \text{if } n \text{ is odd,}\\ \omega_x, & \text{if } n \text{ is even.}\end{cases}$$

Here ω_x is the constant $\mathscr{D}$-module on $\{x\}$ and the shift of Hodge filtration depends on n (see below). Since $i_+\mathcal{P}$ supports on $\{x\}$, we have $\operatorname{gr}^V_0(i_+\mathcal{P})\cong i_+\mathcal{P}$. Therefore

$$(\operatorname{gr}^V_0\mathcal{M}_X, F_\bullet)\oplus i_+(\mathcal{P},F_\bullet)\cong R^0\pi_+(\operatorname{gr}^V_0\mathcal{M}_Y, F_\bullet).$$

We claim that

$$(\mathrm{gr}_0^V \mathcal{M}_Y, F_\bullet) \cong i_{X_m,+}(\omega_{X_m}, F_{\bullet+1}). \tag{6.9}$$

This is deduced in the following way. Since the nilponent operator N on $\mathrm{gr}_{-1}^V \mathcal{M}_Y$ satisfies $N^2 = 0$, so (2.8) in Proposition 2.9 gives

$$\mathrm{gr}_0^V \mathcal{M}_Y = \mathrm{Im}\, N(\mathrm{gr}_{-1}^V \mathcal{M}_Y)(1) \cong \mathrm{gr}_1^W \mathrm{gr}_{-1}^V \mathcal{M}_Y(1) \cong i_{X_m,+}(\mathcal{V}_{-1,J}, F_\bullet)(-1),$$

where $\mathcal{V}_{-1,J}$ is a $\mathscr{D}$-module on X_m and $J = \{\tilde{D}, E\}$. Moreover, (2.10) implies that $(\mathcal{V}_{-1,J}, F_\bullet) \cong (\omega_{X_m}, F_\bullet)$. This proves (6.9). As a consequence, the weight filtration on $\mathrm{gr}_0^V \mathcal{M}_Y$ (hence $\mathrm{gr}_0^V \mathcal{M}_X$) is also trivial and

$$(\mathrm{gr}_0^V \mathcal{M}_X, F_\bullet) \oplus i_+(\mathcal{P}, F_\bullet) = i_+(\mathcal{N}_0', F_\bullet), \quad \mathrm{gr}_{-n+k}^F \mathcal{N}_0' \cong H^{n-1-k,k-1}(X_m, \mathbb{C}),$$

where $(\mathcal{N}_0', F_\bullet) = R^0\pi_{X_m,+}(\omega_{X_m}, F_{\bullet+1})$, a filtered $\mathscr{D}$-module over $\{x\}$. There are two subcases.

(1) If n is odd, then $\mathcal{P} = 0$ and $H^{n-1-k,k-1}(X_m, \mathbb{C}) = H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C})$ (because $H^{n-2}(\mathbf{P}^{n-1}) = 0$).

(2) If n is even, then $\mathcal{P} = \omega_x$ such that $\mathrm{gr}_{-n+(n/2)}^F \mathcal{P} \cong H^{n/2-1,n/2-1}(\mathbf{P}^{n-1})$ and

$$H^{n-1-k,k-1}(X_m, \mathbb{C}) = \begin{cases} H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}), & \text{if } k \neq n/2, \\ H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}) \oplus H^{n/2-1,n/2-1}(\mathbf{P}^{n-1}), & \text{if } k = n/2. \end{cases}$$

Therefore

$$(\mathrm{gr}_0^V \mathcal{M}_X, F_\bullet) \cong i_+(\mathcal{N}_0, F_\bullet), \text{ and } \mathrm{gr}_{-n+k}^F \mathcal{N}_0 \cong H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}), \tag{6.10}$$

where $(\mathcal{N}_0, F_\bullet)$ is a filtered $\mathscr{D}$-module over $\{x\}$.

We claim that in both (6.8) and (6.10), one has

$$\mathrm{gr}_{-n+k}^F \mathcal{N}_\alpha \cong (S/J_F)^{m(k-\alpha)-n}, \text{ whenever } \alpha \in (-1, 0] \text{ and } k \in [0, n-1].$$

Let $\alpha = -p/m$ for some $0 \le p \le m-1$. By Lemma 6.2, if $1 \le p \le m-1$, then

$$\begin{aligned} \mathrm{gr}_{-n+k}^F \mathcal{N}_\alpha &\cong H^{n-1-k,k}_{\mathrm{prim}}(W, \mathbb{C})_{e^{-2\pi i p/m}} \\ &= H^{n-1-k,k}_{\mathrm{prim}}(W, \mathbb{C})_{e^{2\pi i (m-p)/m}} \\ &\cong (S/J_F)^{m(k+1)-n-(m-p)} \\ &= (S/J_F)^{m(k-\alpha)-n}. \end{aligned}$$

If $p = 0$, then

$$\mathrm{gr}_{-n+k}^F \mathcal{N}_\alpha \cong H^{n-1-k,k-1}_{\mathrm{prim}}(X_m, \mathbb{C}) \cong (S/J_F)^{mk-n} = (S/J_F)^{m(k-\alpha)-n}.$$

Summarizing the computation above, for $\alpha \in (-1, 0]$ one has

$$\begin{aligned} (\mathrm{gr}_\alpha^V \mathcal{M}_X, F_\bullet) &= i_+(\mathcal{N}_\alpha, F_\bullet), \\ \mathrm{gr}_{-n+k}^F \mathcal{N}_\alpha &\cong \begin{cases} (S/J_F)^{m(k-\alpha)-n}, & \text{if } 0 \le k \le n-1 \\ 0, & \text{else}, \end{cases} \end{aligned}$$

where $(\mathcal{N}_\alpha, F_\bullet)$ is a filtered $\mathscr{D}$-module over $\{x\}$. Thus by Lemma 6.4 we have

$$\begin{aligned} \mathrm{gr}_{-n+k}^F \mathrm{gr}_\alpha^V \mathcal{M}_X &\cong \bigoplus_{\ell \ge 0} (\mathrm{gr}_{-n+k-\ell}^F \mathcal{N}_\alpha)^{\oplus \binom{n+\ell-1}{\ell}} \\ &= \bigoplus_{\ell \ge 0,\ 0 \le k-\ell \le n-1} ((S/J_F)^{m(k-\alpha-\ell)-n})^{\oplus \binom{n+\ell-1}{\ell}} \end{aligned}$$

Note that $(S/J_F)^p = 0$ if $p < 0$ and $\mathcal{G}_{k,\alpha}(D)_x \cong \mathrm{gr}^F_{-n+k}\mathrm{gr}^V_\alpha \mathcal{M}_X$ (it supports on $\{x\}$), one obtains (6.5).

Moreover, if $m(k-\alpha) - n \le m-1$ and $k \in [0, n-1]$, then there is only term in the formula above so that

$$\mathcal{G}_{k,\alpha}(D)_x \cong (S/J_F)^{m(k-\alpha)-n}.$$

This gives (6.3) and (6.4) (via Lemma 6.2). We finish the proof of this proposition. □

Theorem 6.6. *Let D be an effective divisor on a complex manifold X of dimension $n \ge 2$ and let $x \in D$ be a singular point of multiplicity $m \ge 2$ such that the projectivized tangent cone $\mathbf{P}(C_xD)$ is smooth. Let F be the equation of $\mathbf{P}(C_xD)$ inside $\mathbf{P}(T_xX)$, $(Z_0, \ldots, Z_{n-1})$ be the local algebraic coordinates of $\mathcal{O}_{X,x}$. Let S be the homogeneous coordinate ring of $\mathbf{P}(T_xX)$ and $J_F = S \cdot (\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_{n-1}})$. Let $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ be the maximal ideal. For $\alpha \in \mathbb{Q}, k \in \mathbb{N}$, the following hold.*

(1) If $\alpha > -1$, then

$$\mathcal{G}_{k,\alpha}(D)_x \cong \bigoplus_{\substack{0 \le \ell \le \frac{m(k-\alpha)-n}{m} \\ k-n-1 \le \ell \le k}} \left((S/J_F)^{m(k-\alpha-\ell)-n}\right)^{\oplus\binom{n+\ell-1}{\ell}}. \tag{6.11}$$

where $(S/J_F)^p$ is the degree p component. In addition, if $n \ge 3$, then

$$\mathcal{G}_{k,\alpha}(D)_x \neq 0 \Longleftrightarrow k - \alpha \ge \frac{n}{m} \text{ and } m\alpha \in \mathbb{Z}. \tag{6.12}$$

(2) If $\alpha \ge -1$, then

$$\begin{aligned} \mathcal{I}_{k,\alpha}(D)_x &= \sum_{ma+\deg(v_\gamma)\ge m(k-\alpha)-n} \mathcal{O}_{X,x} \cdot (J_F)^a \cdot v_\gamma \\ &= \sum_{0 \le \ell \le \frac{m(k-\alpha)-n}{m}} \mathcal{O}_{X,x} \cdot (J_F)^\ell \cdot (S/J_F)_{\ge m(k-\alpha-\ell)-n} \end{aligned} \tag{6.13}$$

where $\{v_\gamma\}$ is a monomial basis of the ring S/J_F and $(S/J_F)_{\ge p}$ denotes the polynomials generated by v_γ with $\deg v_\gamma \ge p$ in S. In particular,

$$\mathcal{I}_{k,\alpha}(D)_x = \mathcal{O}_{X,x} \Longleftrightarrow k - \alpha \le \frac{n}{m}, \tag{6.14}$$

and

$$\mathcal{I}_{k,\alpha}(D)_x \subseteq \mathfrak{m}_x^{m(k-\alpha)-n-\lfloor\frac{m(k-\alpha)-n}{m}\rfloor}, \tag{6.15}$$

with equality when $m(k-\alpha) - n \le m - 1$.

Remark 6.7. The dimension assumption $n \ge 3$ is necessary for (6.12), otherwise it will not hold for $D = \mathrm{div}(x^2 + y^2) \subseteq \mathbb{C}^2$, where $\mathcal{G}_{k,-1/2}(D) = 0$ for all $k \in \mathbb{N}$ (see Example 6.19).

Proof. By shrinking X we can assume $X = \mathbb{C}^n$ and work with Set-up 6.1: $D \subseteq \mathbb{C}^n$ is the cone over a smooth hypersurface $X_m \subseteq \mathbf{P}^{n-1}$ of degree m, $x \in D$ is the unique singular point.

Let us analyze $\mathcal{G}_{k,\alpha}(D)_x$ first. If $\alpha \in (-1, 0]$, (6.11) follows from (6.5) in Proposition 6.5. If $\alpha \in (0, k)$, by the repeated use of the property of (5.15), one has

$$\mathcal{G}_{k,\alpha}(D)_x \cong \mathcal{G}_{k-t,\alpha-t}(D)_x,$$

where $t = \lfloor\alpha\rfloor + 1$ such that $\alpha - t \in (-1, 0]$. Note that $k - t - (\alpha - t) = k - \alpha$ and thus (6.11) also holds for α. If $n \ge 3$, then $(S/J_F)^q \neq 0$ if and only if $q \ge 0$. Therefore (6.12) holds.

For $\mathcal{I}_{k,\alpha}(D)$, we use Saito's computation of microlocal V-filtration along F. First, Corollary 5.16 gives

$$\mathcal{I}_{k,\alpha}(D)_x = \tilde{V}^{k-\alpha}\mathcal{O}_{X,x}, \text{ whenever } \alpha \geq -1.$$

Here $\tilde{V}^\bullet\mathcal{O}_X$ is the microlocal V-filtration along D. Let $\{v_\gamma\}$ be a monomial basis of the ring S/J_F and denote by $\partial_i F = \frac{\partial F}{\partial Z_i}$. Then [65, (2.2.4)] implies that

$$\tilde{V}^{k-\alpha}\mathcal{O}_{X,x} = \sum_{m\sum_i \mu_j + \deg(v_\gamma) \geq m(k-\alpha)-n} \mathcal{O}_{X,x} \cdot \prod_j (\partial_j F)^{\mu_j} \cdot v_\gamma. \tag{6.16}$$

This is because the weight for the homogeneous polynomial F is $w = 1/m$, so if $v_\gamma = \prod_i Z_i^{m_i}$, then the requirement in [65, (2.2.4)] is

$$\frac{1}{m}\sum_{i=1}^{n}(m_i+1) + \sum_j \mu_j \geq k-\alpha.$$

Since J_F^a is generated by $\prod_j (\partial_i F)^{\mu_j}$ with $\sum_j \mu_j = a$, (6.16) implies that

$$\tilde{V}^{k-\alpha}\mathcal{O}_{X,x} = \sum_{ma+\deg(v_\gamma)\geq m(k-\alpha)-n} \mathcal{O}_{X,x} \cdot J_F^a \cdot v_\gamma.$$

Then (6.13) and (6.14) follow immediately. For (6.15), one uses that polynomials in $\mathcal{I}_{k,\alpha}(D)_x$ of smallest degrees appear in the term

$$\mathcal{O}_{X,x} \cdot J_F^\ell \cdot (S/J_F)_{\geq m(k-\alpha-\ell)-n},$$

where $\ell = \lfloor \frac{m(k-\alpha)-n}{m} \rfloor$. This is contained in

$$\mathfrak{m}_x^{\ell(m-1)+m(k-\alpha-\ell)-n} = \mathfrak{m}_x^{m(k-\alpha)-n-\lfloor \frac{m(k-\alpha)-n}{m}\rfloor}.$$

□

Remark 6.8. To see that the formula (6.13) for $\mathcal{I}_{k,\alpha}(D)$ implies the formula (6.11) for $\mathcal{G}_{k,\alpha}(D)$, it suffices to note the following fact. Let $\{v_\gamma\}$ be a monomial basis of S/J_F, then

$$\deg v_\gamma \leq (m-1)n, \quad \text{for any } v_\gamma.$$

Remark 6.9. For $\alpha \in (-1, 0]$, (partial cases of) the statement (6.12) can be proved in two more ways. The first way is: since D has an ordinary singularity at x, its minimal exponent $\tilde{\alpha}_{D,x}$ is n/m; this is well-known, for example see [65, (2.5.1)]. Thus Lemma 5.26 implies that

$$\mathcal{G}_{k,\alpha}(D)_x = 0, \quad \text{whenever } k-\alpha < \frac{n}{m}.$$

The second method is to use the comparison with Hodge ideals (Lemma 5.18) and a characterization of vanishing of Hodge ideals for ordinary singularities [46, Corollary 11.8]. We leave the details for interested readers.

Remark 6.10. For weighted homogeneous polynomials with isolated singularities, Saito's formula (6.16) works in the same way, see [65, (2.2.4)]. In particular, one can obtain compute their higher multiplier ideals in a similar fashion.

Corollary 6.11. *With the same notation in Theorem 6.6. Write $n = km + r$ for some $k \in \mathbb{N}$ and $0 \leq r \leq m-1$, then*

$$\mathcal{G}_{k,-p/m}(D)_x = \begin{cases} 0 & \text{if } p \in [0, r-1], \\ \mathfrak{m}_x^{p-r}/\mathfrak{m}_x^{p-r+1} & \text{if } r \leq p \leq \min(m-1, m+r-2), \\ \mathfrak{m}_x^{m-1}/(\mathfrak{m}_x^m, J_F) & \text{if } p = m-1 \text{ and } r = 0, \end{cases}$$

and

$$\mathcal{I}_{k,-p/m}(D)_x = \begin{cases} \mathcal{O}_{X,x} & \text{if } p \in [0, r], \\ \mathfrak{m}_x^{p-r} & \text{if } r \le p \le \min(m, m+r-1), \\ (\mathfrak{m}_x^m, J_F) & \text{if } p = m \text{ and } r = 0, \end{cases} \tag{6.17}$$

Hence

$$\mathcal{G}_{k,-r/m}(D)_x = \mathcal{O}_{X,x}/\mathfrak{m}_x, \quad \text{and} \quad \mathcal{I}_{k,<-r/m}(D)_x = \mathfrak{m}_x.$$

Proof. Note that we have $m(k-(-p/m)) - n = p - r$. So if $0 \le p - r \le m-1$, then (6.11) implies that

$$\mathcal{G}_{k,-p/m}(D)_x \cong (S/J_F)^{p-r}.$$

The computation for $\mathcal{I}_{k,\alpha}(D)_x$ is similar. □

6.2. **Normal crossing divisors.** Now we turn to normal crossing divisors. Let X be a complex manifold of dimension n and let

$$D = \sum_i m_i E_i, \quad m_i \in \mathbb{N},$$

be an effective divisor on X with normal crossing support. Denote by D_{red} the reduced part of D and $(D_{\mathrm{red}})_{\mathrm{sing}}$ the singular locus of D_{red}, i.e.

$$D_{\mathrm{red}} = \sum_i E_i, \quad (D_{\mathrm{red}})_{\mathrm{sing}} = \bigcup_{i \ne j} E_i \cap E_j.$$

Denote by $\mathcal{I}_Z$ the ideal sheaf of a subscheme Z.

Proposition 6.12. *Assume $\alpha < 0$. Then*

$$\begin{aligned} \mathcal{I}_{k,\alpha}(D) &= \mathcal{I}_{0,\alpha}(D) \otimes \mathcal{O}_X(-D + D_{\mathrm{red}})^{\otimes k} \otimes \mathcal{I}^k_{(D_{\mathrm{red}})_{\mathrm{sing}}}, \\ &= \mathcal{O}_X\left(\sum_i (\lceil(\alpha+\epsilon)m_i\rceil - km_i + k)E_i\right) \otimes \mathcal{I}^k_{(D_{\mathrm{red}})_{\mathrm{sing}}}. \end{aligned}$$

Here $(D_{\mathrm{red}})_{\mathrm{sing}} = \varnothing$ when D is smooth. In particular, if D is smooth, then

$$\mathcal{I}_{k,\alpha}(D) = \mathcal{I}_{0,\alpha}(D) = \mathcal{O}_X(\lfloor \alpha + 1 \rfloor D).$$

Remark 6.13. The formulas above for D and mD are compatible with the isomorphism from (5.22):

$$\mathcal{I}_{k,\alpha}(mD) \cong \mathcal{I}_{k,m\alpha}(D) \otimes \mathcal{O}_X(-(m-1)kD).$$

Proof. It suffices to prove the formula locally. Assume X is a polydisk in $\mathbb{C}^n$ with coordinates $x = (x_1, \ldots, x_n)$ and $D = \mathrm{div}(x^m)$ for a monomial x^m, where

$$m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\ge 0}.$$

Consider the graph embedding

$$i : X \to X \times \mathbb{C}_t, \quad x \mapsto (x, x^m).$$

The change of variables $t \mapsto t - x^m$ and $x \mapsto x$ induces an isomorphism of $\mathscr{D}$-modules

$$\widetilde{\omega}_X := i_+\omega_X \cong \sum_{\ell \ge 0} \omega_X \otimes \partial_t^\ell = \omega_X[\partial_t],$$

where the action of $\mathscr{D}_{X\times\mathbb{C}}$ on $\omega_X[\partial_t]$ is as follows: denote by $\partial_i = \partial/\partial_{x_i}$ and $e_i = (0,\dots,1,\dots,0)$ the i-th standard basis vector in $\mathbb{Z}^n_{\geq 0}$, then

$$\begin{aligned}(u\otimes 1)\cdot\partial_i &= u\partial_i\otimes 1 - (m_i u x^{m-e_i})\otimes\partial_t\\ (u\otimes 1)\cdot\partial_t &= u\otimes\partial_t\\ (u\otimes 1)\cdot f(x,t) &= (uf(x,x^m))\otimes 1.\end{aligned}$$

We also have

$$F_{-n+k}\widetilde{\omega}_X \cong \sum_{0\leq\ell\leq k}\omega_X\otimes\partial_t^{\ell}.$$

For any $\alpha\in\mathbb{Q}$, denote by $V_\alpha\widetilde{\omega}_X$ the V-filtration along t. For any $a=(a_1,\dots,a_n)\in\mathbb{Q}^n$, denote $V_a\omega_X := \bigcap_{i=1}^n V_{a_i}\omega_X$, where $V_{a_i}\omega_X$ is the V-filtration along x_i. It is direct to check that

$$V_{a_i}\omega_X = \begin{cases}\omega_X & \text{if } a_i > -1,\\ \mathcal{O}_X\cdot(x_i^{\lceil -a_i-1\rceil}\omega) & \text{if } a_i\leq -1,\end{cases}$$

where ω is a nowhere vanishing canonical form of X. If $\alpha<0$, by [62, Theorem 3.4] one has

$$V_\alpha\widetilde{\omega}_X = (V_{\alpha m}\omega_X\otimes 1)\cdot\mathscr{D}_X,\quad \text{where } \alpha m = (\alpha m_1,\dots,\alpha m_n).$$

Moreover, if we denote by $F_pV_\alpha\widetilde{\omega}_X = F_p\widetilde{\omega}_X\cap V_\alpha\widetilde{\omega}_X$, then [62, Proposition 3.17] implies that

$$\begin{aligned}F_{-n+k}V_\alpha\widetilde{\omega}_X &= \sum_{\ell\geq 0}(F_{-n+k-\ell}V_{\alpha m}\omega_X\otimes 1)\cdot F_\ell(\mathscr{D}_X[t\partial_t]),\\ &= \sum_{0\leq\ell\leq k}(V_{\alpha m}\omega_X\otimes 1)\cdot F_\ell(\mathscr{D}_X[t\partial_t]).\end{aligned}\tag{6.18}$$

Here we view $t\partial_t$ as an degree 1 element and $F_\bullet(\mathscr{D}_X[t\partial_t])$ is the associated order filtration.

With the formulas above, we can now compute $\mathcal{I}_{k,\alpha}(D)$ for $\alpha<0$ using (5.4)

$$(\mathcal{I}_{k,\alpha}(D)\otimes_{\mathcal{O}_X}\omega_X)\otimes\partial_t^k = \mathrm{gr}^F_{-n+k}V_\alpha\widetilde{\omega}_X.$$

For $k=0$,

$$(\mathcal{I}_{0,\alpha}(D)\otimes_{\mathcal{O}_X}\omega_X)\otimes 1 = \mathrm{gr}^F_{-n}V_\alpha\widetilde{\omega}_X = F_{-n}V_\alpha\widetilde{\omega}_X = V_{\alpha m}\omega_X\otimes 1.$$

This gives

$$\begin{aligned}\mathcal{I}_{0,\alpha}(D) &= (V_{\alpha m}\omega_X)\otimes\omega_X^{-1}\\ &= \mathcal{O}_X(\prod x_i^{\lceil -\alpha m_i-1\rceil})\\ &= \mathcal{O}_X(\sum\lceil(\alpha+\epsilon)m_i\rceil)E_i)\end{aligned}$$

Note that $\lceil(\alpha+\epsilon)m_i\rceil$ also equals to $-\lfloor(-\alpha-\epsilon)m_i\rfloor$, therefore this actually gives $\mathcal{I}_{0,\alpha}(D) = \mathcal{J}((-\alpha-\epsilon)D)$.

Next for $k=1$, one has

$$(\mathcal{I}_{1,\alpha}(D)\otimes_{\mathcal{O}_X}\omega_X)\otimes\partial_t = \mathrm{gr}^F_{-n+1}V_\alpha\widetilde{\omega}_X.$$

The right hand side is the image of $F_{-n+1}V_\alpha\widetilde{\omega}_X$ under the projection map

$$F_{-n+1}\widetilde{\omega}_X\to\mathrm{gr}^F_{-n+1}\widetilde{\omega}_X = \omega_X\otimes\partial_t.$$

By (6.18), it is the image of

$$(V_{\alpha m}\omega_X\otimes 1)\cdot\mathscr{T}_X[t\partial_t]\oplus(V_{\alpha m}\omega_X\otimes 1)\cdot\mathcal{O}_X.$$

But only the first term contributes, since the second term projects to 0. Let u be a local section of $V_{\alpha m}\omega_X = \mathcal{I}_{0,\alpha}(D)\otimes\omega_X$. Then by the $\mathscr{D}_{X\times\mathbb{C}}$-action above,

$$(u\otimes 1)\cdot\partial_i = u\partial_i\otimes 1 - (m_i u x^{m-e_i})\otimes\partial_t,$$

which projects to $(m_i u x^{m-e_i})\otimes\partial_t$ in $\omega_X\otimes\partial_t$. Similarly,

$$(u\otimes 1)\cdot t\partial_t = ux^m\otimes\partial_t.$$

Note that we always have

$$x^m \in \sum_{m_i\neq 0}\mathcal{O}_X\cdot x^{m-e_i}.$$

Therefore the action of $t\partial_t$ can be generated using the action of all ∂_i and

$$\begin{aligned}\mathcal{I}_{1,\alpha}(D) &= \sum_{g\in\mathcal{I}_{0,\alpha}(D),m_i\neq 0}\mathcal{O}_X\cdot g\cdot x^{m-e_i}\\ &= \sum_{g\in\mathcal{I}_{0,\alpha}(D),m_i\neq 0}\mathcal{O}_X\cdot g(x^m\prod_{m_i\neq 0}x_i^{-1})(\prod_{j\neq i,m_j\neq 0}x_j).\end{aligned}$$

Note that $D=\operatorname{div}(x^m)$, $D_{\mathrm{red}}=\operatorname{div}(\prod_{m_i\neq 0}x_i)$ and $\mathcal{I}_{(D_{\mathrm{red}})_{\mathrm{sing}}}$ is generated by $\prod_{j\neq i,m_j\neq 0}x_j$. Hence the calculation above can be globalized to any normal crossing divisor on a complex manifold X so that

$$\mathcal{I}_{1,\alpha}(D)=\mathcal{I}_{0,\alpha}(D)\otimes\mathcal{O}_X(-D+D_{\mathrm{red}})\otimes\mathcal{I}_{(D_{\mathrm{red}})_{\mathrm{sing}}}.$$

For higher $k\geq 2$, the argument is similar, where each action of ∂_i contributes to a factor of x^{m-e_i}.

□

Example 6.14. Let D be an effective divisor with a unique ordinary singularity at x and let $\pi: Y\to X$ be the blow up of x with $\pi^*D=\tilde{D}+mE$. Then for $m\alpha\in\mathbb{Z}$ and $\alpha\in[-1,0)$, Proposition 6.12 implies that

$$\begin{aligned}\omega_{Y/X}\otimes\mathcal{I}_{k,\alpha}(\pi^*D) &= \mathcal{O}_Y((n-1)E)\otimes\mathcal{O}_Y((\alpha m+1-km+k)E)\otimes\mathcal{I}^k_{\tilde{D}\cap E}\\ &= \mathcal{O}_Y\left(-(m(k-\alpha)-n+k)E\right)\otimes\mathcal{I}^k_{\tilde{D}\cap E}.\end{aligned}$$

Therefore by (6.13) we have

$$\pi_*(\omega_{Y/X}\otimes\mathcal{I}_{k,\alpha}(\pi^*D))_x=\mathfrak{m}_x^{m(k-\alpha)-n}\subseteq\mathcal{I}_{k,\alpha}(D)_x,\quad\forall\ell\in\mathbb{N}.$$

However for $\ell\gg 0$, this is usually not an equality. For example, if D has multiplicity m at x and $\dim X=km$, then $\mathcal{I}_{k,-1}(D)_x=(J_F,\mathfrak{m}_x^m)$ by (6.13), where J_F is the Jacobian ideal of F_x and F_x is the local equation of D around x. Thus

$$\pi_*(\omega_{Y/X}\otimes\mathcal{I}_{k,-1}(\pi^*D))_x=\mathfrak{m}_x^m\subsetneq(J_F,\mathfrak{m}_x^m)=\mathcal{I}_{k,-1}(D)_x.$$

6.3. **Sum of functions: Thom-Sebastiani-type formula.** There is a Thom-Sebastiani formula for higher multiplier ideals, essentially due to Maxim-Saito-Schürmann [40], which generalizes Mustaţă's summation theorem for multiplier ideals (see [48] and [36, Theorem 9.5.26]). Let X_1 and X_2 be complex manifolds and consider the product $X=X_1\times X_2$ with projections $p_i: X\to X_i$ with $i=1,2$.

Notation 6.15. The box product for coherent sheaves $\mathcal{F}_i$ on X_i are defined to be

$$\mathcal{F}_1\boxtimes\mathcal{F}_2 := p_1^*\mathcal{F}_1\otimes p_2^*\mathcal{F}_2.$$

Proposition 6.16. *Let f_i be nonconstant holomorphic functions on X_i and denote by $D_i = \mathrm{div}(f_i) \subseteq X_i$ the corresponding effective divisor. Define*

$$f := p_1^* f_1 + p_2^* f_2$$

as the summation function and denote $D = \mathrm{div}(f)$. Then for any $k \in \mathbb{N}$ and $\alpha \in [-1, 0)$, we have

$$\begin{aligned}\mathcal{I}_{k,\alpha}(D) &= \sum_{\beta_1+\beta_2=k-\alpha, \beta_1,\beta_2>0} \mathcal{I}_{\lfloor \beta_1-\epsilon\rfloor, \lfloor \beta_1-\epsilon\rfloor-\beta_1}(D_1) \boxtimes \mathcal{I}_{\lfloor \beta_2-\epsilon\rfloor, \lfloor \beta_2-\epsilon\rfloor-\beta_2}(D_2),\\ &= \sum_{\substack{k_1+k_2-\alpha_1-\alpha_2=k-\alpha,\\ k_1,k_2\in\mathbb{N},\alpha_1,\alpha_2\in[-1,0)}} \mathcal{I}_{k_1,\alpha_1}(D_1) \boxtimes \mathcal{I}_{k_2,\alpha_2}(D_2),\end{aligned}$$

by replacing X_i with an open neighborhood of D_i for $i = 1, 2$ so that $D_{\mathrm{sing}} = (D_1)_{\mathrm{sing}} \times (D_2)_{\mathrm{sing}}$ if necessary.

Proof. By [40, (3.2.3)] one has

$$\tilde{V}^{\beta}\mathcal{O}_X = \sum_{\beta_1+\beta_2=\beta} \tilde{V}^{\beta_1}\mathcal{O}_{X_1} \boxtimes \tilde{V}^{\beta_2}\mathcal{O}_{X_2}.$$

The desired formula then follows from (5.27) □

Example 6.17 (Diagonal hypersurfaces)**.** Let $z_1, \ldots, z_n$ be a coordinate on $X = \mathbb{C}^n$ and consider the divisor D defined by $f = \sum_{j=1}^n z_j^{m_j}$, where $m_j \geq 2$ for all $1 \leq j \leq n$. In [40, Example 3.6(ii)], the authors compute the microlocal V-filtrations $\tilde{V}^\bullet\mathcal{O}_X$. By Corollary 5.16, it translates to

$$\mathcal{I}_{k,\alpha}(D) = \sum_{\mu} \mathcal{O}_X \cdot z^{\mu}, \quad \text{for } k \in \mathbb{N}, \alpha \geq -1,$$

where the summation is taken over $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ satisfying

$$\rho(\mu) := \sum_{j=1}^{n} \frac{1}{m_j}\left(\mu_j + 1 + \lfloor \frac{\mu_j}{m_j - 1} \rfloor\right) \geq k - \alpha. \tag{6.19}$$

In particular, one sees that the minimal exponent $\tilde{\alpha}_f$ is $\sum_{j=1}^n \frac{1}{m_j}$.

Example 6.18. Let $D = \mathrm{div}(x^m) \subseteq \mathbb{C}$ and denote by $\mathfrak{m}_x$ the maximal ideal of $0 \in \mathbb{C}$.

- If $m \geq 2$, then
$$\mathcal{I}_{k,\alpha}(D) = \mathfrak{m}_x^{k(m-1)-\lceil(\alpha+\epsilon)m\rceil}, \quad \text{whenever } \alpha < 0.$$
- If $m = 1$, by (5.22) and the previous case we have
$$\mathcal{I}_{k,-\ell}(x = 0) = \mathcal{I}_{k,-1}(x^{\ell} = 0) \otimes \mathcal{O}_{\mathbb{C}}(x)^{(\ell-1)k} = \mathfrak{m}_x^{\ell-1}, \quad \text{whenever } \ell \in \mathbb{N}_{\geq 2}.$$
Using (5.13), it follows that $\mathcal{I}_{k,-1}(x = 0) = \mathcal{I}_{k,-2}(x = 0) \otimes \mathcal{O}_{\mathbb{C}}(x) = \mathcal{O}_{\mathbb{C}}$. Therefore
$$\mathcal{I}_{k,-\ell}(x = 0) = \mathfrak{m}_x^{\ell-1}, \quad \forall \ell \in \mathbb{N}_{\geq 1}.$$

This matches up with Proposition 6.12.

Example 6.19 (Node)**.** Let $D = \mathrm{div}(x^2 + y^2) \subseteq \mathbb{C}^2 = X$. Assume $\alpha \geq -1$, then one has

$$\mathcal{I}_{k,\alpha}(D) = \sum_{a+b\geq k-\alpha-1} \mathcal{O}_X \cdot x^a y^b = \mathfrak{m}_x^{\lceil k-\alpha-1\rceil}.$$

Therefore

$$\mathcal{G}_{k,\alpha}(D) = \begin{cases} 0 & \text{if } \alpha \notin \mathbb{Z}, \\ \mathfrak{m}_x^{k-\alpha-1}/\mathfrak{m}_x^{k-\alpha} & \text{if } \alpha \in \mathbb{Z} \text{ and } \alpha \geq 0. \end{cases}$$

The case of $\alpha < -1$ can be easily deduced from (5.13). Let $t = -\lfloor \alpha \rfloor - 1$ so that $\alpha + t \in [-1, 0)$. Then

$$\begin{aligned}\mathcal{I}_{k,\alpha}(D) &= \mathcal{I}_{k,\alpha+t}(D) \otimes \mathcal{O}_X(-tD)\\ &= \mathcal{O}_X(\lfloor \alpha+1 \rfloor D) \otimes \mathfrak{m}_x^{\lceil k-(\alpha+t)-1 \rceil}\\ &= \mathcal{O}_X(\lfloor \alpha+1 \rfloor D) \otimes \mathfrak{m}_x^{\lceil k-\{\alpha\} \rceil} = \mathcal{O}_X(\lfloor \alpha+1 \rfloor D) \otimes \mathfrak{m}_x^{k}.\end{aligned}$$

In particular,

$$\mathcal{I}_{1,0}(D) = \mathcal{O}_X, \quad \mathcal{I}_{1,<0}(D) = \mathfrak{m}_x, \quad \mathcal{G}_{1,0}(D) = \mathcal{O}_X/\mathfrak{m}_x.$$

Example 6.20 (Cusp)**.** Let $D = \mathrm{div}(x^2 + y^3) \subseteq \mathbb{C}^2 = X$ be the cupsidal singularity. Set $z_1 = x, z_2 = y$ and we list $\rho(\mu)$ from (6.19) in an increasing order:

z^μ	y	x, y^2	xy,y^3	xy^2, x^2	x^2y	x^3
$\rho(\mu)$	7/6	11/6	13/6	17/6	19/6	23/6

Then [40, Example 3.6(ii)] gives

$$\tilde{V}^\beta \mathcal{O}_X = \begin{cases} \mathcal{O}_X & \text{if } \beta \in (0, \frac{5}{6}] \\ (x,y) & \text{if } \beta \in (\frac{5}{6}, \frac{7}{6}] \\ (x, y^2) & \text{if } \beta \in (\frac{7}{6}, \frac{11}{6}] \\ (x^2, xy, y^3) & \text{if } \beta \in (\frac{11}{6}, \frac{13}{6}]. \end{cases}$$

Then we have

$$\mathcal{I}_{0,\alpha}(D) = \begin{cases} \mathcal{O}_X & \text{if } \alpha \in [-\frac{5}{6}, 0] \\ (x,y) & \text{if } \alpha \in [-1, -\frac{5}{6}) \end{cases}, \quad \mathcal{I}_{1,\alpha}(D) = \begin{cases} (x,y) & \text{if } \alpha \in [-\frac{1}{6}, 0) \\ (x,y^2) & \text{if } \alpha \in [-\frac{5}{6}, -\frac{1}{6}) \\ (x^2, xy, y^3) & \text{if } \alpha \in [-1, -\frac{5}{6}) \end{cases}.$$

$$\mathcal{I}_{2,\alpha}(D) = (x^2, xy, y^3), \quad \text{if } \alpha \in [-\frac{1}{6}, 0).$$

In the range above, we conclude that there is an equality

$$\mathcal{I}_{k,\alpha}(D) = I_k(-\alpha D),$$

from the work of Mingyi Zhang [74, Example 3.5]. Furthermore, we can use (5.14) to obtain

$$\mathcal{I}_{1,0}(D) = \mathcal{I}_{0,-1}(D) = (x,y), \quad \mathcal{I}_{2,0}(D) = \mathcal{I}_{1,-1}(D) = (x^2, xy, y^3).$$

More generally, for weighted homogeneous isolated singularity, there is a precise conjecture when the Hodge ideal equal to Saito's microlocal multiplier ideal, see [74, Conjecture E]. Using Corollary 5.16, this corresponds to a conjecture when $\mathcal{I}_{k,\alpha}(D) = I_k(-\alpha D)$ holds.

7. Local properties

In this section, we study the local properties of higher multiplier ideals and use Set-up 5.1 throughout.

7.1. **Birational transformation formula.** Let D be an effective divisor on a complex manifold X of dimension n. Let $\pi : \tilde{X} \to X$ be a log resolution of (X, D). The work of Budur-Saito provides the following birational transformation formula

$$\mathcal{I}_{0,\alpha}(D) = \pi_*(\omega_{\tilde{X}/X} \otimes \mathcal{I}_{0,\alpha}(\pi^* D)), \tag{7.1}$$

see [9, Remark 3.3]. Moreover, they also show that

$$R^i \pi_*(\omega_{\tilde{X}/X} \otimes \mathcal{I}_{0,\alpha}(\pi^* D)) = 0, \text{ whenever } i > 0. \tag{7.2}$$

Using (5.5), these statements are equivalent to the birational formula and local vanishing for the usual multiplier ideals. We give a refinement of (7.1) for the weight filtration $W_\ell\mathcal{I}_{0,\alpha}(D)$ as follows.

Proposition 7.1. *With the notation above. For any $\alpha \in \mathbb{Q}_{<0}$ and $\ell \in \mathbb{Z}$, one has*

$$W_\ell\mathcal{I}_{0,\alpha}(D) = \pi_*(\omega_{\tilde{X}/X} \otimes W_\ell\mathcal{I}_{0,\alpha}(\pi^*D)).$$

Proof. By Definition 5.8 and (7.1), it suffices to prove

$$W_\ell\mathcal{G}_{0,\alpha}(D) = \pi_*(\omega_{\tilde{X}/X} \otimes W_\ell\mathcal{G}_{0,\alpha}(\pi^*D)).$$

The proof is similar to the ones of [9, Proposition 3.2]. Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be the $\mathscr{D}$-modules associated to D and π^*D. Since $\alpha < 0$, by (2.7) one has

$$(\mathrm{gr}^V_\alpha \mathcal{M}, F_\bullet) = R^0\pi_+(\mathrm{gr}^V_\alpha \tilde{\mathcal{M}}, F_\bullet).$$

Furthermore, since $\mathrm{gr}^V_\alpha \tilde{\mathcal{M}}$ underlies a mixed Hodge module, [62, Theorem 2.14] implies that

$$W_\ell \,\mathrm{gr}^V_\alpha \mathcal{M} = \mathrm{Im}(R^0\pi_+(W_\ell \,\mathrm{gr}^V_\alpha \tilde{\mathcal{M}}, F_\bullet) \to R^0\pi_+(\mathrm{gr}^V_\alpha \tilde{\mathcal{M}}, F_\bullet)).$$

Then we obtain the desired statement by taking the lowest pieces gr^F_{-n} of both sides. □

For $\alpha \in [-1, 0)$, we have an explicit formula for the term $W_\ell\mathcal{I}_{0,\alpha}(\pi^*D)$ in Proposition 7.1. Write

$$\pi^*D = \sum_{i\in I} e_iE_i, \quad K_{\tilde{X}/X} = \sum_{i\in I} k_iE_i.$$

Set $I_\alpha = \{i \in I \mid \alpha \cdot e_i \in \mathbb{Z}\}$ and

$$E = \sum_{i\in I_\alpha} E_i, \quad E^k = \bigcup_{\substack{J\subseteq I_\alpha,\\ |J|=k}} E_J, \text{ and } E_J = \bigcap_{j\in J} E_j.$$

Since π^*D is normal crossing and $\dim X = n$, we have $E^{n+1} = \varnothing$.

Proposition 7.2. *With the notation above. For $\alpha \in [-1, 0)$ and $\ell \geq -1$, one has*

$$\begin{aligned} W_\ell\mathcal{I}_{0,\alpha}(D) &= \pi_*(\omega_{\tilde{X}/X} \otimes \mathcal{I}_{0,\alpha}(\pi^*D) \otimes \mathcal{I}_{E^{\ell+2}}), \\ &= \pi_*\left(\mathcal{O}_{\tilde{X}}(\sum_{i\in I}(k_i + \lceil(\alpha+\epsilon)e_i\rceil)E_i) \otimes \mathcal{I}_{E^{\ell+2}}\right). \end{aligned}$$

In particular,

$$W_{-1}\mathcal{I}_{0,\alpha}(D) = \mathcal{I}_{0,<\alpha}(D), \quad W_{n-1}\mathcal{I}_{0,\alpha}(D) = \mathcal{I}_{0,\alpha}(D),$$

and

$$\max\{\ell : E^{\ell+1} \neq \varnothing\} = \min\{\ell : W_\ell\mathcal{I}_{0,\alpha}(X) = \mathcal{I}_{0,\alpha}(X)\}. \tag{7.3}$$

Proof. Since π^*D is normal crossing, the formula for multiplier ideals implies that

$$\mathcal{I}_{0,\alpha}(\pi^*D) = \mathcal{O}_{\tilde{X}}\left(\sum_{i\in I}\lceil(\alpha+\epsilon)e_i\rceil E_i\right), \quad \mathcal{I}_{0,<\alpha}(\pi^*D) = \mathcal{O}_{\tilde{X}}\left(\sum_{i\in I}\lceil\alpha e_i\rceil E_i\right).$$

Hence $\mathcal{I}_{0,<\alpha}(\pi^*D) = \mathcal{I}_{0,\alpha}(\pi^*D) \otimes \mathcal{O}_{\tilde{X}}(-E)$. By Proposition 7.1, it suffices to prove that

$$W_\ell\mathcal{I}_{0,\alpha}(\pi^*D) = \mathcal{I}_{0,\alpha}(\pi^*D) \otimes \mathcal{I}_{E^{\ell+2}}. \tag{7.4}$$

Note that $\mathcal{I}_{0,\alpha}(\pi^*D)$ is a line bundle, we have

$$\mathcal{G}_{0,\alpha}(\pi^*D) = \frac{\mathcal{I}_{0,\alpha}(\pi^*D)}{\mathcal{I}_{0,<\alpha}(\pi^*D)} = \mathcal{I}_{0,\alpha}(\pi^*D) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_E. \tag{7.5}$$

Since the weight filtration on $\mathcal{I}_{0,\alpha}(\pi^*D)$ is induced by the weight filtration on $\mathcal{G}_{0,\alpha}(\pi^*D)$, we reduce the proof of (7.4) to showing

$$W_\ell \mathcal{G}_{0,\alpha}(\pi^*D) = (\mathcal{I}_{0,\alpha}(\pi^*D) \otimes \mathcal{I}_{E^{\ell+2}}) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_E. \tag{7.6}$$

Denote by $\mathcal{M}_{\tilde{X}}$ the $\mathscr{D}$-module associated to the embedding of the divisor π^*D as in Set-up 5.1. Then by (5.7) one has

$$W_\ell \mathcal{G}_{0,\alpha}(\pi^*D) \otimes \omega_{\tilde{X}} = W_\ell \operatorname{gr}^F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}} = W_\ell F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}.$$

The last equality follows from that $F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}$ is the smallest non-zero piece in the Hodge filtration. By (7.5) and (5.8), we can rewrite (7.6) as

$$W_\ell F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}} = F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{I}_{E^{\ell+2}}. \tag{7.7}$$

It remains to prove (7.7). As $\operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}$ underlies a mixed Hodge module and $W_\bullet$ is the weight filtration determined by the nilpotent operator N, one has N is strict with respect to the Hodge filtration $F_\bullet$. Together with the convolution formula $W_\ell = \sum_i \operatorname{Ker} N^{i+1} \cap \operatorname{Im} N^{i-\ell}$ and F_{-n} being the smallest piece, it follows that

$$W_\ell F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}} = \operatorname{Ker}\{N^{\ell+1} : \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}} \to \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}\}.$$

In [13, Corollary 7.7] Chen computes $\operatorname{Ker} N^{\ell+1}$. Using Lemma (2.11) (translation between Chen's result and $\operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}$), we know that $W_\ell F_{-n} \operatorname{gr}^V_\alpha \mathcal{M}_{\tilde{X}}$ is generated by degree $|I_\alpha| - 1 - \ell$ monomials dividing $\prod_{i \in I_\alpha} z_i$, where z_i is the local equation of E_i. Note that these monomials are also generators of $\mathcal{I}_{E^{\ell+2}}$. We conclude that (7.7) holds and therefore we finish the proof of the Proposition. ☐

Remark 7.3. If D is reduced and $\alpha = -1$, the birational formula in Proposition 7.2 recovers Olano's birational formula [51, Proposition 4.3 and Proposition 5.1] that

$$W_\ell \mathcal{I}_{0,-1}(D) = \pi_*(\omega_{\tilde{X}/X} \otimes \mathcal{I}_{0,-1}(\pi^*D) \otimes \mathcal{I}_{E^{\ell+2}}).$$

This follows from (5.32) and $E - \pi^*D = \lfloor (1-\epsilon)\pi^*D \rfloor = \mathcal{I}_{0,-1}(\pi^*D)$ for some $0 < \epsilon \ll 1$.

If $\alpha = -\operatorname{lct}(D)$, where $\operatorname{lct}(D)$ is the log canonical threshold, then we can produce a minimal log canonical center in the sense of Kawamata [28] and Kollár [32, §4] using $W_\ell \mathcal{I}_{0,\alpha}(D)$ for certain ℓ. Consider the following set

$$\operatorname{Lct} = \{i \in I \mid \frac{k_i + 1}{e_i} = \operatorname{lct}(D)\}, \quad I_\alpha = \{i \in I \mid e_i \cdot \alpha \in \mathbb{Z}\}.$$

It is clear that $\operatorname{Lct} \subseteq I_\alpha$ but the latter can be strictly bigger, for example when $\operatorname{lct}(D) = 1$.

Corollary 7.4. *Let D be an effective divisor on a complex manifold X. Let $\alpha = -\operatorname{lct}(D)$. Assume* $\operatorname{Lct} = I_\alpha$, *i.e. every exceptional divisor with $e_i \cdot \operatorname{lct}(D) \in \mathbb{Z}$ must compute the log canonical threshold. Let ℓ be the largest integer such that $E^{\ell+1} \neq \varnothing$, where $E^\ell = \cup_{\substack{J \subseteq I_\alpha \\ |J| = \ell}} \cap_{j \in J} E_j$. Set*

$$Y = \operatorname{Zero}(W_{\ell-1} \mathcal{I}_{0,\alpha}(D)).$$

Then Y is a minimal log canonical center of (X, D).

Proof. By Proposition 7.2 we have

$$W_{\ell-1} \mathcal{I}_{0,\alpha}(D) = \pi_*(\mathcal{O}_{\tilde{X}}(\sum_i (k_i + \lceil (\alpha + \epsilon) e_i \rceil) E_i) \otimes \mathcal{I}_{E^{\ell+1}}).$$

It is well-known that $\operatorname{lct}(D) = \min_{i \in I} \frac{k_i + 1}{e_i}$. Hence

$$\alpha \cdot e_i \geq -k_i - 1, \text{ and } \quad \lceil (\alpha + \epsilon) e_i \rceil \geq -k_i \quad \text{for all } i \in I.$$

Consequently, the term $\mathcal{O}_{\tilde{X}}(\sum_i(k_i+\lceil(\alpha+\epsilon)e_i\rceil)E_i)$ does not contribute under π_* and we have

$$W_{\ell-1}\mathcal{I}_{0,\alpha}(D)=\pi_*(\mathcal{I}_{E^{\ell+1}}),$$

which means that Y is the image of $E^{\ell+1}$ under π. Since every exceptional divisor in I_α computes the log canonical threshold and ℓ is the largest number that $E^{\ell+1}\neq\varnothing$, it follows that Y is minimal with respect to inclusion and therefore Y is a minimal log canonical center of (X,D). □

Remark 7.5. For $k\geq 1$, in general we have

$$\pi_*(\omega_{\tilde{X}/X}\otimes\mathcal{I}_{k,\alpha}(\pi^*D))\neq\mathcal{I}_{k,\alpha}(D),$$

because the right hand side may involves direct images of lower order multiplier ideals. For a concrete case, see Example 6.14. But it seems reasonable to guess that

$$\pi_*(\omega_{\tilde{X}/X}\otimes\mathcal{I}_{k,\alpha}(\pi^*D))\subseteq\mathcal{I}_{k,\alpha}(D).$$

We leave this for future investigation.

7.2. **Restriction and semicontinuity theorems.** We analyze the behavior of higher multiplier ideals under restriction and deformation.

7.2.1. *Restriction theorem.* We first prove the following result.

Theorem 7.6. *Let D be an effective divisor on X and let $i:H\hookrightarrow X$ be the closed embedding of a smooth hypersurface that is not entirely contained in the support of D so that the pullback $D_H=i^*D$ is defined. Then for any $k\in\mathbb{N},\alpha\in\mathbb{Q}$, there exists a morphism*

$$\mathcal{I}_{k,\alpha}(D_H)\to\mathbf{L}i^*\mathcal{I}_{k,\alpha}(D),\tag{7.8}$$

which commutes with the two natural morphisms to $\mathcal{O}_H$. In particular, one has an inclusion

$$\mathcal{I}_{k,\alpha}(D_H)\subseteq\mathcal{I}_{k,\alpha}(D)\cdot\mathcal{O}_H,$$

where the latter is defined as the image of $\{i^*\mathcal{I}_{k,\alpha}(D)\to i^*\mathcal{O}_X=\mathcal{O}_H\}$. Moreover, if H is sufficiently general, then

$$\mathcal{I}_{k,\alpha}(D_H)=\mathcal{I}_{k,\alpha}(D)\cdot\mathcal{O}_H=i^*\mathcal{I}_{k,\alpha}(D).$$

Proof. The inclusion $H\hookrightarrow X$ induces a commutative diagram

$$\begin{tikzcd} L_H \arrow[r, "i"] \arrow[d, "p"] & L \arrow[d, "p"] \\ H \arrow[r, "i"] & X \end{tikzcd}$$

where L_H is the total space of the pullback line bundle i^*L and p are projection maps. Recall from Set-up 5.1 that the effective divisor D induces a closed embedding $s:X\to L$ and a mixed Hodge module $M=s_*\mathbb{Q}^H_X[n]\in\mathrm{MHM}(L)$; analogously, we define $M_H\in\mathrm{MHM}(L_H)$ using the divisor D_H.

Let $j:L\setminus L_H\hookrightarrow L$ be the open embedding. The idea is that the morphism (7.8) will be induced by the distinguished triangle from [62, (4.4.1)]:

$$i_*i^!M=i_!i^!M\to M\to j_*j^*M\to i_!i^!M[1].$$

Since L_H is a smooth hypersurface in L, one has $i^!M[1]=M_H(-1)$ with $F_\bullet\mathcal{M}_H(-1)=F_{\bullet+1}\mathcal{M}_H$, where the shift comes from M_H being the pushforward of the constant Hodge module on H. This gives a short exact sequence of mixed Hodge modules

$$0\to M\to j_*j^*M\to i_*M_H(-1)\to 0.\tag{7.9}$$

Fix $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$. Since both Hodge and V-filtration preserve exactness and commute with the functor i_*, (7.9) induces a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathrm{gr}^F_{-n+k} V_\alpha \mathcal{M} \to \mathrm{gr}^F_{-n+k} V_\alpha(j_*j^*\mathcal{M}) \to i_*\left(\mathrm{gr}^F_{-n+k+1} V_\alpha \mathcal{M}_H\right) \to 0. \tag{7.10}$$

This defines an element of

$$\mathrm{Ext}^1_X\left(i_*(\mathrm{gr}^F_{-n+k+1} V_\alpha \mathcal{M}_H), \mathrm{gr}^F_{-n+k} V_\alpha \mathcal{M}\right).$$

The right adjoint of the functor i_* for coherent sheaves is the functor $i^! = \omega_{H/X} \otimes \mathbf{L}i^*[-1]$, hence (7.10) also determines in a natural way a morphism

$$\mathrm{gr}^F_{-(n-1)+k} V_\alpha \mathcal{M}_H \to i^! \,\mathrm{gr}^F_{-n+k} V_\alpha \mathcal{M}[1] \cong \omega_{H/X} \otimes \mathbf{L}i^* \,\mathrm{gr}^F_{-n+k} V_\alpha \mathcal{M}. \tag{7.11}$$

Substituting in the definition of higher multiplier ideals (5.3), we arrive at

$$\omega_H \otimes L_H^k \otimes \mathcal{I}_{k,\alpha}(D_H) \to \omega_{H/X} \otimes \mathbf{L}i^*(\omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D)),$$

which easily gives the desired morphism

$$\mathcal{I}_{k,\alpha}(D_H) \to \mathbf{L}i^* \mathcal{I}_{k,\alpha}(D). \tag{7.12}$$

Since $\mathrm{gr}^F_{-n+k} \mathcal{M} \cong s_*(\omega_X \otimes L^k)$ and $\mathrm{gr}^F_{-n+k+1} \mathcal{M}_H \cong (s \circ i)_*(\omega_H \otimes L_H^k)$ by Lemma 5.2, it is easy to see that the morphism (7.12) commutes with the two obvious morphisms to $\mathcal{O}_H$. In particular, one has a commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{k,\alpha}(D_H) & \longrightarrow & i^*\mathcal{I}_{k,\alpha}(D) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_H & = & i^*\mathcal{O}_X \end{array}$$

Since the image of $i^*\mathcal{I}_{k,\alpha}(D)$ in $i^*\mathcal{O}_X = \mathcal{O}_H$ is defined to be $\mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_H$, we conclude that there is an injection $\mathcal{I}_{k,\alpha}(D_H) \hookrightarrow \mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_H$.

Now suppose H is sufficiently transverse to D. To prove that $\mathcal{I}_{k,\alpha}(D_H) = \mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_H$, we need to show that the morphism (7.11) is an isomorphism. This is a local problem, and so we may assume without loss of generality that D is defined, in the graph embedding, by a holomorphic function t whose divisor is a smooth hypersurface. By passing to subquotients, the morphism in (7.11) determines, for each $\alpha \in \mathbb{R}$, a morphism

$$\mathrm{gr}^F_{-(n-1)+k} \mathrm{gr}^V_\alpha \mathcal{M}_H \to \omega_{H/X} \otimes i^* \,\mathrm{gr}^F_{-n+k} \mathrm{gr}^V_\alpha \mathcal{M}. \tag{7.13}$$

Here we are allowed to replace $\mathbf{L}i^*$ by i^* because H, being sufficiently transverse to D, is noncharacteristic with respect to the mixed Hodge module $\mathrm{gr}^V_\alpha \mathcal{M}$ for every α.

We claim that it is enough to prove that (7.13) is an isomorphism for every α. Indeed, this implies that the kernel and cokernel of (7.11) are trivial modulo $V_\alpha \mathcal{M}$ (respectively $V_\alpha \mathcal{M}_H$) for all $\alpha \ll 0$. But by the definition of mixed Hodge modules, multiplication by t induces isomorphisms

$$t\colon \mathrm{gr}^F_p V_\alpha \mathcal{M} \to \mathrm{gr}^F_p V_{\alpha-1} \mathcal{M} \quad \text{and} \quad t\colon \mathrm{gr}^F_p V_\alpha \mathcal{M}_H \to \mathrm{gr}^F_p V_{\alpha-1} \mathcal{M}_H$$

for every $\alpha < 0$. Since $\mathrm{gr}^F_p V_\alpha \mathcal{M}$ is a coherent $\mathscr{O}_X$-module, the result that we want now follows from Krull's intersection theorem (with respect to the ideals generated by the powers of t) and Nakayama's lemma.

Now the right-hand side of (7.13) is the noncharacteristic restriction of the mixed Hodge module $\mathrm{gr}^V_\alpha \mathcal{M}$ to the hypersurface H. Since morphisms of mixed Hodge modules are strict with respect to the Hodge filtration, it is therefore enough to show that

$$\mathrm{gr}^V_\alpha \mathcal{M}_H \to \omega_{H/X} \otimes i^* \,\mathrm{gr}^V_\alpha \mathcal{M} \tag{7.14}$$

is an isomorphism of mixed Hodge modules. But this follows from Corollary 2.7, again due to the fact that H is sufficiently transverse to D. □

Remark 7.7. One can use Theorem 7.6 to obtain similar statement for restriction to smooth subvarieties of higher codimension as in [36, Corollary 9.5.6].

Corollary 7.8. *Let X and T be complex manifolds, and*

$$p : X \to T$$

a smooth surjective morphism. Consider an effective Cartier divisor D on X whose support does not contain any of the fibers $X_t = p^{-1}(t)$, so that for each $t \in T$ the restriction $D_t = D|_{X_t}$ is defined. Then there is a non-empty Zariski open set $U \subseteq T$ such that

$$\mathcal{I}_{k,\alpha}(D_t) = \mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_{X_t},$$

for every $\alpha \leq 0$ and every $t \in U$, where $\mathcal{I}_{k,\alpha}(D) \cdot \mathcal{O}_{X_t}$ denotes the restriction of the indicated ideal to the fiber X_t.

Proof. We can choose hyperplanes $H_1, \ldots, H_n$ on T such that X_t is a connected component of $f^*(H_1) \cap \ldots \cap f^*(H_n)$. By Theorem 7.6, "generically" higher multiplier ideals commute with restriction. Apply this several times, we obtain the desired statement. □

Theorem 7.9 (Restriction theorem for graded pieces)**.** *With the same assumption in Theorem 7.6. For any $k \in \mathbb{N}, \alpha \in \mathbb{Q}, \ell \in \mathbb{Z}$, we have a natural morphism*

$$W_\ell \mathcal{G}_{k,\alpha}(D_H) \to \mathbf{L}i^* W_\ell \mathcal{G}_{k,\alpha}(D).$$

If H is sufficiently general, then it induces an isomorphism

$$W_\ell \mathcal{G}_{k,\alpha}(D_H) \cong W_\ell \mathcal{G}_{k,\alpha}(D) \otimes_{\mathcal{O}_X} \mathcal{O}_H.$$

The same statements hold for $W_\ell \mathcal{I}_{k,\alpha}(D)$.

Proof. The proof is quite close to Theorem 7.6 and we just give a sketch. Using that the weight filtration also preserves exactness, (7.9) induces a short exact sequence

$$0 \to W_\ell \operatorname{gr}^F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M} \to W_\ell \operatorname{gr}^F_{-n+k} \operatorname{gr}^V_\alpha (j_* j^* \mathcal{M}) \to i_* \left(W_\ell \operatorname{gr}^F_{-n+k+1} \operatorname{gr}^V_\alpha \mathcal{M}_H \right) \to 0.$$

As in the proof of Theorem 7.6, using Ext-group and $i^! = \omega_{H/X} \otimes \mathbf{L}i^*$, this sequence induces a natural morphism

$$W_\ell \operatorname{gr}^F_{-n+k+1} \operatorname{gr}^V_\alpha \mathcal{M}_H \to \omega_{H/X} \otimes \mathbf{L}i^* W_\ell \operatorname{gr}^F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M},$$

and hence the desired morphism.

If H is sufficiently general, the desired isomorphism follows from that fact that (7.14) is an isomorphism of mixed Hodge modules. For $W_\ell \mathcal{I}_{k,\alpha}(D)$, the desired statements follow from (5.9) and that the construction is functorial with respect to $\mathbf{L}i^*$. □

Remark 7.10. After the first draft of this manuscript, Mustaţă suggests that the equality case of Theorem 7.6 may also be proved using a different argument similar to [16, Lemma 4.2]; see also [65, (1.3.11)].

7.2.2. *Semicontinuity Theorem.* As in the theory of multiplier ideals, we can use the Restriction Theorem 7.6 to study how higher multiplier ideals behave in families. One can obtain the following semicontinuity theorems. Let $p : X \to T$ be a smooth morphism of relative dimension n between arbitrary varieties X and T, and let $s : T \to X$ be a morphism such that $p \circ s = \mathrm{id}_T$. Suppose that D is an effective Cartier divisor on X, relative over T. For every $t \in T$, denote by $X_t := p^{-1}(t)$ the fiber over t and D_t the restriction of D on X_t. For every $x \in X$, we denote by $\mathfrak{m}_x$ the ideal defining x in $X_{p(x)}$.

Theorem 7.11 (Weaker verision of semicontinuity)**.** *With the notation above, if*

$$s(t) \in \operatorname{Zeroes}(\mathcal{I}_{k,\alpha}(D_t)) \quad \textit{for all } t \neq 0 \in T,$$

then $s(0) \in \operatorname{Zeroes}(\mathcal{I}_{k,\alpha}(D_0))$.

Proof. Argue as in [36, Theorem 9.5.39] and use Theorem 7.6, Corollary 7.8. □

Theorem 7.12 (Stronger version of semicontinuity)**.** *With the notation above, for every $q \geq 1$, the set*

$$V_q := \left\{ t \in T \mid \mathcal{I}_{k,\alpha}(D_t) \not\subseteq \mathfrak{m}^q_{s(t)} \right\},$$

is open in T.

Proof. Since we already have Theorem 7.6 and Corollary 7.8, the same proof of [44, Theorem E] gives the desired statement. □

Remark 7.13. It is direct to generalize Theorem 7.6, Theorem 7.11 and Theorem 7.12 to $\mathbb{Q}$-divisors for $\alpha \leq 0$.

7.3. **Numerical criterion of nontriviality.** We study how the presence of very singular points contributes to the nontriviality of certain higher multiplier ideals. This recovers and expands the corresponding phenomenon for usual multiplier ideals.

As a motivation, let us first review the related facts for multiplier ideals and later they will be recovered using Theorem 7.17, see Remark 7.18. Let $D = \sum a_i D_i$ be an effective $\mathbb{Q}$-divisor on a complex manifold X of dimension n, and let $x \in X$ be a fixed point. The *multiplicity* $\mathrm{mult}_x D$ is the rational number

$$\mathrm{mult}_x D := \sum a_i \cdot \mathrm{mult}_x D_i.$$

More generally, given any irreducible subvariety $Z \subseteq X$, $\mathrm{mult}_Z D = \sum a_i \cdot \mathrm{mult}_Z D_i$, where $\mathrm{mult}_Z D_i$ denotes the multiplicity of D_i at a generic point of Z. The p^{th} *symbolic power* $\mathcal{I}_Z^{\langle p \rangle}$ of $\mathcal{I}_Z$ is the ideal sheaf consisting of germs of functions that have multiplicity $\geq p$ at a general point of Z.

Proposition 7.14. [36, Proposition 9.5.13 and Example 9.3.5] *The following holds.*

- *If* $\mathrm{mult}_x D < 1$*, then* $\mathcal{J}(X, D)_x = \mathcal{O}_{X,x}$*.*
- *Let $Z \subseteq X$ be an irreducible variety. If* $\mathrm{mult}_Z D \geq \mathrm{codim}_X(Z)$*, then* $\mathcal{J}(D) \neq \mathcal{O}_X$*. Moreover, let $q \geq 1$ be an integer, then*

$$\mathcal{J}(D) \subseteq \mathcal{I}_Z^{\langle q \rangle}, \quad \textit{whenever } \mathrm{mult}_Z D \geq \mathrm{codim}_X(Z) + q - 1.$$

Using the relation $\mathcal{I}_{0,<\alpha}(D) = \mathcal{J}(X, -\alpha D)$ from (5.5), Proposition 7.14 implies the following.

Corollary 7.15. *Let D be an effective divisor on X.*

- *If D has multiplicity $m \geq 2$ at x, then*

$$\mathcal{I}_{0,<\alpha}(D)_x = \mathcal{O}_{X,x}, \quad \textit{whenver } \alpha > -\frac{1}{m}. \tag{7.15}$$

- *Let $Z \subseteq X$ be an irreducible variety such that* $\mathrm{mult}_Z D = m \geq 2$*, then*

$$\mathcal{I}_{0,<\alpha}(D) \subseteq \mathcal{I}_Z^{\langle q \rangle}, \quad \textit{whenever } -1 < \alpha \leq -\frac{\mathrm{codim}_X(Z) + q - 1}{m} \textit{ for some } q \geq 1. \tag{7.16}$$

Remark 7.16. Note that $\mathcal{I}_{0,<-1}(D) = \mathcal{O}_X(-D)$, so we need $\alpha > -1$ in (7.16).

Now we prove the numerical criterion for higher multiplier ideals.

Theorem 7.17. *Let D be an effective divisor on a complex manifold X of dimension n. Suppose $Z \subseteq \mathrm{Sing}_m(D)$ is an irreducible component of dimension d. Write*

$$n - d = km + r, \quad \textit{with } k \in \mathbb{N} \textit{ and } 0 \leq r \leq m - 1.$$

Then

$$\mathcal{I}_{k,<\alpha}(D) \neq \mathcal{O}_X, \quad \textit{for some } \alpha \geq -r/m. \tag{7.17}$$

Moreover, if $\alpha = -\frac{r+q-1}{m}$ *for some* $q \geq 1$*, then*

$$\mathcal{I}_{k,<\alpha}(D) \subseteq \mathcal{I}_Z^{\langle q\rangle}, \quad \text{if } q \leq \max(m-r, m-1), \tag{7.18}$$

where $\mathcal{I}_Z^{\langle q\rangle} = \mathcal{O}_X$ *if* $q \leq 0$*. For arbitrary integer* $\ell \in \mathbb{N}$ *and* $\alpha \geq -1$*, we have*

(7.19)

$$\mathcal{I}_{\ell,\alpha}(D) \subseteq \mathcal{I}_Z^{\langle p\rangle}, \quad \text{where } p = m(\ell-\alpha) - \operatorname{codim}_X(Z) - \lfloor \frac{m(\ell-\alpha) - \operatorname{codim}_X(Z)}{m} \rfloor.$$

Proof. The problem is local, and so we may assume that X is an open subset of $\mathbb{C}^n$, and that D is the zero locus of a holomorphic function f with $f(0) = 0$. By cutting with d generic chosen hyperplanes and using the Restriction Theorem 7.6, we can assume $\dim X = n - d$, where $d = \dim Z$ and $0 \in D$ is an isolated singular point of multiplicity m.

The idea is to deform to the case of ordinary singularities. Let $\mathbf{A}^N$ be the affine space parametrizing the coefficients of homogeneous polynomials of degree m, with coordinates c_v, for $v = (v_1, \ldots, v_{n-d}) \in \mathbf{Z}_{\geq 0}^{n-d}$, with $|v| := \sum_i v_i = m$. Let us consider the effective divisor F on $X \times \mathbf{A}^N$ defined by $f + \sum_{|v|=m} c_v x^v$. It is direct to see that there is an open neighborhood $U \subseteq \mathbf{A}^N$ of 0, consisting of those $t \in \mathbf{A}^N$ such that

$$D_t := F \cap (X \times \{t\})$$

is a reduced divisor on $X \times \{t\} \cong X$ and D_t has an isolated singularity at 0 with nonsingular projective tangent cone for $t \neq 0$.

Now we can prove the desired statement. Let $\alpha = -\frac{r+q-1}{m}$ for some $1 \leq q \leq \max(m-r, m-1)$, since $n - d = km + r$ for $k \in \mathbb{N}$ and $0 \leq r \leq m-1$, then (6.17) gives

$$\mathcal{I}_{k,<\alpha}(D_t)_0 = \mathcal{I}_{k,-(r+q)/m}(D_t)_0 = \mathfrak{m}_0^q.$$

The first equality uses that all jumping numbers of $\mathcal{I}_{k,<\alpha}(D_t)_0$ satisfy $m\alpha \in \mathbb{Z}$ from (6.12). By the semicontinuity Theorem 7.12 we must have

$$\mathcal{I}_{k,<-r/m}(D)_0 \neq \mathcal{O}_{X,0}, \quad \mathcal{I}_{k,<\alpha}(D)_0 \subseteq \mathfrak{m}_0^q.$$

Thus, we have (7.17) and (7.18). A similar argument using (6.15) gives

$$\mathcal{I}_{\ell,\alpha}(D)_0 \subseteq \mathfrak{m}_0^{m(\ell-\alpha)-(n-d)-\lfloor \frac{m(\ell-\alpha)-(n-d)}{m} \rfloor},$$

for any ℓ and $\alpha \geq -1$. This proves (7.19). ☐

Remark 7.18. Theorem 7.17 recovers the nontriviality statement (7.16) for multiplier ideals. Let D be an effective divisor on X and let $Z \subseteq X$ be an irreducible subvariety such that $\operatorname{codim}_X(Z) = r \geq 1$ and $\operatorname{mult}_Z D = m$. In particular, Z is an irreducible component of $\operatorname{Sing}_m(D)$. Let $\alpha = -\frac{r+q-1}{m} > -1$ and $q \geq 1$. The condition $\alpha > -1$ implies that $m > r + (q-1) \geq r$ and so we can write

$$r = 0 \cdot m + r.$$

Since $r \geq 1$ and $q \leq \min(m-1, m-r)$, therefore $\mathcal{I}_{0,<\alpha}(D) \subseteq \mathcal{I}_Z^{\langle q\rangle}$ by (7.18).

Remark 7.19. For $k = 0$, one can also recover the triviality statement (7.15) using the Restriction Theorem 7.6 and the fact that $\mathcal{I}_{0,<\alpha}(D) = \mathfrak{m}_x^{\lfloor -\alpha m\rfloor}$ for $D = \{x^m = 0\} \subseteq \mathbb{C}$. However, for $k \geq 1$, there is no such analogous statement in terms of multiplicities. Here is the reason: from Example 6.18 one has

$$\mathcal{I}_{k,<\alpha}(x^m = 0) = \mathfrak{m}_x^{k(m-1)+\lfloor -\alpha m\rfloor}, \quad \text{whenever } \alpha < 0.$$

But for $\alpha < 0$, $k \geq 1$ and $m \geq 2$, we always have $k(m-1) + \lfloor -\alpha m\rfloor \geq k(m-1) \geq 1$, so $\mathcal{I}_{k,<\alpha}(x^m = 0) \neq \mathcal{O}_X$.

Remark 7.20. For $\alpha = -1$, from (7.19) one can deduce that

$$\mathcal{I}_{\ell,-1}(D) \subseteq \mathcal{I}_Z^{\langle p \rangle}, \quad \text{where } p = \min(m-1, m(\ell+1) - \operatorname{codim}_X(Z)).$$

This is because for any $m \geq 2$, one has

$$m(\ell+1) - \operatorname{codim}_X(Z) - \lfloor \frac{m(\ell+1) - \operatorname{codim}_X(Z)}{m} \rfloor \geq \min(m-1, m(\ell+1) - \operatorname{codim}_X(Z)).$$

Therefore we obtain a similar numeric bound with the ones of Hodge ideals [45, Theorem E], in view of $\mathcal{I}_{\ell,-1}(D) \equiv I_\ell(D)$ modulo $\mathcal{I}_D$ in (1.7). As an application, we can give alternative proofs of several results on theta divisors in[45, §29-§30] obtained using higher multiplier ideals, see §9.5.

Corollary 7.21. *Let D be an effective divisor on X. Then*

$$\tilde{\alpha}_D \leq \min \frac{\operatorname{codim}_X(Z)}{m}, \tag{7.20}$$

where the minimum run through all $m \geq 2$ and all irreducible components Z of $\operatorname{Sing}_m(D)$.

Proof. Write $\operatorname{codim}_X(Z) = mk + r$ for $k \in \mathbb{N}$ and $0 \leq r \leq m-1$. Since $\mathcal{I}_{k,<\alpha}(D) \neq \mathcal{O}_X$ for $\alpha = -r/m \in (-1, 0]$, Lemma 5.26 implies that

$$\tilde{\alpha}_D \leq k - \alpha = \frac{\operatorname{codim}_X(Z)}{m}.$$

$\square$

Corollary 7.22. *Let D be an effective divisor on a complex manifold X. If Z is an irreducible closed subset of X of codimension r such that $m = \operatorname{mult}_Z(D) \geq 2$, then*

$$\mathcal{I}_{k,-1}(D) \subseteq \mathcal{I}_Z, \quad \textit{for all } k \geq \frac{r + 1 - \lceil \frac{r}{m} \rceil}{m-1} - 1.$$

Proof. By (7.19), one needs k such that

$$m(k+1) - r - \lfloor \frac{m(k+1) - r}{m} \rfloor \geq 1.$$

Using $-\lfloor a \rfloor = \lceil -a \rceil$ for any $a \in \mathbb{Q}$, this is

$$m(k+1) - r - (k+1) + \lceil \frac{r}{m} \rceil \geq 1,$$

which is equivalent to the desired bound. $\square$

Remark 7.23. Our bound is comparable with the ones of [45, Corollary 21.3], where

$$I_k(D) \subseteq \mathcal{I}_Z, \quad \text{for all } k \geq \frac{r+1}{m} - 1,$$

assuming D is a reduced effective divisor. It gives better bounds when apply to singularities of hypersurfaces in projective spaces, see §8.4.

7.3.1. *Minimal exponents via log resolutions.* Using Corollary 7.21, one can compute the minimal exponent of a divisor D using a log resolution in some cases.

Proposition 7.24. *Let D be an effective divisor on a complex manifold X. Assume there exists a log resolution $\pi : \tilde{X} \to X$ of (X, D) satisfying the following conditions:*

- *the proper transform $\tilde{D}$ is smooth, $\pi^* D$ has simple normal crossing support,*
- *the morphism π is the iterated blow up of X along (the proper transform of) all irreducible components of* $\operatorname{Sing}_m(D)$ *for all $m \geq 2$, and all the blow-up centers are smooth.*

Then

$$\tilde{\alpha}_D = \min \frac{\operatorname{codim}_X(Z)}{m},$$

where the minimum runs through all $m \geq 2$ *and all irreducible components* Z *of* $\mathrm{Sing}_m(D)$*.*

Proof. Let us write

$$\pi^* D = \tilde{D} + \sum_{i \in I} a_i D_i, \quad K_{\tilde{X}/X} = \sum_{i \in I} k_i D_i.$$

Then we have

$$\min_{i \in I} \frac{k_i + 1}{a_i} \leq \tilde{\alpha}_D \leq \min_{m \geq 2} \frac{\operatorname{codim}_X(\mathrm{Sing}_m(D))}{m}. \tag{7.21}$$

The lower bound is a result of Mustaţă-Popa [47, Corollary D] (see also [18, Corollary 1.5]). The upper bound follows from (7.20).

Now by the assumption that one only blows up the proper transforms of irreducible components of multiplicity loci $\mathrm{Sing}_m(D)$, then we can rearrange the index so that

$$\frac{k_m + 1}{a_m} = \frac{\operatorname{codim}_X(\mathrm{Sing}_m(D))}{m}, \quad \forall m \in I.$$

Here we use the following fact: let $\pi : Y \to X$ be the blow-up of a smooth subvariety Z inside a smooth variety X, then

$$K_Y = \pi^* K_X + (\operatorname{codim}_X(Z) - 1)E.$$

Therefore by (7.21) we conclude the proof. $\square$

Example 7.25. Let $D \subseteq X$ be the generic determinantal hypersurface in the space of n by n matrices, it is proved in [25, Chapter 4] (see also [72]) that one can obtain a log resolution of (X, D) by successively blowing up $\mathrm{Sing}_m(D)$. Therefore we conclude that

$$\tilde{\alpha}_D = \min_{m \geq 2} \frac{\operatorname{codim}_X(\mathrm{Sing}_m(D))}{m} = \min_{m \geq 2} \frac{m^2}{m} = 2.$$

This can also be verified directly using the Bernstein-Sato polynomial.

On the other hand, let C be a smooth projective curve embedded in $X = \mathbf{P}H^0(C, M) = \mathbf{P}^{2n+2}$ by a line bundle M and let $D = \mathrm{Sec}^n(C)$ be the n-th secant variety, which is also a hypersurface in X. By the work of Bertram [4], one can also achieve a log resolution by blowing up all $\mathrm{Sing}_m(D)$ and thus

$$\tilde{\alpha}_D = \min_{m \geq 2} \frac{2m - 1}{m} = \frac{3}{2}.$$

This seems difficult to obtain directly using Bernstein-Sato polynomials. For the computation related to theta divisors of curves of principally polarized abelian varieties, see Theorem 9.6.

7.3.2. *Dimension of multiplicity loci and projectivized tangent cone.* Theorem 7.17 recovers the following well-known inequality.

Corollary 7.26. *Let* D *be an effective divisor on* X*. For any* $m \geq 2$ *and* $x \in D$*, we have*

$$\dim \mathrm{Sing}_m(D)_x \leq \dim \mathbf{P}(C_x D)_{\mathrm{Sing}} + 1,$$

where $\mathrm{Sing}_m(D)_x = \mathrm{Sing}_m(D)_x \cap U$*,* U *is a small open neighborhood of* x *in* X *and* $\mathbf{P}(C_x D)_{\mathrm{Sing}}$ *is the singular locus of the tangent cone of* D *at* x*.*

Proof. By Theorem 7.17 and [47, Theorem E], one has

$$\frac{n - \dim \mathbf{P}(C_x D)_{\mathrm{Sing}} - 1}{m} \leq \tilde{\alpha}_{D,x} \leq \frac{n - \dim Z}{m},$$

where Z is any irreducible component of $\mathrm{Sing}_m(D)$ containing x. □

7.4. **The center of minimal exponent.** Let D be an effective divisor on a complex manifold X. We use Set-up 5.1. In the previous section, we see that higher multiplier ideals lead to a better understanding of the minimal exponent $\tilde{\alpha}_D$ (see Corollary 7.21). In this section, we push this idea a bit further and construct several subschemes of D associated to $\tilde{\alpha}_D$, which generalize the notion of log canonical centers and minimal log canonical centers [32, §4]. Recall from Lemma 5.26 that

$$\begin{aligned}\tilde{\alpha}_D &= \min\{k - \alpha, k \in \mathbb{N}, \alpha \in (-1, 0] \mid \mathcal{G}_{k,\alpha}(D) \neq 0\},\\ &= \min\{k - \alpha, k \in \mathbb{N}, \alpha \in (-1, 0] \mid \mathcal{I}_{k,<\alpha}(D) \subsetneq \mathcal{O}_X\}.\end{aligned}$$

Let us write

$$\tilde{\alpha}_D = k - \alpha, \quad \text{for a unique } k \in \mathbb{N} \text{ and a unique } \alpha \in (-1, 0]. \tag{7.22}$$

Then

$$\mathcal{I}_{k,<\alpha}(D) \subsetneq \mathcal{O}_X, \quad \mathcal{G}_{k,\alpha}(D) \neq 0, \text{ and } \mathcal{G}_{\ell,\alpha}(D) = 0 \quad \text{whenever } \ell < k. \tag{7.23}$$

Therefore $\mathcal{I}_{k,<\alpha}(D)$ is the first non-trivial higher multiplier ideal, whose zero locus would carry interesting information of (X, D).

7.4.1. *The zero locus of* $\mathcal{I}_{k,<\alpha}(D)$. Let us first study $\mathcal{I}_{k,<\alpha}(D)$.

Definition 7.27. With the notation above. Let $Z \subseteq X$ be the subscheme such that $\mathcal{I}_Z = \mathcal{I}_{k,<\alpha}(D)$, or equivalently we have $\mathcal{O}_Z = \mathcal{G}_{k,\alpha}(D)$.

Remark 7.28. By definition, we have

$$\tilde{\alpha}_D = \min_{x \in D} \tilde{\alpha}_{D,x}.$$

Therefore Lemma 5.26 implies the following set-theoretical equality

$$Z = \{x \in D \mid \tilde{\alpha}_{D,x} = \tilde{\alpha}_D\},$$

the locus where the global minimal exponent is achieved.

Remark 7.29. If $\tilde{\alpha}_D \leq 1$, then the scheme Z recovers the notion of locus of log canonical singularities. By assumption, $\mathrm{lct}(D) = \min\{1, \tilde{\alpha}_D\} = \tilde{\alpha}_D$, so in (7.22) one has

$$k = 0, \quad \text{and} \quad \alpha = -\tilde{\alpha}_D = -\mathrm{lct}(D).$$

Then (5.5) gives

$$\mathcal{I}_Z = \mathcal{I}_{0,<\alpha}(D) = \mathcal{J}(X, \mathrm{lct}(D) \cdot D),$$

and thus Z is the *locus of log canonical singularities* of D (see [36, Definition 10.4.7]). In [36, Example 9.3.18], it is shown that $\mathcal{J}(X, \mathrm{lct}(D) \cdot D)$ is radical, so Z is reduced. The next lemma is a generalization of this fact to $\tilde{\alpha}_D > 1$.

Lemma 7.30. *We have* $Z \subseteq D_{\mathrm{sing}}$ *and* Z *is reduced.*

Proof. By (5.6), we have

$$\omega_X \otimes L^k \otimes \mathcal{O}_Z = \omega_X \otimes L^k \otimes \mathcal{G}_{k,\alpha}(D) = \mathrm{gr}^F_{-n+k} \mathrm{gr}^V_\alpha \mathcal{M}.$$

By (7.23) and (5.6) again, one has $F_{-n+\ell} \mathrm{gr}^V_\alpha \mathcal{M} = 0$ for any $\ell < k$. This means that $\mathcal{O}_Z$ is isomorphic to $F_{-n+k} \mathrm{gr}^V_\alpha \mathcal{M} \otimes (\omega_X \otimes L^{-k})$, which is locally the first step in the Hodge filtration of a complex mixed Hodge module. It follows that Z is reduced: let h be a local holomorphic function vanishing on Z, then locally $\operatorname{supp} \operatorname{gr}^V_\alpha \mathcal{M} \subseteq h^{-1}(0)$. By [61, Lemme 3.2.6] (which also works for complex mixed Hodge modules), locally one has

$$\mathcal{O}_Z \cdot h \cong (F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M}) \cdot h \subseteq F_{-n+(k-1)} \operatorname{gr}^V_\alpha \mathcal{M} = 0,$$

so Z is reduced.

Since we have shown that Z is reduced, then

$$Z = \operatorname{supp} F_{-n+k} \operatorname{gr}^V_\alpha \mathcal{M} = \operatorname{supp} \operatorname{gr}^V_\alpha \mathcal{M}.$$

Since $\alpha \in (-1, 0]$, by (2.2) one knows that locally the support of the vanishing cycle $\mathscr{D}$-module $\operatorname{gr}^V_\alpha \mathcal{M}$ is contained in the singular locus of f, where $D = \operatorname{div}(f)$. Thus we conclude that $Z \subseteq D_{\mathrm{sing}}$.

□

Example 7.31 (Plane curves)**.** If $D = \operatorname{div}(xy) \subseteq \mathbb{C}^2$, then $\tilde{\alpha}_D = 1$ and $\mathcal{G}_{1,0}(D) = \mathcal{O}_x$, where x is the unique singularity of D. Therefore the corresponding Z is $\{x\}$. If $D = \operatorname{div}(x^2 + y^3)$, then $\tilde{\alpha}_D = 5/6$ and Z is also equals to $\{x\}$, the unique singularity of D.

Example 7.32 (Generic determinantal hypersurface)**.** Let X be the space of n by n matrices and let D be the generic determinantal hypersurface, cut out by $f = \det$. Example 7.25 implies that $\tilde{\alpha}_D = 2$. It is proved in [37] that $\mathcal{I}_{2,<0}(D) = \mathcal{I}_{D_{\mathrm{Sing}}}$ and so the corresponding Z is the singular locus D_{Sing}, which consists of matrices of rank $<= n-2$.

Example 7.33 (Theta divisors)**.** Let C be a smooth projective curve and let Θ be the theta divisor on $\operatorname{Jac}(C)$. If C is hyperelliptic, Theorem 9.6 says that $\tilde{\alpha}_\Theta = 3/2$ and Theorem 9.8 implies that the corresponding Z is Θ_{Sing}. If C is a Brill-Noether general curve, then by a result of Budur and Doan [8, Theorem 1.6(iii)], one has $\tilde{\alpha}_\Theta = 2$ and $Z = \Theta_{\mathrm{Sing}}$.

7.4.2. *The support of* $\operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D)$*: the center of minimal exponent.* Let D be an effective divisor on a complex manifold X and write $\tilde{\alpha}_D = k - \alpha$ for unique $k \in \mathbb{N}$ and $\alpha \in (-1, 0]$. In the previous section, we study the subscheme Z such that $\mathcal{O}_Z = \mathcal{G}_{k,\alpha}(D)$ and show it is reduced. Now we construct a subscheme of Z with better singularities. Let $W_\bullet \mathcal{G}_{k,\alpha}(D)$ be the weight filtration in Definition 5.7 and let ℓ be the largest integer such that $\operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) \neq 0$. Then there is a surjection

$$\mathcal{O}_Z = \mathcal{G}_{k,\alpha}(D) \twoheadrightarrow \operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D). \tag{7.24}$$

Definition 7.34. The *center of minimal exponent* of (X, D) is defined to be the subscheme $Y \subseteq X$ satisfying

$$\operatorname{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) = \mathcal{O}_Y,$$

where ℓ, k, α are defined above.

We have $Y \subseteq Z$ and the ideal sheaf of Y fits into the following short exact sequence

$$0 \to \mathcal{I}_{k,<\alpha}(D) \to \mathcal{I}_Y \to W_{\ell-1}\mathcal{G}_{k,\alpha}(D) \to 0. \tag{7.25}$$

Remark 7.35. If $\tilde{\alpha}_D \leq 1$, then in Corollary 7.42 we will show that Y is a minimal log canonical center under additional assumptions. Therefore this notion of "center of minimal exponent" is a generalization of minimal log canonical center.

The following result explains why Y has a better singularity compared to Z.

Theorem 7.36. *Every connected component of* Y *is irreducible, reduced, normal and has at worst rational singularities.*

Proof. The isomorphism (5.8) gives

$$\omega_X \otimes L^k \otimes \mathcal{O}_Y = \omega_X \otimes L^k \otimes \mathrm{gr}_\ell^W \mathcal{G}_{k,\alpha}(D) \cong \mathrm{gr}^F_{-n+k} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M}.$$

By (7.23) and (5.8), we have

$$\mathrm{gr}^F_{-n+k'} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} \cong \mathrm{gr}_\ell^W \mathcal{G}_{k',\alpha}(D) = 0, \quad \forall k' < k.$$

Therefore we conclude that locally $\mathcal{O}_Y \cong F_{-n+k} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M}$ is the first step in the Hodge filtration of a complex Hodge module. Then the desired properties of Y follow from Proposition 7.37 below. □

Proposition 7.37. *Let M be a complex Hodge module on a complex manifold X. Suppose that the underlying filtered $\mathscr{D}$-module $\mathcal{M}$ satisfies $F_{p-1}\mathcal{M} = 0$. If $F_p\mathcal{M} \cong \mathcal{O}_Y$ for a connected closed subscheme Y, then Y is irreducible, reduced, normal and has at worst rational singularities.*

Proof. Being a Hodge module, M admits a decomposition by strict support. But since Y is connected, $\mathcal{O}_Y$ is indecomposable, and so Y must be irreducible and there is a unique summand of M that has strict support equal to Y such that its lowest piece is $\mathcal{O}_Y$. Without loss of generality, we can assume M has strict support equal to Y. As in the proof of Lemma 7.30, we also find that Y must be reduced, because it is the first step in the Hodge filtration of a complex mixed Hodge module. By the structure theorem for complex Hodge modules, M is generically a variation of complex polarized Hodge structure $\mathcal{V}$ on Y. Let $\mu : \tilde{Y} \to Y$ be a resolution of singularities that makes the singular locus of the CVHS $\mathcal{V}$ into a normal crossing divisor D. On $\tilde{Y}$, we then get a complex Hodge module $\tilde{M}$ by uniquely extending $\mathcal{V}$ from $\tilde{Y} \setminus D$ to $\tilde{Y}$ such that $F_p\tilde{\mathcal{M}}$ is a line bundle. The direct image theorem for complex Hodge modules implies that

$$\mathcal{O}_Y = F_p\mathcal{M} \xrightarrow{\sim} \mathbf{R}\mu_* F_p\tilde{\mathcal{M}}.$$

By adjunction we get a morphism $\mathcal{O}_{\tilde{Y}} \to F_p\tilde{\mathcal{M}}$. Taking direct image gives

$$\mathbf{R}\mu_*\mathcal{O}_{\tilde{Y}} \to \mathbf{R}\mu_* F_p\tilde{\mathcal{M}} \xrightarrow{\sim} F_p\mathcal{M} = \mathcal{O}_Y,$$

which is a splitting of the natural morphism $\mathcal{O}_Y \to \mathbf{R}\mu_*\mathcal{O}_{\tilde{Y}}$. By a result of Kovács [34, Theorem 1], this implies that Y is normal with at worst rational singularities. □

Remark 7.38. The essential difference between Z and Y is that Z is the support of the first step of Hodge filtration in a mixed Hodge module; by passing to the highest weight, one obtains a pure Hodge module, where we have much better Hodge theory results (e.g. decomposition Theorem).

Example 7.39. Let X be the space of n by n matrices and let D be the generic determinantal hypersurface. It is proved in [37] that the center of minimal exponent of (X, D) is D_{Sing}. Compared to Example 7.32, we do not obtain a smaller subscheme compared to Z. By [8, Theorem 1.4 and Theorem 1.5], the same statement then holds for the theta divisor of a generic curve C, i.e. the center of minimal exponent of $(\mathrm{Jac}(C), \Theta)$ is Θ_{Sing}. Note that both D_{Sing} and Θ_{Sing} are normal and with at worst rational singularities, due to a result of Kempf [2].

7.5. **Rationality and normality criterions.** We use Proposition 7.37 to recover several well-known results in birational geometry about rationality and normality criterion, due to Ein-Lazarsfeld and Kawamata. Let us first deduce a useful lemma, which will be used several times.

Lemma 7.40. *Let D be an effective divisor on a complex manifold X. Suppose there exists $\ell \in \mathbb{Z}, \alpha \in \mathbb{Q}$ and a subscheme $Z \subseteq X$ such that*

$$\mathrm{gr}_\ell^W \mathcal{G}_{0,\alpha}(D) = \mathcal{O}_Z.$$

Then every connected component of Z is reduced, irreducible, normal and has at worst rational singularities.

Proof. Let $\mathcal{M}$ be the $\mathscr{D}$-module associated to D in Set-up 5.1. By construction

$$F_{-n-1}\,\mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} = 0,$$

therefore by (5.8) $\mathcal{O}_Z$ is locally isomorphic to the first step of the complex Hodge module $\mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M}$. Then one can apply Proposition 7.37 to obtain the desired results. □

We first recover the criterion of rationality and normality via adjoint ideals, see [36, Proposition 9.3.48] or [19, Proposition 3.1].

Corollary 7.41 (Ein-Lazarsfeld)**.** *Let D be a reduced effective divisor on a complex manifold X. If the adjoint ideal* $\mathrm{adj}(D)$ *is trivial, i.e.* $\mathrm{adj}(D) = \mathcal{O}_X$*, then D is normal and has at worst rational singularities.*

Proof. By Corollary 5.24 one has

$$W_{-1}\mathcal{G}_{0,-1}(D) = 0, \quad W_0\mathcal{G}_{0,-1}(D) = \mathrm{adj}(D)/\mathcal{O}_X(-D) = \mathcal{O}_X/\mathcal{O}_X(-D).$$

This implies that

$$\mathrm{gr}_0^W \mathcal{G}_{0,-1}(D) = \mathcal{O}_D.$$

Since D is connected, we get the desired statement by Lemma 7.40. □

The second corollary concerns Kawamata's subadjunction theorem [28] on the singularity property of minimal log canonical centers. Let D be an effective divisor on a complex manifold X. Let $\pi : \tilde{X} \to X$ be a log resolution of (X, D). Write

$$\pi^* D = \sum_{i\in I} e_i E_i, \quad K_{\tilde{X}/X} = \sum_{i \in I} k_i E_i.$$

Let $\alpha = -\mathrm{lct}(D)$ and we have

$$\mathcal{I}_{0,\alpha}(D) = \mathcal{J}((-\alpha - \epsilon)D) = \mathcal{O}_X.$$

We show that one can construct a minimal log canonical center using the weight filtration on the multiplier ideal $\mathcal{I}_{0,\alpha}(D)$ and recover partially Kawamata's theorem.

Corollary 7.42. *Assume every exceptional divisor E_i with $e_i \cdot \mathrm{lct}(D) \in \mathbb{Z}$ must compute the log canonical threshold, i.e. $(k_i + 1)/e_i = \mathrm{lct}(D)$. Let ℓ be the smallest number such that $W_\ell \mathcal{I}_{0,\alpha}(D) = \mathcal{I}_{0,\alpha}(D)$ and set*

$$Y = \mathrm{Zero}(W_{\ell-1}\mathcal{I}_{0,\alpha}(D)).$$

Then Y is a minimal log canonical center of (X, D) and every connected component of Y is irreducible, reduced, normal and has at worst rational singularities.

Proof. Let $I_\alpha = \{i \in I \mid e_i \cdot \alpha \in \mathbb{Z}\}$ and let

$$E^k = \bigcup_{\substack{J \subseteq I_\alpha \\ |J| = k}} \bigcap_{j \in J} E_j$$

be the union of all k-fold intersection of components of $\sum_{i\in I_\alpha} E_i$. By (7.3) in Proposition 7.2, we know that ℓ is also the largest number such that $E^{\ell+1} \neq \varnothing$. Our assumption says that

$$\{i \in I \mid \frac{k_i+1}{e_i} = \mathrm{lct}(D)\} = I_\alpha,$$

thus Corollary 7.4 implies that Y is a minimal log canonical center. On the other hand,

$$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y = \frac{W_\ell \mathcal{I}_{0,\alpha}(D)}{W_{\ell-1}\mathcal{I}_{0,\alpha}(D)} = \mathrm{gr}_\ell^W \mathcal{G}_{0,\alpha}(D),$$

so the singularity properties of Y follow from Lemma 7.40. □

8. Global properties

In this section, we study the global perspective of higher multiplier ideals and prove several Nadel-type vanishing theorems. Throughout this section, we use Set-up 5.1 and notations in §5.

8.1. **Vanishing for higher multiplier ideals.** We deduce some vanishing results concerning the sheaf $\mathcal{G}_{k,\alpha}(D)$. In this section, we let D be an effective divisor on a projective complex manifold X of dimension n and let $k \in \mathbb{N}, \ell \in \mathbb{Z}, \alpha \in \mathbb{Q}$. We use Set-up 5.1: $\mathcal{M}$ is the $\mathscr{D}$-module associated to the total embedding of D and $L = \mathcal{O}_X(D)$. Recall from (5.6) and (5.8) we have

(8.1)
$$\omega_X \otimes L^k \otimes \mathcal{G}_{k,\alpha}(D) = \mathrm{gr}^F_{-n+k}\mathrm{gr}^V_\alpha \mathcal{M}, \quad \omega_X \otimes L^k \otimes \mathrm{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) = \mathrm{gr}^F_{-n+k}\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M}.$$

Theorem 8.1. *With the notation above and assume $\alpha \in [-1,0]$. Let B be an effective divisor such that the $\mathbb{Q}$-divisor $B+\alpha D$ is ample. Then*

$$H^i\left(X, \mathrm{gr}^F_{-n+k}\mathrm{DR}(\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M}) \otimes \mathcal{O}_X(B)\right) = 0, \quad \textit{for every } i > 0.$$

If Ω^1_X is trivial, then

$$H^i(X, \mathrm{gr}^F_{-n+k}\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M} \otimes \mathcal{O}_X(B)) = 0, \quad \textit{for every } i > 0.$$

Proof. Proposition 3.12 implies that the pair

$$(\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M}, F_{\bullet+\lfloor\alpha\rfloor}\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M})$$

is the filtered αL-twisted $\mathscr{D}$-module that underlies an αL-twisted polarizable Hodge module. Therefore, we can apply the vanishing Theorem 4.7 to get the desired statement. □

Corollary 8.2. *With the same assumption of Proposition 8.1. In addition, assume that*

$$\mathcal{G}_{\ell,\alpha}(D) = 0, \quad \textit{for all } 0 \le \ell \le k-1.$$

Then

(8.2) $$H^i(X, \omega_X \otimes \mathrm{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) \otimes L^k \otimes \mathcal{O}_X(B)) = 0, \quad \textit{for all } i>0,\ \ell \in \mathbb{Z}.$$

In particular, if we write

$$\tilde{\alpha}_D = k - \alpha$$

for unique $k \in \mathbb{N}$ and $\alpha \in (-1,0]$, then (8.2) holds.

Proof. By (8.1), the assumption implies that

$$\mathrm{gr}^F_\ell \mathrm{gr}^V_\alpha \mathcal{M} = 0, \quad \text{for all } \ell < k.$$

Then $\mathrm{gr}^F_{-n+k}\mathrm{DR}_X(\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M}) \cong \mathrm{gr}^F_{-n+k}\mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M}$ and the desired vanishing follows from Theorem 8.1.

If $\tilde{\alpha}_D = k-\alpha$, then Lemma 5.26 implies that $\mathcal{G}_{\ell,\alpha}(D) = 0$ for all $0 \le \ell \le k-1$. □

Corollary 8.3. *Let D be an effective divisor on a projective complex manifold X of dimension n. Assume $\alpha \in [-1,0]$. Let B be an effective divisor such that $B+\beta D$ is ample for any $\beta \in (\alpha, \alpha+1]$ and $B+kL$ is big and nef. Assume*

$$\mathcal{I}_{0,\alpha}(D) = \mathcal{I}_{1,\alpha}(D) = \cdots = \mathcal{I}_{k-1,\alpha}(D) = \mathcal{O}_X, \tag{8.3}$$

then

$$H^i(X, \omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D) \otimes \mathcal{O}_X(B)) = 0, \quad \textit{for all } i \geq 2. \tag{8.4}$$

Moreover

$$H^1(X, \omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D) \otimes \mathcal{O}_X(B)) = 0 \tag{8.5}$$

holds if $\mathcal{I}_{k-1,-1}(D) = \mathcal{O}_X$, D is reduced, $B+pD$ is ample for all integers $-1 \leq p \leq k-1$ and rational numbers $p \in (-1, \alpha]$ and

$$H^{k-p}(X, \Omega_X^{n-(k-p)} \otimes L^p \otimes \mathcal{O}_X(B)) = 0, \quad \textit{for all } 0 \leq p \leq k-1.$$

Proof. First we claim that the assumption (8.3) implies that

$$\mathcal{I}_{\ell,-1}(D) = 0, \quad \text{for all } 0 \leq \ell \leq k-2. \tag{8.6}$$

This is because the property of (5.14) and the assumption (8.3) give

$$\mathcal{I}_{\ell,-1}(D) = \mathcal{I}_{\ell+1,0}(D) \supseteq \mathcal{I}_{\ell+1,\alpha}(D) = \mathcal{O}_X, \text{ for any } \ell+1 \leq k-1.$$

For the vanishing of degree $i \geq 2$, we look at the short exact sequence

$$0 \to \mathcal{I}_{k,\alpha}(D) \to \mathcal{I}_{k,\alpha+1}(D) = \mathcal{O}_X \to \mathcal{I}_{k,\alpha+1}(D)/\mathcal{I}_{k,\alpha}(D) \to 0,$$

where the property of (5.14) implies that $\mathcal{I}_{k,\alpha+1}(D) = \mathcal{I}_{k-1,\alpha}(D) = \mathcal{O}_X$. Since $B+kL$ is big and nef, we have

$$H^i(X, \omega_X \otimes L^k \otimes \mathcal{O}_X(B)) = 0, \quad \text{for all } i \geq 1,$$

by Kawamata-Viehweg vanishing. To prove (8.4), it suffices to prove

$$H^i(X, \omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha+1}(D)/\mathcal{I}_{k,\alpha}(D) \otimes \mathcal{O}_X(B)) = 0, \quad \text{for } i \geq 1. \tag{8.7}$$

Note that $\mathcal{I}_{k,\alpha+1}(D)/\mathcal{I}_{k,\alpha}(D)$ is a finite extension of $\mathcal{G}_{k,\beta}(D)$ for $\beta \in (\alpha, \alpha+1]$. There are two cases.

(1) If $\beta \in (\alpha, 0]$, then the assumption (8.3) implies that

$$\mathcal{G}_{\ell,\beta}(D) = 0, \quad \text{for all } 0 \leq \ell \leq k-1.$$

Since $B+\beta D$ is ample, Corollary 8.2 gives

$$H^i(X, \omega_X \otimes \mathcal{G}_{k,\beta}(D) \otimes L^k \otimes \mathcal{O}_X(B)) = 0, \quad \text{for all } i \geq 1.$$

(2) If $\beta \in (0, \alpha+1]$, then

$$\mathcal{G}_{\ell,\beta-1}(D) = 0, \quad \text{whenever } 0 \leq \ell \leq k-2,$$

as a consequence of (8.6) and $\beta - 1 > -1$. Since $(B+D)+(\beta-1)D = B+\beta D$ is ample for all $\beta \in (0, \alpha+1] \subseteq (\alpha, \alpha+1]$, Corollary 8.2 implies that

$$\begin{aligned} &H^i(X, \omega_X \otimes \mathcal{G}_{k,\beta}(D) \otimes L^k \otimes \mathcal{O}_X(B)) \\ \cong &H^i(X, \omega_X \otimes \mathcal{G}_{k-1,\beta-1}(D) \otimes L^{k-1} \otimes \mathcal{O}_X(B+D)) = 0, \text{ for } i \geq 1. \end{aligned}$$

Here we use the property of (5.15) to obtain $\mathcal{G}_{k,\beta}(D) = \mathcal{G}_{k-1,\beta-1}(D) \otimes \mathcal{O}_X(D)$.

Hence we obtain the vanishing (8.7) and thus the vanishing (8.4).

For the vanishing (8.5), the plan is to use the short exact sequence (5.16) to reduce to the vanishing theorems of Hodge ideals. Consider the following exact sequence

$$0 \to \mathcal{I}_{k,-1}(D) \to \mathcal{I}_{k,\alpha}(D) \to \mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,-1}(D) \to 0.$$

We reduce (8.5) to the vanishings involving the first term and third term. First, let us show that

$$H^1(X, \omega_X \otimes L^k \otimes \mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,-1}(D) \otimes \mathcal{O}_X(B)) = 0. \tag{8.8}$$

By (8.6) and the extra assumption $\mathcal{I}_{k-1,-1}(D) = \mathcal{O}_X$, one has

$$\mathcal{I}_{\ell,-1}(D) = \mathcal{O}_X, \quad \text{whenever } 0 \leq \ell \leq k-1, \tag{8.9}$$

and

$$\mathcal{G}_{\ell,\beta}(D) = 0, \quad \text{for all } 0 \leq \ell \leq k-1 \text{ and } \beta > -1. \tag{8.10}$$

Now note that $\mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,-1}(D)$ is a finite extension of $\mathcal{G}_{k,\beta}(D)$ for $\beta \in (-1, \alpha]$, thus (8.8) holds by Corollary 8.2 and the extra assumption that $B + \beta D$ is ample for $\beta \in (-1, \alpha]$.

To deal with the term $\mathcal{I}_{k,-1}(D)$, consider the short exact sequence from (5.16):

$$0 \to \omega_X \otimes L^{k-1} \otimes \mathcal{I}_{k,0}(D) \to \omega_X \otimes L^k \otimes \mathcal{I}_{k,-1}(D) \to \mathrm{gr}^F_{-n+k}\, \omega_X(*D) \otimes L^{-1} \to 0.$$

The first term above has no first cohomology because $\mathcal{I}_{k,0}(D) = \mathcal{I}_{k-1,-1}(D) = \mathcal{O}_X$ by the property of (5.14) and $B + (k-1)D$ is ample, so that one can use Kodaira vanishing. On the other hand, note that $\mathrm{gr}^F_{-n+k}\, \omega_X(*D) \otimes L^{-1}$ fits into another short exact sequence

$$0 \to \omega_X \otimes L^{k-1} \otimes I_{k-1}(D) \to \omega_X \otimes L^k \otimes I_k(D) \to \mathrm{gr}^F_{-n+k}\, \omega_X(*D) \otimes L^{-1} \to 0,$$

where $I_\ell(D)$ is the ℓ-th Hodge ideal of the reduced divisor D. Lemma 5.18 implies that $I_\ell(D) = \mathcal{O}_X$ if and only if $\mathcal{I}_{\ell,-1}(D) = \mathcal{O}_X$. Thus (8.9) gives

$$I_0(D) = I_1(D) = \cdots = I_{k-1}(D) = \mathcal{O}_X.$$

Then [12, Theorem 1.1] (by setting $\mathcal{L} = \mathcal{O}_X(B)$) gives

$$H^1(X, \omega_X \otimes L^k \otimes I_k(D) \otimes \mathcal{O}_X(B)) = 0.$$

Finally, $H^2(X, \omega_X \otimes L^{k-1} \otimes I_{k-1}(D) \otimes \mathcal{O}_X(B)) = 0$ because $I_{k-1}(D) = \mathcal{O}_X$ so we can use Kodaira vanishing. We conclude that $\mathrm{gr}^F_{-n+k}\, \omega_X(*D) \otimes L^{-1} \otimes \mathcal{O}_X(B)$ has no first cohomology and hence (8.5) follows. ☐

Remark 8.4. The statement of Corollary 8.3 is similar to the vanishing theorem of Hodge ideals for $\mathbb{Q}$-divisors, which is first proved in [46, Theorem 12.1] with a global assumption and later removed by Bingyi Chen in [12, Theorem 1.1].

Corollary 8.5. *Let D be an effective divisor on a projective complex manifold X. Let $\alpha \in [-1, 0], k \in \mathbb{N}$ and let B be an effective divisor such that $B + \beta D$ is ample for any $\beta \in (\alpha, 1)$ and $B + kD$ is big and nef. Assume that*

$$\mathcal{I}_{0,\alpha}(D) = \mathcal{I}_{1,\alpha}(D) = \cdots = \mathcal{I}_{k-1,\alpha}(D) = \mathcal{O}_X.$$

Then for any $\gamma \in (\alpha, 0]$ and any $\ell \in \mathbb{Z}$, we have

$$H^i(X, \omega_X \otimes L^k \otimes W_\ell \mathcal{I}_{k,\gamma}(D) \otimes \mathcal{O}_X(B)) = 0, \quad \textit{for all } i \geq 2.$$

Proof. Fix $\gamma \in (\alpha, 0]$. Recall the short exact sequence (5.9)

$$0 \to \mathcal{I}_{k,<\gamma}(D) \to W_\ell \mathcal{I}_{k,\gamma}(D) \to W_\ell \mathcal{G}_{k,\gamma}(D) \to 0.$$

The vanishing assumption implies that $\mathcal{G}_{\ell,\gamma}(D) = 0$ for all $0 \leq \ell \leq k-1$, thus

$$H^i(X, \omega_X \otimes L^k \otimes W_\ell \mathcal{I}_{k,\gamma}(D) \otimes \mathcal{O}_X(B)) = 0, \text{ for } i \geq 2$$

by Corollary 8.2. On the other hand, note that $\mathcal{I}_{k,<\gamma}(D) = \mathcal{I}_{k,\gamma-\epsilon}(D)$ for some $0 < \epsilon \ll 1$. As $(\gamma - \epsilon) + 1 < 1$, we have $B + \beta D$ is ample for any $(\gamma - \epsilon, \gamma - \epsilon + 1]$. Hence it follows from Corollary 8.3 that

$$H^i(X, \omega_X \otimes L^k \otimes \mathcal{I}_{k,<\gamma}(D) \otimes \mathcal{O}_X(B)) = 0, \quad \text{for all } i \geq 2.$$

These two statements imply the desired vanishing. □

Remark 8.6. One can give a similar vanishing for H^1 with extra Nakano-type vanishing assumptions as in Corollary 8.3, and we leave the details to interested readers. There are vanishing theorems for weighted Hodge ideals for divisors with isolated singularities, see [51, Theorem E] and [52, Theorem C]. It is not clear to us how to obtain a similar statement for $\gamma = -1$.

8.2. **Vanishing on abelian varieties.** On abelian varieties, we have a much better vanishing theorem.

Theorem 8.7. *Let D be an effective divisor on an abelian variety A such that the line bundle $L = \mathcal{O}_A(D)$ is ample. For any line bundle $\rho \in \mathrm{Pic}^0(A)$ and $i \geq 1$, we have*

(1) $H^i\left(A, L^{k+1} \otimes W_\ell \mathcal{G}_{k,\alpha}(D) \otimes \rho\right) = 0$ for $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$ and $\alpha \in (-1, 0]$.
(2) $H^i(A, L^k \otimes \mathcal{I}_{k,0}(D) \otimes \rho) = 0$ for $k \geq 1$.
(3) $H^i(A, L^{k+1} \otimes \mathcal{I}_{k,\alpha}(D) \otimes \rho) = 0$ for $k \in \mathbb{N}$ and $\alpha \in [-1, 0)$.

Proof. Since $\omega_A = \mathcal{O}_A$, by (5.8), we have

$$L^{k+1} \otimes \mathrm{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) \otimes \rho \cong \mathrm{gr}^F_{-n+k} \mathrm{gr}^W_\ell \mathrm{gr}^V_\alpha \mathcal{M} \otimes (L \otimes \rho).$$

Since $(L \otimes \rho) \otimes \mathcal{O}_A(\alpha D)$ is ample for any $\alpha \in (-1, 0]$ and $\rho \in \mathrm{Pic}^0(A)$, also Ω^1_A is trivial, Theorem 8.1 implies that

$$H^i(A, L^{k+1} \otimes \mathrm{gr}^W_\ell \mathcal{G}_{k,\alpha}(D) \otimes \rho) = 0, \quad \text{for all } i \geq 1.$$

Then the first statement follows because the weight filtration is finite. In particular, we have

$$H^i(A, L^{k+1} \otimes \mathcal{G}_{k,\alpha}(D) \otimes \rho) = 0, \quad \text{whenever } \alpha \in (-1, 0], \, i \geq 0. \tag{8.11}$$

For the vanishing of $\mathcal{I}_{k,\alpha}(D)$, let us treat the cases $\alpha = -1$ and $\alpha = 0$ together, by induction on $k \geq 0$. The base case is $k = 0$, we have $\mathcal{I}_{0,-1}(D) = \mathcal{J}(A, (1-\epsilon)D)$ by (5.5) for some $0 < \epsilon \ll 1$. So the desired vanishing is a consequence of Nadel vanishing theorem (see [36, Theorem 9.4.8]). There is nothing to prove for $\alpha = 0$. Assume the vanishing holds for some $k - 1$ with $k \geq 1$. By (5.14), one has

$$\mathcal{I}_{k,0}(D) \cong \mathcal{I}_{k-1,-1}(D).$$

Consequently, the vanishing for $L^k \otimes \mathcal{I}_{k,0}(D) \otimes \rho$ follows from the induction hypothesis. Now consider the short exact sequence from Proposition 5.10

$$0 \to L^k \otimes \mathcal{I}_{k,0}(D) \to L^{k+1} \otimes \mathcal{I}_{k,-1}(D) \to \mathrm{gr}^F_{-n+k} \omega_A(*D) \to 0.$$

Since $A \setminus D$ is affine and Ω^1_A, it is direct to deduce that $H^i(A, \mathrm{gr}^F_{-n+k} \omega_A(*D) \otimes \rho) = 0$ for any $i \geq 1$ and any $\rho \in \mathrm{Pic}^0(A)$ (see [45, Theorem 28.2]). Therefore we conclude that $L^{k+1} \otimes \mathcal{I}_{k,-1}(D) \otimes \rho$ has no higher cohomology as well and this finish the inductive proof for $\alpha = -1$ and $\alpha = 0$.

It is then deal with the remaining case $\alpha \in (-1, 0)$. Indeed, (8.11) implies that $L^{k+1} \otimes \mathcal{G}_{k,\beta}(D) \otimes \rho$ has no higher cohomology for $\beta \in (-1, \alpha)$; this suffices, because we already know the result in the case $\alpha = -1$ and $\mathcal{I}_{k,\alpha}(D)/\mathcal{I}_{k,-1}(D)$ is a finite extension of $\mathcal{G}_{k,\beta}(D)$ for $\beta \in (-1, \alpha)$.

□

8.3. **Vanishing on projective spaces.** First, we give a refinement of Theorem 4.7 in the case of projective spaces.

Theorem 8.8. *Let $L = \mathcal{O}_{\mathbf{P}^n}(d)$ be a line bundle on $\mathbf{P}^n$. Let $\alpha \in \mathbb{Q}$ and let M be an αL-twisted polarizable Hodge module on X. If m is an integer such that $m + \alpha d > 0$, then for any $k \in \mathbb{Z}$, we have*

$$H^i(\mathbf{P}^n, \operatorname{gr}_k^F \mathcal{M} \otimes \mathcal{O}_{\mathbf{P}^n}(m)) = 0, \quad \textit{for every } i > 0.$$

Proof. Consider the complex $\operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes \mathcal{O}_{\mathbf{P}^n}(m)$, which is denoted by

$$\mathcal{F}^\bullet = \left[\operatorname{gr}_{k-n}^F \mathcal{M} \otimes \bigwedge^n \mathscr{T}_{\mathbf{P}^n} \to \cdots \to \operatorname{gr}_{k-1}^F \mathcal{M} \otimes \mathscr{T}_{\mathbf{P}^n} \to \operatorname{gr}_k^F \mathcal{M}\right][n] \otimes \mathcal{O}_{\mathbf{P}^n}(m),$$

concentrated in degree $-n$ up to 0. Consider the hypercohomology spectral sequence such that

$$E_1^{p,q} = H^q(\mathbf{P}^n, \mathcal{F}^p) \Longrightarrow H^{p+q}(\mathbf{P}^n, \mathcal{F}^\bullet).$$

Fix $i \geq 1$. Since $m + \alpha d > 0$, Theorem 4.7 implies that $H^i(\mathbf{P}^n, \mathcal{F}^\bullet) = 0$, and hence $E_\infty^{0,i} = 0$. The vanishing we want is equivalent to $E_1^{0,i} = 0$. Since $\mathcal{F}^j = 0$ for all $j \geq 1$, it suffices to prove that $E_1^{-r,i+r-1} = 0$ for all $r \geq 1$, which is equivalent to

$$H^{i+r-1}(\mathbf{P}^n, \operatorname{gr}_{k-r}^F \mathcal{M} \otimes \bigwedge^r \mathscr{T}_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(m)) = 0, \quad \text{for all } r, i \geq 1. \tag{8.12}$$

We induct on k. If k is the lowest weight in the Hodge filtration, then $\operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) = \operatorname{gr}_k^F \mathcal{M}$, and therefore the vanishing follows from Theorem 4.7.

For k general, assume the vanishing holds for all $k' < k$. From the analysis above, it suffices to prove (8.12) for all $r, i \geq 1$. There is a Koszul resolution

$$0 \to \bigoplus \mathcal{O}_{\mathbf{P}^n} \to \bigoplus \mathcal{O}_{\mathbf{P}^n}(1) \to \cdots \to \bigoplus \mathcal{O}_{\mathbf{P}^n}(r) \to \bigwedge^r \mathscr{T}_{\mathbf{P}^n} \to 0.$$

By the inductive assumption, for $r, i, j \geq 1$ we have

$$H^{i+r-1+j}(\mathbf{P}^n, \operatorname{gr}_{k-r}^F \mathcal{M} \otimes \mathcal{O}_{\mathbf{P}^n}(j+m)) = 0.$$

Using the Koszul resolution, this implies (8.12). □

Theorem 8.9. *Let D be a reduced hypersurface of degree d in $\mathbf{P}^n$. For $k \in \mathbb{N}$ and $i \geq 1$, we have*

(1) $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes W_\ell \mathcal{G}_{k,\alpha}(D)) = 0$ for $\alpha \in (-1, 0]$, $\ell \in \mathbb{Z}$ and $m > d(k-\alpha)-n-1$.
(2) $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{I}_{k,0}(D)) = 0$ for $k \geq 1$ and $m \geq kd - n - 1$.
(3) $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{I}_{k,\alpha}(D)) = 0$ for $\alpha \in [-1, 0)$ and $m \geq (k+1)d - n - 1$.

Proof. Let $L = \mathcal{O}_{\mathbf{P}^n}(D)$. The first statement follows from Theorem 8.8 and Proposition 1.8 that $\operatorname{gr}_\ell^W \operatorname{gr}_\alpha^V \mathcal{M}$ is an αL-twisted Hodge module on $\mathbf{P}^n$. In particular, we have

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{G}_{k,\alpha}(D)) = 0, \quad \text{for } i > 0,\ \alpha \in (-1, 0] \text{ and } m > d(k-\alpha) - n - 1. \tag{8.13}$$

The vanishing theorem for $\mathcal{I}_{k,\alpha}(D)$ is proved in a similar way as the proof of Theorem 8.7. We first prove the cases $\alpha = -1$ and $\alpha = 0$ together, and then deal with $\alpha \in (-1, 0)$. The base case is $k = 0$: the vanishing for $\alpha = -1$ follows from Nadel's vanishing, since $\mathcal{I}_{0,-1}(D) = \mathcal{J}(\mathbf{P}^n, (1-\epsilon)D)$ and $m - (1-\epsilon)d + n + 1 \geq \epsilon d > 0$. There is nothing to prove for $\alpha = 0$.

Assume the vanishing hold for some $k - 1$ and $k \geq 1$. By (5.14) and the induction hypothesis, one has

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^m}(m) \otimes \mathcal{I}_{k,0}(D)) \cong H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^m}(m) \otimes \mathcal{I}_{k-1,-1}(D)) = 0.$$

for $i \geq 1$ and $m \geq kd - n - 1 = (k - 1 + 1)d - n - 1$. For $\alpha = -1$, set

$$m_0 = m - (d(k+1) - n - 1)$$

and consider the short exact sequence from Proposition 5.10 twisted by $\mathcal{O}_{\mathbf{P}^n}(m_0)$:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(m-d) \otimes \mathcal{I}_{k,0}(D) \to \mathcal{O}_{\mathbf{P}^n}(m) \otimes \mathcal{I}_{k,-1}(D) \to \mathrm{gr}^F_{-n+k}\, \omega_{\mathbf{P}^n}(*D) \otimes \mathcal{O}_{\mathbf{P}^n}(m_0) \to 0.$$

Since $m - d \geq dk - n - 1$, $\mathcal{O}_{\mathbf{P}^n}(m-d) \otimes \mathcal{I}_{k,0}(D)$ has no higher cohomology by what we just proved. On the other hand, because $m_0 \geq 0$, $\mathrm{gr}^F_{-n+k}\, \omega_{\mathbf{P}^n}(*D) \otimes \mathcal{O}_{\mathbf{P}^n}(m_0)$ also has no higher cohomology. This follows from [45, Theorem 25.3] and the short exact sequence from the proof of [45, Theorem 25.3]:

$$0 \to \omega_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(kd) \otimes I_{k-1}(D) \to 0 \to \omega_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}((k+1)d) \otimes I_k(D) \to \mathrm{gr}^F_{-n+k}\, \omega_{\mathbf{P}^n}(*D) \to 0,$$

where $I_k(D)$ is the k-th Hodge ideal of D and note that the last term is isomorphic to $\omega_{\mathbf{P}^n} \otimes \mathrm{gr}^F_k\, \mathcal{O}_{\mathbf{P}^n}(*D)$. This finishes the inductive proof for the case of $\alpha = -1$ and $\alpha = 0$.

Finally, the case $\alpha \in (-1, 0)$ follows from the case $\alpha = -1$ and (8.13). □

Remark 8.10 (Toric varieties)**.** One can obtain a simple vanishing statement of higher multiplier ideals on smooth projective toric varieties as in [45, Corollary 25.1]. We leave the details to interested readers.

8.4. **Application to hypersurfaces in projective spaces.**

Proof of Corollary 1.13. Denote by S the set of isolated singularities of D of multiplicity m. We know from Corollary 7.22 that

$$\mathcal{I}_{k,-1}(D) \subseteq \mathcal{I}_S, \quad \text{where } k = \lceil \frac{n+1-\lceil \frac{n}{m} \rceil}{m} \rceil - 1.$$

Then the result follows from Theorem 1.6, which gives a surjection

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}((k+1)d - n - 1)) \twoheadrightarrow \mathcal{O}_W,$$

where W is the 0-dimensional part of the zero locus of $\mathcal{I}_{k,-1}(D)$. □

Remark 8.11. Our bound $\lceil \frac{n+1-\lceil n/m \rceil}{m} \rceil d - n - 1$ is better than the bound $(\lfloor \frac{n}{m} \rfloor + 1)d - n - 1$ from [45, Corollary H]: write $n = pm + r$ for $p \in \mathbb{N}$ and $0 \leq r \leq m-1$, then we have

$$\lceil \frac{n+1-\lceil n/m \rceil}{m-1} \rceil = \begin{cases} p + \frac{1}{m-1} & \text{if } r = 0, \\ p + \frac{r}{m-1} & \text{if } r > 0. \end{cases}$$

In both cases, it is smaller than $\lfloor \frac{n}{m} \rfloor + 1 = p + 1$. When $m = 2$, our bound coincide with theirs and give $(\lfloor \frac{n}{2} + 1) \rfloor d - n - 1$, thus is still not the best bound $\frac{n}{2}d - n$ for even n from [17, Corollary 2.2].

9. Application: singularities of theta divisors

In this section, we give some applications of higher multiplier ideals to the singularities of theta divisors on principally polarized abelian varieties. Let A be a principally polarized abelian variety (p.p.a.v.) of dimension $g \geq 1$, and let Θ be a *symmetric* theta divisor. We assume that Θ is irreducible, or equivalently, that (A, Θ) is indecomposable. By the work of Kollár [30, Theorem 17.13] and Ein-Lazarsfeld [19], it is known that Θ is normal with at worst rational singularities; moreover, if we define

$$\mathrm{Sing}_m(\Theta) = \{x \in \Theta \mid \mathrm{mult}_x \Theta \geq m\},$$

then

$$\dim \mathrm{Sing}_m(\Theta) \leq g - m - 1, \quad \forall m \geq 2, \tag{9.1}$$

where for a scheme Z, we define $\dim Z$ to be the dimension of the largest component of Z.

We are interested in the following conjecture by Casalaina-Martin [10, Question 4.7].

Conjecture 9.1. If (A, Θ) is an indecomposable p.p.a.v., then

$$\dim \mathrm{Sing}_m(\Theta) \leq g - 2m + 1$$

for every $m \geq 2$.

This conjecture holds for theta divisors on the Jacobians of smooth projective curves (Marten's theorem [2]) and Prym theta divisors associated to etale double covers by the work of Casalaina-Martin [11]. In particular, it holds when $\dim A \leq 5$. The only known cases where equality holds in Conjecture 9.1 are Jacobians of hyperelliptic curves, and intermediate Jacobians of cubic threefolds (where $g = 5$ and Θ has a unique singular point of multiplicity 3). This leads to the following stronger conjecture, which characterizes the boundary cases in Conjecture 9.1 and would give a solution to the Riemann-Schottky problem.

Conjecture 9.2. If (A, Θ) is an indecomposable p.p.a.v. and is not a hyperelliptic Jacobian or the intermediate Jacobian of a smooth cubic threefold, then

$$\dim \mathrm{Sing}_m(\Theta) \leq g - 2m$$

for every $m \geq 2$.

When $m = 2$, Conjecture 9.2 is due to Debarre [15], proposed by Grushevsky [23, Conjecture 5.5], which says that any indecomposable p.p.a.v. (A, Θ) whose theta divisor has codimension 3 singularity in A must be a hyperelliptic Jacobian. Conjecture 9.1 implies the following conjecture by Grushevsky [23, Conjecture 5.12].

Conjecture 9.3 (Grushevsky)**.** Let (A, Θ) be an indecomposable p.p.a.v. and let $x \in \Theta$ be any point, then

$$\mathrm{mult}_x \Theta \leq \frac{g+1}{2}.$$

Assuming Θ has only isolated singularities, Conjecture 9.3 is proved by Mustaţă and Popa [45, Theorem I] using the theory of Hodge ideals. We give an alternative proof of their result using higher multiplier ideals in §9.5. Our main result provides the first instance of Conjecture 9.2 for non-isolated singularities.

Theorem 9.4. *Let (A, Θ) be an indecomposable principally polarized abelian variety. Assume the center of minimal exponent Y of (A, Θ) in the sense of Definition 7.34 is a one dimension scheme, then Conjecture 9.1 holds and Y must be a smooth hyperelliptic curve. Moreover, if there exists $m \geq 2$ such that*

$$\dim \mathrm{Sing}_m(\Theta) = g - 1 - 2m,$$

then one of the following holds

1. *either* $(A, \Theta) = (\mathrm{Jac}(Y), \Theta_{\mathrm{Jac}(Y)})$,
2. *or* $g(Y) = 2m$, $\dim A = 2m - 1$, *the minimal exponent of* Θ *is* $\frac{2m-1}{m}$ *and* Θ *has a unique singular point of multiplicity* m.

In §9.4, we also prove the following general statement, due to Popa [55].

Proposition 9.5. *A modified version of the Conjecture A of Pareschi and Popa* [53] *implies Conjecture 9.2.*

9.1. **Minimal exponents of theta divisors.** Before the proof of Theorem 9.4, let us make some preliminary discussion on minimal exponents of theta divisors. Ein and Lazarsfeld [19, Theorem 1] show that if (A, Θ) is indecomposable, then Θ is rational. By [63, Theorem 0.4], Θ has rational singularities is equivalent to

$$\tilde{\alpha}_\Theta > 1. \tag{9.2}$$

Since the minimal exponent provide new information compared to the log canonical threshold, it will be instructive to compute the minimal exponent of theta divisors appearing in the boundary case of Conjecture 9.2. We would like to thank Mircea Mustaţă for the discussion of this point.

Theorem 9.6. *Let (A, Θ) be a principally polarized abelian variety.*

(1) If $A = \mathrm{Jac}(C)$ where C is an arbitrary smooth projective curve, then $1 < \tilde{\alpha}_\Theta \leq 2$.
- *(Budur-Doan) If C is a Brill-Noether general curve, then $\tilde{\alpha}_\Theta = 2$.*
- *If C is a smooth hyperelliptic curve, then $\tilde{\alpha}_\Theta = \frac{3}{2}$.*

(2) If A is the intermediate Jacobian of a smooth cubic threefold, then $\tilde{\alpha}_\Theta = \frac{5}{3}$.

Remark 9.7. This provides some numerical evidences for Conjecture 9.2: assume there exists a $m \geq 2$ such that $\dim \mathrm{Sing}_m(\Theta) \geq g - 2m + 1$, then Theorem 7.17 implies that

$$\tilde{\alpha}_\Theta \leq \frac{2m-1}{m} < 2,$$

which is indeed satisfied by all the boundary examples. This is the starting point of our proof of Theorem 9.4.

Proof of Theorem 9.6. If C is a smooth projective curve of genus g and $A = \mathrm{Jac}(C)$, by the Riemann Singularity Theorem we have

$$\mathrm{Sing}_m(\Theta) \cong W^{m-1}_{g-1}(C), \quad \forall m \geq 2.$$

Since $m-1 \geq (g-1)-g$, by [2, Chapter IV, Lemma (3.3)] we know that every component of $\mathrm{Sing}_m(\Theta)$ has dimension greater or equal to the Brill-Noether number

$$\rho = g - (m-1+1)(g-(g-1)+m-1) = g - m^2.$$

Therefore Theorem 7.17 implies that

$$\tilde{\alpha}_\Theta \leq \min_{m \geq 2} \frac{m^2}{m} = 2.$$

If A is the intermediate Jacobian of a smooth cubic threefold, Mumford [43, p. 348] proved that Θ has a unique isolated singular point of multiplicity 3 (see also [3]), whose projectivized tangent is smooth is the original cubic threefold. Therefore one can blow up the unique singular point along A to obtain a log resolution of (A, Θ). By Proposition 7.24, one has

$$\tilde{\alpha}_\Theta = \frac{\mathrm{codim}_A(\mathrm{Sing}_3(\Theta))}{3} = \frac{5}{3}.$$

In fact, the same argument works for any isolated ordinary singularities of multiplicity m in a smooth n-dimensional space so that the minimal exponent is n/m.

If $A = \mathrm{Jac}(C)$ for a smooth hyperelliptic curve, then by [2, Chapter IV, Theorem 5.1] one has $\mathrm{Sing}_m(\Theta)$ is an irreducible variety of dimension $g - 2m + 1$ for each $m \geq 2$. By [70] there exists a log resolution of (A, Θ) such that one can do an iterative blow up of (the proper transform of) $\mathrm{Sing}_m(\Theta)$ from $m = \lfloor \frac{g+1}{2} \rfloor$ to $m = 2$. Then by Proposition 7.24, one has

$$\tilde{\alpha}_\Theta = \min_{m \geq 2} \frac{\mathrm{codim}_A(\mathrm{Sing}_m(\Theta))}{m} = \min_{m \geq 2} \frac{2m-1}{m} = \frac{3}{2}.$$

If C is a Brill-Noether general curve, in [70, Problem 9.2] we predicted that there exists a log resolution for (A, Θ) by iteratively blowing up all Brill-Noether subvarieties of Θ starting from the deepest stratum. This is recently confirmed by Budur and Doan [8, Theorem 1.6]. In this case, the Brill-Noether number gives

$$\mathrm{codim}_A \mathrm{Sing}_m(\Theta) = m^2, \quad \forall m \geq 2,$$

then Proposition 7.24 implies that

$$\tilde{\alpha}_\Theta = \min_{m\geq 2} \frac{m^2}{m} = 2.$$

□

For hyperelliptic curves, one can obtain a stronger result.

Theorem 9.8. *Let C be a smooth hyperelliptic curve and $(A, \Theta) = (\mathrm{Jac}(C), \Theta_{\mathrm{Jac}(C)})$, then*

$$\tilde{\alpha}_{\Theta,x} = \frac{3}{2}, \quad \forall x \in \Theta_{\mathrm{Sing}},$$

where $\tilde{\alpha}_{\Theta,x}$ is the minimal exponent of the local defining equation of Θ near x and Θ_{Sing} denotes the singular locus of Θ, which is also $\mathrm{Sing}_2(\Theta)$.

Therefore we have

$$\mathcal{I}_{1,<-1/2}(\Theta) = \mathcal{I}_{\Theta_{\mathrm{Sing}}} \quad \textit{and} \quad \mathcal{G}_{1,-1/2}(\Theta) = \mathcal{O}_{\Theta_{\mathrm{Sing}}}. \tag{9.3}$$

Remark 9.9. In general, the minimal exponent of an effective divisor D with nonempty singular locus on a complex manifold X satisfies

$$\tilde{\alpha}_D := \min_{x\in D} \tilde{\alpha}_{D,x} = \min_{x\in D_{\mathrm{Sing}}} \tilde{\alpha}_{D,x}.$$

This is because if x is a smooth point of D, then $\tilde{\alpha}_{D,x} = +\infty$. Theorem 9.8 means that in the case of hyperelliptic theta divisors, this minimum is actually achieved at every point in the singular locus of Θ. If C is Brill-Noether general, this is also true by [8, Theorem 1.6].

Proof. First, we claim that it suffices to prove

$$\tilde{\alpha}_{\Theta,x} = \frac{3}{2}, \quad \forall x \in \mathrm{Sing}_2(\Theta) - \mathrm{Sing}_3(\Theta). \tag{9.4}$$

Grant (9.4) for now. Let g be the genus of C, using the isomorphism $\mathrm{Sing}_m(\Theta) \cong W^{m-1}_{g-1}(\Theta)$ and $W^r_d(C) \cong W^{r-1}_{d-2}(C)$ because C is hyperelliptic, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sing}_3(\Theta) \arrow[r, "\sim"] \arrow[d, hook] & W^2_{g-1}(C) \arrow[r, "\sim"] \arrow[d, hook] & W^1_{g-3}(C) \arrow[d, hook] \\
\Theta_{\mathrm{Sing}} = \mathrm{Sing}_2(\Theta) \arrow[r, "\sim"] & W^1_{g-1}(C) \arrow[r, "\sim"] & W_{g-3}(C)
\end{tikzcd}$$

By [2, Chapter IV, Corollary 4.5] $\mathrm{Sing}_3(\Theta) \cong W^1_{g-3}(C)$ is exactly the singular locus of $W_{g-3}(C) \cong \mathrm{Sing}_2(\Theta)$. Theorem 9.6 implies that

$$\frac{3}{2} = \tilde{\alpha}_\Theta = \min_{x\in\Theta_{\mathrm{Sing}}} \tilde{\alpha}_{\Theta,x}.$$

Then (9.4) implies that $\tilde{\alpha}_\Theta = \frac{3}{2}$ is achieved by a Zariski open subset of Θ_{Sing}. Therefore the lower semicontinuity of minimal exponent $\tilde{\alpha}_{\Theta,x}$ with respect to x [47, Theorem E(2)] gives

$$\tilde{\alpha}_{\Theta,x} = \frac{3}{2}, \quad \forall x \in \Theta_{\mathrm{Sing}}.$$

To prove (9.3), let Z be the subscheme such that $\mathcal{G}_{1,-1/2}(\Theta) = \mathcal{O}_Z$. Lemma 7.30 implies that Z must be reduced and $Z \subseteq \Theta_{\mathrm{Sing}}$. Since $\tilde{\alpha}_{\Theta,x} = \frac{3}{2}, \forall x \in \Theta_{\mathrm{Sing}}$ and $\Theta_{\mathrm{Sing}} \cong W^1_{g-1}(C)$ is reduced by [2, Chapter IV, Corollary 4.5], Lemma 5.26 implies that

$$\Theta_{\mathrm{Sing}} \subseteq \operatorname{supp} \mathcal{G}_{1,-1/2}(\Theta) = Z.$$

We conclude that $Z = \Theta_{\mathrm{Sing}}$ and hence (9.3) holds.

It remains to prove (9.4). The theta divisor of $\mathrm{Jac}(C)$ has the following chain of multiplicity loci

$$\mathrm{Jac}(C) \supseteq \Theta \supseteq \Theta_{\mathrm{Sing}} = \mathrm{Sing}_2(\Theta) \supseteq \cdots \supseteq \mathrm{Sing}_\ell(\Theta) \supseteq \mathrm{Sing}_{\ell+1}(\Theta) = \varnothing, \tag{9.5}$$

where $\ell = \lfloor \frac{g+1}{2} \rfloor$. Since C is hyperelliptic, we know $\mathrm{Sing}_m(\Theta) = W^{m-1}_{g-1}(C)$ is irreducible,

$$\operatorname{codim}_{\mathrm{Jac}(C)} \mathrm{Sing}_m(\Theta) = 2m-1, \quad m \geq 2,$$

and the singular locus of $\mathrm{Sing}_m(\Theta)$ is exactly $\mathrm{Sing}_{m+1}(\Theta)$. By Theorem 7.17 one has

$$\tilde{\alpha}_{\Theta,x} \leq \frac{3}{2}, \quad \forall x \in \mathrm{Sing}_2(\Theta) - \mathrm{Sing}_3(\Theta). \tag{9.6}$$

Now the plan is to apply Proposition 7.24 where we need to argue that there exists a log resolution of

$$(\mathrm{Jac}(C) - \mathrm{Sing}_3(\Theta), \Theta - \mathrm{Sing}_3(\Theta))$$

which only blows up $\mathrm{Sing}_2(\Theta) - \mathrm{Sing}_3(\Theta)$.

The log resolution of $(\mathrm{Jac}(C), \Theta)$ in [70] is achieved by the following procedure. Note that in (9.5) the deepest strata $\mathrm{Sing}_\ell(\Theta)$ is either the hyperelliptic curve or a point, hence is always smooth.

Step 1: Blow up $\mathrm{Jac}(C)$ along the deepest strata $\mathrm{Sing}_\ell(\Theta)$, then the proper transform of $\mathrm{Sing}_{\ell-1}(\Theta)$ becomes smooth and is transverse to the exceptional divisor denoted by $E_{\ell-1}$.

Step 2: Blow up along the proper transform of $\mathrm{Sing}_{\ell-1}(\Theta)$, then the proper transform of $\mathrm{Sing}_{\ell-2}(\Theta)$ becomes smooth and is transverse to the exceptional divisor $E_{\ell-2}$ and (proper transform of) $E_{\ell-1}$.

Step $\ell-1$: Blow up along the proper transform of $\mathrm{Sing}_2(\Theta)$, then the proper transform of Θ becomes smooth and is transverse to the exceptional divisor $E_1, \ldots, E_{\ell-1}$.

The conclusion is the we have a proper birational map $\pi : \tilde{A} \to A = \mathrm{Jac}(C)$ with

$$\pi^*\Theta = \tilde{\Theta} + 2E_1 + 3E_2 + \ldots + \ell E_{\ell-1}. \tag{9.7}$$

is a divisor with normal crossing support and

$$K_{\tilde{A}/A} = 2E_1 + 4E_2 + \ldots + (2\ell-2)E_{\ell-1}.$$

Note that because the singular locus of $\mathrm{Sing}_m(\Theta)$ is exactly $\mathrm{Sing}_{m+1}(\Theta)$ for any $m \geq 2$, the procedure above actually produces a log resolution

$$\pi_m : \tilde{A}_m \to \mathrm{Jac}(C) - \mathrm{Sing}_{m+1}(\Theta)$$

for the pair

$$\big(\mathrm{Jac}(C) - \mathrm{Sing}_{m+1}(\Theta), \Theta - \mathrm{Sing}_{m+1}(\Theta)\big)$$

by iterated blowing up of $\mathrm{Sing}_j(\Theta) - \mathrm{Sing}_{m+1}(\Theta)$ from $j = m$ to $j = 2$. The data of log resolution is

$$\begin{aligned} \pi_m^*(\Theta - \mathrm{Sing}_{m+1}(\Theta)) &= \tilde{\Theta} + 2E_1 + 3E_2 + \ldots + mE_{m-1}, \\ K_{\tilde{A}_m/\mathrm{Jac}(C) - \mathrm{Sing}_{m+1}(\Theta)} &= 2E_1 + 4E_2 + \ldots + (2m-2)E_{m-1}. \end{aligned}$$

Then we can apply Proposition 7.24 to show that

$$\min_{x\in \mathrm{Sing}_2(\Theta)-\mathrm{Sing}_3(\Theta)} \tilde{\alpha}_{\Theta,x} = \tilde{\alpha}_{\Theta-\mathrm{Sing}_3(\Theta)} = \frac{3}{2}.$$

By (9.6) we conclude that

$$\tilde{\alpha}_{\Theta,x} = \frac{3}{2}, \quad \forall x \in \mathrm{Sing}_2(\Theta) - \mathrm{Sing}_3(\Theta),$$

which is what we want. □

9.2. **Properties of the center of minimal exponent.** Let (A,Θ) be an indecomposable principally polarized abelian variety. In this section, we deduce some special properties of the higher multiplier ideal sheaf for $\mathcal{I}_{1,\alpha}(\Theta)$ for $\alpha \in [-1,0)$. We start by reinterpreting the work of Kollár and Ein-Lazarsfeld discussed in §9.1 in our language. Using the relation $\mathcal{I}_{0,\alpha}(\Theta) = \mathcal{J}((-\alpha-\epsilon)\Theta)$, Kollár's result [30, Theorem 17.13] $\mathcal{J}((1-\epsilon)\Theta) = \mathcal{O}_A$ can be translated to

$$\mathcal{I}_{0,-1}(\Theta) = \mathcal{O}_A, \quad \mathcal{G}_{0,\alpha}(\Theta) = 0, \quad \forall -1 < \alpha \leq 0, \quad \mathcal{G}_{0,-1}(\Theta) = \mathcal{O}_\Theta. \tag{9.8}$$

Lemma 9.10. *We have N acts trivally on $\mathcal{G}_{0,-1}(\Theta)$, and $\mathcal{G}_{1,0}(\Theta) = 0$. Therefore, the weight filtration on $\mathcal{G}_{0,-1}(\Theta)$ is trivial.*

Proof. By Corollary 5.24, we have

$$W_0(\mathcal{G}_{0,-1}(\Theta)) = \mathrm{adj}(\Theta)/\mathcal{O}_A(-\Theta), \quad W_{-1}(\mathcal{G}_{0,-1}(\Theta)) = 0,$$

where $\mathrm{adj}(\Theta)$ is the adjoint ideal. By [19, Theroem 3.3] we know $\mathrm{adj}(\Theta) = \mathcal{O}_A$. Therefore

$$W_0(\mathcal{G}_{0,-1}(\Theta)) = \mathcal{O}_A/\mathcal{O}_A(-\Theta) = \mathcal{G}_{0,-1}(\Theta),$$

which implies that N acts trivially on $\mathcal{G}_{0,1}(\Theta)$.

Now let us prove $\mathcal{G}_{1,0}(\Theta) = 0$. Note that on $\mathcal{G}_{0,-1}(\Theta) \cong \mathrm{gr}_0^F \mathrm{gr}_{-1}^V \mathcal{M}$, where $\mathcal{M}$ is the $\mathscr{D}$-module associated to the graph embedding $A \hookrightarrow \mathbb{L}$, we have locally $N = t\partial_t + 1 = \partial_t t$. This gives the following diagram

$$\begin{tikzcd}
\mathcal{G}_{0,-1}(\Theta) \arrow[r] \arrow[d, equal] & \mathcal{G}_{1,0}(\Theta) \arrow[r] \arrow[d, equal] & \mathcal{G}_{1,-1}(\Theta) \arrow[d, equal] \\
\mathrm{gr}_0^F \mathrm{gr}_{-1}^V \mathcal{M} \arrow[r, "\partial_t"] \arrow[rr, bend right, "N"'] & \mathrm{gr}_1^F \mathrm{gr}_0^V \mathcal{M} \arrow[r, "t"] & \mathrm{gr}_1^F \mathrm{gr}_{-1}^V \mathcal{M}
\end{tikzcd}$$

We represent the bottom arrows locally by ∂_t and t. Since $\mathcal{M}$ has strict support, ∂_t is surjective and t is injective. By the discussion above, we also know that $N = \partial_t t$ acts trivially on $\mathrm{gr}_0^F \mathrm{gr}_{-1}^V \mathcal{M}$. This implies that

$$(\mathrm{gr}_1^F \mathrm{gr}_0^V \mathcal{M}) \cdot t = (\mathrm{gr}_0^F \mathrm{gr}_{-1}^V \mathcal{M}) \cdot \partial_t t = 0.$$

The injectivity of t means that $\mathrm{gr}_0^F \mathrm{gr}_0^V \mathcal{M} = 0$.

Here is another way of deducing $\mathcal{G}_{1,0}(\Theta) = 0$. Lemma 5.26 and (9.2) give

$$\min\{k-\alpha, k\in\mathbb{N}, \alpha\in(-1,0] \mid \mathcal{G}_{k,\alpha}(\Theta) \neq 0\} = \tilde{\alpha}_\Theta > 1.$$

This implies that $\mathcal{G}_{1,0}(\Theta) = 0$.

□

In view of (9.8) and Lemma 9.10, we see that there is no interesting information left in the usual multiplier ideals of Θ and $\mathcal{G}_{1,0}(\Theta)$. The idea is to use the first higher multiplier ideal $\mathcal{I}_{1,\alpha}(\Theta)$ to get more information on the singularities of the theta divisor.

Lemma 9.11. *We have* $\mathcal{I}_{1,<0}(\Theta) = \mathcal{O}_A$ *and* $\mathcal{I}_{1,<-1}(\Theta) = \mathcal{O}_A(-\Theta)$.

Proof. By (5.15) in Proposition 5.10, $\mathcal{G}_{0,\alpha-1}(\Theta) \to \mathcal{G}_{1,\alpha}(\Theta)$ is an isomorphism for $\alpha > 0$. Using (9.8) and Lemma 9.10, we have

$$\mathcal{G}_{1,\alpha}(\Theta) = 0, \quad \forall \alpha \in [0,1).$$

Since $\mathcal{I}_{1,<1}(\Theta) = \mathcal{O}_A$ by Proposition 5.10, this gives $\mathcal{I}_{1,<0}(\Theta) = \mathcal{O}_A$. By (5.13) in Proposition 5.10, one also has

$$\mathcal{I}_{1,<-1}(\Theta) = \mathcal{I}_{1,<0}(\Theta) \otimes \mathcal{O}_A(-\Theta) = \mathcal{O}_A(-\Theta).$$

□

It follows that the chain of coherent sheaves

$$\mathcal{I}_{1,\alpha}(\Theta) \otimes \mathcal{O}_A(2\Theta), \quad \text{for } \alpha \in [-1,0),$$

interpolates between $\mathcal{O}_A(\Theta)$ and $\mathcal{O}_A(2\Theta)$. They also satisfy the following vanishing theorem from Theorem 8.7.

Lemma 9.12. *For* $\alpha \in [-1,0)$, *we have*

$$H^i\left(A, \mathcal{I}_{1,\alpha}(\Theta) \otimes \mathcal{O}_A(2\Theta) \otimes \rho\right) = 0, \quad \forall i > 0, \rho \in \mathrm{Pic}^0(A).$$

Now we analyze the subscheme Z from §7.4 and the center of minimal exponent of (A,Θ) under the extra assumption that $1 < \tilde{\alpha}_\Theta < 2$. The left inequality automatically holds by (9.2) and the right inequality will be satisfied if the assumption of Conjecture 9.1 does not hold.

Lemma 9.13. *Assume* $\tilde{\alpha}_\Theta < 2$ *and write*

$$\tilde{\alpha}_\Theta = 1 - \alpha, \ \textit{for some } \alpha \in (-1,0).$$

Let Z be the closed subscheme defined by the ideal $\mathcal{I}_{1,<\alpha}(\Theta)$. Then the subvariety Z is reduced, connected and generates A as an abelian variety. The line bundle $\mathcal{O}_Z(2\Theta|_Z)$ is base-point-free and not very ample. The map induced by $|2\Theta|_Z|$

$$Z \to \mathbb{P}H^0(Z, 2\Theta|_Z)$$

is two to one and the image of Z is non-degenerate.

Proof. The reducedness of Z follows from Lemma 7.30. For the connectedness, let us study the properties of $2\Theta|_Z$ first. Since $\alpha \in (-1,0)$, we can exploit the vanising theorem from Lemma 9.12, which implies that

$$H^0\left(A, \mathcal{O}_A(2\Theta) \otimes L\right) \twoheadrightarrow H^0\left(Z, (\mathcal{O}_A(2\Theta) \otimes L)|_Z\right) \tag{9.9}$$

is surjective for every $L \in \mathrm{Pic}^0(A)$. It is known that $|2\Theta|$ is base-point-free; moreover, when (A,Θ) is indecomposable, the resulting morphism

$$A \to \mathbb{P}^{2^g-1} = \mathbb{P}H^0(A, 2\Theta)$$

factors through the quotient of A by the involution $x \mapsto -x$ and thus is a $2:1$ map. Then (9.9) implies that the linear system $|2\Theta|_Z|$ is also base-point-free, and the resulting map is two to one and the image of Z is non-degenerate.

Now suppose Z is not connected. Since we have the equality of sets

$$\{\mathcal{O}_A(2\Theta) \otimes L \mid L \in \mathrm{Pic}^0(A)\} = \{t_a^*\mathcal{O}_A(2\Theta) \mid a \in A, \ t_a \text{ denotes the translation by } a\},$$

we could translate Z so that one connected component contains a point x_0 and another point $-x_0$, and (9.9) gives

$$H^0(A, \mathcal{O}_A(2\Theta)) \twoheadrightarrow H^0(Z, \mathcal{O}_A(2\Theta)|_Z).$$

This would then contradict the fact that $|2\Theta|$ does not separate these two points.

Finally let us prove the generation statement. Let B be the subtorus generated by Z. We first claim that $2\Theta|_Z$ is not very ample: the argument above shows that $2\Theta|_Z$ cannot separate x_0 and $-x_0$, thus not very ample; an alternative argument is we assume Θ is symmetric in the beginning of §9 and thus $Z \subseteq \Theta_{\mathrm{Sing}}$ must be symmetric as well. Therefore $2\Theta|_Z$ cannot separate x_0 and $-x_0$ for some $x_0 \in Z$. By the surjectivity in (9.9), we conclude that $2\Theta|_B$ is also not very ample. In [50, Theorem A], Ohbuchi showed that if $2\Theta|_B$ is not very ample for some sub abelian variety $B \subseteq A$, then B is isomorphic to a product of polarized abelian varieties with at least one positive dimensional principally polarized factor. It follows that $B = A$ because (A, Θ) is indecomposable as principally polarized abelian varieties.

For a slightly different argument of $B = A$, one can use [49, Theorem 1] as follows: assume B is a proper subtorus of A. Since (A, Θ) is indecomposable as p.p.a.v., then we must have $h^0(B, \Theta|_B) \geq 2$ and $(B, \Theta|_B)$ cannot have nontrivial principally polarized factor. Then [49, Theorem 1] implies that $2\Theta|_B$ is very ample, which is a contradiction. Therefore $B = A$ and Z generates A. $\square$

We still assume $1 < \tilde{\alpha}_\Theta < 2$. Let Y be the center of minimal exponent for (A, Θ) in the sense of Definition 7.34, i.e. write $\tilde{\alpha}_\Theta = 1 - \alpha$ for some $\alpha \in (-1, 0)$. Let $\ell \geq 0$ be the maximal integer such that $\mathrm{gr}^W_\ell \mathcal{G}_{1,\alpha}(\Theta) \neq 0$, then

$$\mathcal{O}_Y = \mathrm{gr}^W_\ell \mathcal{G}_{1,\alpha}(\Theta).$$

Lemma 9.14. *Assume $1 < \tilde{\alpha}_\Theta < 2$ and let Y be the center of minimal exponent for (A, Θ). Then Y is reduced, connected and generates A as an abelian variety. The line bundle $\mathcal{O}_Y(2\Theta|_Y)$ is base-point-free, but cannot separate x and $-x$ for some $x \in Y$, therefore not very ample. The map induced by $|2\Theta|_Y|$*

$$Y \to \mathbb{P}H^0(Y, 2\Theta|_Y)$$

is two to one and the image of Y is non-degenerate. Moreover, Y is irreducible and normal with at worst rational singularities.

Proof. We use the notation above. By (7.25), there is a short exact sequence

$$0 \to \mathcal{I}_Z = \mathcal{I}_{1,<\alpha}(\Theta) \to \mathcal{I}_Y \to W_{\ell-1}\mathcal{G}_{1,\alpha}(\Theta) \to 0.$$

Since $\alpha \in (-1, 0)$, Theorem 8.7 also gives

$$H^i(A, \mathcal{O}_A(2\Theta) \otimes W_{\ell-1}\mathcal{G}_{1,\alpha}(\Theta) \otimes L) = 0, \quad \forall i > 0, L \in \mathrm{Pic}^0(A).$$

Then combined with Lemma 9.12, we have $H^i(A, \mathcal{I}_Y \otimes \mathcal{O}_A(2\Theta) \otimes L) = 0$ for all $i > 0$ and $L \in \mathrm{Pic}^0(A)$. Therefore we still have

$$H^0\left(A, \mathcal{O}_A(2\Theta) \otimes L\right) \twoheadrightarrow H^0\left(Y, (\mathcal{O}_A(2\Theta) \otimes L)|_Y\right)$$

is surjective for any $L \in \mathrm{Pic}^0(A)$. Moreover Y must be symmetric because Θ is symmetric. These properties are good enough to ensure the proof of Lemma 9.13 works and we obtain the properties of Y except for the irreducibility, normality and rationality, which follows from Y being the center of minimal exponent of (A, Θ), connected and Theorem 7.36. $\square$

Lemma 9.15. *With the same assumption as in Lemma 9.14, then we have*

$$H^i(Y, \mathcal{O}_A(2\Theta)|_Y \otimes L) = 0, \quad \forall i > 0, L \in \mathrm{Pic}^0(A).$$

Moreover, if Y is smooth, then $2\Theta|_Y$ is the smallest piece of the Hodge filtration in a $(1+\alpha)\Theta|_Y$-twisted Hodge module on Y and

$$H^i(Y, \mathcal{O}_A(2\Theta)|_Y \otimes L_Y) = 0, \quad \forall i > 0,$$

and for all line bundle L_Y on Y such that $\mathcal{O}_A((1+\alpha)\Theta)|_Y \otimes L_Y$ is ample on Y.

Proof. By the construction of Y as $\mathcal{O}_Y = \mathrm{gr}_\ell^W \mathcal{G}_{1,\alpha}(\Theta)$, $\omega_A = \mathcal{O}_A$ and (5.8), we have

$$\mathcal{O}_A(\Theta) \otimes \mathcal{O}_Y = \mathrm{gr}^F_{-g+1} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M}.$$

By construction, $\mathrm{gr}_k^F \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} = 0$ for all $k \leq -g-1$. On the other hand, since $\alpha \in (-1, 0)$, using (5.8) and (9.8) one has

$$\mathrm{gr}^F_{-g} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} = \mathcal{O}_A(\Theta) \otimes \mathcal{G}_{0,\alpha}(\Theta) = 0.$$

Therefore $\mathcal{O}_A(\Theta) \otimes \mathcal{O}_Y$ is the smallest piece in the Hodge filtration of the $\alpha\Theta$-twisted Hodge module $\mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M}$ (c.f. Proposition 3.12). Since twisting by $\mathcal{O}_A(\Theta)$ changes a $\beta\Theta$-twisted Hodge module into a $(\beta+1)\Theta$-twisted Hodge module (see Remark 3.3 and Lemma 3.14), we have

$$\mathcal{O}_A(2\Theta) \otimes \mathcal{O}_Y = \mathrm{gr}^F_{-g+1}((\mathrm{gr}_k^W \mathrm{gr}_\alpha^V \mathcal{M}) \otimes \mathcal{O}_A(\Theta))$$

is the smallest piece in the Hodge filtration of a $(1+\alpha)\Theta$-twisted Hodge module. Because $\alpha > -1$, the $\mathbb{Q}$-divisor $(1+\alpha)\Theta + L$ is ample on A for all $L \in \mathrm{Pic}^0(A)$. Since Ω_A^1 is trivial, the vanishing Theorem 4.7 implies that

$$H^i(Y, \mathcal{O}_A(2\Theta)|_Y \otimes L) = H^i(A, \mathcal{O}_A(2\Theta) \otimes \mathcal{O}_Y \otimes L) = 0,$$

for all $i > 0$ and $L \in \mathrm{Pic}^0(A)$.

Assume Y is smooth. We go further by exploiting the fact that $\mathcal{O}_A(2\Theta)|_Y$ is the first step in the Hodge filtration of an $(1+\alpha)\Theta$-twisted Hodge module supported on Y. Since Y is smooth, the twisted form of Kashiwara's equivalence (Theorem 3.10) shows that

$$\mathcal{O}_A(2\Theta)|_Y$$

is the direct image of the first step in the Hodge filtration of an $(1+\alpha)\Theta|_Y$-twisted Hodge module on Y. Therefore Theorem 4.7 implies that

$$H^i(Y, \mathcal{O}_A(2\Theta)|_Y \otimes L_Y) = 0$$

for every line bundle L_Y on Y such that $L_Y \otimes \mathcal{O}_A((1+\alpha)\Theta)|_Y$ is ample on Y. □

9.3. **The case of one dimensional center.** Now we give the proof of Theorem 9.4. By the way of contradiction, suppose that for some $m \geq 2$, we have

$$d := \dim \mathrm{Sing}_m(\Theta) \geq g - 2m + 1. \tag{9.10}$$

The Ein-Lazarsfeld bound (9.1) gives $d \leq g - m - 1$, thus we get $g - d = m + r$ for a unique

$$1 \leq r \leq m - 1. \tag{9.11}$$

According to Theorem 7.17, we can conclude that

$$\mathcal{I}_{1,<\beta}(\Theta) \subsetneq \mathcal{O}_A, \quad \text{for some } \beta \geq -r/m > -1.$$

So one of the higher multiplier ideals $\mathcal{I}_{1,<\alpha}(\Theta)$ in the range $\alpha \in (-1, 0)$ must be nontrivial; moreover, a greater defect in (9.10) means a smaller value of r, and hence a jump in $\mathcal{I}_{1,<\alpha}(\Theta)$ for a larger value of α.

Let α be the maximal value for which $\mathcal{I}_{1,<\alpha}(\Theta) \subsetneq \mathcal{O}_A$. Lemma 9.11 gives $\mathcal{I}_{1,<0}(\Theta) = \mathcal{O}_A$. Then we have

$$-1 < -r/m \leq \alpha < 0. \tag{9.12}$$

Equivalently, let $\tilde{\alpha}_\Theta$ be the minimal exponent of Θ. Corollary 7.21 implies that

$$\tilde{\alpha}_\Theta \leq \frac{\mathrm{codim}_A(\mathrm{Sing}_m(\Theta))}{m} \leq \frac{2m-1}{m} < 2.$$

Thus Lemma 5.26 implies that $\tilde{\alpha}_\Theta = 1 - \alpha$, and together with (9.2) one has

$$1 < \tilde{\alpha}_\Theta \leq \frac{g-d}{m} < 2. \tag{9.13}$$

Recall that Y is the center of minimal exponent of (A, Θ). As $1 < \tilde{\alpha}_\Theta < 2$, one can apply Lemma 9.14 to conclude that Y is irreducible and normal. Since we assume that $\dim Y = 1$, hence Y must be a smooth projective curve. For the rest of the proof, set

$$C := Y, \quad e := \Theta \cdot C = \deg_C(\Theta|_C), \quad g(C) := \text{genus of } C.$$

Proposition 9.16. *The curve C must be hyperelliptic and $e = g(C)$.*

Proof. Since $1 < \tilde{\alpha}_\Theta < 2$, Lemma 9.15 gives $H^1(C, \mathcal{O}_A(2\Theta)|_C) = 0$. Therefore by Riemann-Roch we have

$$\dim H^0(C, \mathcal{O}_A(2\Theta)|_C) = 2e - g(C) + 1.$$

It follows from Lemma 9.14 that the morphism defined by $|2\Theta|_C|$ maps C two-to-one to a non-degenerate curve of degree e in $\mathbb{P}^{2e-g(C)}$. By Castelnuovo's theorem on degrees of non-degenerate curves, this gives us the degree bound

$$e \geq 2e - g(C), \quad \text{or equivalently } g(C) \geq e. \tag{9.14}$$

On the other hand, for any $L_C \in \mathrm{Pic}^0(C)$, we have $(1+\alpha)\Theta|_C \otimes L_C$ is ample on C, because $\alpha > -1$ by (9.12). Hence Theorem 4.7 gives

$$H^1(C, \mathcal{O}_A(2\Theta)|_C \otimes L_C) = 0, \quad \forall L_C \in \mathrm{Pic}^0(C).$$

By Serre duality, this is equivalent to

$$H^0(C, \omega_C \otimes \mathcal{O}_A(-2\Theta)|_C \otimes L_C) = 0, \quad \forall L_C \in \mathrm{Pic}^0(C).$$

By Lemma 9.17 below, we deduce that

$$2e = \deg_C(2\Theta|_C) > \deg_C(\omega_C) = 2g(C) - 2,$$

and hence that $e \geq g(C)$. We therefore have equality in the Castelnuovo bound (9.14), and so the image of C in $\mathbb{P}^{2e-g(C)} = \mathbb{P}^e$ must be a rational normal curve. We conclude that C must be hyperelliptic, as a 2:1 cover of a rational normal curve. □

Lemma 9.17. *Let C be a smooth projective curve, if P is a line bundle on C such that $H^0(C, P \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(C)$, then $\deg P < 0$.*

Proof. Assume $\deg P > g(C)$, one can choose a $L \in \mathrm{Pic}^0(C)$ such that $P \otimes L$ is effective. Then by geometric Riemann-Roch one has $\dim H^0(C, P \otimes L) = \deg P - g(C) + 1 > 1$, which is a contradiction.

Assume $0 \leq d = \deg P \leq g(C)$, then by [2, Chapter IV, Lemma (3.3)], the dimension of the Brill-Noether variety $W^0_d(C)$ is at least $g - (g-d) = d \geq 0$. Hence there exists at least one $L \in \mathrm{Pic}^0(C)$ so that $\dim H^0(C, P \otimes C) \geq 1$, which is also a contradiction. Therefore we finish the proof. □

The next proposition finishes the proof of Theorem 9.4.

Proposition 9.18. *With the notation above, the following hold.*

1. *The inequality $\dim \mathrm{Sing}_m(\Theta) \leq g - 2m + 1$ holds for every $m \geq 2$.*
2. *Moreover, $\dim \mathrm{Sing}_m(\Theta) = g - 2m + 1$ can only happen if either $A = J(C)$ is the Jacobian of the hyperelliptic curve C, or if $g = 2m-1, g(C) = 2m, \alpha = -(m-1)/m$, $\tilde{\alpha}_\Theta = \frac{2m-1}{m}$ and Θ has a unique singular point of multiplicity m; moreover $m \geq 4$.*

Proof. Since $C = Y$ is smooth, Lemma 9.15 implies that $\mathcal{O}_A(2\Theta)|_C$ is the smallest piece in the Hodge filtration of a $(1+\alpha)\Theta|_C$-twisted Hodge module on C. Then the twisted version of the Arakelov inequalities for variation of Hodge structure (see Corollary 9.20) implies that

$$2e = \deg_C(\mathcal{O}_A(2\Theta)|_C) \geq \deg \omega_C + (1+\alpha)\deg_C(\mathcal{O}_A(\Theta)|_C) = 2e - 2 + (1+\alpha)e.$$

Here we use Lemma 9.16 to get $\deg \omega_C = 2g(C) - 2 = 2e - 2$. This implies that

$$(1+\alpha)e \leq 2. \tag{9.15}$$

On the other hand, Lemma 9.14 says that C generates A (as an effective 1-cycle). Then the Matsusaka-Ran theorem [57] gives $e = \Theta \cdot C \geq \dim A = g$, with equality if and only if $A = \mathrm{Jac}(C)$. Recall that $d = \dim \mathrm{Sing}_m(\Theta) \geq 0$ and $r = g - d - m$. Then (9.11) and (9.12) give

$$\alpha \geq -r/m, \quad 1 \leq r \leq m-1.$$

Combining with the inequality (9.15), we see that

$$m + r \leq m + r + d = g \leq e \leq \frac{2}{1+\alpha} \leq \frac{2m}{m-r}. \tag{9.16}$$

This implies that

$$m^2 - 2m \leq r^2.$$

Suppose $r \leq m-2$, then

$$m^2 - 2m \leq m^2 - 4m + 4.$$

Therefore, we have $m \leq 2$ and $r \leq 0$, which contradicts with $1 \leq r \leq m-1$. In particular, we conclude that

$$r = m - 1.$$

Plugging $r = m-1$ back to (9.16), we see that

$$2m + d - 1 \leq 2m.$$

Therefore $d = 0$ or $d = 1$, and $e = g$ or $e = g+1$.

- If $e = g$, then we see that the abelian variety A contains a curve C generating A with
$$C \cdot \Theta = e = g = \dim A.$$
By the Matsusaka-Ran criterion for Jacobians [57], we conclude that
$$A = \mathrm{Jac}(C),$$
i.e. A is the Jacobian of a hyperelliptic curve.
- If $e = g+1$, then $d = 0$, $g = 2m-1$, $g(C) = e = 2m$, $\alpha = -\frac{m-1}{m}$ and $\tilde{\alpha}_\Theta = \frac{2m-1}{m}$. In this case, Θ has a singular point with multiplicity m. Finally, let us rule out the case where $m = 3$. If $m = 3$, then by Casalaina-Martin's result [10, Proposition 3.5], either A is the intermediate Jacobian of a cubic threefold or A is the Jacobian of a hyperelliptic curve. The first case is impossible because then $\Theta_{\mathrm{Sing}} \supseteq C$, which contradicts with the known result that the theta divisor on the intermediate Jacobian of a cubic threefold has only one isolated singularity. The second case is also impossible because on the one hand $\tilde{\alpha}_\Theta = (2\times 3 - 1)/3$, on the other hand $\tilde{\alpha}_\Theta = 3/2$ by Theorem 9.6, which causes a contradiction! We conclude that $m \geq 4$.

$\square$

Lemma 9.19 ([54])**.** *Let C be a smooth projective curve of genus $g(C)$ and let $\mathcal{M}$ be a $\mathscr{D}$-module underlying a polarized complex Hodge module with strict support C. Let p be the smallest integer such that $F_p\mathcal{M} \neq 0$. Then*

$$\deg F_p\mathcal{M} \geq 2g(C) - 2.$$

Proof. Let $\mathcal{V}$ be the polarized CVHS on some open subset C_0. Then

$$\deg F_p\mathcal{M} \geq 2g(C) - 2 + \deg F^{-p}\mathcal{V} \geq 2g(C) - 2.$$

□

Corollary 9.20. *Let C be a smooth projective curve of genus $g(C)$. Let D be an effective divisor on C and set $L = \mathscr{O}_C(D)$. Let M be a αL-twisted Hodge module with strict support C. Let p be the smallest integer such that $F_p\mathcal{M} \neq 0$, where $\mathcal{M}$ is the underlying twisted $\mathscr{D}$-module. Then*

$$\deg F_p\mathcal{M} \geq 2g(C) - 2 + \alpha \cdot \deg D.$$

Proof. If $\alpha \in \mathbb{Z}$, then $\mathcal{M} \otimes L^{-\alpha}$ underlies a polarized complex Hodge module by Lemma 3.14 and then apply Lemma 9.19. □

Remark 9.21. Using Reider's theorem on Fujita conjecture for surfaces [58], one can deduce some partial results when the center of minimal exponent is two dimensional. But we will leave this for future investigation.

9.4. **General case.** In this section, we relate [53, Conjecture A] with Conjecture 9.2. We need to modify this conjecture slightly: let (A, Θ) be an indecomposable p.p.a.v. and let $Y \subseteq A$ be a closed reduced subscheme of A of pure dimension $d \leq g - 2$ which generates A as an abelian variety. If $\mathcal{I}_Y(2\Theta)$ satisfies the IT_0-property, then (A, Θ) must be a Jacobian of curve or a Jacobian of a smooth cubic threefold. The original conjecture assumes Y is geometrically non-degenerate instead of the generation condition, the latter condition is a priori weaker.

Proof of Proposition 9.5. We prove by contradiction. Let (A, Θ) be an indecomposable p.p.a.v. so that it is not a hyperelliptic Jacobian or the Jacobian of a smooth cubic threefold. Assume there exists $m \geq 2$ so that

$$\dim \mathrm{Sing}_m(\Theta) \geq g - 2m + 1.$$

Then by (9.2) and Corollary 7.21, we must have

$$1 < \tilde{\alpha}_\Theta \leq \frac{\mathrm{codim}_A \mathrm{Sing}_m(\Theta)}{m} \leq \frac{2m-1}{m} < 2.$$

Let Y be the center of minimal exponent of (A, Θ). By Lemma 9.14, Y is a closed, reduced and irreducible subscheme of A and Y generates A. In particular, Y is of pure dimension. Lemma 9.15 implies that the sheaf $\mathcal{I}_Y(2\Theta)$ has the IT_0-property in the sense of Pareschi-Popa [53]. Since $\dim Y \leq \dim \Theta_{\mathrm{Sing}} \leq g - 2$, then the modified version of [53, Conjecture A] (see above) implies that (A, Θ) must be a Jacobian of curve or a Jacobian of a smooth cubic threefold. By a result of Martens, for any non-hyperelliptic curve, one must have

$$\dim \mathrm{Sing}_k(\Theta) \leq g - 2k, \quad \forall k \geq 2.$$

Therefore (A, Θ) must be a hyperelliptic Jacobian or a Jacobian of a smooth cubic threefold. This causes a contradiction! We conclude that Conjecture 9.2 holds. □

9.5. **Theta divisors with isolated singularities.** In this section, we give different proofs of some results obtained by Mustaţă-Popa [45], not using Hodge ideals.

Theorem 9.22. [45, Theorem I] *Let (A, Θ) be an indecomposable p.p.a.v. of dimension g such that Θ has isolated singularities. Then*

(1) *For every $x \in \Theta$ we have $\mathrm{mult}_x(\Theta) \leq \frac{g+1}{2}$.*
(2) *Moreover, there is at most one point $x \in \Theta$ with $\mathrm{mult}_x(\Theta) = \frac{g+1}{2}$.*

Proof. Assume $m = \mathrm{mult}_x(\Theta) \geq \frac{g+2}{2}$, then (7.19) implies that

$$\mathcal{I}_{1,-1}(\Theta) \subseteq \mathfrak{m}_x^{2m-g-\lfloor \frac{2m-g}{m} \rfloor} \subseteq \mathfrak{m}_x^2.$$

This is because the Ein-Lazarsfeld bound (9.1) implies that $m \geq g-1$, so $2 \leq 2m-g \leq m-1$. Consider the short exact sequence

$$0 \to \mathcal{O}_A(2\Theta) \otimes \mathcal{I}_{1,-1}(\Theta) \to \mathcal{O}_A(2\Theta) \otimes \mathfrak{m}_x^2 \to \mathcal{O}_A(2\Theta) \otimes \mathfrak{m}_x^2/\mathcal{I}_{1,-1}(\Theta) \to 0.$$

Note that $\mathrm{supp}\, \mathfrak{m}_x^2/\mathcal{I}_{1,-1}(\Theta) \subseteq \{x\}$, so $\mathcal{O}_A(2\Theta) \otimes \mathfrak{m}_x^2/\mathcal{I}_{1,-1}(\Theta) \otimes \rho$ has no higher cohomology for any $\rho \in \mathrm{Pic}^0(A)$. Combining with Lemma 9.12, one has

$$H^i(A, \mathcal{O}_A(2\Theta) \otimes \mathfrak{m}_x^2 \otimes \rho) = 0, \quad \text{for all } i > 0.$$

Then one can argue as in the proof of [45, Theorem I] to obtain a contradiction with the fact that $|2\Theta|$ is $2:1$ and ramified at 2-torsion points. Therefore $\mathrm{mult}_x(\Theta) \leq \frac{g+1}{2}$.

Assume there are two points x, y such that $m = \mathrm{mult}_x(\Theta) = \mathrm{mult}_y(\Theta) = \frac{g+1}{2}$, then (7.19) gives

$$\mathcal{I}_{1,-1}(\Theta) \subseteq \mathfrak{m}_x \otimes \mathfrak{m}_y,$$

because $2m - g - \lfloor (2m-g)/m \rfloor = 1$. Then a similar argument as above will contradict that fact that $|2\Theta|$ does not separate z and $-z$ for $z \neq 0$. □

We can obtain similar bounds of multiplicities in terms of jet separation, as in [45, Theorem 29.5]. Following [45], we denote by $s(\ell, x)$ the largest integer s such that the linear system $|\ell\Theta|$ separates s-jets at x and we denote

$$s_\ell = \min\{s(\ell, x) \mid x \in A\}.$$

Let $\epsilon(\Theta)$ be the Seshadri constant.

Theorem 9.23. *Let (A, Θ) be a p.p.a.v. of dimension g such that Θ has isolated singularities. Then for every $x \in \Theta$ and every $k \geq 1$, we have*

$$\mathrm{mult}_x(\Theta) < \frac{s_{k+1} + g + k + 3 + \sqrt{(s_{k+1} + g + k + 3)^2 - 4g(k+1)}}{2(k+1)}. \tag{9.17}$$

In particular, for every $x \in \Theta$ we have

$$\mathrm{mult}_x(\Theta) \leq \epsilon(\Theta) + 1 \leq \sqrt[g]{g!} + 1.$$

Proof. Assume (9.17) does not hold, then using the elementary fact that if $a > 0$ and $m \geq \frac{-b+\sqrt{b^2-4ac}}{2a}$, then $am^2 + bm + c \geq 0$, we have

$$(k+1)m^2 - (k + g + s_{k+1} + 3)m + g \geq 0,$$

which can be rearranged as

$$m(k+1) - g - (\frac{m(k+1) - g}{m} + 1) \geq s_{k+1} + 2.$$

Since $\lfloor m(k+1) - gm \rfloor \leq (\frac{m(k+1)-g}{m} + 1$, according to (7.19) it follows that

$$\mathcal{I}_{k,-1}(\Theta) \subseteq \mathfrak{m}_x^{2+s_{k+1}}.$$

Using Theorem 8.7, one then argue as in the proof of [45, Theorem 29.5] to derive a contradiction. □

Remark 9.24. Our precise bound (9.17) is better than those of [45, Theorem 29.5], but asymptotically is only 1 better.

Remark 9.25. One can apply similar arguments to recover results in [45, §30] for singular points on ample divisors on abelian varieties, using Theorem 8.7. We leave the details to interested readers.

10. Questions and open problems

We finish this work with some questions and open problems. Let D be an effective divisor on a complex manifold X.

Question 10.1. Let $\pi : \tilde{X} \to X$ be a log resolution of (X, D). Is it true that

$$\pi_*(\omega_{\tilde{X}/X} \otimes \mathcal{I}_{k,\alpha}(\pi^* D)) \subseteq \mathcal{I}_{k,\alpha}(D)?$$

See Remark 7.5 and Example 6.14.

Problem 10.2. Does there exist a simple formula for $\mathcal{I}_{k,\alpha}(D)$ in terms of some log resolution of (X, D)?

Question 10.3. Let D_1, D_2 are two effective divisors. Assume $D_1 \geq D_2$, i.e. $D_1 - D_2$ is effective. For $k = 0$, one has

$$\mathcal{I}_{0,\alpha}(D_1) \subseteq \mathcal{I}_{0,\alpha}(D_2).$$

Is there any similar relation between $\mathcal{I}_{k,\alpha}(D_1)$ and $\mathcal{I}_{k,\alpha}(D_2)$ for $k \geq 1$?

Problem 10.4. The equality (5.22) expresses $\mathcal{I}_{k,\alpha}(mD)$ in terms of $\mathcal{I}_{k,\alpha}(D)$. Find a relation between $\mathcal{I}_{k,\alpha}(D_1 + D_2)$ and $\mathcal{I}_{k,\bullet}(D_1), \mathcal{I}_{k,\bullet}(D_2)$ in the spirit of the subaddivity theorem for usual multiplier ideals.

For any p.p.a.v (A, Θ), the work of Kollár [30] implies that the minimal exponent satisfies $\tilde{\alpha}_\Theta \geq 1$. Furthermore, if (A, Θ) is indecomposable, one has $\tilde{\alpha}_\Theta > 1$ by the work of Ein-Lazarsfeld [19], see (9.2). Inspired by Conjecture 9.2 and Theorem 9.6, we ask

Question 10.5. Let (A, Θ) be an indecomposable principally polarized abelian variety. Does one always have

$$\tilde{\alpha}_\Theta \geq \frac{3}{2}?$$

If it is true, do hyperelliptic Jacobians characterize the equality case?

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