CORRECTION TO "HEIGHT BOUNDS AND THE SIEGEL PROPERTY"

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Abstract. This is a correction to the paper "Height bounds and the Siegel property" (Orr 2018). We correct an error in the proof of Theorem 4.1. Theorem 4.1 as stated in the original paper is correct, but the correction affects additional information about the theorem which is important for applications.

There is an error in the proof of [Orr18, Theorem 4.1]. The statement of [Orr18, Theorem 4.1] is correct, but [Orr18, Lemma 4.4] is incorrect under the conditions on $K_{\mathbf{G}}$ stated above it.

Subsequent applications [BKT20, Theorem 1.1(2)], [DO21, Lemma 2.3] have required greater control of the maximal compact subgroup $K_{\mathbf{G}}$ than is given by the statement of [Orr18, Theorem 4.1]. As a result of the error in the proof, the choice of $K_{\mathbf{G}}$ is more constrained than it appears in [Orr18]. We therefore state a version of [Orr18, Theorem 4.1], extended to correctly describe the constraints on $K_{\mathbf{G}}$.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\mathbf{H} \subset \mathbf{G}$. Let $\mathfrak{S}_{\mathbf{H}}$ be a Siegel set in $\mathbf{H}(\mathbb{R})$ with respect to the Siegel triple $(\mathbf{P}_{\mathbf{H}}, \mathbf{S}_{\mathbf{H}}, K_{\mathbf{H}})$. Let $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})$ be a maximal compact subgroup such that*

(i) $K_{\mathbf{H}} \subset K_{\mathbf{G}}$; and

(ii) the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$ stabilises $\mathbf{S}_{\mathbf{H}}$.

Then there exist subgroups $\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}} \subset \mathbf{G}$ forming a Siegel triple $(\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}}, K_{\mathbf{G}})$, a Siegel set $\mathfrak{S}_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \mathbf{G}(\mathbb{Q})$ such that

$$\mathfrak{S}_{\mathbf{H}} \subset C.\mathfrak{S}_{\mathbf{G}}.$$

Furthermore $R_u(\mathbf{P}_{\mathbf{H}}) \subset R_u(\mathbf{P}_{\mathbf{G}})$ and $\mathbf{S}_{\mathbf{H}} = \mathbf{S}_{\mathbf{G}} \cap \mathbf{H}$.

Remark 2. In the setting of Theorem 1, let Θ be the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$. We now compare (ii) with

(ii′) Θ stabilises $\mathbf{H}$.

If (i) and (ii′) are satisfied, then the restriction $\Theta_{|\mathbf{H}}$ is the Cartan involution of $\mathbf{H}$ associated with $K_{\mathbf{H}}$. Hence by the definition of Siegel triple, (ii) is satisfied. However, if (i) and (ii) are satisfied, then (ii′) does not necessarily hold. This may be seen in the example $\mathbf{G} = \mathrm{SL}_2$, $\mathbf{H} = \left\{ \left(\begin{smallmatrix} a & b \\ db & a \end{smallmatrix}\right) : a^2 - db^2 = 1 \right\}$ where d is a non-square positive rational number, $K_{\mathbf{G}} = \mathrm{SO}_2(\mathbb{R})$, $\mathbf{S}_{\mathbf{H}} = \{1\}$, $K_{\mathbf{H}} = \{1\}$.

In this note, we will explain how to correct the proof of [Orr18, Theorem 4.1] and prove Theorem 1. We will also give examples showing that condition (ii) of Theorem 1 cannot be deleted from the statement of the theorem: first an example in which $\mathbf{H}$ is a torus, then a more sophisticated example in which $\mathbf{H}$ is semisimple. At the end of the note, we correct some unrelated minor errors in [Orr18].

A. **Correction to proof of [Orr18, Theorem 4.1].** On [Orr18, p. 470], item (2) (the choice of $K_{\mathbf{G}}$) should be replaced by

(2) $K_{\mathbf{G}}$, a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ containing $K_{\mathbf{H}}$, such that the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$ stabilises $\mathbf{S_H}$.

Paragraph 1 of the proof of [Orr18, Lemma 4.4] is incorrect: neither the original constraint on $K_{\mathbf{G}}$, nor the corrected constraint, are sufficient to guarantee that Θ restricts to an involution of $\mathbf{H}$ (see Remark 2). With the corrected constraint, that paragraph can be ignored and paragraph 2 of the proof of [Orr18, Lemma 4.4] is valid. Hence [Orr18, Lemma 4.4] is true under the corrected constraint on $K_{\mathbf{G}}$.

The remainder of the proof of [Orr18, Theorem 4.1] is valid without any changes related to the choice of $K_{\mathbf{G}}$ (but see unrelated minor corrections in section E of this note). No further conditions are imposed on $K_{\mathbf{G}}$, so this proves Theorem 1.

In order to establish [Orr18, Theorem 4.1], it is necessary to verify the existence of $K_{\mathbf{G}}$ satisfying (2) above. To show this, choose a faithful representation $\rho\colon \mathbf{G}_{\mathbb{R}} \to \mathrm{GL}(V)$ for some real vector space V. By [Mos55, Theorem 7.3], there exists a positive definite symmetric form ψ on V with respect to which the groups $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{R}) \subset \mathbf{G}(\mathbb{R}) \subset \mathrm{GL}(V)$ are simultaneously self-adjoint. In other words, if Θ denotes the Cartan involution of $\mathrm{GL}(V)$ associated with ψ, then Θ restricts to Cartan involutions of $\mathbf{G}$, $\mathbf{H}$ and $K_{\mathbf{H}}$.

Letting $K_{\mathbf{G}}$ denote the stabiliser of ψ in $\mathbf{G}(\mathbb{R})$, we obtain $K_{\mathbf{H}} \subset K_{\mathbf{G}}$. By Remark 2, Θ stabilises $\mathbf{S_H}$.

B. **Counter-example in which condition (ii) of Theorem 1 is not satisfied: a torus.** Let $\mathbf{G} = \mathrm{SL}_2$ and let $(\mathbf{P}_0, \mathbf{S}_0, K_{\mathbf{G}})$ be the standard Siegel triple for $\mathbf{G}$, that is, $\mathbf{P}_0$ is the subgroup of upper triangular matrices in $\mathbf{G}$, $\mathbf{S}_0$ is the subgroup of diagonal matrices in $\mathbf{G}$ and $K_{\mathbf{G}} = \mathrm{SO}_2(\mathbb{R})$.

Let
$$\mathbf{H} = \left\{ \begin{pmatrix} x & x^{-1} - x \\ 0 & x^{-1} \end{pmatrix} \right\} \subset \mathbf{G}.$$

This is a $\mathbb{Q}$-split torus so it possesses a unique Siegel triple, namely $\mathbf{P_H} = \mathbf{S_H} = \mathbf{H}$, $K_{\mathbf{H}} = \{\pm 1\}$, and a unique Siegel set, $\mathfrak{S}_{\mathbf{H}} = \mathbf{H}(\mathbb{R})$.

Clearly $K_{\mathbf{H}} = \{\pm 1\} \subset K_{\mathbf{G}}$. Thus $K_{\mathbf{G}}$ satisfies condition (i) of Theorem 1. However by [Orr18, Lemma 2.1], $\mathbf{S}_0$ is the only $\mathbb{Q}$-split torus in $\mathbf{P}_0$ stabilised by the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$. Hence this Cartan involution does not stabilise $\mathbf{S_H}$. In other words, $K_{\mathbf{G}}$ does not satisfy condition (ii) of Theorem 1.

Now we shall show that this $\mathfrak{S}_{\mathbf{H}}$ and $K_{\mathbf{G}}$ do not satisfy the conclusion of Theorem 1. Suppose for contradiction that there exist subgroups $\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}} \subset \mathbf{G}$ forming a Siegel triple $(\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}}, K_{\mathbf{G}})$, a Siegel set $\mathfrak{S}_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \mathbf{G}(\mathbb{Q})$ such that $\mathfrak{S}_{\mathbf{H}} \subset C.\mathfrak{S}_{\mathbf{G}}$.

By [BT65, Théorème 4.13], there exists $g \in \mathbf{G}(\mathbb{Q})$ such that $\mathbf{P}_0 = g\mathbf{P}_{\mathbf{G}}g^{-1}$. Writing $g = pk$ where $p \in \mathbf{P}_0(\mathbb{R})$ and $K \in K_{\mathbf{G}}$, $(\mathbf{P}_0, k\mathbf{S}_{\mathbf{G}}k^{-1}, K_{\mathbf{G}})$ is a Siegel triple and $g\mathfrak{S}_{\mathbf{G}}$ is a Siegel set with respect to $(\mathbf{P}_0, k\mathbf{S}_{\mathbf{G}}k^{-1}, K_{\mathbf{G}})$. Hence we can replace $\mathbf{P}_{\mathbf{G}}$ by $\mathbf{P}_0$, $\mathbf{S}_{\mathbf{G}}$ by $k\mathbf{S}_{\mathbf{G}}k^{-1}$, $\mathfrak{S}_{\mathbf{G}}$ by $g\mathfrak{S}_{\mathbf{G}}$ and C by Cg^{-1}. We can thus assume that $\mathbf{P}_{\mathbf{G}} = \mathbf{P}_0$. By the uniqueness of the torus in a Siegel triple, this implies that $\mathbf{S}_{\mathbf{G}} = \mathbf{S}_0$ and $\mathfrak{S}_{\mathbf{G}}$ is a standard Siegel set in $\mathbf{G}(\mathbb{R})$.

The image of $\mathfrak{S}_{\mathbf{H}} = \mathbf{S}_{\mathbf{H}}(\mathbb{R})$ in $\mathbf{G}(\mathbb{R})/K_0$, identified with the upper half-plane, is the ray

$$R = \{(1-y) + yi : y \in \mathbb{R}_{>0}\}.$$

Write $\mathcal{F}_{\mathbf{G}}$ for the image of $\mathfrak{S}_{\mathbf{G}}$ in the upper half-plane.

Since $R \subset C\mathcal{F}_{\mathbf{G}}$ and C is finite, there exists $\gamma \in C \subset \mathbf{G}(\mathbb{Q})$ such that $R \cap \gamma\mathcal{F}_{\mathbf{G}}$ contains points z where both $\operatorname{Im} z, |\operatorname{Re} z| \to \infty$. But this is impossible because:

(i) If $\gamma \notin \mathbf{P}_0(\mathbb{Q})$, then $\gamma\mathcal{F}_{\mathbf{G}}$ lies below a horizontal line.
(ii) If $\gamma \in \mathbf{P}_0(\mathbb{Q})$, then $\gamma\mathcal{F}_{\mathbf{G}}$ lies within a vertical strip of finite width.

C. **Counter-example in which condition (ii) of Theorem 1 is not satisfied: a semisimple subgroup.** Let $\mathbf{G} = \mathrm{SL}_3$ and let $(\mathbf{P}_0, \mathbf{S}_0, K_{\mathbf{G}})$ be the standard Siegel triple for $\mathbf{G}$. Let

$$\mathbf{H}_0 = \mathrm{SO}_3(J) \text{ where } J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let Q_J denote the quadratic form on $\mathbb{R}^3$ represented by J. This form is negative definite on the 1-dimensional subspace $L = \mathbb{R}(1,0,-1)^t \subset \mathbb{R}^3$ and positive definite on the 2-dimensional subspace $M = \mathbb{R}(0,1,0)^t + \mathbb{R}(1,0,1)^t$. Let

$$K_{\mathbf{H}} = \{h \in \mathbf{H}_0(\mathbb{R}) : h(L) = L \text{ and } h(M) = M\}.$$

This is a maximal compact subgroup of $\mathbf{H}_0(\mathbb{R})$ and is isomorphic to $\mathrm{O}_2(\mathbb{R})$ via restriction to its action on M.

Let $c \in \mathbb{Q} \setminus \{0, \pm 1\}$. Let $a \in \mathrm{GL}_3(\mathbb{Q})$ be the linear map which acts as multiplication by c on L and as the identity on M. Explicitly,

$$a = \begin{pmatrix} \frac{1}{2}(1+c) & 0 & \frac{1}{2}(1-c) \\ 0 & 1 & 0 \\ \frac{1}{2}(1-c) & 0 & \frac{1}{2}(1+c) \end{pmatrix}.$$

Let

$$\mathbf{H} = a\mathbf{H}_0a^{-1} = \mathrm{SO}_3(aJa^t).$$

By construction, a centralises $K_{\mathbf{H}}$. It follows that $aK_{\mathbf{H}}a^{-1} = K_{\mathbf{H}} = K_{\mathbf{G}} \cap \mathbf{H}(\mathbb{R})$ and $K_{\mathbf{H}}$ is a maximal compact subgroup of $\mathbf{H}(\mathbb{R})$.

Let Q_0 denote the standard quadratic form on $\mathbb{R}^3$. The spaces L and M are orthogonal with respect to Q_0 and $Q_{0|M} = Q_{J|M}$. Hence $K_{\mathbf{H}} \subset \mathrm{SO}_3(Q_0) = K_{\mathbf{G}}$. Thus condition (i) of Theorem 1 is satisfied.

Let $\mathbf{P}_{\mathbf{H}} = a(\mathbf{P}_0 \cap \mathbf{H}_0)a^{-1}$ and $\mathbf{S}_{\mathbf{H}} = a(\mathbf{S}_0 \cap \mathbf{H}_0)a^{-1}$. As in [Bor69, 11.16], $\mathbf{P}_0 \cap \mathbf{H}_0$ is a minimal $\mathbb{Q}$-parabolic subgroup of $\mathbf{H}_0$ so $(\mathbf{P}_{\mathbf{H}}, \mathbf{S}_{\mathbf{H}}, K_{\mathbf{H}})$ is a Siegel triple in $\mathbf{H}$. Let $\mathfrak{S}_{\mathbf{H}} = \Omega_{\mathbf{H}} A_{\mathbf{H},t} K_{\mathbf{H}}$ be a Siegel set in $\mathbf{H}(\mathbb{R})$ with respect to this Siegel triple.

We shall show that $\mathfrak{S}_{\mathbf{H}}$ and $K_{\mathbf{G}}$ do not satisfy the conclusion of Theorem 1. Suppose for contradiction that there exist subgroups $\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}} \subset \mathbf{G}$ forming a Siegel triple $(\mathbf{P}_{\mathbf{G}}, \mathbf{S}_{\mathbf{G}}, K_{\mathbf{G}})$, a Siegel set $\mathfrak{S}_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \mathbf{G}(\mathbb{Q})$ such that $\mathfrak{S}_{\mathbf{H}} \subset C.\mathfrak{S}_{\mathbf{G}}$. By the same argument as in section B, we may assume that $\mathbf{P}_{\mathbf{G}} = \mathbf{P}_0$ and $\mathbf{S}_{\mathbf{G}} = \mathbf{S}_0$.

Let $\sigma_s = \mathrm{diag}(s, 1, s^{-1})$ for $s \in \mathbb{R}_{>0}$. Now

$$\{a\sigma_s a^{-1} : s \geq t\} = A_{\mathbf{H},t} \subset \mathfrak{S}_{\mathbf{H}} \subset C\mathfrak{S}_{\mathbf{G}}.$$

Since C is finite, there exists some $\gamma \in C$ such that $\gamma\mathfrak{S}_{\mathbf{G}}$ contains elements of the form $a\sigma_s a^{-1}$ for arbitrarily large s. Consequently $a^{-1}\gamma\mathfrak{S}_{\mathbf{G}} a$ contains σ_s for arbitrarily large s. Furthermore the standard Siegel set $\mathfrak{S}_{\mathbf{G}}$ contains $\{\sigma_s : s \geq t'\}$ for some $t' \in \mathbb{R}_{>0}$.

Let χ_1, χ_2 denote the simple roots of $\mathbf{G}$ with respect to $\mathbf{S}_0$, using the ordering induced by $\mathbf{P}_0$. Then $\chi_1(\sigma_s) = \chi_2(\sigma_s) = s$ so the previous paragraph shows that $\mathfrak{S}_{\mathbf{G}} \cap a^{-1}\gamma\mathfrak{S}_{\mathbf{G}} a$ contains elements σ_s with arbitrarily large values for χ_1 and χ_2. Applying Lemma 5 below (with $\Omega_{\mathbf{G}} = K_{\mathbf{G}} \cup K_{\mathbf{G}} a^{-1}$), we deduce that $a^{-1}\gamma$ is contained in the standard parabolic subgroup ${}_{\mathbb{Q}}\mathbf{P}_{0,\emptyset} = \mathbf{P}_0$.

Let $\mathbf{U}_0 = R_u(\mathbf{P}_0)$. Write the Iwasawa decomposition of a^{-1} as

$$a^{-1} = \mu\alpha\kappa \text{ where } \mu \in \mathbf{U}_0(\mathbb{R}), \alpha \in \mathbf{S}_0(\mathbb{R}), \kappa \in K_{\mathbf{G}}.$$

For arbitrarily large real numbers s, we have

$$\sigma_s\mu\sigma_s^{-1}.\sigma_s\alpha.\kappa = \sigma_s a^{-1} \in \mathfrak{S}_{\mathbf{G}} a^{-1} \cap a^{-1}\gamma\mathfrak{S}_{\mathbf{G}} \subset a^{-1}\gamma\mathfrak{S}_{\mathbf{G}}. \tag{1}$$

By the definition of Siegel sets and since $a^{-1}\gamma \in \mathbf{P}_0(\mathbb{R})$, the $\mathbf{U}_0(\mathbb{R})$-component in the Iwasawa decomposition of every element of $a^{-1}\gamma\mathfrak{S}_{\mathbf{G}}$ is bounded. Thus $\sigma_s\mu\sigma_s^{-1}$ lies in a bounded set for arbitrarily large real numbers s. By direct calculation, this implies that $\mu = 1$. (This is the opposite situation to [Bor69, Lemme 12.2], adapted to our conventions about Siegel sets.) Hence $a^{-1} = \alpha\kappa \in \mathbf{S}_0(\mathbb{R})K_{\mathbf{G}}$.

It follows that $a^t a = (\alpha^{-1})^t(\kappa^{-1})^t\kappa^{-1}\alpha^{-1} = \alpha^{-2}$ is diagonal. But $a^t a$ is not diagonal, as can be seen either by direct calculation or by noting that a is symmetrical so $a^t a = a^2$ has L as a 1-dimensional eigenspace yet L is not a coordinate axis.

D. **Siegel sets with non-compact intersection.** In this section, we prove a generalisation of [Bor69, Proposition 12.6], replacing a Siegel set $\mathfrak{S} = \Omega_{\mathbf{P}} A_t K$ by a set of the form $\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}}$ where $\Omega_{\mathbf{G}}$ may be any compact subset of $\mathbf{G}(\mathbb{R})$. This generalisation, namely Lemma 5, is used in section C.

Throughout this section, let $\mathbf{G}$ be a reductive $\mathbb{Q}$-algebraic group. Let $\mathbf{P}$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{S}$ and let $\mathbf{M}$ be the maximal $\mathbb{Q}$-anisotropic subgroup of $Z_{\mathbf{G}}(\mathbf{S})$. Let t be a positive real number and let A_t be the subset of $\mathbf{S}(\mathbb{R})$ defined in [Orr18, section 2B].

Let Δ be the set of simple roots of $\mathbf{G}$ with respect to $\mathbf{S}$, using the ordering induced by $\mathbf{P}$. For $\theta \subset \Delta$, let Ψ_θ denote the set of roots ϕ such that the expression of ϕ as a linear combination of elements of Δ has a positive coefficient for at least one element of θ. For any $\mathbb{Q}$-root ϕ of $\mathbf{G}$ with respect to $\mathbf{S}$, write $\mathbf{U}_{(\phi)}$ for the associated relative root group.

Lemma 3. *Let $\chi \in \Delta$ and let $\phi \in \Psi_\Theta$. Let (α_n) be a sequence of elements of A_t such that $\chi(\alpha_n) \to +\infty$ as $n \to \infty$. Then $\alpha_n^{-1}\nu\alpha_n \to 1$ for all $\nu \in \mathbf{U}_{(\phi)}(\mathbb{R})$.*

Proof. Write ϕ as an integer combination of simple roots: $\phi = \sum_{\psi\in\Delta} m_\psi \psi$. By hypothesis, $m_\chi > 0$. Since all elements of Ψ_Θ are positive roots, $m_\psi \geq 0$ for all $\psi \in \Delta$. Therefore, since $\alpha \in A_t$, we have

$$\phi(\alpha_n) \geq \Bigl(\prod_{\psi\in\Delta\setminus\{\chi\}} t^{m_\psi}\Bigr)\chi(\alpha_n)^{m_\phi} \to +\infty.$$

Let $\lambda \colon \mathbb{G}_m \to \mathbf{S}$ be a cocharacter such that $\langle\phi,\lambda\rangle > 0$. By the dynamical description of root groups [Con20, p. 327, paragraph after proof of Lemma V.2.3],

$$\mathbf{U}_{(\phi)} = \{g \in Z_{\mathbf{G}}(\ker(\phi)) : \lambda(t)g\lambda(t)^{-1} \to 0 \text{ as } t \to 0\}. \tag{2}$$

Let

$$t_n = \phi(\alpha_n)^{-1/\langle\phi,\lambda\rangle} \in \mathbb{R}_{>0}.$$

Since $\chi(\alpha_n) \to +\infty$, $t_n \to 0$. Hence by (2), $\lambda(t_n)\nu\lambda(t_n)^{-1} \to 1$.

We can calculate that $\phi(\alpha_n\lambda(t_n)) = 1$ for all n. Since $\mathbf{U}_{(\phi)}(\mathbb{R})$ is contained in the centraliser of $\ker(\phi)$, it follows that $\alpha_n^{-1}\nu\alpha_n = \lambda(t_n)\nu\lambda(t_n)^{-1}$ for all n. ☐

Lemma 4. *Let $g \in \mathbf{G}(\mathbb{R})$. If there exists a sequence (α_n) of elements of A_t such that $\alpha_n^{-1}g\alpha_n \to 1$, then $g \in \mathbf{U}(\mathbb{R})$.*

Proof. Let $\mathbf{T}$ be a maximal $\mathbb{R}$-split torus in $\mathbf{G}$ and let $\mathbf{Q}$ be a minimal parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ such that $\mathbf{S} \subset \mathbf{T} \subset \mathbf{Q} \subset \mathbf{P}$. Let $\mathbf{N} = N_{\mathbf{G}}(\mathbf{T})$ and let $\mathbf{V} = R_u(\mathbf{Q})$. By Bruhat decomposition [BT65, Théorème 5.15], we can write $g = \nu_1 n \nu_2$ where $\nu_1, \nu_2 \in \mathbf{V}(\mathbb{R})$ and $w \in \mathbf{N}(\mathbb{R})$.

Since $\mathbf{V} \subset \mathbf{U}$ and using [Bor69, Lemme 12.2] (adapted to our definition of Siegel sets), $\{a^{-1}\nu_1 a : a \in A_t\}$ is compact. Therefore, after passing to a subsequence, we may assume that $\alpha_n^{-1}\nu_1\alpha_n$ converges, say to μ_1. Since $\mathbf{S}$ normalises $\mathbf{V}$ and $\mathbf{V}(\mathbb{R})$ is closed, $\mu_1 \in \mathbf{V}(\mathbb{R})$. Similarly, after again passing to a subsequence, $\alpha_n^{-1}\nu_2\alpha_n$ also converges, say to $\mu_2 \in \mathbf{V}(\mathbb{R})$. Then

$$\alpha_n^{-1}w\alpha_n = (\alpha_n^{-1}\nu_1\alpha_n)^{-1}(\alpha_n^{-1}g\alpha_n)(\alpha_n^{-1}\nu_2\alpha_n)^{-1} \to \mu_1^{-1}\mu_2^{-1} \in \mathbf{V}(\mathbb{R}).$$

Since $\mathbf{N}(\mathbb{R})$ is closed, $\mu_1^{-1}\mu_2^{-1} \in \mathbf{N}(\mathbb{R})$. By [BT65, Théorème 5.15, equation (2)], $\mathbf{N} \cap \mathbf{V} = \{1\}$. Hence $\mu_1^{-1}\mu_2^{-1} = 1$.

Since w normalises $\mathbf{T}$, $\alpha_n^{-1} w \alpha_n \in w\mathbf{T}(\mathbb{R})$ for all n. Since $w\mathbf{T}(\mathbb{R})$ is closed and $\alpha_n^{-1} w \alpha_n \to \mu_1^{-1}\mu_2^{-1} = 1$, it follows that $w \in \mathbf{T}(\mathbb{R})$. But then $\alpha_n^{-1} w \alpha_n = w$ for all n, so $w = 1$. Thus $g = \nu_1\nu_2 \in \mathbf{V}(\mathbb{R}) \subset \mathbf{U}(\mathbb{R})$. □

For each character $\chi \in X^*(\mathbf{S})$, there is a unique continuous group homomorphism $\mathbf{P}(\mathbb{R}) \to \mathbb{R}_{>0}$, which we denote f_χ, with the properties $f_\chi(s) = |\chi(s)|$ for all $s \in \mathbf{S}(\mathbb{R})$ and $f_\chi = 1$ on $\mathbf{U}(\mathbb{R})\mathbf{M}(\mathbb{R})$. (This is because $\mathbf{S}(\mathbb{R}) \cap \mathbf{U}(\mathbb{R})\mathbf{M}(\mathbb{R})$ is finite, so $|\chi(s)| = 1$ for all $s \in \mathbf{S}(\mathbb{R}) \cap \mathbf{U}(\mathbb{R})\mathbf{M}(\mathbb{R})$, and $\mathbf{S}$ normalises $\mathbf{UM}$.) Choose a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$. Then $f_\chi(\mathbf{P}(\mathbb{R}) \cap K)$ is a compact subgroup of $\mathbb{R}_{>0}$, so is trivial. Therefore we can extend f_χ to a continuous function $\mathbf{G}(\mathbb{R}) = \mathbf{P}(\mathbb{R})K \to \mathbb{R}_{>0}$ by setting $f_\chi(pk) = f_\chi(p)$ for all $p \in \mathbf{P}(\mathbb{R})$ and $k \in K$. These functions f_χ are not necessarily “of type $(\mathbf{P}, \chi)$” as defined in [Bor69, 14.1] because $\chi \in X^*(\mathbf{S})$ might not extend to a character of $\mathbf{P}$, but the argument in [Bor69, 14.2 (c)] still applies to the functions f_χ.

Lemma 5. *Let $\Omega_{\mathbf{P}}$ and $\Omega_{\mathbf{G}}$ be compact subsets of $\mathbf{P}(\mathbb{R})$ and $\mathbf{G}(\mathbb{R})$ respectively. Let $\gamma \in \mathbf{G}(\mathbb{R})$. If $\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}} \cap \gamma\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}}$ is non-compact, then γ is contained in a proper parabolic $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$ containing $\mathbf{P}$. More precisely, let*

$$\theta = \{\chi \in \Delta : f_\chi \text{ is bounded above on } \Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}} \cap \gamma\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}}\}.$$

Then γ lies in the standard parabolic subgroup ${}_{\mathbb{Q}}\mathbf{P}_\theta$ in the notation of [BT65, 5.12].

Proof. Let

$$\Omega = \Big(\bigcup_{a \in A_t} a^{-1}\Omega_{\mathbf{P}} a\Big)\Omega_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R}).$$

By [Bor69, Lemme 12.2], Ω is compact. From the definitions, $\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}} \subset A_t\Omega$. Hence, for all $\chi \in \Delta \setminus \theta$, f_χ is unbounded on $A_t\Omega \cap \gamma A_t\Omega$.

Let

$$U = \{\nu \in \mathbf{G}(\mathbb{R}) : \xi_n^{-1}\nu\xi_n \to 1 \text{ for some sequence } (\xi_n) \text{ in } A_t\Omega \cap \gamma A_t\Omega\}.$$

Let $\langle U \rangle$ denote the subgroup of $\mathbf{G}(\mathbb{R})$ generated by U. Let ${}_{\mathbb{Q}}\mathbf{U}_\theta$ denote the unipotent radical of ${}_{\mathbb{Q}}\mathbf{P}_\theta$. We shall show that

$$ {}_{\mathbb{Q}}\mathbf{U}_\theta(\mathbb{R}) \subset \langle U \rangle \subset \gamma\mathbf{U}(\mathbb{R})\gamma^{-1}. \tag{3}$$

To prove the first inclusion of (3), let $\phi \in \Psi_{\Delta\setminus\theta}$. Then $\phi \in \Psi_\chi$ for some simple root $\chi \in \Delta \setminus \theta$. By the definition of θ, f_χ is unbounded on $\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}} \cap \gamma\Omega_{\mathbf{P}} A_t \Omega_{\mathbf{G}} \subset A_t\Omega \cap \gamma A_t\Omega$. Choose a sequence (ξ_n) in $A_t\Omega \cap \gamma A_t\Omega$ such that $f_\chi(\xi_n) \to +\infty$. Write $\xi_n = \alpha_n\kappa_n$ where $\alpha_n \in A_t$ and $\kappa_n \in \Omega$.

The argument of [Bor69, 14.2 (c)] shows that $f_\chi(\xi_n)/f_\chi(\alpha_n)$ is bounded both above and below independently of n. Hence

$$|\chi(\alpha_n)| = f_\chi(\alpha_n) \to +\infty.$$

Since $\alpha_n \in A_t$, $\chi(\alpha_n) > 0$ for all n so $\chi(\alpha_n) \to +\infty$. Hence by Lemma 3, for every $\nu \in \mathbf{U}_{(\phi)}(\mathbb{R})$, we have $\alpha_n^{-1}\nu\alpha_n \to 1$. Since Ω is compact, after replacing (ξ_n) by a subsequence, we may assume that κ_n converges, say to $\kappa \in \Omega$. Then $\xi_n^{-1}\nu\xi_n \to \kappa 1 \kappa^{-1} = 1$. Thus $\nu \in U$.

By standard facts about algebraic groups [Bor91, 21.9, 21.11], ${}_{\mathbb{Q}}\mathbf{U}_\theta$ is directly spanned by the $\mathbb{Q}$-root groups $\mathbf{U}_{(\phi)}$ as ϕ runs over the non-divisible elements of $\Psi_{\Delta\backslash\theta}$. Thus ${}_{\mathbb{Q}}\mathbf{U}_\theta(\mathbb{R})$ is generated by $\mathbf{U}_{(\phi)}(\mathbb{R})$ for $\phi \in \Psi_{\Delta\backslash\theta}$. Hence ${}_{\mathbb{Q}}\mathbf{U}_\theta(\mathbb{R}) \subset \langle U \rangle$.

To prove the second inclusion of (3), consider an element $\nu \in U$. Let (ξ_n) be a sequence in $A_t\Omega \cap \gamma A_t \Omega$ such that $\xi_n^{-1}\nu\xi_n \to 1$. Write $\xi_n = \gamma\beta_n\lambda_n$ with $\beta_n \in A_t$, $\lambda_n \in \Omega$. Since Ω is compact, after replacing (ξ_n) by a subsequence, we may assume that λ_n converges, say to $\lambda \in \Omega$. Then

$$\beta_n^{-1}\gamma^{-1}\nu\gamma\beta_n \to \lambda 1 \lambda^{-1} = 1.$$

By Lemma 4, this implies that $\gamma^{-1}\nu\gamma \in \mathbf{U}(\mathbb{R})$. In other words, $\nu \in \gamma\mathbf{U}(\mathbb{R})\gamma^{-1}$ for all $\nu \in U$. Thus $\langle U \rangle \subset \gamma\mathbf{U}(\mathbb{R})\gamma^{-1}$.

We have proved both parts of (3). Thus

$${}_{\mathbb{Q}}\mathbf{U}_\theta \subset \gamma\mathbf{U}\gamma^{-1} \subset \gamma\mathbf{P}\gamma^{-1} \subset \gamma\, {}_{\mathbb{Q}}\mathbf{P}_\theta\, \gamma^{-1}.$$

By [BT65, Corollaire 4.5], it follows that ${}_{\mathbb{Q}}\mathbf{P}_\theta = \gamma\, {}_{\mathbb{Q}}\mathbf{P}_\theta\, \gamma^{-1}$. Since a parabolic subgroup of $\mathbf{G}$ is its own normaliser, we conclude that $\gamma \in {}_{\mathbb{Q}}\mathbf{P}_\theta(\mathbb{R})$. □

E. **Additional minor corrections to [Orr18].**

[Orr18, p. 461, section 2D] (F2) should begin "For every $g \in \mathbf{G}(\mathbb{Q})$"

[Orr18, p. 474, proof of Proposition 4.7] On the fifth line from the end, should say "$\chi_{|\mathbf{S_H}} \in \Phi_\alpha \cup \{0\}$" instead of "$\chi_{|\mathbf{S_H}} \in \Phi_\alpha$"

[Orr18, p. 474, proof of Lemma 4.10] First paragraph should say "Let $\mathbf{T_G}$ be a maximal $\mathbb{R}$-split torus in $\mathbf{G}$ which contains $\mathbf{S_G}$ and is stabilised by the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$." This is necessary to apply [BT65, section 14].

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