SURFACES WITH BIG ANTICANONICAL CLASS

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1. Introduction

One aspect of the Minimal Model Program is the classification of algebraic varieties based on the behavior of the canonical divisor class K_X. Two classes of varieties, at opposite ends of the spectrum, are of particular importance:

(1) Varieties of general type, where K_X is big.
(2) Fano varieties, where $-K_X$ is ample.

Together with many other important results about the Minimal Model Program, C. Birkar, P. Cascini, C. Hacon, and J. McKernan [4] have recently proved the following theorem about the structure of the second class of varieties.

Theorem 1 (Birkar, Cascini, Hacon, and McKernan). *Let (X, Δ) be a pair, consisting of a $\mathbb{Q}$-factorial and normal projective variety X, and an effective $\mathbb{Q}$-divisor Δ. Assume that $K_X + \Delta$ is dlt, and that $-(K_X + \Delta)$ is ample. Then X is a Mori dream space.*

Mori dream spaces were introduced by Y. Hu and S. Keel [9]; they are natural generalizations of toric varieties. We recall the definition. Let X be a $\mathbb{Q}$-factorial and normal projective variety, such that $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = N^1(X)$. Let $D_1, \dots, D_r$ be a collection of divisors that give a basis for $\operatorname{Pic}(X)$, and whose affine hull contains the pseudoeffective cone. The *Cox ring* of X is the multi-graded section ring

$$\operatorname{Cox}(X) = \bigoplus_{a \in \mathbb{N}^r} H^0\big(X, \mathscr{O}_X(a_1 D_1 + \cdots + a_r D_r)\big).$$

Then X is called a *Mori dream space* if $\operatorname{Cox}(X)$ is finitely generated as a $\mathbb{C}$-algebra.

The purpose of this paper is to study varieties whose anticanonical class is big. The following easy corollary of Theorem 1 shows that this is an interesting condition.

Proposition 2. *Let X be a projective variety with only klt singularities and such that $-K_X$ is big and nef. Then X is a Mori dream space.*

Proof. Since $-K_X$ is big and nef, there is an effective divisor D such that $-K_X - \varepsilon D$ is ample for all sufficiently small values of $\varepsilon > 0$ (see [11, Example 2.2.19 on p. 145] for details). Since X is klt, the pair $(X, \varepsilon D)$ remains klt when ε is small, and the assertion is therefore a direct consequence of Theorem 1. □

Problem. Let X be a projective variety with klt singularities and such that $-K_X$ is big. Is X a Mori dream space?

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One of the motivations to raise this problem comes from the study of the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. The canonical class $K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)}$ was computed by Pandharipande [13]. Moreover, Coskun, Harris and Starr [6] worked out the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. It is not hard to check that $-K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)}$ is big in this case. When $d = 3$, $-K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)}$ is actually ample, so $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is a Mori dream space. In [5], all the Mori chambers and birational models of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ were described explicitly. Therefore, it would be of interest to know if $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a Mori dream space in general.

Unfortunately the above problem has a negative answer, at least when $\dim X$ is bigger than two. A counterexample was told to the authors by Coskun and Ein. Let X be the blow-up of $\mathbb{P}^n$ at twelve general points on a plane cubic C, $n > 2$. Denote H as the pullback of the hyperplane class and E as the sum of the exceptional divisors. It is not hard to check that $-K_X = (n+1)H - (n-1)E$ is big. Now, consider the line bundle $L = \mathcal{O}_X(4H - E)$. L is big and nef. Moreover, it contains the proper transform of C in its stable base locus. Therefore, the section ring of L is not finitely generated as a consequence of Wilson's Theorem [11, Theorem 2.3.15 on p. 165].

When $n = 2$, $-K_X$ is not big in the above construction. So the question remains for the surface case. Firstly, we need to rule out the case when $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \neq N^1(X)$. For instance, let X be a ruled surface, i.e. a $\mathbb{P}^1$ bundle over a smooth genus g curve C such that $g > 0$ and $C \cdot C = -e$. Also let F denote a fiber class in $N^1(X)$. In this case, $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ is generated by C and F. Moreover, $-K_X = 2C + (e - 2g + 2)F$ is big if $e > 2g - 2$. Take two degree e divisors $\mathbf{b}_1$ and $\mathbf{b}_2$ on C such that $\mathbf{b}_1$ is linear equivalent to $N^*_{C/X}$ but $\mathbf{b}_2 \otimes N_{C/X}$ is non-torsion, where $N_{C/X}$ is the normal bundle of C in X. Consider the line bundle $L_i = \mathscr{O}_X(C + \mathbf{b_i}F)$, $i = 1, 2$. Note that $L_1 =_{\mathrm{num}} L_2$ in $N^1(X)$. Both L_1 and L_2 are big and nef. However, mL_1 is base-point-free for any $m > 0$ but C is contained in the stable base locus of L_2. In particular, the section ring of L_1 is finitely generated while the section ring of L_2 is not.

To avoid this pathology, we further impose the condition $H^1(\mathscr{O}_X) = 0$ since in that case $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = N^1(X)$. Recall Castelnuovo's rationality criterion, which says that a smooth surface X is rational if and only if $H^1(\mathscr{O}_X) = H^0(2K_X) = 0$. So $H^1(\mathscr{O}_X) = 0$ along with the bigness of $-K_X$ force X to be a rational surface. One of the results we will prove in this note is the following.

Theorem 3. *Let X be a smooth projective rational surface with big anticanonical class $-K_X$. Then X is a Mori dream space.*

2. A criterion for the finite generation of the Cox ring

C. Galindo and F. Monserrat [8, Corollary 1 on p. 95] proved the following condition for the Cox ring of a smooth projective surface to be finitely generated.

Theorem 4 (Galindo and Monserrat)**.** *Let X be a smooth projective surface, satisfying the following two conditions:*

(1) The cone of curves $\overline{NE}(X)$ is polyhedral.
(2) Every nef divisor on X is semiample.

Then $\mathrm{Cox}(X)$ *is a finitely generated* $\mathbb{C}$*-algebra.*

3. Results about surfaces with big anticanonical class

Throughout this section, X will be a smooth projective rational surface with big anticanonical class $-K_X$. Evidently, no positive multiple of K_X can have any sections. In particular, we see that

$$H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0.$$

Now recall that there exists a unique Zariski decomposition [11, Theorem 2.3.19 on p. 167] $-K_X = P + N$, with following three properties:

(1) P is a nef $\mathbb{Q}$-divisor.

(2) $N = \sum_{i=1}^r a_i E_i$ is an effective $\mathbb{Q}$-divisor, and the intersection matrix

$$\|(E_i \cdot E_j)\|$$

determined by the components of N is negative definite.

(3) P is orthogonal to N, which implies that $P \cdot E_i = 0$ for all $i = 1, \ldots, r$.

Since $-K_X$ is big, it follows that the positive part P is big and nef [11, Corollary 2.3.22 on p. 169]. Given any big and nef $\mathbb{Q}$-divisor B, we let $\mathrm{Null}(B)$ be the set of irreducible curves $C \subseteq X$ whose classes are orthogonal to B, meaning that $B \cdot [C] = 0$. Obviously, each component of N belongs to $\mathrm{Null}(P)$.

Lemma 5. *Let B be any big and nef $\mathbb{Q}$-divisor on X. Then $\mathrm{Null}(B)$ consists of finitely many smooth rational curves. More generally, any purely one-dimensional subscheme Z supported on $\mathrm{Null}(B)$ satisfies $H^1(Z, \mathscr{O}_Z) = 0$.*

Proof. That $\mathrm{Null}(B)$ has only finitely many irreducible components is proved in [12, Lemma 10.3.6 on p. 249] (see also [3, Lemma 1 on p. 237]). Now let Z be any purely one-dimensional subscheme of X supported on the set $\mathrm{Null}(B)$; we will show that $H^1(Z, \mathscr{O}_Z) = 0$. Let $D = [Z]$ be the class of the subscheme; then $B \cdot D = 0$ by assumption.

Starting from the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-D) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Z \longrightarrow 0$$

for the subscheme Z, we can take cohomology to obtain the four-term exact sequence

$$H^1(X, \mathscr{O}_X) \longrightarrow H^1(Z, \mathscr{O}_Z) \longrightarrow H^2\big(X, \mathscr{O}_X(-D)\big) \longrightarrow H^2(X, \mathscr{O}_X).$$

Noting that the first and last term are zero, because X is a rational surface, and using Serre duality, we find that

$$H^1(Z, \mathscr{O}_Z) \simeq H^2\big(X, \mathscr{O}_X(-D)\big) \simeq \mathrm{Hom}_{\mathbb{C}}\Big(H^0\big(X, \mathscr{O}_X(K_X + D)\big), \mathbb{C}\Big).$$

But the space on the right-hand side is zero, because the line bundle $\mathscr{O}_X(K_X + D)$ cannot have any sections. Indeed, using that B is nef, we compute that

$$B \cdot (K_X + D) = B \cdot K_X = B \cdot (-P - N) \leq -B \cdot P.$$

By the Hodge Inequality, $(B \cdot P)^2 \geq B^2 \cdot P^2 \geq 1$, since both B and P are big and nef. Thus $B \cdot (K_X + D) < 0$, which means that $K_X + D$ cannot be effective. This shows that $H^1(Z, \mathscr{O}_Z) = 0$.

Specializing to the case when Z is a curve, it follows that any irreducible curve $C \in \mathrm{Null}(B)$ has arithmetic genus zero, and is therefore a smooth rational curve. This completes the proof. $\square$

In particular, $\mathrm{Null}(P)$ is a finite union of smooth rational curves.

Lemma 6. *The cone of curves $\overline{NE}(X)$ is polyhedral, and is generated by the classes of finitely many smooth rational curves.*

Proof. Let H be a fixed ample divisor on X, and $\varepsilon > 0$ a small rational number. Recall that the stable base locus $\mathbf{B}(-K_X - \varepsilon H)$ is independent of $\varepsilon > 0$, provided that ε is sufficiently small [12, Lemma 10.3.1 on p. 247]. It is called the *augmented base locus*, and denoted by $\mathbf{B}_+(-K_X)$. We assume from now on that $\varepsilon > 0$ is small enough to guarantee that

$$\mathbf{B}(-K_X - 2\varepsilon H) = \mathbf{B}_+(-K_X).$$

By [7, Example 1.11 on p. 1708], we have $\mathbf{B}_+(-K_X) = \mathrm{Null}(P)$, since X is a surface.

According to the Cone Theorem [10, Theorem 1.24 on p. 22], there is a decomposition

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

of the cone of curves; each C_i is a smooth rational curve, whose class spans an extremal ray for $\overline{NE}(X)$.

Now let C be an irreducible curve such that $(K_X + \varepsilon H) \cdot [C] \geq 0$. Then we have $-(K_X + 2\varepsilon H) \cdot [C] < 0$, and so

$$C \in \mathbf{B}(-K_X - 2\varepsilon H) = \mathbf{B}_+(-K_X) = \mathrm{Null}(P).$$

Thus $\overline{NE}(X)$ is generated by the finitely many extremal rays $[C_i]$, together with the classes $[C]$ for $C \in \mathrm{Null}(P)$. But according to Lemma 5, this is a finite set of smooth rational curves, and so the assertion is proved. □

Lemma 7. *Let B be a big and nef divisor on X. Then B is semiample.*

Proof. This can be proved very quickly by applying a result of X. Benveniste [3, Proposition on p. 237]; note that Lemma 5 is exactly the condition needed to apply his result. For the convenience of the reader, we include a slightly different proof.

To begin with, note that since B is big and nef, we have

$$\mathbf{B}(B) \subseteq \mathbf{B}_+(B) = \mathrm{Null}(B),$$

and so the stable base locus is contained in $\mathrm{Null}(B)$. By Lemma 5, any purely one-dimensional subscheme Z supported on $\mathrm{Null}(B)$ satisfies $H^1(Z, \mathscr{O}_Z) = 0$. In particular, every irreducible component of $\mathrm{Null}(B)$ is a smooth rational curve.

By the Hodge Index Theorem, the intersection pairing is negative definite on the subset of $N^1(X)$ spanned by the curves in $\mathrm{Null}(B)$. The proof of [2, Proposition 2 on p. 130] shows that it is possible to find an effective divisor E, with support exactly equal to $\mathrm{Null}(B)$, such that $E \cdot C < 0$ for every irreducible curve $C \in \mathrm{Null}(B)$.

It is then possible to choose sufficiently large integers n and m, such that the divisor

$$A = mB - K_X - nE$$

becomes ample. Indeed, we can first take n sufficiently large so that $A \cdot C > 0$ for every irreducible curve $C \in \mathrm{Null}(B)$. By subsequently making m large, we can guarantee that $A \cdot C > 0$ for every irreducible curve $C \notin \mathrm{Null}(B)$ (since $mB - K_X - nE$ is effective for large m, only finitely many curves need to be considered), and that $A^2 > 0$. By Kleiman's Criterion, A is then ample.

Now let Z be the subscheme corresponding to the effective divisor nE. Note that the line bundle $\mathscr{O}_X(mB)$ has degree zero on each component of Z, since Z is supported on $\mathrm{Null}(B)$. We also have $H^1(Z, \mathscr{O}_Z) = 0$, and so [1, Theorem 1.7 on p. 489] implies that the restriction of $\mathscr{O}_X(mB)$ to Z is the trivial line bundle. We thus have an exact sequence

$$0 \longrightarrow \mathscr{O}_X(mB - nE) \longrightarrow \mathscr{O}_X(mB) \longrightarrow \mathscr{O}_Z \longrightarrow 0.$$

From the long exact sequence in cohomology, we then get exactness of

$$H^0\big(X, \mathscr{O}_X(mB)\big) \longrightarrow H^0\big(Z, \mathscr{O}_Z\big) \longrightarrow H^1\big(X, \mathscr{O}_X(mB - nE)\big).$$

But now $H^1\big(X, \mathscr{O}_X(mB-nE)\big) = H^1\big(X, \mathscr{O}_X(K_X+A)\big) = 0$ by Kodaira's Vanishing Theorem. The restriction map

$$H^0\big(X, \mathscr{O}_X(mB)\big) \longrightarrow H^0\big(Z, \mathscr{O}_Z\big)$$

is therefore surjective, and so $\mathscr{O}_X(mB)$ has a section that does not vanish at *any* point of Z. But by construction, the support of Z contains the stable base locus of B; the only possible conclusion is that $\mathbf{B}(B) = \emptyset$, which means exactly that B is semiample. □

Next, we study nef divisors that are not big.

Lemma 8. *Let B be a nef divisor on X with $B^2 = 0$. Then either $B = 0$, or $h^0\big(X, \mathscr{O}_X(B)\big) \geq 2$.*

Proof. Let us assume that $B \neq 0$; we will deduce from this that $K_X \cdot B < 0$. Using the Zariski decomposition for $-K_X$, we have

$$B \cdot K_X = B \cdot (-P - N) \leq -B \cdot P \leq 0;$$

The possibility that $B \cdot P = 0$ is ruled out by the Hodge Index Theorem. Indeed, suppose we had $B \cdot P = 0$. The intersection pairing on $N^1(X)$ has exactly one positive eigenvalue; because $P^2 > 0$, while $B^2 = 0$, we conclude that B would have to be numerically trivial. But on the rational surface X, numerical equivalence and linear equivalence coincide, and so $B = 0$. Thus if $B \neq 0$, we conclude that $K_X \cdot B < 0$.

Now we apply the Riemann-Roch Theorem to estimate $h^0\big(X, \mathscr{O}_X(B)\big)$. Using that X is rational, we have

$$\begin{aligned} h^0\big(X, \mathscr{O}_X(B)\big) - h^1\big(X, \mathscr{O}_X(B)\big) + h^2\big(X, \mathscr{O}_X(B)\big) \\ = \frac{B \cdot (B - K_X)}{2} + \chi(\mathscr{O}_X) = -\frac{K_X \cdot B}{2} + 1 \geq 2. \end{aligned}$$

By Serre duality, $h^2\big(X, \mathscr{O}_X(B)\big) = h^0\big(X, \mathscr{O}_X(K_X - B)\big)$, and this quantity is zero, because $B \cdot (K_X - B) = B \cdot K_X < 0$ shows that $K_X - B$ cannot be effective. It follows that $h^0\big(X, \mathscr{O}_X(B)\big) \geq 2$, as claimed. □

Lemma 9. *Let B be a nef divisor on X with $B^2 = 0$. Then B is base point free.*

Proof. If $B = 0$, then B is trivially base point free; for the remainder of the argument, we assume that $B \neq 0$. By Lemma 8, the linear system $|B|$ is non-empty. Let F be the (divisorial) fixed part of $|B|$; then we have a decomposition $B = F + D$, where D is effective and $|D|$ has only finitely many base points. Since X is a surface, it follows that D is nef.

Since F and D are both effective, and B is nef, the identity

$$0 = B^2 = B \cdot F + B \cdot D$$

implies that $B \cdot F = B \cdot D = 0$. Similarly,

$$0 = D \cdot B = D \cdot F + D^2$$

implies that $D \cdot F = D^2 = 0$. Thus we have $B \cdot C = D \cdot C = 0$ for every irreducible component C of the support of F.

This last fact implies that the fixed part F is itself nef; indeed, we have $F \cdot C = B \cdot C - D \cdot C = 0$ whenever C is in the support of F. But now

$$0 = F \cdot B = F^2 + F \cdot D = F^2,$$

and so Lemma 8 implies that either $h^0\big(X, \mathscr{O}_X(F)\big) \geq 2$, or $F = 0$. The first option would contradict the fact that F is the fixed part of $|B|$, and so we conclude that $F = 0$. Thus $B = D$ has only finitely many base points. But then $B^2 = 0$ implies that B is actually free. □

Theorem 10. *Let X be a smooth projective rational surface with big anticanonical class $-K_X$. Then X is a Mori dream space.*

Proof. By Lemma 6, the cone of curves on X is polyhedral. Moreover, any nef divisor B on X is semiample: if $B^2 > 0$, this follows from Lemma 7; and if $B^2 = 0$, from Lemma 9. We can therefore apply the criterion of Galindo-Monserrat in Theorem 4 to conclude that $\mathrm{Cox}(X)$ is finitely generated. □

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