Morihiko Saito, Christian Schnell

A variant of Néron models over curves

Abstract. We study a variant of the Néron models over curves which has recently been found by the second named author in a more general situation using the theory of Hodge modules. We show that its identity component is a certain open subset of an iterated blow-up along smooth centers of the Zucker extension of the family of intermediate Jacobians and that the total space is a complex Lie group over the base curve and is Hausdorff as a topological space. In the unipotent monodromy case, the image of the map to the Clemens extension coincides with the Néron model defined by Green, Griffiths and Kerr. In the case of families of Abelian varieties over curves, it coincides with the Clemens extension, and hence with the classical Néron model in the algebraic case (even in the non-unipotent monodromy case).

0. Introduction

Let $\mathbf{H}$ be a polarizable variation of Hodge structure of weight -1 on a punctured disk Δ^*. Let $(\mathcal{L}_{\Delta^*}, F)$ be its underlying filtered $\mathcal{O}_{\Delta^*}$-module, and $(\mathcal{L}^{\geq 0}, F)$ be its Deligne extension over Δ where the eigenvalues of the residue are contained in $[0, 1)$. Let $\mathcal{V}$ be the vector bundle corresponding to the locally free sheaf $\mathcal{L}^{\geq 0}/F^0\mathcal{L}^{\geq 0}$. Let $\Gamma \subset \mathcal{V}$ denote the subset corresponding to the subsheaf $j_*\mathbf{H_Z}$ where $\mathbf{H_Z}$ denotes here the underlying $\mathbf{Z}$-local system of $\mathbf{H}$ and $j : \Delta^* \hookrightarrow \Delta$ is the inclusion. Then the Zucker extension [15] of the family of intermediate Jacobians is defined by

$$J_\Delta^Z(\mathbf{H}) = \mathcal{V}/\Gamma.$$

It is a complex analytic Lie group over Δ, and is the identity component $J_\Delta^C(\mathbf{H})^0$ of the Clemens extension $J_\Delta^C(\mathbf{H})$, i.e. the latter is constructed by gluing copies of $J_\Delta^Z(\mathbf{H})$ [3,11]. However, $J_\Delta^Z(\mathbf{H})$ and hence $J_\Delta^C(\mathbf{H})$ are not necessarily Hausdorff in general. In the unipotent monodromy case it has been pointed out by M. Green, P. Griffiths and M. Kerr [8] that it is more natural to consider a subset $J_\Delta^{\mathrm{GGK}}(\mathbf{H})^0$ of $J_\Delta^Z(\mathbf{H})$ whose fiber over the origin is the Jacobian of $\mathrm{Ker}\, N \subset H_\infty$ where H_∞

M. Saito: RIMS, Kyoto University, Kyoto 606-8502, Japan.
e-mail: msaito@kurims.kyoto-u.ac.jp

C. Schnell (✉): Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607, USA. e-mail: cschnell@math.uic.edu

is the limit mixed Hodge structure and $N = \log T$ with T the monodromy. Indeed, it is a Hausdorff topological space ([12]) although it is not a complex analytic Lie group over Δ in the usual sense.

Recently the second named author [13] found a variant of the Néron model $J_S^{\mathrm{Sch}}(\mathbf{H})$ in a more general situation where S is a complex manifold which is a partial compactification of S^* on which the variation of Hodge structure $\mathbf{H}$ is defined. In order to define $J_S^{\mathrm{Sch}}(\mathbf{H})$, consider the polarizable Hodge module on S naturally extending $\mathbf{H}$, and let $(\mathcal{M}, F)$ be its underlying filtered left $\mathcal{D}$-module. This corresponds by the de Rham functor to the intersection complex with coefficients in the local system $\mathbf{H_C}$ [10]. Then the identity component $J_S^{\mathrm{Sch}}(\mathbf{H})^0$ is defined by taking a quotient of the analytic space associated to the symmetric algebra of $F_0\mathcal{M}$ [13]. In our case where $S = \Delta$, $\mathcal{M}$ is a $\mathcal{D}_\Delta$-submodule of the union of the Deligne extensions $\mathcal{L}^{>-\infty}$, and $F_0\mathcal{M}$ is a free sheaf on Δ. Then $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ is defined by replacing $\mathcal{L}^{\geq 0}/F^0\mathcal{L}^{\geq 0}$ in the definition of $J_\Delta^Z(\mathbf{H})$ with the dual of $F_0\mathcal{M}$ which is identified with a free subsheaf of $\mathcal{L}^{\geq 0}/F^0\mathcal{L}^{\geq 0}$ using the polarization of $\mathbf{H}$.

Set

$$H_\infty^{\mathrm{van}} = H_\infty/H_\infty^{\mathrm{inv}} \quad \text{with} \quad H_\infty^{\mathrm{inv}} := \mathrm{Ker}(T - id) \subset H_\infty.$$

Here the limit mixed Hodge structure H_∞ is defined by using the base change associated to a cyclic ramified covering of Δ, and H_∞^{inv} is a mixed Hodge structure.

Theorem 1. *Let $a = \max\{p \in \mathbf{N} \mid F^p H_{\infty,\mathbf{C}}^{\mathrm{van}} \neq 0\}$, and $d_k = \dim F^k H_{\infty,\mathbf{C}}^{\mathrm{van}}$. There is a sequence of morphisms of complex Lie groups $\sigma_k : Y_k \to Y_{k-1}$ over Δ for $k = 1, \ldots, a$ such that $Y_0 = J_\Delta^Z(\mathbf{H})$, $Y_a = J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$, and Y_k is a complex manifold defined by $\mathcal{V}_k/\Gamma$ where $\mathcal{V}_k$ are vector bundles over Δ for $k \in [0, a]$. Moreover, σ_k is obtained by taking the blow-up of Y_{k-1} along a smooth center of codimension $d_k + 1$ which is contained in the fiber $Y_{k,0}$ over $0 \in \Delta$, and by restricting it to the complement of the strict transform of $Y_{k-1,0}$.*

Theorem 2. *The image I_k of the composition $\sigma_1 \circ \cdots \circ \sigma_k : Y_k \to Y_0$ is independent of $k = 1, \ldots, a$. Moreover, the unipotent monodromy part of the fiber $(I_k)_0$ of I_k over $0 \in \Delta$ coincides with the image of $H_{\infty,\mathbf{C}}^{\mathrm{inv}}$ in $J_\Delta^Z(\mathbf{H})_0$. In particular, I_k for $k > 0$ coincides with the identity component of the Néron model $J_\Delta^{\mathrm{GGK}}(\mathbf{H})$ of Green, Griffiths and Kerr in the unipotent monodromy case.*

Here the unipotent monodromy part of $(I_k)_0$ means its image by the surjection from $J_\Delta^Z(\mathbf{H})_0$ to $J_\Delta^Z(\mathbf{H})_0^{\mathrm{unip}} := H_{\infty,\mathbf{C},1}/(F^0 H_{\infty,\mathbf{C},1} + H_{\infty,\mathbf{Z}}^{\mathrm{inv}})$ where $H_{\infty,\mathbf{C},1}$ is the unipotent monodromy part of $H_{\infty,\mathbf{C}}$. Using an argument on perfect pairings of locally free sheaves, the proofs are reduced to the calculation of $F_0\mathrm{Gr}_V^\alpha\mathcal{M}$ where V is the filtration of Kashiwara and Malgrange, and this is calculable in terms of the limit mixed Hodge structure by the theory of Hodge modules ([10], 3.2). Using Theorems 1 and 2 together with [11,12], we get moreover

Theorem 3. *With the above notation, the Y_k for $k \in [0, a]$ and hence $Y_a = J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ are complex Lie groups over Δ. Moreover these are Hausdorff topological spaces if $k > 0$.*

For $k = 0$, $Y_0 = J_\Delta^Z(\mathbf{H})$ is not necessarily Hausdorff, although it is Hausdorff on a neighborhood of the image of σ_1 [11,12]. Note that Hausdorff property is not included in the definition of complex manifold in loc. cit. and also in this paper. From Theorem 1 we can deduce

Corollary. *In case of families of Abelian varieties, $J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ coincides with the Clemens extension $J_\Delta^C(\mathbf{H})$, and hence with the classical Néron model in the algebraic case.*

Indeed, the first assertion follows from Theorem 1 since $a = 0$ in this case. The last assertion follows from [11], 4.5. Here we do not assume the monodromy unipotent.

To illustrate Theorem 3, we describe what happens for variations of Hodge structure on Δ^* of "mirror quintic type" up to a Tate twist. This means that the rank of the local system is 4 and the nonzero Hodge numbers of the general fibers are given by $h^{p,q} = 1$ for $p = -2, -1, 0, 1$ with $q = -p - 1$. We assume that the monodromy around the origin is unipotent. In this case, $\mathcal{V}$ is a vector bundle of rank 2, and the central fiber $\mathcal{V}_0$ is $H_{\infty,\mathbf{C}}/F^0 H_{\infty,\mathbf{C}}$.

In the notation of [7], there are three types of degenerations. When $N^2 = 0$ and rk $N = 1$ (type II$_1$), we have $F^0 H_{\infty,\mathbf{C}}^{\mathrm{van}} = 0$, i.e. Ker N surjects onto $\mathcal{V}_0/\Gamma_0$, and so $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0 = J_\Delta^Z(\mathbf{H})$. When $N^2 = 0$ and rk $N = 2$ (type II$_2$), or when $N^3 \neq 0$ (type I), the subspace $F^0 H_{\infty,\mathbf{C}}^{\mathrm{van}}$ is one-dimensional, i.e. the image of Ker N in $J_\Delta^Z(\mathbf{H})_0 = \mathcal{V}_0/\Gamma_0$ has codimension 2 in $J_\Delta^Z(\mathbf{H})$. So $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ is obtained from the Zucker extension $J_\Delta^Z(\mathbf{H})$ by blowing up along the image of Ker N in $\mathcal{V}_0/\Gamma_0$ and deleting the strict transform of the central fiber $\mathcal{V}_0/\Gamma_0$ in this case.

In the non-unipotent monodromy case, the situation is similar. Tensoring the above examples with the local system of rank 1 with monodromy -1, we get examples with non-unipotent monodromy. We get $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ in the same way as above except that the center of the blow-up in the last two cases is not the image of Ker N.

Finally we note a few remarks on $J_S^{\mathrm{Sch}}(\mathbf{H})$ and other Néron models in the higher dimensional case:

For families of Abelian varieties defined outside a divisor with normal crossings, $F_0\mathcal{M}$ is a free subsheaf of a Deligne extension whose quotient is also free, and $J_S^{\mathrm{Sch}}(\mathbf{H})^0$ coincides with the generalized Zucker extension. A generalization of the Néron model in this case has been constructed by A. Young [14] assuming the local monodromies unipotent and using a different method.

In general $F_0\mathcal{M}$ is not necessarily free or reflexive even in the normal crossing case, and $\sigma^* F_0\mathcal{M}$ may have torsion for a blow-up σ. If we replace $F_0\mathcal{M}$ with its reflexive hull, i.e. the double dual $(F_0\mathcal{M})^{\vee\vee}$, then the latter is reflexive and the morphism $(F_0\mathcal{M})^\vee \to (F_0\mathcal{M})^{\vee\vee\vee}$ is an isomorphism [9]. In this way, it may be possible to extend some of the arguments using the pairings in this paper at least to the normal crossing case.

It is shown in [13] that there is a natural surjection (a kind of 'blow-down') from $J_S^{\mathrm{Sch}}(\mathbf{H})$ onto the generalized Néron model $J_S^{\mathrm{BPS}}(\mathbf{H})$ defined in [2]; for families of Abelian varieties over curves, this was noted in [2], Remark 2.7(i).

Here is a brief outline of the paper: In Section 1 we review some facts related to vector bundles and locally free sheaves over curves including Deligne extensions and V-filtrations. In Section 2 we prove the main theorems using these. In Section 3 we give some remarks for the case where the base space S is not a curve.

1. Preliminaries

1.1. Vector bundle case

We first consider the vector bundle case (before dividing out by Γ in the introduction). In general, let $\mathcal{E}$ be a free sheaf of rank r on Δ, and let $\mathcal{V}(\mathcal{E})$ be the corresponding vector bundle over Δ such that

$$\mathcal{E} = \mathcal{O}_\Delta(\mathcal{V}(\mathcal{E})), \quad \mathcal{V}(\mathcal{E}) = \mathcal{S}pecan_\Delta(\mathrm{Sym}^\bullet_\mathcal{O}\mathcal{E}^\vee), \tag{1.1.1}$$

where $\mathcal{O}_\Delta(\mathcal{V}(\mathcal{E}))$ is the sheaf of local sections of $\mathcal{V}(\mathcal{E})$, $\mathcal{E}^\vee := \mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{O}_\Delta)$ is the dual free sheaf of $\mathcal{E}$, and $\mathrm{Sym}^\bullet_\mathcal{O}\mathcal{E}^\vee$ is the symmetric algebra.

Let $\mathcal{E}'$ be a free subsheaf of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{E}'$ is t-torsion, where t is the coordinate of Δ. Then there are nonnegative integers a_i such that

$$\mathcal{E}/\mathcal{E}' = \bigoplus_{i=1}^r \big(\mathbf{C}[t]/t^{a_i}\mathbf{C}[t]\big).$$

More precisely, by the theory of modules over principal ideal domains, there are integers a_i and bases $v_1, \ldots, v_r$ and $v'_1, \ldots, v'_r$ of $\mathcal{E}$ and $\mathcal{E}'$ respectively (shrinking Δ if necessary) such that

$$v'_i = t^{a_i} v_i \quad \text{and} \quad a_i \geq a_{i+1} \geq 0. \tag{1.1.2}$$

Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ be respectively the coordinates of the vector bundles $\mathcal{V}(\mathcal{E})$, $\mathcal{V}(\mathcal{E}')$ associated to the bases $v_1, \ldots, v_r$ and $v'_1, \ldots, v'_r$, i.e. $\langle x_i, v_j \rangle = \delta_{i,j}$ and similarly for y_i, v'_j, where the x_i and y_i are identified with sections of $\mathcal{E}^\vee$ and $\mathcal{E}'^\vee$. Then

$$x_i = t^{a_i} y_i. \tag{1.1.3}$$

Set $a = \max\{a_i\}$, and $J_k = \{i \mid a_i \geq k\}$ $(k = 0, \ldots, a)$. Then $J_k = \{1, \ldots, m_k\}$ with $m_k \geq m_{k+1}$. Define

$$\mathcal{E}_k = \mathcal{E}' + t^k \mathcal{E} \quad (k = 0, \ldots, a). \tag{1.1.4}$$

Then $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{E}_a = \mathcal{E}'$, and $\mathcal{E}_k$ is the free subsheaf of $\mathcal{E}$ generated by

$$t^{c_{k,i}} v_i \quad \text{with} \quad c_{k,i} = \min(a_i, k) \ (i \in [1, r]).$$

Note that $c_{k,i} \leq k$, and $c_{k,i} = k$ if and only if $i \leq m_k$. Let $x_1^{(k)}, \ldots, x_r^{(k)}$ be the coordinates of the vector bundle $\mathcal{V}(\mathcal{E}_k)$ corresponding to the basis $t^{c_{k,1}} v_1, \ldots, t^{c_{k,r}} v_r$. Then for $k \in [1, a]$

$$x_i^{(k-1)} = \begin{cases} t x_i^{(k)} & \text{if } i \leq m_k, \\ x_i^{(k)} & \text{if } i > m_k. \end{cases} \tag{1.1.5}$$

Thus $\mathcal{V}(\mathcal{E}_k)$ is an open subset of the blow-up of $\mathcal{V}(\mathcal{E}_{k-1})$ along a center of dimension $r - m_k$. More precisely, $\mathcal{V}(\mathcal{E}_k)$ is the complement of the proper transform of the fiber $\mathcal{V}(\mathcal{E}_{k-1})_0$ over $0 \in \Delta$.

1.2. Duality

Let $\mathcal{E}$ be a free sheaf on Δ, and $\mathcal{K} := \mathcal{E}[t^{-1}]$ be the localization by the coordinate t of Δ. In this case we say that $\mathcal{E}$ is a lattice of $\mathcal{K}$. We will fix $\mathcal{K}$ and consider lattices $\mathcal{E}$ of $\mathcal{K}$.

In this subsection, we denote the dual free sheaf by

$$\mathbf{D}(\mathcal{E}) := \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}_\Delta).$$

There is a perfect pairing

$$\langle *, * \rangle : \mathbf{D}(\mathcal{E}) \otimes_{\mathcal{O}} \mathcal{E} \to \mathcal{O}_\Delta,$$

corresponding to the canonical isomorphism

$$\mathbf{D}(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}_\Delta).$$

The above pairing is extended to

$$\langle *, * \rangle : \mathbf{D}(\mathcal{K}) \otimes_{\mathcal{O}} \mathcal{K} \to \mathcal{O}_\Delta[t^{-1}],$$

where

$$\mathbf{D}(\mathcal{K}) := \mathcal{H}om_{\mathcal{O}[t^{-1}]}(\mathcal{K}, \mathcal{O}_\Delta[t^{-1}]).$$

(Here $\mathcal{K}$ is not a $\mathcal{D}$-module in general, and this is quite different from the dual as a $\mathcal{D}$-module in [10] in case $\mathcal{K}$ has a structure of a $\mathcal{D}$-module.) We have

$$\mathbf{D}(\mathbf{D}(\mathcal{K})) = \mathcal{K}, \quad \mathbf{D}(\mathbf{D}(\mathcal{E})) = \mathcal{E}, \tag{1.2.1}$$

$$\mathbf{D}(\mathcal{E}) = \{\xi \in \mathbf{D}(\mathcal{K}) \mid \langle \xi, v \rangle \in \mathcal{O}_\Delta \quad \text{for any } v \in \mathcal{E}\}. \tag{1.2.2}$$

Note also that for any basis $v_1 \ldots, v_r$ of $\mathcal{E}$, there is a unique dual basis $v_1^*, \ldots, v_r^*$ of $\mathbf{D}(\mathcal{E})$ such that

$$\langle v_i^*, v_j \rangle = \delta_{i,j}.$$

Using (1.2.1–2), we have for lattices $\mathcal{E}, \mathcal{E}'$ of $\mathcal{K}$

$$\mathbf{D}(\mathcal{E}) \cap \mathbf{D}(\mathcal{E}') = \mathbf{D}(\mathcal{E} + \mathcal{E}'), \quad \mathbf{D}(\mathcal{E}) + \mathbf{D}(\mathcal{E}') = \mathbf{D}(\mathcal{E} \cap \mathcal{E}'). \tag{1.2.3}$$

For a t-torsion $\mathcal{O}_\Delta$-module $\mathcal{E}''$, define

$$\mathbf{D}^1(\mathcal{E}'') := \mathcal{E}xt_{\mathcal{O}}^1(\mathcal{E}'', \mathcal{O}_\Delta).$$

This is defined by taking a free resolution $0 \to E_1 \to E_0 \to \mathcal{E}'' \to 0$. Note that

$$\mathcal{E}xt_{\mathcal{O}}^i(\mathcal{E}'', \mathcal{O}_\Delta) = 0 \quad \text{for} \quad i \neq 1.$$

For a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

such that $\mathcal{E}', \mathcal{E}$ are lattices (and hence $\mathcal{E}''$ is t-torsion), we have the dual exact sequence

$$0 \to \mathbf{D}(\mathcal{E}) \to \mathbf{D}(\mathcal{E}') \to \mathbf{D}^1(\mathcal{E}'') \to 0. \tag{1.2.4}$$

Moreover, if $\mathcal{E}''$ is annihilated by t, then we have an isomorphism as $\mathbf{C}$-vector spaces

$$\mathbf{D}^1(\mathcal{E}'') = \mathbf{D}_{\mathbf{C}}(\mathcal{E}'') := \mathcal{H}om_{\mathbf{C}}(\mathcal{E}'', \mathbf{C}), \tag{1.2.5}$$

using a free resolution $0 \to \mathcal{O}_\Delta \otimes \mathcal{E}'' \xrightarrow{t} \mathcal{O}_\Delta \otimes \mathcal{E}'' \to \mathcal{E}'' \to 0$.

1.3. Relation between F and V

Let $(\mathcal{M}, F)$ be a filtered left $\mathcal{D}_\Delta$-module underlying a polarizable Hodge module with strict support Δ. The condition on strict support is equivalent to the condition that $\mathcal{M}$ has no nontrivial submodule or quotient module supported on points, and is further equivalent to the condition that it corresponds by the de Rham functor to an intersection complex ([1]) with local system coefficients, see [10], 5.1.3. Let $\mathcal{M}[t^{-1}]$ be the localization of $\mathcal{M}$ by the coordinate t of Δ.

Shrinking Δ if necessary we may assume that the restriction $\mathcal{L}_{\Delta^*} := \mathcal{M}|_{\Delta^*}$ is locally free over $\mathcal{O}_{\Delta^*}$, i.e. $\mathcal{L}_{\Delta^*}$ corresponds to a local system on Δ^*. Let $\mathcal{L}^{\geq\alpha}$ (resp. $\mathcal{L}^{>\alpha}$) denote the Deligne extension of $\mathcal{L}_{\Delta^*}$ such that the eigenvalues of the residue of the connection are contained in $[\alpha, \alpha+1)$ (resp. $(\alpha, \alpha+1]$) [4]. These are identified with the V-filtration of Kashiwara and Malgrange on $\mathcal{M}[t^{-1}]$ indexed by $\mathbf{Q}$. Here we use left $\mathcal{D}$-modules, and assume that the action of $t\partial_t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha \mathcal{M}$. Note that $\mathcal{M}[t^{-1}]$ is identified with the union of Deligne extensions $\mathcal{L}^{>-\infty}$, and there are canonical isomorphisms

$$\mathcal{L}^{\geq\alpha} = V^\alpha \mathcal{L}^{>-\infty}, \quad \mathcal{L}^{>\alpha} = V^{>\alpha} \mathcal{L}^{>-\infty}.$$

It is also well-known that $F^p \mathcal{L}_{\Delta^*}$ are extended as locally free subsheaves $F^p \mathcal{L}^{\geq\alpha}$ of $\mathcal{L}^{\geq\alpha}$ such that the $\mathrm{Gr}_p^F \mathcal{L}^{\geq\alpha}$ are locally free for any $\alpha \in \mathbf{Q}$. We have similar assertions with $\mathcal{L}^{\geq\alpha}$ replaced by $\mathcal{L}^{>\alpha}$.

There are canonical inclusions

$$\mathcal{L}^{>-1} \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{L}^{>-\infty},$$

and moreover we have by the definition of Hodge modules ([10], 3.2)

$$F_p \mathcal{M} = \sum_{i \geq 0} \partial_t^i (F^{i-p} \mathcal{L}^{>-1}), \tag{1.3.1}$$

and $\mathcal{M} = \mathcal{D}_\Delta \mathcal{L}^{>-1}$ for $p \to \infty$, where $\partial_t := \partial/\partial t$ and $F_p = F^{-p}$. Note that (1.3.1) implies

$$F_p \mathcal{M} \cap \mathcal{L}^{>-1} = F^{-p} \mathcal{L}^{>-1}, \tag{1.3.2}$$

and assuming (1.3.2), condition (1.3.1) is equivalent to the surjections ([10], 3.2.2)

$$\partial_t : F_p \mathrm{Gr}_V^\alpha \mathcal{M} \to F_{p+1} \mathrm{Gr}_V^{\alpha-1} \mathcal{M} \quad \text{for any} \quad p \in \mathbf{Z}, \alpha \leq 0. \tag{1.3.3}$$

Note also that the morphism (1.3.3) for $\alpha < 0$ is injective (and hence bijective) since it is an isomorphism if F_p and F_{p+1} are omitted. So we get the isomorphisms

$$\partial_t : F_p \mathrm{Gr}_V^\alpha \mathcal{M} \xrightarrow{\sim} F_{p+1} \mathrm{Gr}_V^{\alpha-1} \mathcal{M} \quad \text{for any} \quad p \in \mathbf{Z}, \alpha < 0. \tag{1.3.4}$$

Then (1.3.3) is equivalent to (1.3.4) together with

$$\partial_t : \mathrm{Gr}_V^0(\mathcal{M}, F) \to \mathrm{Gr}_V^{-1}(\mathcal{M}, F[-1]) \quad \text{is strictly surjective}, \tag{1.3.5}$$

where $(F[k])_p = F_{p-k}$.

1.4. Relation with $(H_{\infty,\mathbf{C}}, F)$

By definition, $H_{\infty,\mathbf{C}}$ is identified with

$$\mathcal{L}^{\geq 0}(0) = \mathcal{L}^{\geq 0}/\mathcal{L}^{\geq 1}.$$

However, this is not compatible with the Hodge filtration F in general. We have to use the following isomorphism (which is a special case of [10], 3.4.12)

$$H_{\infty,\mathbf{C}} = \bigoplus_{\alpha \in [0,1)} \mathrm{Gr}_V^{-\alpha} \mathcal{L}^{>-1} = \bigoplus_{\alpha \in [0,1)} \mathrm{Gr}_V^{-\alpha} \mathcal{M}, \tag{1.4.1}$$

where the last isomorphism follows from (1.3.2). Set

$$H_{\infty,\mathbf{C},\lambda} := \mathrm{Ker}(T_s - \lambda) \subset H_{\infty,\mathbf{C}},$$

where T_s is the semisimple part of the monodromy T. Then (1.4.1) induces

$$(H_{\infty,\mathbf{C},\lambda}, F) = \mathrm{Gr}_V^{-\alpha}(\mathcal{M}, F) \quad \text{for} \quad \alpha \in [0, 1), \ \lambda = e^{2\pi i \alpha}, \tag{1.4.2}$$

using the unipotent base change as in (1.5) below, where $F_p = F^{-p}$.

Combined with (1.3.3) (or (1.3.4–5)), (1.4.2) induces the isomorphisms

$$(H_{\infty,\mathbf{C},\lambda}^{\mathrm{van}}, F) \xrightarrow{\sim} \mathrm{Gr}_V^{-\alpha+j}(\mathcal{M}, F[-j]) \quad \text{for} \quad \alpha \in [0, 1),$$
$$\lambda = e^{2\pi i \alpha}, \quad j \in \mathbf{Z}_{>0}. \tag{1.4.3}$$

Indeed, this is clear for $\alpha \in (0, 1)$, and the case $\alpha = 0$ is reduced to the case $j = 1$ by (1.3.4). In the last case, $-N/2\pi i$ is identified with the composition of

$$\partial_t : \mathrm{Gr}_V^0 \mathcal{M} \to \mathrm{Gr}_V^{-1} \mathcal{M} \quad \text{and} \quad t : \mathrm{Gr}_V^{-1} \mathcal{M} \to \mathrm{Gr}_V^0 \mathcal{M}, \tag{1.4.4}$$

and the kernel of the first morphism is identified by (1.4.2) with $\mathrm{Ker}\, N \subset H_{\infty,\mathbf{C},1}$ since the last morphism of (1.4.4) is injective, see [10], 5.1.3. Then (1.4.3) follows in this case from the strict surjectivity of (1.3.5) together with (1.4.2).

1.5. Deligne extension and the unipotent base change

Let L be a local system on Δ^* with a quasi-unipotent monodromy, and $\mathcal{L}^{\geq 0}$ be the Deligne extension of L such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. Let $T = T_u T_s$ be the Jordan decomposition of the monodromy T where T_s and T_u are respectively the semisimple and unipotent part of T. Set $N = \log T_u$. For a multivalued section u of L, let $u = \sum_j u_j$ be the decomposition such that $T_s u_j = \exp(-2\pi i \alpha_j) u_j$ with $\alpha_j \in [0, 1)$. Then we have a corresponding section $\widehat{u}$ of the Deligne extension $\mathcal{L}^{\geq 0}$ defined by

$$\widehat{u} = \sum_j \exp\left(-\frac{\log t}{2\pi i} N\right) t^{\alpha_j} u_j. \tag{1.5.1}$$

Let $\pi : \widetilde{\Delta} \to \Delta$ be a unipotent base change. By definition it is an m-fold ramified covering of open disks such that $\pi^* t = \widetilde{t}^m$ and the monodromy $\widetilde{T}$ on $\widetilde{\Delta}$ is unipotent, i.e. $\widetilde{T}_s = id$, where t and $\widetilde{t}$ are the coordinates of Δ and $\widetilde{\Delta}$ respectively. Set $\widetilde{N} = \log \widetilde{T}$. Since $\widetilde{N}$ corresponds to $mN = m \log T_u$, we get

$$\pi^* \widehat{u} = \sum_j \exp\left(-\frac{\log \widetilde{t}}{2\pi i} \widetilde{N}\right) \widetilde{t}^{m\alpha_j} u_j, \tag{1.5.2}$$

where $m\alpha_j \in \mathbf{N}$ by hypothesis. This implies that $\pi^* \mathcal{L}^{\geq 0}$ is naturally identified with a subsheaf of the Deligne extension $\widetilde{\mathcal{L}}^{\geq 0}$, and the V-filtration on $\mathcal{L}^{\geq 0}$ is induced by the $\widetilde{t}$-adic filtration on $\widetilde{\mathcal{L}}^{\geq 0}$.

If L underlies a polarizable variation of Hodge structure of weight -1, then we can define as in the introduction

$$\mathcal{E} = \mathcal{L}^{\geq 0} / F^0 \mathcal{L}^{\geq 0}, \quad \Gamma \subset \mathcal{V} := \mathcal{V}(\mathcal{E}).$$

We can repeat the same for the pullback to $\widetilde{\Delta}$, and get

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{L}}^{\geq 0} / F^0 \widetilde{\mathcal{L}}^{\geq 0}, \quad \widetilde{\Gamma} \subset \widetilde{\mathcal{V}} := \mathcal{V}(\widetilde{\mathcal{E}}).$$

If π^* denote also the base change by the morphism π, then we have canonical morphisms

$$\pi^* \Gamma \hookrightarrow \widetilde{\Gamma}, \quad \pi^* \mathcal{V} \to \widetilde{\mathcal{V}},$$

since $\pi^* \mathcal{E}$ is a subsheaf of $\widetilde{\mathcal{E}}$ and $\pi^* \mathcal{V}$ is associated to $\pi^* \mathcal{E}$ as in (1.1.1). Moreover, Γ and $\mathcal{V}$ are obtained by taking the quotient of $\pi^* \Gamma$ and $\pi^* \mathcal{V}$ by the action of the covering transformation group G. So the conditions (2.3.1–2) for $\Gamma \subset \mathcal{V}$ are reduced to those for $\widetilde{\Gamma} \subset \widetilde{\mathcal{V}}$. Moreover, (2.3.2) is satisfied for X if it is satisfied for the pullback of $X \cap \text{Ker}(T_s - id)$. Indeed, the non-unipotent monodromy part causes no problem by (1.5.2) since $m\alpha_j \geq 1$ if $\alpha_j \neq 0$.

2. Proof of the main theorems

2.1. Construction

Let $\mathbf{H}$ be a polarizable variation of Hodge structure of weight -1 on Δ^*, and $\mathcal{L}^{\geq\alpha}$, $\mathcal{L}^{>\alpha}$ be the Deligne extensions as in (1.3). Let $(\mathcal{M}, F)$ be the filtered left $\mathcal{D}_\Delta$-module underlying the polarizable Hodge module extending $\mathbf{H}$ over Δ. This is determined by $\mathcal{L}^{>-1}$ as in (1.3.1). In this paper we use left $\mathcal{D}$-modules whereas right $\mathcal{D}$-modules are used in [10]. The transformation between left and right $\mathcal{D}_X$-modules on a complex manifold X is given by

$$(\mathcal{M}, F) \mapsto (\omega_X, F) \otimes_\mathcal{O} (\mathcal{M}, F)$$

for filtered left $\mathcal{D}_X$-modules $(\mathcal{M}, F)$, where $\omega_X = \Omega_X^{\dim X}$ is the dualizing sheaf of X and the Hodge filtration F on ω_X is defined by $\mathrm{Gr}_F^p \omega_X = 0$ for $p \neq -\dim X$.

Let $\mathbf{D}(F_0\mathcal{M})$ be the dual sheaf as in (1.2), and $\mathcal{V}(\mathbf{D}(F_0\mathcal{M}))$ be the associated vector bundle over Δ as in (1.1). Here $F_0\mathcal{M} \subset \mathcal{L}^{>-\infty}$ is torsion-free, and is hence a free $\mathcal{O}_\Delta$-module.

Using a polarization of the variation of Hodge structure $\mathbf{H}$, we have perfect pairings for $\alpha \in \mathbf{Q}$

$$\mathcal{L}^{\geq\alpha} \otimes_\mathcal{O} \mathcal{L}^{>-\alpha-1} \to \mathcal{O}_\Delta,$$
$$\left(\mathcal{L}^{\geq\alpha}/F^0\mathcal{L}^{\geq\alpha}\right) \otimes_\mathcal{O} F^0\mathcal{L}^{>-\alpha-1} \to \mathcal{O}_\Delta.$$

Define

$$\mathcal{E}^{\geq\alpha} := \mathcal{L}^{\geq\alpha}/F^0\mathcal{L}^{\geq\alpha}, \quad \mathcal{E} := \mathcal{E}^{\geq 0}. \qquad (2.1.1)$$

In the notation of (1.2), the above perfect pairings imply the identifications

$$\mathbf{D}(\mathcal{L}^{>-\alpha-1}) = \mathcal{L}^{\geq\alpha}, \quad \mathbf{D}(F^0\mathcal{L}^{>-\alpha-1}) = \mathcal{E}^{\geq\alpha},$$

and similarly with $> -\alpha - 1$ and $\geq \alpha$ replaced respectively by $\geq -\alpha - 1$ and $> \alpha$.
Since (1.3.1) implies

$$F^0\mathcal{L}^{>-1} \subset F_0\mathcal{M},$$

we have by (1.2.4) the inclusion

$$\mathcal{E}' = \mathbf{D}(F_0\mathcal{M}) \subset \mathcal{E}.$$

We define the identity component $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ by

$$J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0 = \mathcal{V}(\mathbf{D}(F_0\mathcal{M}))/\Gamma,$$

where Γ is the subspace of $\mathcal{V}(\mathbf{D}(F_0\mathcal{M}))$ corresponding to the subsheaf

$$j_*\mathbf{H}_\mathbf{Z} \hookrightarrow \mathbf{D}(F_0\mathcal{M}) \subset \mathcal{E} := \mathcal{L}^{\geq 0}/F^0\mathcal{L}^{\geq 0}.$$

Here we have the inclusion over Δ^* since $\mathbf{H}$ has weight -1. For the inclusion over $0 \in \Delta$, we have to show

$$\langle u, v \rangle \in \mathcal{O}_\Delta \quad \text{for any} \quad u \in j_* \mathbf{H}_{\mathbf{Z}}, \ v \in F_0 \mathcal{M}. \tag{2.1.2}$$

Since we have the injection $j_* \mathbf{H}_{\mathbf{Z}} \hookrightarrow \mathcal{E}$ and $\partial_t u$ vanishes, (2.1.2) follows from (1.3.1) using

$$\partial_t \langle u, v \rangle = \langle \partial_t u, v \rangle + \langle u, \partial_t v \rangle \quad \text{for any} \quad u, v \in \mathcal{L}^{> -\infty}, \tag{2.1.3}$$

where $\mathcal{L}^{> -\infty} = \mathcal{M}[\frac{1}{t}] = \mathcal{L}^{> -1}[\frac{1}{t}]$. (It is shown in [13] that (2.1.2) holds for any $u \in j_* \mathbf{H}_{\mathbf{Z}}$ and $v \in \mathcal{M}$ in a more general situation.)

Set

$$G = H^1(\Delta^*, \mathbf{H}_{\mathbf{Z}})_{\text{tor}}.$$

For any $g \in G$, let ν_g be an admissible normal function whose cohomology class is g [11]. We can then define $J_\Delta^{\text{Sch}}(\mathbf{H})$ in the same way as in loc. cit., i.e.

$$J_\Delta^{\text{Sch}}(\mathbf{H}) = \bigcup_{g \in G} J_\Delta^{\text{Sch}}(\mathbf{H})^g \quad \text{with} \quad J_\Delta^{\text{Sch}}(\mathbf{H})^g := \nu_g + J_\Delta^{\text{Sch}}(\mathbf{H})^0. \tag{2.1.4}$$

By Proposition (2.2) below, the g-component $J_\Delta^{\text{Sch}}(\mathbf{H})^g$ is independent of the choice of ν_g. Since G is torsion, we may assume here

$$\nu_g \in \left(\tfrac{1}{m} \Gamma|_{\Delta^*} \right) / \Gamma|_{\Delta^*} \quad \text{for some} \quad m \in \mathbf{N}. \tag{2.1.5}$$

The following proposition is a refinement of [6], and is proved in a more general situation in [13] using the theory of duality of mixed Hodge modules [10]. We give here a proof in the curve case using (1.2–4).

Proposition 2.2. *Let ν be any admissible normal function whose cohomology class vanishes. Then it extends to a section of $J_\Delta^{\text{Sch}}(\mathbf{H})^0$.*

Proof. Corresponding to an admissible normal function ν, we have a short exact sequence of admissible variations of mixed Hodge structures

$$0 \to \mathbf{H} \to \mathbf{H}^e \to \mathbf{Z}_{\Delta^*} \to 0,$$

and the cohomology class of ν is defined by the extension class of the underlying $\mathbf{Z}$-local systems. This short exact sequence is easily extended to a short exact sequence of mixed Hodge modules since H has weight -1 (and the intermediate extension of perverse sheaves corresponds to the minimal extension of $\mathcal{D}$-modules). In particular, we have a short exact sequence of underlying filtered $\mathcal{D}_\Delta$-modules

$$0 \to (\mathcal{M}, F) \to (\mathcal{M}^e, F) \xrightarrow{\rho} (\mathcal{O}_\Delta, F) \to 0.$$

Note that ρ is bistrictly compatible with F, V where V is the filtration of Kashiwara and Malgrange indexed by $\mathbf{Q}$. Indeed, $\text{Gr}_V^\alpha \rho$ underlies a morphism of mixed Hodge modules and hence is strictly surjective, see [10], 3.3.3. So we have a splitting σ_F which preserves F and V (shrinking Δ if necessary). On the other hand, we have a splitting $\sigma_{\mathbf{Z}}$ of ρ defined over $\mathbf{Z}$ since the cohomology class of ν vanishes. It

preserves the filtration V since it is a morphism of $\mathcal{D}$-modules. Thus the normal function ν is represented by

$$\nu' := \sigma_F(1) - \sigma_{\mathbf{Z}}(1) \in \mathcal{L}^{\geq 0}.$$

We have to show $\nu' \in \mathbf{D}(F_0\mathcal{M})$, i.e.

$$\langle \nu', v \rangle \in \mathcal{O}_\Delta \quad \text{for any} \quad v \in F_0\mathcal{M}.$$

By the Griffiths transversality we have

$$\partial_t^i \nu'|_{\Delta^*} \in F^{-i}\mathcal{L}_{\Delta^*} \quad \text{for} \quad i > 0.$$

Since $\mathbf{H}$ has weight -1, we have by the definition of polarization

$$\langle F^i\mathcal{L}_{\Delta^*}, F^j\mathcal{L}_{\Delta^*} \rangle = 0 \quad \text{if} \quad i + j \geq 0.$$

So the assertion follows from (1.3.1) using (2.1.3).

Remarks 2.3. (i) With the notation of the introduction, the construction in (1.1) is compatible with taking the quotient by Γ if $\Gamma \subset \mathcal{V}(\mathcal{E}')$ and the following condition is satisfied.

There is an open neighborhood U of $0 \in \mathcal{V}_0$ in $\mathcal{V}$ such that $U \cap \Gamma \subset 0_\Delta$.

$$(2.3.1)$$

Here 0_Δ denotes the zero section. Condition (2.3.1) is preserved under blowing up as in Theorem 1. Note that (2.3.1) is equivalent to the condition that, for any point p of $\mathcal{V}_0$, there exists a neighborhood U_p of p in $\mathcal{V}$ such that the map $U_p \to \mathcal{V}/\Gamma$ is injective. Indeed, if (2.3.1) is satisfied, then take U_p such that $U_p - U_p \subset U$. If the latter condition is satisfied, then U can be taken to be the intersection of U_0 with the pull-back of $U_0 \cap 0_\Delta$.

So (2.3.1) is equivalent to the condition that the quotient has the induced structure of a complex manifold (or a complex analytic Lie group over the base space in this case) although the Hausdorff property is unclear. (In this paper, a complex manifold means a ringed space which is locally isomorphic to $(\Delta^n, \mathcal{O}_{\Delta^n})$, and the Hausdorff property is not assumed.) A similar argument is noted in [11], Remark after 3.4.

(ii) With the above notation, let X be a vector subspace of $\mathcal{V}_0$. Then the Hausdorff property for any distinct two points of $X/(\Gamma_0 \cap X)$ in $\mathcal{V}/\Gamma$ is equivalent to the following.

For any $p \in X \setminus \Gamma_0$, there is an neighborhood U_p in $\mathcal{V}$ with $U_p \cap \Gamma = \emptyset$.

$$(2.3.2)$$

Indeed, the Hausdorff property for the images of p_1, $p_2 \in X$ in $\mathcal{V}/\Gamma$ is equivalent to the existence of neighborhoods U_{p_i} of p_i in $\mathcal{V}$ $(i = 1, 2)$ such that $(U_{p_1} - U_{p_2}) \cap \Gamma = \emptyset$. If (2.3.2) is satisfied, then take neighborhoods U_{p_i} such that $U_{p_1} - U_{p_2} \subset U_p$ where $p = p_1 - p_2$. If the latter condition is satisfied, then apply this to p and 0 so that $U_p \cap \Gamma \subset (U_p - U_0) \cap \Gamma = \emptyset$ where we may replace U_p by its intersection with the pull-back of $U_0 \cap 0_\Delta$.

The combination of (2.3.1) and (2.3.2) is thus equivalent to the condition that $\mathcal{V}/\Gamma$ has the induced structure of a Hausdorff complex manifold on a neighborhood of $X/X \cap \Gamma_0$, and is equivalent to the following single condition:

For any $p \in X$, there is an neighborhood U_p in $\mathcal{V}$ with $U_p \cap \Gamma \subset \Gamma(p)$.
$$(2.3.3)$$

Here $\Gamma(p)$ is the section of Γ passing through $p \in \mathcal{V}_0$ if $p \in \Gamma$, and is empty otherwise.

Note that (2.3.3) is also stable by blowing-ups as in (1.1). More precisely, (2.3.3) is satisfied by $\mathcal{V}(\mathcal{E}_k)$ in (1.1) if (2.3.3) is satisfied on a neighborhood of the image of $\mathcal{V}(\mathcal{E}_k)$ in $\mathcal{V}(\mathcal{E}_{k-1})$.

Using the above equivalence together with the finiteness of the cohomology classes of admissible normal functions in the curve case, we can reduce the proof of the property that $J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ is a Hausdorff complex manifold to that for $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$. Indeed, the assertion is clear for the property that $J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ is a complex manifold by definition. As for the Hausdorff property note that the normal function ν_g in (2.1.4) is represented by a multivalued section u of $\frac{1}{m}\Gamma|_{\Delta^*}$ as in (2.1.5). Hence any point q of the g-component $\nu_g + J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ over $0 \in \Delta$ is formally represented by $u + p$ with $p \in \mathcal{V}_0$. This means that if we consider a neighborhood U_p of p in Δ, then the restriction over Δ^* of the corresponding neighborhood of q is represented by $u + U_p|_{\Delta^*}$. For the proof of the Hausdorff property, it is then sufficient to show the following:

For any $p_1, p_2 \in \mathcal{V}_0$ and for any multivalued sections u_1, u_2 of $\frac{1}{m}\Gamma|_{\Delta^*}$ such that
$$u_1 - u_2 \notin \Gamma|_{\Delta^*} \quad \text{or} \quad p_1 - p_2 \notin \Gamma,$$
there are respectively open neighborhoods U_1, U_2 of p_1, p_2 in $\mathcal{V}$ such that
$$(U_1 - U_2) \cap (u_2 - u_1 + \Gamma|_{\Delta^*}) \subset \Gamma(p_1 - p_2).$$

(Note that the two points represented by $u_1 + p_1$ and $u_2 + p_2$ are in the same g-component if and only if $u_1 - u_2 \in \Gamma|_{\Delta^*}$.) Here we may replace
$$u_1 - u_2 + \Gamma|_{\Delta^*} \quad \text{with} \quad \tfrac{1}{m}\Gamma,$$
since $u_1 - u_2 \subset \frac{1}{m}\Gamma|_{\Delta^*}$. We may then replace further
$$\Gamma(p_1 - p_2) \quad \text{with} \quad \left(\tfrac{1}{m}\Gamma\right)(p_1 - p_2).$$

So the proof of the Hausdorff property for $J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ is reduced to the case $m = 1$ by replacing Γ with $\frac{1}{m}\Gamma$, and then follows from (2.3.3) if $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$ has the induced structure of a Hausdorff complex manifold.

2.4. *Proof of Theorem 1.* Let V denote also the quotient filtration on
$$\mathcal{E} := \mathcal{L}^{\geq 0}/F^0\mathcal{L}^{\geq 0}.$$

By [10], (3.2.1.2), we have in the notation of (2.1.1)

$$V^\alpha \mathcal{E} = \mathcal{E}^{\geq \alpha}, \quad V^{>\alpha} \mathcal{E} = \mathcal{E}^{>\alpha} \quad (\alpha \geq 0).$$

This implies

$$\mathrm{Gr}_V^\alpha(\mathcal{E}/\mathbf{D}(F_0\mathcal{M})) = \frac{(\mathbf{D}(F_0\mathcal{M}) + \mathcal{E}^{\geq \alpha})/\mathbf{D}(F_0\mathcal{M})}{(\mathbf{D}(F_0\mathcal{M}) + \mathcal{E}^{>\alpha})/\mathbf{D}(F_0\mathcal{M})},$$

Considering the definition (1.1.4), it is then sufficient to show

$$d_{j+1} = \sum_{0 \leq \alpha < 1} \dim \mathrm{Gr}_V^{\alpha+j}(\mathcal{E}/\mathbf{D}(F_0\mathcal{M})) \quad (j \in \mathbf{N}). \tag{2.4.1}$$

Indeed, we have

$$t V^\alpha \mathcal{E} = V^{\alpha+1} \mathcal{E} \quad (\alpha \geq 0),$$

and hence $V^\alpha \mathcal{E}$ is a refinement of the t-adic filtration $V^i \mathcal{E} = t^i \mathcal{E}$ ($i \in \mathbf{N}$).

By (1.2.3) we have

$$\mathbf{D}(F_0\mathcal{M}) + \mathcal{E}^{\geq \alpha} = \mathbf{D}(F_0\mathcal{M} \cap F^0\mathcal{L}^{>-\alpha-1}) = \mathbf{D}(F_0 V^{>-\alpha-1}\mathcal{M}),$$
$$\mathbf{D}(F_0\mathcal{M}) + \mathcal{E}^{>\alpha} = \mathbf{D}(F_0\mathcal{M} \cap F^0\mathcal{L}^{\geq-\alpha-1}) = \mathbf{D}(F_0 V^{-\alpha-1}\mathcal{M}).$$

Combining these with (1.2.4–5), we get thus

$$\mathbf{D_C}(F_0\mathrm{Gr}_V^{-\alpha-1}\mathcal{M}) = \mathrm{Gr}_V^\alpha(\mathcal{E}/\mathbf{D}(F_0\mathcal{M})) \quad (\alpha \geq 0). \tag{2.4.2}$$

So (2.4.1) follows from (1.4.3). This finishes the proof of Theorem 1.

2.5. Proof of Theorem 2. In the notation of (1.1) we have

$$x_i = t^{c_{k,i}} x_i^{(k)}.$$

Then $(I_k)_0 \subset \mathcal{V}_0$ is given by

$$x_i = 0 \quad \text{for} \quad c_{k,i} > 0.$$

However, this is independent of $k \in [1, a]$ by definition (i.e. $c_{k,i} = \min(a_i, k)$). So the first assertion follows.

For the second assertion we first show that the two spaces have the same dimension. With the notation of (1.4) we have

$$\mathbf{D_C}\big(\mathrm{Im}(N : F^{j+1} H_{\infty,\mathbf{C},1} \to F^j H_{\infty,\mathbf{C},1})\big)$$
$$= \mathrm{Coim}(N : H_{\infty,\mathbf{C},1}/F^{-j} H_{\infty,\mathbf{C},1} \to H_{\infty,\mathbf{C},1}/F^{-j-1} H_{\infty,\mathbf{C},1})$$
$$= H_{\infty,\mathbf{C},1}/(F^{-j} H_{\infty,\mathbf{C},1} + \mathrm{Ker}\, N). \tag{2.5.1}$$

Here the first isomorphism is induced by a polarization, and the last isomorphism follows from the strict compatibility of

$$N : (H_{\infty,\mathbf{C},1}, F) \to (H_{\infty,\mathbf{C},1}, F[-1]),$$

since N is a morphism of mixed Hodge structure of type $(-1, -1)$, [5]. The last term of (2.5.1) for $j = 0$ is further isomorphic to

$$\frac{H_{\infty,\mathbf{C},1}/F^0 H_{\infty,\mathbf{C},1}}{(F^0 H_{\infty,\mathbf{C},1} + \operatorname{Ker} N)/F^0 H_{\infty,\mathbf{C},1}}$$

$$= \frac{\operatorname{Gr}_V^0 \mathcal{E}}{\operatorname{Im}(H_{\infty,\mathbf{C}}^{\mathrm{inv}} \to H_{\infty,\mathbf{C},1}/F^0 H_{\infty,\mathbf{C},1})}.$$

By (2.4.2) and (1.4.3) for $\alpha = 0$ and $j = p = 0$, the dimension of the first term of (2.5.1) for $j = 0$ coincides with that of

$$\operatorname{Gr}_V^0(\mathcal{E}/\mathbf{D}(F_0\mathcal{M})) = \frac{\operatorname{Gr}_V^0 \mathcal{E}}{\operatorname{Gr}_V^0 \mathbf{D}(F_0\mathcal{M})}.$$

The dimension of $\operatorname{Gr}_V^0 \mathbf{D}(F_0\mathcal{M})$ coincides with the dimension of the image of $(I_1)_0$ in the unipotent monodromy part since it can be defined by using the V-filtration as in (1.4.2). So we get the coincidence of the dimensions of the two spaces in the second assertion, and it is enough to show an inclusion.

The unipotent monodromy part of $(I_1)_0$ is identified with the image of $\mathbf{D}(F_0\mathcal{M})$ in $\operatorname{Gr}_V^0 \mathcal{E}/H_{\infty,\mathbf{Z}}^{\mathrm{inv}}$. Here we have to divide $\operatorname{Gr}_V^0 \mathcal{E}$ by $\Gamma_0 = H_{\infty,\mathbf{Z}}^{\mathrm{inv}}$. To show an inclusion, it is then sufficient to show that the image of $\mathbf{D}(F_0\mathcal{M})$ in $\operatorname{Gr}_V^0 \mathcal{E}$ contains the image of $\operatorname{Ker} N \subset H_{\infty,\mathbf{C},1}$ in

$$\operatorname{Gr}_V^0 \mathcal{E} = H_{\infty,\mathbf{C},1}/F^0 H_{\infty,\mathbf{C},1},$$

i.e.,

$$\langle u, v \rangle \in \mathcal{O}_\Delta \quad \text{if} \quad u \in \operatorname{Ker} t\partial_t, \ v \in F_0\mathcal{M},$$

since N corresponds to $-t\partial_t$ on $\operatorname{Gr}_V^0 \mathcal{M}$. (Note that the V-filtration in the one-variable case splits by the action of $t\partial_t$.) But the above assertion follows from (1.3.1). So the second assertion is proved. The last assertion then follows from the definition of the Néron model of Green, Griffiths and Kerr [GKK]. This finishes the proof of Theorem 2.

2.6. Proof of Theorem 3. The assertion follows from Theorems 1 and 2 together with (1.5) and (2.3) by reducing to [11,12].

Remark 2.7. In the notation of the last part of the introduction, there is an injection $J_\Delta^{\mathrm{BPS}}(\mathbf{H}) \hookrightarrow J_\Delta^{\mathrm{Sch}}(\mathbf{H})$ using horizontal sections passing through the monodromy invariant part of $J_\Delta^C(\mathbf{H})_0$ [13]. Even in the unipotent monodromy case, however, this cannot be continuous unless it is an isomorphism (e.g. the level is 1). Here we cannot use admissible normal functions as in the abelian scheme case explained in the introduction since Griffiths transversality gives a strong restriction. However, the topology of $J_\Delta^{\mathrm{BPS}}(\mathbf{H})$ is induced by the inclusion $J_\Delta^{\mathrm{BPS}}(\mathbf{H}) \hookrightarrow J_\Delta^C(\mathbf{H})$ by definition [2] (in the unipotent monodromy case), and this implies a contradiction by Theorem 2 if the morphism is continuous and the surjection $J_\Delta^{\mathrm{Sch}}(\mathbf{H}) \to J_\Delta^{\mathrm{BPS}}(\mathbf{H}) = J_\Delta^{\mathrm{GGK}}(\mathbf{H})$ is not bijective.

Remark 2.8. Theorem 3 shows that the first blow-up Y_1 is already a Hausdorff complex manifold. As is remarked by the referee, Proposition 2.2 holds for that space as well, and so Y_1, instead of $J_\Delta^{\mathrm{Sch}}(\mathbf{H})^0$, could be used as the identity component of a Néron model. However, it is not easy to generalize this to the case $\dim S > 1$ even in the normal crossing divisor case.

3. Remarks about the higher dimensional case

In this section we give some remarks for the case where the base space S is not a curve.

Remark 3.1. In the case $\dim S > 1$, $J_S^{\mathrm{Sch}}(\mathbf{H})$ may have singularities, caused by the fact that $F_0\mathcal{M}$ is not always locally free. This can happen even when $\mathbf{H}$ is a nilpotent orbit on $S^* = (\Delta^*)^2$. For example, consider the case where the limit mixed Hodge structure H_∞ has rank 4 with type

$$(1, -1), \ (-1, 1), \ (0, -2), \ (-2, 0),$$

and N_1, N_2 are nonzero. Then $J_S^{\mathrm{Sch}}(\mathbf{H})$ is locally defined by

$$x_1 t_1 = x_2 t_2,$$

in an open neighborhood of $0 \in \mathbf{C}^5$ with coordinates x_1, x_2, x_3, t_1, t_2. Indeed, we have

$$F_0\mathcal{M} = \mathcal{I}_0 \oplus \mathcal{O}_S,$$

as an $\mathcal{O}_S$-module where $\mathcal{I}_0$ denotes the sheaf of ideals of $0 \in S = \Delta^2$. (More precisely, x_1, x_2 respectively correspond to the two generators t_1, t_2 of $\mathcal{I}_0$, and x_3 to the generator 1 of $\mathcal{O}_S$.) Hence $F_0\mathcal{M}$ is not a locally free sheaf in this case; in fact, it is even non-reflexive since

$$\mathcal{I}_0^\vee := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_0, \mathcal{O}_S) = \mathcal{O}_S,$$

using the short exact sequence $0 \to \mathcal{I}_0 \to \mathcal{O}_S \to \mathbf{C}_0 \to 0$. This calculation implies that the reflexive hull, i.e. the double dual $(F_0\mathcal{M})^{\vee\vee}$, is free in this case.

Remark 3.2. The torsion-freeness of $F_0\mathcal{M}$ is not stable by the pull-back under morphisms of base spaces. For example, if $\sigma : S' \to S$ is the blow-up along the origin with S, $(\mathcal{M}, F)$ as above, then the pull-back $\sigma^* F_0\mathcal{M}$ has torsion. Indeed, $\mathcal{I}_0$ is quasi-isomorphic to the mapping cone of

$$(t_1, t_2) : \mathcal{O}_S \to \mathcal{O}_S \oplus \mathcal{O}_S,$$

and its pull-back by σ is locally the mapping cone of

$$(t_1', t_1' t_2') : \mathcal{O}_{S'} \to \mathcal{O}_{S'} \oplus \mathcal{O}_{S'},$$

where t_1', t_2' are local coordinates of S' such that $\sigma^* t_1 = t_1'$, $\sigma^* t_2 = t_1' t_2'$. Then the cokernel of the morphism has nontrivial t_1'-torsion.

Remark 3.3. The freeness of the reflexive hull in the above example does not hold in general even in the normal crossing case, e.g. if $\mathbf{H}$ is a nilpotent orbit of three variables such that H_∞ has dimension 8 with the same type as above and the images of $F^1 H_\infty$ by N_1, N_2, N_3 are 1-dimensional and are not compatible subspaces. The last condition is equivalent to the condition that they are distinct to each other and span a 2-dimensional subspace. So there are u_1, u_2 such that u_1, u_2 and $u_1 + u_2$ respectively generate the images of $F^1 H_\infty$ by N_1, N_2 and N_3. Then $F_0 \mathcal{M}$ is a direct sum of a free $\mathcal{O}_S$-module of rank 2 and a coherent $\mathcal{O}_S$-module $\mathcal{M}'$ which is generated by $t_1^{-1} \widehat{u}_1, t_2^{-1} \widehat{u}_2$ and $t_3^{-1}(\widehat{u}_1 + \widehat{u}_2)$ where $S = \Delta^3$, and $\widehat{u}_1, \widehat{u}_2$ are defined as in (1.5.1). This implies that $\mathcal{M}'$ has a free resolution defined by

$$(t_1, t_2, t_3) : \mathcal{O}_S \to \mathcal{O}_S \oplus \mathcal{O}_S \oplus \mathcal{O}_S.$$

Indeed, if $\mathcal{M}''$ denote the cokernel of this morphism, then $H^k_{\{0\}} \mathcal{M}'' = 0$ for $k \leq 1$, and hence $\mathcal{M}'' = j_{0*} j_0^* \mathcal{M}''$ where $j_0 : S \setminus \{0\} \to S$ is the inclusion. It is easy to show that $j_0^* \mathcal{M}' = j_0^* \mathcal{M}''$. So there are morphisms $\mathcal{M}'' \to \mathcal{M}' \to j_{0*} j_0^* \mathcal{M}'' = \mathcal{M}''$ whose composition is the identity. Then the kernel of the surjection $\mathcal{M}' \to \mathcal{M}''$ vanishes, and we get the isomorphism $\mathcal{M}'' = \mathcal{M}'$.

The above resolution is the first two terms of the Koszul complex $K^\bullet(\mathcal{O}_S; t_1, t_2, t_3)$ which will be denoted by $K^\bullet$. This implies that $\mathcal{M}'$ is self-dual, i.e.

$$\mathbf{D}\mathcal{M}' = \mathcal{H}^{-1}\mathbf{D}(\sigma_{\leq 1} K^\bullet) = \mathcal{H}^2(\sigma_{\geq 2} K^\bullet) = \mathcal{H}^1(\sigma_{\leq 1} K^\bullet) = \mathcal{M}',$$

using the self-duality and the exactness at the middle terms of the Koszul complex where $\mathbf{D}(*) := \mathcal{H}om_{\mathcal{O}_S}(*, \mathcal{O}_S)$. (For the truncation $\sigma_{\leq p}$ [5].) So $F_0 \mathcal{M}$ is self-dual and hence reflexive. Note that $F_0 \mathcal{M}$ cannot be free since the freeness implies a free resolution of $\mathbf{C}$ over $\mathcal{O}_S$ with three terms (shrinking S if necessary).

References

[1] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris (1982)
[2] Brosnan, P., Pearlstein, G., Saito, M.: A generalization of the Néron models of Green, Griffiths and Kerr, preprint (arXiv:0809.5185)
[3] Clemens, H.: The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities. Publ. Math. IHES **58**, 5–18 (1983)
[4] Deligne, P.: Equations différentiellesà points singuliers réguliers, Lect. Notes in Math. vol. 163, Springer, Berlin, 1970
[5] Deligne, P.: Théorie de Hodge II. Publ. Math. IHES **40**, 5–58 (1971)
[6] El Zein, F., Zucker, S.: Extendability of normal functions associated to algebraic cycles. In topics in transcendental algebraic geometry. Ann. Math. Stud. vol. 106, pp. 269–288. Princeton University Press, Princeton, NJ (1984)
[7] Green, M., Griffiths, P., Kerr, M.: Neron models and boundary components for degenerations of Hodge structures of mirror quintic type. In: Alexeev, V. (ed.) Curves and Abelian Varieties. Contemp. Math. vol. 465, pp. 71–145, AMS, USA (2007)
[8] Green, M., Griffiths, P., Kerr, M.: Néron models and limits of Abel-Jacobi mappings. Compositio Math. **146**, 288–366 (2010)

[9] Okonek, C., Schneider, M., Spindler, H.: Vector bundles on complex projective spaces. Progress in Mathematics, Vol. 3, Birkhäuser Boston, MA (1980)

[10] Saito, M.: Modules de Hodge polarisables. Publ. RIMS, Kyoto Univ. **24**, 849–995 (1988)

[11] Saito, M.: Admissible normal functions. J. Algebraic Geom. **5**, 235–276 (1996)

[12] Saito, M.: Hausdorff property of the Néron models of Green, Griffiths and Kerr, preprint (arXiv:0803.2771)

[13] Schnell, C.: Complex analytic Néron models for arbitrary families of intermediate Jacobians, preprint (arXiv:0910.0662)

[14] Young, A.: Complex analytic Néron models for degenerating Abelian varieties over higher dimensional parameter spaces, Ph. D. Thesis, Princeton University, NJ, Sept 2008

[15] Zucker, S.: Generalized intermediate Jacobians and the theorem on normal functions. Inv. Math. **33**, 185–222 (1976)