The De Giorgi-Nash-Moser Estimates

We are going to discuss the the equation

$$Lu \equiv -D_i(a_{ij}(x)D_ju) = 0 \quad \text{in } B_4 \subset \mathbb{R}^n. \tag{1}$$

The a_{ij}, with $i, j \in \{1, \ldots, n\}$, are functions on the ball B_4. Here and in the following doubly occurring indices are always understood to indicate summation. We assume that the coefficients a_{ij} satisfy the following *ellipticity condition*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \quad \text{for all } \xi \in \mathbb{R}^n \text{ and all } x \in B_4. \tag{2}$$

with a positive constant λ. The equation $Lu = 0$ is then a second-degree elliptic equation. We also require the a_{ij} to be bounded and measurable, satisfying $\|a_{ij}\|_{L^\infty(B_4)} \leq \Lambda$ with another constant $\Lambda > 0$. (In case you wonder, the radius '4' of the ball is to avoid fractions. Most of our estimates will be of the kind "some expression on $B_1 \leq$ another expression on B_4 " and it would be unconvenient to have things like 1/8 as a radius.)

It is clear that under these assumptions the equation (1) does not make sense, for the a_{ij} need not be differentiable. In fact, we shall use it as an abbreviation and really talk about so called weak solutions. Let us introduce these terms. A function $u \in H^1(B_4)$ is a *weak solution* to $Lu = 0$ if for all $\phi \in H_0^1(B_4)$,

$$\int a_{ij}D_juD_i\phi = 0. \tag{3}$$

Conventional or strong solutions are obviously weak solutions as well. The notion of weak solutions has come up because it provides a good way of attacking equations. The question of solvability splits in two parts—first, show that a weak solution exists; second, find out how nice (continuous, differentiable) solutions are. Besides, there are equations without strong solutions.

A function u is called a *subsolution* if

$$\int a_{ij}D_juD_i\phi \leq 0 \tag{4}$$

holds for all $\phi \in H_0^1(B_4)$ with $\phi \geq 0$. If the inequality in (4) is reversed, u is a *supersolution*.

Our purpose is to carry out part of the second step described above. We will demonstrate that a weak solution of $Lu = 0$ is not just an element of the Sobolev space $H^1(B_4)$ but is in fact in $C^\alpha(B_4)$, i.e., Hölder continuous with some exponent α. Recall what this means: On a domain, a function u is Hölder continuous with exponent $\alpha \in [0, 1]$ if it is continuous and

$$\sup_{x,y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

for every compact subset K of the domain.

Results of this kind can of course be obtained under different assumptions on the a_{ij}; the mildest ones are probably those stated above. That solutions to more general second-order equations with bounded coefficients a_{ij} are Hölder continuous was first proved by De Giorgi (1957, for the elliptic case) and independently by Nash (1958, for the parabolic case). De Giorgi's arguments were then much simplified and extended by Moser (1960, 1961).

The methods used involve what are called *a priori estimates*. One assumes that an equation has a weak solution, usually in some Sobolev space, and then tries to obtain general estimates on the solution, as in the case of Theorem 1 below. That way, regularity is proved for whole classes of solutions, only from the fact that they are solutions; one does not consider individual properties. (Hence the word 'a priori'.)

One more word about notation. As usual, there are lots of constants. Rather than lump them all together under one letter 'C' we label them successively by $C_1, C_2, \ldots$ so that one can work out the individual values if needed. It is easier to check the dependence on the parameters that way.

1 Local Boundedness of Solutions

The first step in proving Hölder continuity is to show that solutions u to $Lu = 0$ are locally bounded. This involves estimating the supremum of a solution in terms of its L^2-norm. But how in the world does one do that? Because of the Sobolev inequality, u is in L^q for $q = 2n/(n-2)$. It turns out that, u being a solution, these "gains" can be amplified by an iterative procedure. The iteration used in the proof is due to Moser (and is named after him).

Theorem 1. *Let $u \in H^1(B_4)$ be a subsolution to the equation, i.e., assume that u satisfies* (3)*. Then there is a positive constant C_4, depending only on n and Λ/λ, such that*

$$\sup_{B_2} u^+ \leq C_4 \|u\|_{L^2(B_4)}.$$

Proof. Let us first give the general idea of the proof. By inserting a suitable test function ϕ in the equation and playing around a bit, we can bound the L^{p_1}-norm of u in a smaller ball B_{r_1} by the L^{p_2}-norm in a larger ball B_{r_2}, where $p_1 > p_2 \geq 2$. In (13) below, we will have an estimate of the form

$$\|u^+\|_{L^{p_1}(B_{r_1})} \leq C\|u^+\|_{L^{p_2}(B_{r_2})},$$

some kind of reversed Hölder inequality. We will iterate this, choosing r_i and p_i carefully, to get our result.

To begin with, introduce the following two functions. For positive numbers k and m, set $\bar{u} = k + u^+$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \geq m \end{cases}$$

The point is that $\bar{u}_m$ is still an element of $H^1(B_4)$, but bounded from below by k and from above by $(k+m)$. Also note that $D\bar{u}_m = 0$ whenever $u < 0$ or $u > m$ and that $\bar{u}_m = \bar{u}$ at all other points. The function $\bar{u}$ is always positive and $D\bar{u} = Du$ if $u \geq 0$. Both k and m are needed to make the argument work; in the end, we will let $k \to 0+$ and $m \to \infty$, so that both $\bar{u}$ and $\bar{u}_m$ converge to u^+.

We start from the fact that u is a subsolution. In the inequality (4), we use a test function of the form $\phi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1})$, where $\beta \geq 0$ is an arbitray real number and $\eta \in C_0^1(B_4)$ a nonnegative cut-off function to be chosen later on. This function ϕ is an element of $H_0^1(B_4)$ because $\bar{u}_m$ is bounded; it is also nonnegative and can therefore be used as a test function. An explicit calculation with the weak derivatives gives

$$\begin{aligned} D\phi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + D\bar{u}\eta^2\bar{u}_m^\beta + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}), \end{aligned} \tag{5}$$

where we used positivity of $\bar{u}_m$ and the fact that $\bar{u} = \bar{u}_m$ whenever $D\bar{u}_m \neq 0$.

Now insert this expression into the inequality (4) to obtain

$$
\begin{aligned}
0 &\geq \int a_{ij} D_j u D_i \phi \\
&= \int a_{ij} D_j \bar{u} \eta^2 \bar{u}_m^\beta (\beta D_i \bar{u}_m + D_i \bar{u}) + 2 \int a_{ij} D_j \bar{u} D_i \eta \eta (\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \\
&\geq \int \eta^2 \bar{u}_m^\beta \big(\beta\lambda |D\bar{u}_m|^2 + \lambda |D\bar{u}|^2\big) - 2 \int \eta \big|a_{ij} D_j u D_i \eta\big| \big|\bar{u}_m^\beta \bar{u} - k^{\beta+1}\big|.
\end{aligned}
\tag{6}
$$

Notice that $D\phi = 0$ whenever $u < 0$, so that all integrals are effectively over the set $\{u \geq 0\}$ only. This allowed us to replace $D_j u$ by $D_j \bar{u}$, since the two are equal if $u \geq 0$. We also used the ellipticity condition (2) and the fact that $D_i \bar{u} = D_i \bar{u}_m$ whenever the latter is nonzero.

Let us estimate the second integral. From Cauchy's inequality,

$$
2 \int \eta \big|a_{ij} D_j \bar{u} D_i \eta\big| \big|\bar{u}_m^\beta \bar{u} - k^{\beta+1}\big| \leq 2 \int \eta \cdot \Lambda n |D\bar{u}| |D\eta| \cdot |\bar{u}_m^\beta \bar{u} - k^{\beta+1}|,
$$

and since $\bar{u}_m^\beta \bar{u} - k^{\beta+1} \geq 0$,

$$
\leq 2n\Lambda \int \eta |D\bar{u}| |D\eta| \bar{u}_m^\beta \bar{u} = \int \big(|D\bar{u}| \eta \bar{u}_m^{\beta/2}\big) \big(2n\Lambda |D\eta| \bar{u} \bar{u}_m^{\beta/2}\big).
$$

Apply Young's inequality $\big(ab \leq \frac{\lambda}{2} a^2 + \frac{1}{2\lambda} b^2\big)$ to the two bracketed factors to get

$$
\leq \frac{\lambda}{2} \int |D\bar{u}|^2 \eta^2 \bar{u}_m^\beta + \frac{2n^2\Lambda^2}{\lambda} \int |D\eta|^2 \bar{u}^2 \bar{u}_m^\beta.
$$

Combining this result with (6) and simplifying slightly, we obtain

$$
\begin{aligned}
\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 &\leq \frac{4n^2\Lambda^2}{\lambda^2} \int |D\eta|^2 \bar{u}^2 \bar{u}_m^\beta \\
&= C_1 \int |D\eta|^2 \bar{u}^2 \bar{u}_m^\beta
\end{aligned}
\tag{7}
$$

where $C_1 = 4n^2(\Lambda/\lambda)^2$.

We define an additional function $w \in H^1(B_4)$ by $w = \bar{u}_m^{\beta/2} \bar{u}$. As before, one calculates that $Dw = \bar{u}_m^{\beta/2}(\beta/2 \cdot D\bar{u}_m + D\bar{u})$; therefore

$$
\begin{aligned}
|Dw|^2 = \bar{u}_m^\beta \Big|\frac{\beta}{2} D\bar{u}_m + D\bar{u}\Big|^2 &= \bar{u}_m^\beta \left(\frac{\beta^2}{4} |D\bar{u}_m|^2 + \beta D\bar{u}_m D\bar{u} + |D\bar{u}|^2\right) \\
&= \bar{u}_m^\beta \left(\beta(\beta/4 + 1) |D\bar{u}_m|^2 + |D\bar{u}|^2\right) \\
&\leq \bar{u}_m^\beta (\beta + 1) \left(\beta |D\bar{u}_m|^2 + |D\bar{u}|^2\right).
\end{aligned}
\tag{8}
$$

In combination with (7), we have

$$\int |Dw|^2\eta^2 \le (\beta+1)\left(\beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2 + \int \eta^2\bar{u}_m^\beta|D\bar{u}|^2\right)$$
$$\le (\beta+1)C_1\int |D\eta|^2 w^2$$

and so

$$\begin{aligned}\int |D(\eta w)|^2 &\le 2\int |D\eta|^2 w^2 + |Dw|^2\eta^2 \le 2\big(1+C_1(\beta+1)\big)\int |D\eta|^2 w^2\\ &\le 4C_1(\beta+1)\int |D\eta|^2 w^2.\end{aligned} \tag{9}$$

Remember that we want to estimate stronger L^p-norms by weaker ones. Here is how we do it. From the Sobolev inequality, with $\chi = n/(n-2) > 1$ for $n > 2$ and any fixed $\chi > 2$ for $n = 2$, we get

$$\left(\int (\eta w)^{2\chi}\right)^{\frac{1}{\chi}} \le C(n)\int |D(\eta w)|^2 \le 4C_1C(n)(\beta+1)\int |D\eta|^2 w^2. \tag{10}$$

Now choose a suitable cut-off function. For $0 < r < R \le 4$, take $\eta \in C_0^1(B_4)$ with $\eta \equiv 1$ in B_r and $|D\eta| \le 2/(R-r)$. Then

$$\begin{aligned}\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} &\le \left(\int (\eta w)^{2\chi}\right)^{\frac{1}{\chi}} \le 4C_1C(n)(\beta+1)\int |D\eta|^2 w^2\\ &\le 16C_1C(n)\frac{(\beta+1)}{(R-r)^2}\int_{B_R} w^2 = C_2\frac{(\beta+1)}{(R-r)^2}\int_{B_R} w^2,\end{aligned} \tag{11}$$

and, if we let $\gamma = \beta+2 \ge 2$, recall the definition of w and use that $\bar{u}_m \le \bar{u}$,

$$\begin{aligned}\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} &= \left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} = \left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \le C_2\frac{(\beta+1)}{(R-r)^2}\int_{B_R} w^2\\ &\le C_2\frac{(\gamma-1)}{(R-r)^2}\int_{B_R} \bar{u}^2\bar{u}_m^\beta \le C_2\frac{(\gamma-1)}{(R-r)^2}\int_{B_R} \bar{u}^\gamma.\end{aligned} \tag{12}$$

Finally, let $m \to \infty$ and $k \to 0+$ (use Fatou's Lemma) to obtain the crucial estimate

$$\|u^+\|_{L^{\gamma\chi}(B_r)} \le \left(C_2\frac{(\gamma-1)}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|u^+\|_{L^\gamma(B_R)}. \tag{13}$$

Observe how the stronger $L^{\gamma\chi}$-norm is estimated by the weaker L^{γ}-norm. As a trade-off, we have to increase the ball from B_r to the larger B_R. As we said above, the two ingredients were the Sobolev inequality and the equation itself.

The key observation is that (13) is valid for all $0 < r < R \le 4$ and for all $\gamma \ge 2$. This suggests an iteration, taking successively the values $\gamma = 2, 2\chi, 2\chi^2, \ldots$. Define, for all $i = 0, 1, 2, \ldots$,

$$\gamma_i = 2\chi^i \quad \text{and} \quad r_i = 2 + \frac{1}{2^{i-1}}.$$

For any $i \ge 0$, insert $r = r_{i+1}$, $R = r_i$ and $\gamma = \gamma_i$ into (13),

$$\|u^+\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \le \left(C_2 \frac{(\gamma_i - 1)}{(1/2^i)^2} \right)^{\frac{1}{\gamma_i}} \|u^+\|_{L^{\gamma_i}(B_{r_i})} \le C_3^{\frac{i}{\chi^i}} \|u^+\|_{L^{\gamma_i}(B_{r_i})}.$$

C_3 depends only on n and Λ/λ because χ is a function of n alone. By iteration,

$$\|u^+\|_{L^{\gamma_i}(B_{r_i})} \le C_3^{\sum \frac{i}{\chi^i}} \|u^+\|_{L^2(B_4)} = C_4 \|u\|_{L^2(B_4)},$$

valid for all $i \ge 1$; since all r_i are greater than 2, one arrives at

$$\|u^+\|_{L^{\gamma_i}(B_2)} = C_4 \|u\|_{L^2(B_4)}.$$

Now let $i \to \infty$; this entails $\gamma_i \to \infty$ and gives us

$$\sup_{B_2} u^+ \le C_4 \|u\|_{L^2(B_4)}.$$

We have proved the estimate. □

There are technical reasons for stating the theorem in terms of subsolutions: this helps in the proofs of the next section. For solutions u of (3), we can immediately derive a stronger result, namely boundedness on compact subsets of B_4.

Corollary 1. *Let $u \in H^1(B_4)$ be a weak solution to $Lu = 0$ in the ball B_4. Then u satisfies*

$$\|u\|_{L^\infty(B_2)} \le C_4 \|u\|_{L^2(B_4)}$$

with the same constant C_4 as in Theorem 1. Moreover, u is bounded on each compact subset K of B_4.

Proof. If u is a solution to (3), then both u and $(-u)$ are subsolutions. In addition to the estimate in the theorem, we therefore have

$$\sup_{B_2} u^- \leq C_4 \|u\|_{L^2(B_4)}.$$

If we combine the two, we get

$$\|u\|_{L^\infty(B_2)} \leq C_4 \|u\|_{L^2(B_4)}. \tag{14}$$

Now one quickly sees that u is bounded on each compact subset K. Consider first the case of a ball $B_s(a)$ contained in B_4, with $s > 0$. If we define $v(x) = u(a + sx)$, v is an element of $H^1(B_4)$, and by taking test functions with support in $B_s(a)$ in the original equation (3) and changing coordinates, we see that v satisfies

$$\int \tilde{a}_{ij} D_j v D_i \varphi = 0$$

for all $\varphi \in H^1_0(B_4)$, where $\tilde{a}_{ij}(x) = a_{ij}(a + sx)$. Inequality (14) above, when applied to v, gives us

$$\|u\|_{L^\infty(B_{s/2}(a))} \leq \frac{C_4}{\sqrt{s}} \|u\|_{L^2(B_s(a))} \leq \frac{C_4}{\sqrt{s}} \|u\|_{L^2(B_4)}.$$

An arbitrary compact subset K can be covered by finitely many open balls $B(a_i, s_i/2)$ of positive radius s_i, such that $B(a_i, s_i) \subset B_4$. Then

$$\|u\|_{L^\infty(K)} \leq \max_i \|u\|_{L^\infty(B_{s_i/2}(a_i))} \leq \max_i \frac{C_4}{\sqrt{s_i}} \|u\|_{L^2(B_4)}$$

which is finite. Therefore u is bounded on K.

□

2 Hölder Continuity of Solutions

In this section, we show that solutions to $Lu = 0$ are Hölder continuous. On the way we meet with the very pretty Theorem 2 which gives a *lower* bound on positive solutions. The following lemma on subsolutions and supersolutions is helpful.

Lemma 1. *Let Φ be a convex and locally Lipschitz continuous function on some interval I.*

1. *If u is a subsolution with values in I and $\Phi' \geq 0$, then $v = \Phi(u)$ is also a subsolution, provided it is in $H^1_{loc}(B_4)$.*
2. *If u is a supersolution with values in I and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution, provided it is in $H^1_{loc}(B_4)$.*

Proof. Let us prove the second statement only, since it is the one used in Theorem 2 below (the first one is dealt with in an analogous manner). Since C^1_0 is dense in H^1_0, it is enough to consider test functions $\phi \in C^1_0$. If one assumes that $\Phi \in C^2_{\text{loc}}(I)$, then $\Phi'(t) \leq 0$ and $\Phi''(t) \geq 0$. Take any nonnegative $\phi \in C^1_0(B_4)$. A direct calculation gives

$$
\begin{aligned}
\int a_{ij} D_j v D_i \phi = \int a_{ij} \Phi'(u) D_j u D_i \phi = \\
&= -\int a_{ij} D_j u D_i(-\Phi'(u)\phi) - \int (a_{ij} D_j u D_i u)\phi \Phi''(u) \\
&\leq -\int a_{ij} D_j u D_i(-\Phi'(u)\phi) - \lambda \int |Du|^2 \phi \Phi''(u) \leq 0
\end{aligned}
$$

because $-\Phi'(u)\phi \in H^1_0(B_4)$ is nonnegative. Therefore $\Phi(u)$ is a subsolution.

In general, let ρ_ε be the standard mollifier and set $\Phi_\varepsilon(t) = \rho_\varepsilon * \Phi(t)$. Then $\Phi'_\varepsilon(t) = \rho_\varepsilon * \Phi'(t) \leq 0$ and $\Phi''_\varepsilon(t) = \rho_\varepsilon * \Phi''(t) \geq 0$. By what we have just proved, $\Phi_\varepsilon(t)$ is a subsolution. Because $\Phi'_\varepsilon(t) \to \Phi'(t)$ a.e. as $\varepsilon \to 0+$ and because ϕ has compact support, the dominated convergence theorem implies that

$$
0 \geq \int a_{ij} \Phi'_\varepsilon(u) D_j u D_i \phi \to \int a_{ij} \Phi'(u) D_j u D_i \phi = \int a_{ij} D_j v D_i \phi,
$$

which gives the result. □

We use this lemma in the following way. The function $\Phi(t) = (\log t)^-$ is convex and satisfies local Lipschitz conditions on $(0, \infty)$. If $u \in H^1(B_4)$ is a positive supersolution to the equation, then $\Phi(u) = (\log u)^-$ is a subsolution, provided it is still in $H^1_{\text{loc}} B_4$, which is the case if, say, u is bounded from below by a positive number.

The next two theorems show that solutions to the equation (3) cannot oscillate too much. This is reminiscent of the behavior of harmonic functions, for example of Harnack's inequality for positive solutions to $\Delta u = 0$ on a domain Ω. It states that for any compact subset K of Ω, there is an absolute constant, depending only on K and Ω, such that

$$\sup_K u \leq C \inf_K u$$

holds for any positive harmonic function u on Ω.

But maybe these connections come as no surprise, for the Laplace equation is a special case of (1). In fact, Moser was able to prove an analogue of Harnack's inequality for weak solutions to (3). For more details, see [1], Chapter 4.4.

Theorem 2 (Density Theorem)**.** *Suppose that $u \in H^1(B_4)$ is a positive supersolution with*

$$\mathrm{m}\big(\{x \in B_2 : u \geq 1\}\big) \geq \varepsilon \mathrm{m}\big(B_2\big)$$

for some $\varepsilon > 0$. Then there exists a constant $C_7 = C_7(\varepsilon, n, \Lambda/\lambda) \in (0,1)$ such that

$$\inf_{B_1} u \geq C_7.$$

Proof. Since one can always add a small constant to u, we can assume that $u \geq \delta > 0$ (let $\delta \to 0+$ in the end). By Lemma 1, the function $v = (\log u)^-$ is a subsolution, bounded by $\log 1/\delta$. After a dilatation, Theorem 1 tells us that

$$\sup_{B_1} v \leq \frac{C_4}{\sqrt{2}} \|v\|_{L^2(B_2)}.$$

Since $\mathrm{m}\big(\{x \in B_2 : v = 0\}\big) = \mathrm{m}\big(\{x \in B_2 : u \geq 1\}\big) \geq \varepsilon \mathrm{m}\big(B_2\big)$, one of the versions of the Poincaré inequality (as discussed in class) implies

$$\sup_{B_1} v \leq \frac{C_4 C(\varepsilon, n)}{\sqrt{2}} \|Dv\|_{L^2(B_2)}. \tag{15}$$

To show that the right-hand side is bounded, use a test function $\phi = \zeta^2/u$ (with $\zeta \in C_0^1(B_2)$),

$$0 \leq \int a_{ij} D_j u D_i \left(\frac{\zeta^2}{u}\right) = -\int \zeta^2 \frac{a_{ij} D_j u D_i u}{u^2} + 2\int \frac{\zeta a_{ij} D_j u D_i \zeta}{u}.$$

From the ellipticity condition and Hölder inequality, one gets

$$\lambda \int \zeta^2 |D \log u|^2 \leq 2\Lambda n \left(\int \zeta^2 |D \log u|^2\right)^{\frac{1}{2}} \left(\int |D\zeta|^2\right)^{\frac{1}{2}},$$

which implies

$$\int \zeta^2 |D \log u|^2 \leq \frac{4\Lambda^2 n^2}{\lambda^2} \int |D\zeta|^2.$$

If we take a fixed $\zeta \in C_0^1(B_4)$ with $\zeta \equiv 1$ in B_2, we have

$$\int |D \log u|^2 \leq \frac{4\Lambda^2 n^2}{\lambda^2} \int |D\zeta|^2 = C_5. \tag{16}$$

where C_5 is a constant depending on n and Λ/λ.

Along with (15) we obtain

$$\sup_{B_1} v = \sup_{B_1} (\log u)^- \leq \frac{C_4 C(\varepsilon, n)}{\sqrt{2}} C_5 = C_6,$$

which gives

$$\inf_{B_1} u \geq e^{-C_6} = C_7 > 0.$$

□

Theorem 3 (Oscillation Theorem)**.** *Suppose that u is a solution of $Lu = 0$ in B_4. Then there exists a number $\gamma = \gamma(n, \Lambda/\lambda) \in (1/2, 1)$ such that*

$$\operatorname*{osc}_{B_{\frac{1}{2}}} u \leq \gamma \operatorname*{osc}_{B_2} u$$

Proof. In Corollary 1, it was shown that u is bounded on compact subsets of B_4. We may thus define

$$\alpha_1 = \sup_{B_2} u \quad \text{and} \quad \beta_1 = \inf_{B_2} u$$

as well as

$$\alpha_2 = \sup_{B_{\frac{1}{2}}} u \quad \text{and} \quad \beta_2 = \inf_{B_{\frac{1}{2}}} u.$$

Excluding the trivial case of constant u, the two functions

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \quad \text{and} \quad \frac{\alpha_1 - u}{\alpha_1 - \beta_1}$$

are positive solutions to the equation on B_2. Note the following two equivalences:

$$u \geq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2}$$
$$u \leq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq \frac{1}{2}$$

Depending on whether u is "generally big" or not, there are two possibilities.

Case 1. Suppose that

$$\mathrm{m}\Big(\Big\{x \in B_1 : 2\frac{u - \beta_1}{\alpha_1 - \beta_1} \geq 1\Big\}\Big) \geq \frac{1}{2}\mathrm{m}\big(B_1\big).$$

Apply the Density Theorem (with $\varepsilon = 1/2$) to the function $2\frac{u-\beta_1}{\alpha_1-\beta_1} \geq 0$, but in B_2 instead of in B_4. For some constant $C_7 \in (0,1)$, we have

$$\inf_{B_{\frac{1}{2}}} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{C_7}{2}$$

from which we obtain the estimate

$$\beta_2 = \inf_{B_{\frac{1}{2}}} u \geq \beta_1 + \frac{C_7}{2}(\alpha_1 - \beta_1).$$

Case 2. Now suppose that

$$\mathrm{m}\Big(\Big\{x \in B_1 : 2\frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq 1\Big\}\Big) \geq \frac{1}{2}\mathrm{m}\big(B_1\big).$$

Here, the result is

$$\alpha_2 = \sup_{B_{\frac{1}{2}}} u \leq \alpha_1 - \frac{C_7}{2}(\alpha_1 - \beta_1).$$

with the same constant C_7.

Since clearly $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$, we have in both cases

$$\alpha_2 - \beta_2 \leq \left(1 - \frac{C_7}{2}\right)(\alpha_1 - \beta_1),$$

which is our inequality with $\gamma = 1 - C_7/2$. □

After all that work, we are now able to come to the following, triumphant conclusion (called De Giorgi's theorem).

Theorem 4. *Suppose that $u \in H^1(B_4)$ is a weak solution of the equation $Lu = 0$ in B_4. Then there holds*

$$\sup_{x \in B_2} |u(x)| + \sup_{x,y \in B_2} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C_8(n, \Lambda/\lambda)\|u\|_{L^2(B_4)}$$

with some real number $\alpha = \alpha(n, \Lambda/\lambda) \in (0,1)$. Moreover, $u \in C^\alpha(B_4)$.

Proof. One half of the estimate, namely

$$\sup_{x \in B_2} |u(x)| \leq C_4(n, \Lambda/\lambda)\|u\|_{L^2(B_4)}, \tag{17}$$

is given by Theorem 1.

For the second half, fix two arbitray distinct points $x, y \in B_2$ and set $r = |x-y|$. For some $n \geq 0$, we have $4^{-n+1} > r \geq 4^{-n}$. Let us first consider the interesting case $n > 0$. By applying the oscillation theorem several times on suitable dilates of u, we obtain

$$\operatorname*{osc}_{B_r(x)} u \leq \gamma^{n-1} \operatorname*{osc}_{B_{4^{n-1}r}(x)} u \leq \gamma^{n-1} 2C_4\|u\|_{L^2(B_4)},$$

invoking Theorem 1 in the last step. In particular,

$$|u(x) - u(y)| \leq \gamma^{n-1} 2C_4\|u\|_{L^2(B_4)}.$$

Now $r \geq 4^{-n}$ and so $r^\alpha \geq 4^{-n\alpha}$ for any $\alpha \in (0,1)$. We get

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \gamma^{n-1} 4^{n\alpha} 2C_4\|u\|_{L^2(B_4)} \leq 8C_4(4^\alpha\gamma)^{n-1}\|u\|_{L^2(B_4)}.$$

Take α such that $4^\alpha\gamma = 1$; since $\gamma \in (1/2, 1)$, α actually falls in the range $(0, 1/2)$.

The case $n = 0$ is somewhat easier. Because $r \geq 1$, we directly conclude that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 2C_4 \|u\|_{L^2(B_4)}.$$

We see that the choice $C_8 = 8C_4$ gives the inequality in the theorem.

To prove that u is an element of $C^\alpha(B_4)$, that is to say, Hölder continuous on each compact subset of B_4, one can proceed in exactly the same way as in Corollary 1. We omit this argument. □

References

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