COMPUTING COHOMOLOGY OF LOCAL SYSTEMS

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1. Statement of the result

Let $\mathscr{V}$ be a holomorphic vector bundle on a complex manifold M, with a flat connection ∇. We shall make the following three assumptions:

(1) M is an open subset of a bigger complex manifold $\bar{M}$.
(2) The boundary $D = \bar{M} \setminus M$ is a divisor with normal crossing singularities.
(3) The connection ∇ is unipotent along D.

Under these assumptions, $\mathscr{V}$ has a canonical extension to a vector bundle $\bar{\mathscr{V}}$ on $\bar{M}$. Since ∇ has at worst logarithmic poles along D, it extends to a map

$$\nabla\colon \bar{\mathscr{V}} \to \bar{\mathscr{V}} \otimes \Omega^1_{\bar{M}}(\log D).$$

Moreover, the residue of ∇ along each component of D is a nilpotent operator.

In this note, we study the cohomology of the local system $\mathcal{H} = \ker \nabla$ of flat sections. In particular, we shall look at four complexes of quasi-coherent analytic sheaves on $\bar{M}$ that are built from the canonical extension, and that compute cohomology groups related to $\mathcal{H}$. The simplest one is the de Rham complex for $\mathscr{V}$ itself,

$$\mathscr{V} \xrightarrow{\nabla} \mathscr{V} \otimes \Omega^1_M \xrightarrow{\nabla} \mathscr{V} \otimes \Omega^2_M \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{V} \otimes \Omega^n_M,$$

n being the dimension of $\bar{M}$. This is a complex of vector bundles on M; pushed forward via the inclusion $i\colon M \to \bar{M}$, it becomes a complex of quasi-coherent sheaves on $\bar{M}$. To save space, we shall abbreviate its terms as

$$\mathscr{E}^p = i_*\Big(\mathscr{V} \otimes \Omega^p_M\Big) \tag{1}$$

for all $p \geq 0$.

Sections of $\mathscr{E}^p$ are allowed to have essential singularities along D, and so we shall also consider subcomplexes with better behavior. In the subcomplex with terms

$$\mathscr{E}^p(\infty D) = \bar{\mathscr{V}} \otimes \Omega^p_{\bar{M}}(\infty D), \tag{2}$$

only poles along D are allowed; in the subcomplex with terms

$$\mathscr{E}^p(\log D) = \bar{\mathscr{V}} \otimes \Omega^p_{\bar{M}}(\log D), \tag{3}$$

this is further restricted to just logarithmic poles.

Finally, the smallest complex that will be used has terms

$$\mathscr{E}^p_{hol} \subseteq \bar{\mathscr{V}} \otimes \Omega^p_{\bar{M}}, \tag{4}$$

where a section ω of $\bar{\mathscr{V}} \otimes \Omega^p_{\bar{M}}$ is in $\mathscr{E}^p_{hol}$ whenever both ω and $\nabla\omega$ are holomorphic.

Each of (1)–(4) defines a complex of quasi-coherent sheaves, with the differential given by the connection ∇. The terms in the complex (3) are actually holomorphic vector bundles, while those in (4) are coherent sheaves. (This is immediate, since the condition that ω and $\nabla\omega$ be holomorphic is stable under multiplication by holomorphic functions.)

The following two theorems explain the usefulness of the four complexes.

Theorem 1. *The hypercohomology of $\mathscr{E}^\bullet$, of $\mathscr{E}^\bullet(\infty D)$, and of $\mathscr{E}^\bullet(\log D)$ computes the cohomology of the local system $\mathcal{H}$ on M; in other words,*

$$H^p(M,\mathcal{H}) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet(\log D)\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet(\infty D)\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet\big).$$

Theorem 2. *Let $i\colon M \to \bar{M}$ be the inclusion map. The complex $\mathscr{E}^\bullet_{hol}$ is a resolution of the sheaf $i_*\mathcal{H}$ on $\bar{M}$; in particular, we have*

$$H^p\big(\bar{M}, i_*\mathcal{H}\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet_{hol}\big)$$

for all $p \geq 0$.

The proof of both theorems naturally falls into two parts—the first one a local computation of the cohomology of each complex; the second one formal arguments with hypercohomology. We shall carry out the local computations in the next section, and complete the proofs in Section 3.

2. Local computations

We begin our proof of Theorems 1 and 2 by doing some computations in local coordinates. Let $n = \dim \bar{M}$ be the dimension of the complex manifold $\bar{M}$. At an arbitrary point of $\bar{M}$, we choose a small open neighborhood isomorphic to Δ^n, with holomorphic coordinates $t_1, t_2, \dots, t_n$. Since $D = \bar{M} \setminus M$ is a divisor with normal crossings, this may be done in such a way that $D \cap \Delta^n$ is given by the equation $t_1 \cdots t_r = 0$. Thus we have

$$M \cap \Delta^n = (\Delta^*)^r \times \Delta^{n-r}.$$

To simplify the exposition, we shall only consider the case when $r = n$; the general case is no different from this special one, except for more cumbersome notation.

Canonical extension. Over Δ^n, the canonical extension $\bar{\mathscr{V}}$ is generated by a class of "distinguished" sections, whose construction is as follows. Let V be a general fiber of the vector bundle $\mathscr{V}$; it is a finite-dimensional complex vector space. The local system $\mathcal{H}$ has monodromy operators $T_1, \dots, T_n$, with T_j given by moving in a counter-clockwise direction around the hyperplane $t_j = 0$. By assumption, each T_j is a unipotent endomorphism of V, and so we can define its (nilpotent) logarithm

$$N_j = -\log T_j = \sum_{m=1}^{\infty} \frac{1}{m}(\mathrm{id} - T_j)^m.$$

On the universal cover

$$\mathbb{H}^n \to (\Delta^*)^n, \qquad (z_1, \dots, z_n) \mapsto \big(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}\big),$$

every element $v \in V$ now defines a holomorphic map $\tilde{s}\colon \mathbb{H}^n \to V$ by the rule

$$\tilde{s}(z) = e^{\sum z_j N_j} v.$$

Each $\tilde{s}$ descends to a holomorphic section s of $\mathscr{V}$ on $(\Delta^*)^n$, and these generate the vector bundle $\bar{\mathscr{V}}$ over Δ^n.

A formula for ∇s is easily obtained from this description. Indeed, we have

$$\nabla \tilde{s} = \sum_{k=1}^{n} N_k e^{z_j N_j} v \otimes dz_k,$$

and since $2\pi i \cdot dz_k = d \log t_k$, we find that

$$\nabla s = \frac{1}{2\pi i} \sum_{k=1}^{n} N_k s \otimes d \log t_k. \tag{5}$$

Of course, $N_k s$ is the section corresponding to the vector $N_k v$.

Sections and differential. An arbitrary section σ of the quasi-coherent sheaf $\mathscr{E}^p$ can now be written in the form

$$\sigma = \sum_{I,\alpha} \sigma_I(\alpha) \cdot t^\alpha \otimes \big(d \log t\big)_I \tag{6}$$

for a suitable choice of distinguished sections $\sigma_I(\alpha)$. In the summation, $\alpha = (\alpha_1, \dots, \alpha_n)$ runs over all multi-indices in $\mathbb{Z}^n$, and I over all subsets of $\{1, \dots, n\}$ of size $|I| = p$. Moreover, we are using the convenient abbreviations

$$t^\alpha = \prod_{i=1}^{n} t_i^{\alpha_i}$$

and

$$\big(d \log t\big)_I = \prod_{i \in I} d \log t_i = \prod_{i \in I} \frac{dt_i}{t_i}.$$

Evidently, σ is a section of $\mathscr{E}^p(\infty D)$ whenever $\sigma_I(\alpha) = 0$ for $|\alpha| \ll 0$; it is a section of the smaller bundle $\mathscr{E}^p(\log D)$, if $\sigma_I(\alpha) = 0$ unless $\alpha \geq 0$. We shall see later the condition for being a section of $\mathscr{E}^p_{hol}$.

From (5), we now get a formula for the differential ∇ in the complex. Namely, if σ is as in (6), then

$$\nabla \sigma = \sum_{I,\alpha,k} \left(\alpha_k + \frac{1}{2\pi i} N_k\right) \sigma_I(\alpha) \cdot t^\alpha \otimes \Big(d \log t_k \wedge \big(d \log t\big)_I\Big).$$

Thus we can write

$$\nabla \sigma = \sum_{J,\alpha} \tau_J(\alpha) \cdot t^\alpha \otimes \big(d \log t\big)_J,$$

the summation being over subsets $J \subseteq \{1, 2, \dots, n\}$ of size $(p+1)$. The coefficients are given by the formula[1]

$$\tau_J(\alpha) = \sum_{k \in J} \left(\alpha_k + \frac{1}{2\pi i} N_k\right) \sigma_{J \setminus \{k\}}(\alpha) \cdot (-1)^{\operatorname{pos}(k,J)}. \tag{7}$$

A nice feature of (7) is that the index α is unchanged by the differential, allowing us to treat each value of α by itself. Also note that, N_k being nilpotent, the operator

$$B_k = \alpha_k + \frac{1}{2\pi i} N_k$$

is invertible if, and only if, $\alpha_k \neq 0$.

[1] The symbol $\operatorname{pos}(k, J)$ denotes the position of k in the set J; if $J = \{j_0 < j_1 < \cdots < j_p\}$, then we have $j_{\operatorname{pos}(k,J)} = k$.

Exactness. To compute the cohomology of the complexes in question, it is best to abstract slightly. Thus we consider, in general, a complex of the form

$$M^0 \xrightarrow{\nabla} M^1 \xrightarrow{\nabla} M^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} M^n.$$

We shall assume that elements of M^p are given by

$$\sigma = \bigl(\sigma_I\bigr)_{|I|=p},$$

indexed by subsets $I \subseteq \{1, 2, \dots, n\}$ of size p, and that the differential ∇ is given by the formula

$$\nabla\sigma = \bigl(\tau_J\bigr)_{|J|=p+1}$$

with

$$\tau_J = \sum_{k\in J} (-1)^{\operatorname{pos}(k,J)} B_k \sigma_{J\setminus\{k\}}.$$

In this general setting, B_k is allowed to be an arbitrary operator. The following lemma gives the condition for the complex $M^\bullet$ to be exact.

Lemma 3. *If at least one of the operators B_k is invertible, then $\bigl(M^\bullet, \nabla\bigr)$ is an exact complex.*

Proof. Renumbering, if necessary, we may assume that B_1 is invertible. We are going to prove that the complex is, in fact, contractible. A contracting homotopy $\varepsilon\colon M^p \to M^{p-1}$ may be defined[2] by the following rule, for $p \geq 1$:

$$\varepsilon(\sigma) = \Bigl(\bigl[1 \notin J\bigr] \cdot B_1^{-1} \sigma_{J\cup\{1\}}\Bigr)_{|J|=p-1}$$

A short computation shows that

$$\begin{aligned} \nabla\varepsilon(\sigma) &= \left(\sum_{k\in J} (-1)^{\operatorname{pos}(k,J)} \bigl[1 \notin J \setminus \{k\}\bigr] \cdot B_k B_1^{-1} \sigma_{J\cup\{1\}\setminus\{k\}}\right)_{|J|=p} \\ &= \left(\bigl[1 \in J\bigr] \cdot \sigma_J + \bigl[1 \notin J\bigr] \sum_{k\in J} (-1)^{\operatorname{pos}(k,J)} B_k B_1^{-1} \sigma_{J\cup\{1\}\setminus\{k\}}\right)_{|J|=p}, \end{aligned}$$

while

$$\begin{aligned} \varepsilon(\nabla\sigma) &= \left(\bigl[1 \notin J\bigr] \cdot B_1^{-1} \sum_{k\in J\cup\{1\}} (-1)^{\operatorname{pos}(k,J)} B_k \sigma_{J\cup\{1\}\setminus\{k\}}\right)_{|J|=p} \\ &= \left(\bigl[1 \notin J\bigr] \cdot \sigma_J - \bigl[1 \notin J\bigr] \sum_{k\in J} (-1)^{\operatorname{pos}(k,J)} B_k B_1^{-1} \sigma_{J\cup\{1\}\setminus\{k\}}\right)_{|J|=p}. \end{aligned}$$

It follows that $\nabla\varepsilon + \varepsilon\nabla = \mathrm{id}$, and this shows that the complex is contractible, hence exact. □

In the case of our complexes, with differential given by (7), the operator B_k is invertible precisely when $\alpha_k \neq 0$. Applying Lemma 3 to this situation, it follows

[2] We are using the notation $\bigl[\langle\text{condition}\rangle\bigr]$, which is defined as 1 if $\langle\text{condition}\rangle$ is true, and as 0 if $\langle\text{condition}\rangle$ is false.

that each complex is exact whenever $\alpha \neq 0$. For $\alpha = 0$, we get a complex with terms

$$M^p = \bigoplus_{|I|=p} V$$

and differential

$$\nabla\big(\sigma_I\big)_{|I|=p} = \left(\frac{1}{2\pi i}\sum_{k\in J}(-1)^{\mathrm{pos}(k,J)} N_k \sigma_{J\setminus\{k\}}\right)_{|J|=p+1} \tag{8}$$

from the description above. Therefore, the cohomology on Δ^n of $\mathscr{E}^\bullet$, of $\mathscr{E}^\bullet(\infty D)$, and of $\mathscr{E}^\bullet(\log D)$ is the same, and agrees with that of the complex for $\alpha = 0$ just given.

Group cohomology. We shall now compute the cohomology of the complex for $\alpha = 0$; it will turn out to be equal to the group cohomology $H^*(G, V)$, where $G = \mathbb{Z}^n$ is the fundamental group of $(\Delta^*)^n$, acting by the monodromy operators $T_1, \dots, T_n$ on the vector space V.

Lemma 4. *The cohomology of the complex* (8) *is the group cohomology* $H^*(G, V)$.

Proof. The group cohomology is easy to describe in this case; by definition,

$$H^*(G, V) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, V),$$

and since $\mathbb{Z}G \simeq \mathbb{Z}[T_1, \dots, T_n]$, a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module is given by the Koszul complex for $\big(T_1 - 1, \dots, T_n - 1\big)$. Thus $H^*(G, V)$ is the cohomology of the complex with terms

$$\hat{M}^p = \bigoplus_{|I|=p} V$$

and differential

$$\hat{\nabla}\big(\sigma_I\big)_{|I|=p} = \left(\sum_{k\in J}(-1)^{\mathrm{pos}(k,J)} (T_k - \mathrm{id}) \sigma_{J\setminus\{k\}}\right)_{|J|=p+1} \tag{9}$$

similar to (8). Noting that we have

$$T_k - \mathrm{id} = \frac{1}{2\pi i} N_k \cdot R_k$$

with R_k invertible, we can define an isomorphism $M^p \to \hat{M}^p$ between the two complexes by

$$\sigma_I \mapsto \prod_{i\in I} R_i \cdot \sigma_I.$$

It is easily seen to be compatible with ∇ and $\hat{\nabla}$, proving our claim. □

Conclusion. To conclude the local computations, we need to know the cohomology of the local system $\mathcal{H}$ on $(\Delta^*)^n$.

Lemma 5. *We have* $H^p\big((\Delta^*)^n, \mathcal{H}\big) \simeq H^p(G, V)$ *for all* $p \geq 0$.

Proof. Since the universal covering space $\mathbb{H}^n$ of $(\Delta^*)^n$ is contractible, the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}G}\Big(H_q\big(\mathbb{H}^n, \mathbb{Z}\big), V\Big) \Longrightarrow H^{p+q}\big((\Delta^*)^n, \mathcal{H}\big)$$

degenerates at the E_2-page. This gives isomorphisms

$$H^p\big((\Delta^*)^n, \mathcal{H}\big) \simeq \operatorname{Ext}^p_{\mathbb{Z}G}(\mathbb{Z}, V) = H^p(G, V),$$

and thus proves the lemma. □

Combining the results of Lemma 4 and of Lemma 5, we get the following statement.

Proposition 6. *On a suitable neighborhood Δ^n of each point in $\bar{M}$, the cohomology of each of the three complexes in (1)–(3) is isomorphic to $H^*\big(M \cap \Delta^n, \mathcal{H}\big)$.*

3. Proof of the two theorems

Theorem 1 follows from the local analysis in the previous section, with just a small dose of formal arguments about hypercohomology.

Proof of Theorem 1. Let us write $\mathscr{A}^p_M$ for the sheaf of smooth differential p-forms on the complex manifold M. It is a fine sheaf, and in consequence, has trivial higher cohomology groups. Since ∇ is flat, it is a well-known result that the complex

$$\mathscr{V} \xrightarrow{\nabla} \mathscr{V} \otimes \mathscr{A}^1_M \xrightarrow{\nabla} \mathscr{V} \otimes \mathscr{A}^2_M \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{V} \otimes \mathscr{A}^n_M$$

is a fine resolution of the local system $\mathcal{H} = \ker \nabla$ of flat sections.

Now consider the push-forward of that complex to $\bar{M}$, with terms

$$\mathscr{E}^p_{\mathscr{A}} = i_*\Big(\mathscr{V} \otimes \mathscr{A}^p_M\Big);$$

this is a complex of fine sheaves on $\bar{M}$. On each neighborhood Δ^n considered in the previous section, its cohomology clearly equals $H^*\big(M \cap \Delta^n, \mathcal{H}\big)$; it follows from Proposition 6 that the complex is quasi-isomorphic to each of the three subcomplexes in (1)–(3). Thus we have

$$\mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet(\log D)\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet(\infty D)\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet\big) \simeq \mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet_{\mathscr{A}}\big).$$

But, at the same time,

$$\mathbb{H}^p\big(\bar{M}, \mathscr{E}^\bullet_{\mathscr{A}}\big) \simeq H^p\Big(H^0\big(M, \mathscr{V} \otimes \mathscr{A}^\bullet_M\big), \nabla\Big) \simeq H^p\big(M, \mathcal{H}\big),$$

since the complex consists of fine sheaves. Combining both isomorphisms now gives the desired result. □

Finally, we give the proof of Theorem 2.

Proof of Theorem 2. We need to show that the complex with terms $\mathscr{E}^p_{hol}$ is a resolution of the sheaf $i_*\mathcal{H}$ on $\bar{M}$. This is clearly a local question, and so we consider a neighborhood Δ^n of an arbitrary point of $\bar{M}$, as above. A section σ as in (6) belongs to $\mathscr{E}^p_{hol}$ if, and only if, both σ and $\nabla\sigma$ are holomorphic. The first condition means that $\sigma_I(\alpha) = 0$, unless each $\alpha_k \geq 0$, and $\alpha_k \geq 1$ for all $k \in I$. By our analysis in Lemma 3, the complex in question is therefore always exact if $I \neq \emptyset$, and hence in all positive degrees.

In degree zero, on the other hand, the complex can only fail to be exact for $\alpha = 0$, which means that cohomology can only occur when σ is itself a distinguished section, associated to some element $v \in V$. In that case, $\nabla\sigma$ can only be holomorphic if $N_k v = 0$ for all k, and so the cohomology in degree zero is precisely the subspace $V^G \subseteq V$ of G-invariants. Since we also have

$$H^0\big(\Delta^n, i_*\mathcal{H}\big) = H^0\big((\Delta^*)^n, \mathcal{H}\big) = V^G,$$

it follows that the complex $\mathscr{E}^\bullet_{hol}$ is indeed a resolution of $i_*\mathcal{H}$. The assertion about hypercohomology follows from this. $\square$