

DEFORMATIONS OF COMPLEX STRUCTURES

CHRISTIAN SCHNELL

1. COMPARING ALMOST COMPLEX STRUCTURES

In this section, we discuss briefly how to compare different almost complex structures on a manifold. Let M be a fixed compact manifold, TM its tangent bundle. Let $J \in \text{End}(TM)$ be an almost complex structure on M , with associated decomposition

$$TM \otimes \mathbb{C} \simeq TM_{0,1} \oplus TM_{1,0},$$

and projections $\pi_{0,1}$ and $\pi_{1,0}$ to the two summands.

Now suppose J' is a second almost complex structure. The reader should think of J' as a small deformation of the fixed almost complex structure J . If J' is “sufficiently close” to J , then $\pi_{0,1}$ gives an isomorphism between $TM'_{0,1}$ and $TM_{0,1}$, and we thus get a map

$$TM_{0,1} \xrightarrow{(\pi_{0,1})^{-1}} TM'_{0,1} \xrightarrow{\pi_{1,0}} TM_{1,0};$$

this map may conveniently be viewed as a $(0,1)$ -form with values in the bundle $TM_{1,0}$, in other words, as an element ξ of $A^{0,1}(TM_{1,0})$. Conversely, this element determines the subspace $TM'_{0,1}$, and thus J' .

Local coordinates. In local coordinates, the situation is especially easy to understand. Assume, for simplicity, that J actually defines a complex structure, and that z^1, \dots, z^n are local holomorphic coordinates on M . Then ξ may be written in the form

$$\xi = \sum_{i,u} h_i^u(z) d\bar{z}^i \otimes \frac{\partial}{\partial z^u};$$

the corresponding subspace $TM'_{0,1}$ is spanned by the images of $\partial/\partial \bar{z}^i$, in other words, by the n vector fields

$$\frac{\partial}{\partial \bar{z}^i} + \sum_i h_i^u(z) \frac{\partial}{\partial z^u} \quad (i = 1, \dots, n).$$

2. FAMILIES OF COMPACT COMPLEX MANIFOLDS

From now on, we let M be a fixed compact complex manifold of dimension n , and Δ a small ball centered at the origin in \mathbb{C}^m .

Let $\pi: \mathfrak{X} \rightarrow \Delta$ be a *family of deformations* of M over Δ ; this means that π should be a proper and submersive holomorphic map from the complex manifold \mathfrak{X} to Δ , and that $M = \pi^{-1}(0)$. All fibers of the map π are themselves compact complex manifolds; while they are all diffeomorphic to M , they typically carry different complex structures. The reader should think of them as being small deformations of the *central fiber* $M = \pi^{-1}(0)$.

By Ehresmann's fibration theorem, we have $\mathfrak{X} \simeq M \times \Delta$ as differentiable manifolds (but not, in general, as complex manifolds). Actually, it can be shown that a family $\mathfrak{X} \rightarrow \Delta$ always admits a *transversely holomorphic trivialization*

$$(\phi, \pi): \mathfrak{X} \rightarrow M \times \Delta;$$

in other words, one can always find a second map $\phi: \mathfrak{X} \rightarrow M$ (not usually holomorphic) such that

- (1) (ϕ, π) is a diffeomorphism;
- (2) ϕ is the identity when restricted to $\pi^{-1}(0) = M$; and
- (3) the fibers of ϕ are holomorphic submanifolds of \mathfrak{X} , transverse to M .

The reader will note that any fiber of ϕ is then biholomorphic to Δ via the holomorphic map π .

An example, first part. Along with our general discussion, we shall work out one special example, the family of elliptic curves.

In the example, we let M be the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot i)$. The complex structure on M can be varied by changing the lattice, moving i to a different point $\tau \in \mathbb{H}$ of the upper half-plane. In this spirit, we consider the family $\pi: \mathfrak{X} \rightarrow \mathbb{H}$ obtained by taking the quotient of the complex manifold $\mathbb{C} \times \mathbb{H}$ by the two relations $(w, \tau) \sim (w + 1, \tau)$ and $(w, \tau) \sim (w + \tau, \tau)$. Since $\mathbb{C} \times \mathbb{H} \rightarrow \mathfrak{X}$ is a covering space map, \mathfrak{X} is still a complex manifold. Let π be the projection to \mathbb{H} , which is still a holomorphic map.

There is a very simple transversely holomorphic trivialization

$$F: \mathfrak{X} \rightarrow M \times \mathbb{H}$$

in this case. It is easiest to define the inverse first; we can let $\mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C} \times \mathbb{H}$ be the map that takes a point $(z = x + iy, \tau)$ to $(w = x + \tau y, \tau)$. This then descends to a diffeomorphism $M \times \mathbb{H} \rightarrow \mathfrak{X}$, and F is the inverse.

The formulas are not hard to write down either. Let us write points on \mathfrak{X} as $[w, \tau]$, and points of M as $[z]$ (these being the images of (w, τ) and z , respectively); then

$$F^{-1}([z], \tau) = \left[\frac{z + \bar{z}}{2} + \tau \cdot \frac{z - \bar{z}}{2i}, \tau \right],$$

from which one finds that

$$F([w, \tau]) = \left(\left[\frac{\tau - i}{\tau - \bar{\tau}} \cdot \bar{w} - \frac{\bar{\tau} - i}{\tau - \bar{\tau}} \cdot w \right], \tau \right).$$

It is quite evident that F is transversely holomorphic; indeed, the formula for F^{-1} shows that if $[z] \in M$ is held fixed, the point $[w, \tau]$ moves holomorphically with $\tau \in \mathbb{H}$. This is all we are going to say for the time being; we shall return to this example later on.

Families as sections of a bundle. We are now going to show how a family $\pi: \mathfrak{X} \rightarrow \Delta$ (actually, the family together with the transversely holomorphic trivialization) can be represented by a section of a certain bundle on $M \times \Delta$.

Using the diffeomorphism (ϕ, π) , we can transfer the complex structure from \mathfrak{X} to a complex structure J' on the product $M \times \Delta$. Items (1)–(3) above imply that all vertical and horizontal slices are complex submanifolds; each horizontal slice of the form $\{z\} \times \Delta$ carries the same complex structure as Δ itself, while the complex

structure on the vertical slices $M \times \{t\}$ varies with t and, in general, only agrees with that of M for $t = 0$. The second projection pr_Δ is still a holomorphic map.

This said, we compare J' to the product complex structure J as described in Section 1. To get J' to be sufficiently close to J , we might have to shrink Δ , but then we can completely describe J' by an element $(\xi(t), \eta(t))$, say, of

$$A^{0,1}(M \times \Delta, T(M \times \Delta)_{1,0}) = A^{0,1}(M \times \Delta, TM_{1,0}) \oplus A^{0,1}(M \times \Delta, T\Delta_{1,0}).$$

But since J' is obtained from complex structure on \mathfrak{X} through the diffeomorphism (ϕ, π) , we have

$$T(M \times \Delta)'_{0,1} = (\phi, \pi)_* T\mathfrak{X}_{0,1} = \phi_* T\mathfrak{X}_{0,1} \oplus T\Delta_{0,1},$$

using that π is holomorphic and submersive, and that each fiber of ϕ is a holomorphic submanifold of \mathfrak{X} , isomorphic to Δ via the map π . It follows that $\eta(t)$ is identically zero, while $\xi(t)$ is zero on all tangent vectors in $T\Delta_{0,1}$. Up to isomorphism, the original family is thus faithfully represented by the datum of

$$\xi(t) \in A^{0,1}(M \times \Delta, TM_{1,0}) \quad \text{such that} \quad \xi(t) \text{ is zero on } T\Delta_{1,0} \text{ and } \xi(0) = 0.^1$$

Equivalently, if $\mathcal{A}^{0,1}(TM_{1,0})$ is the bundle of $TM_{1,0}$ -valued $(0,1)$ -forms on M , the form $\xi(t)$ may be viewed as a smooth section of the pullback bundle $pr_M^* \mathcal{A}^{0,1}(TM_{1,0})$ with $\xi(0) = 0$.

Local coordinates. To allow computations, we again give a description in local coordinates. Thus let z^1, \dots, z^n be local holomorphic coordinates on M , and let t^1, \dots, t^m be holomorphic coordinates on the disk Δ . Of course, $z^1, \dots, z^n, t^1, \dots, t^m$ are not usually holomorphic coordinates for the complex structure J' ; however, together with their conjugates, they do give a smooth coordinate system which can be used to describe vector fields and differential forms on $M \times \Delta$.

We may thus write $\xi(t)$ in the form

$$\xi(t) = \sum_{i,u} h_i^u(z, t) dz^i \otimes \frac{\partial}{\partial z^u},$$

where $h_i^u(z, 0) = 0$; at the point $(z, t) \in M \times \Delta$, the corresponding $(0,1)$ -subspace for J' is then spanned by the $m+n$ vectors

$$\frac{\partial}{\partial \bar{z}^i} + \sum_u h_i^u(z, t) \frac{\partial}{\partial z^u} \quad \text{and} \quad \frac{\partial}{\partial t^k} \quad (1 \leq i \leq n, 1 \leq k \leq m).$$

An example, second part. Returning to the example from above, let us figure out what ξ has to be. In order to do this, we need to know which vector fields span the $(0,1)$ -subbundle of the complexified tangent bundle on $M \times \mathbb{H}$. There is, obviously, the vector field $\partial/\partial \bar{\tau}$ (from \mathbb{H}); on the other hand, there is the image $F_* \partial/\partial \bar{w}$ of the vector field $\partial/\partial \bar{w}$ on \mathfrak{X} .

Maybe we should first convince ourselves that $\partial/\partial \bar{w}$ is really well-defined. Consider a function ϕ on \mathfrak{X} . If ϕ is pulled back to a the function ψ on $\mathbb{C} \times \mathbb{H}$, then

$$\psi(w+1, \tau) = \psi(w, \tau) \quad \text{and} \quad \psi(w+\tau, \tau) = \psi(w, \tau).$$

Differentiating with respect to w shows that $\partial\psi/\partial w$ satisfies the same identities, and thus descends to a function on \mathfrak{X} that we write as $\partial\phi/\partial w$. The vector fields $\partial/\partial w$ and $\partial/\partial \bar{w}$ on \mathfrak{X} are to be understood in this way.

¹We have $\xi(0) = 0$ because the complex structure on the central fiber $M \times \{0\}$ is the same for both J and J' .

The image of $\partial/\partial\bar{w}$ is now readily computed as

$$F_* \frac{\partial}{\partial\bar{w}} = \frac{\partial z}{\partial\bar{w}} \cdot \frac{\partial}{\partial z} + \frac{\partial\bar{z}}{\partial\bar{w}} \cdot \frac{\partial}{\partial\bar{z}} = \frac{\tau - i}{\tau - \bar{\tau}} \cdot \frac{\partial}{\partial z} + \frac{\tau + i}{\tau - \bar{\tau}} \cdot \frac{\partial}{\partial\bar{z}},$$

using the formula for $F([w, \tau]) = ([z], \tau)$ above. Thus a vector field spanning, together with $\partial/\partial\bar{\tau}$, the $(0, 1)$ -subspace at each point is, for instance,

$$\frac{\tau - i}{\tau + i} \cdot \frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}.$$

By comparing this to our explicit formula in local coordinates, we see that

$$\xi(\tau) = \frac{\tau - i}{\tau + i} d\bar{z} \otimes \frac{\partial}{\partial z}$$

for the family in question.

As the reader will have noticed, the base of the family was the upper half-plane, and not a disk. We can remedy this defect by aid of the biholomorphic map

$$\mathbb{H} \rightarrow \Delta, \quad \tau \mapsto t = \frac{\tau - i}{\tau + i}$$

from the upper half-plane to the unit disk; composing with this map, we may view \mathfrak{X} as a family over the disk. The map conveniently sends the point $\tau = i$ to the point $t = 0$, thus centering our family at the origin of Δ . With respect to the new parameter t , the expression for ξ now becomes especially simple,

$$\xi(t) = t \cdot d\bar{z} \otimes \frac{\partial}{\partial\bar{z}}.$$

We shall see later how the general theory we are going to develop also gives this result.

Integrability. We return to the general situation. At this point, we have described an arbitrary deformation of the complex structure on M as an *almost* complex structure. In general, a section $\xi(t)$ of the bundle above gives rise only to an almost complex structure on $M \times \Delta$, and might not come from a complex structure. Thus if we want $\xi(t)$ to correspond to an actual family of complex manifolds, we have to decide when the almost complex structure defined by $\xi(t)$ on $M \times \Delta$ is integrable.

In order to do this, we first need to define the following bracket operation on $A^{0,1}(TM_{1,0})$ (and, by a similar formula, on $A^{0,q}(TM_{1,0})$ for all q). The bracket $[\phi, \psi]$ of two bundle-valued forms $\phi, \psi \in A^{0,1}(TM_{1,0})$ is an element of $A^{0,2}(TM_{1,0})$ —in local coordinates, if

$$\phi = \sum g_i^u d\bar{z}^i \otimes \frac{\partial}{\partial z^u} \quad \text{and} \quad \psi = \sum h_j^v d\bar{z}^j \otimes \frac{\partial}{\partial z^v},$$

then

$$\begin{aligned} [\phi, \psi] &= \sum d\bar{z}^i \wedge d\bar{z}^j \otimes \left[g_i^u \frac{\partial}{\partial z^u}, h_j^v \frac{\partial}{\partial z^v} \right] \\ &= \sum d\bar{z}^i \wedge d\bar{z}^j \otimes \left(g_i^u \frac{\partial h_j^v}{\partial z^u} \frac{\partial}{\partial z^v} - h_j^v \frac{\partial g_i^u}{\partial z^v} \frac{\partial}{\partial z^u} \right) \in A^{0,2}(TM_{1,0}). \end{aligned}$$

We also define the del-bar operator $\bar{\partial}$ by

$$\bar{\partial}\psi = \sum \frac{\partial h_j^v}{\partial\bar{z}^i} d\bar{z}^i \wedge d\bar{z}^j \otimes \frac{\partial}{\partial z^v} \in A^{0,2}(TM_{1,0}).$$

The reader should check that both definitions are independent of the choice of local holomorphic coordinates. The two operations also make sense on sections of $pr_M^* \mathcal{A}^{0,q}(TM_{1,0})$ since they “do not notice the t .” To emphasize that the del-bar operator acts only in the direction of M , we shall write $\bar{\partial}_M$ when applying it to sections of $pr_M^* \mathcal{A}^{0,q}(TM_{1,0})$.

Here is the condition for integrability.

Theorem 1. *A smooth section $\xi(t)$ of $\mathcal{A}^{0,1}(TM_{1,0})$ corresponds to an integrable almost complex structure on $M \times \Delta$ if, and only if,*

$$\bar{\partial}_M \xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0 \quad \text{and} \quad \xi(t) \text{ is holomorphic in } t.$$

Such $\xi(t)$ will be called integrable.

Proof. We use the criterion of the Newlander-Nirenberg theorem to check integrability. Thus we need to show that the $(0,1)$ -distribution on $M \times \Delta$ corresponding to $\xi(t)$ is closed under taking brackets (of tangent vectors, not in the sense above) exactly when the two conditions stated in the theorem hold.

In local coordinates z, t as above, the $(0,1)$ -distribution has basis

$$\frac{\partial}{\partial \bar{z}^i} + \sum_u h_i^u(z, t) \frac{\partial}{\partial z^u} \quad \text{and} \quad \frac{\partial}{\partial t^k} \quad (1 \leq i \leq n, 1 \leq k \leq m);$$

the one-forms that vanish on this basis are thus exactly

$$dz^u - \sum_i h_i^u(z, t) d\bar{z}^i \quad \text{and} \quad dt^k \quad (1 \leq u \leq n, 1 \leq k \leq m).$$

Now we compute, using the summation convention to make the formulas more transparent. First,

$$\left[\frac{\partial}{\partial t^k}, \frac{\partial}{\partial \bar{z}^i} + h_i^u \frac{\partial}{\partial z^u} \right] = \frac{\partial h_i^u}{\partial t^k} \frac{\partial}{\partial z^u}$$

which is clearly again in the distribution if, and only if, h_i^u is holomorphic in t for all i, u . Thus this part of the integrability condition is exactly that $\xi(t)$ should be holomorphic in t .

Next,

$$\left[\frac{\partial}{\partial \bar{z}^i} + h_i^u \frac{\partial}{\partial z^u}, \frac{\partial}{\partial \bar{z}^j} + h_j^v \frac{\partial}{\partial z^v} \right] = \left(\frac{\partial h_j^v}{\partial \bar{z}^i} + h_i^u \frac{\partial h_j^v}{\partial z^u} \right) \frac{\partial}{\partial z^v} - \left(\frac{\partial h_i^u}{\partial \bar{z}^j} + h_j^v \frac{\partial h_i^u}{\partial z^v} \right) \frac{\partial}{\partial z^u};$$

for this to be again in the distribution, each form $dz^w - h_k^w d\bar{z}^k$ should vanish on it; in other words, for all i, j, w , we should have

$$\left(\frac{\partial h_j^w}{\partial \bar{z}^i} + h_i^u \frac{\partial h_j^w}{\partial z^u} \right) - \left(\frac{\partial h_i^w}{\partial \bar{z}^j} + h_j^v \frac{\partial h_i^w}{\partial z^v} \right) = 0.$$

If these equations are multiplied by $d\bar{z}^i \wedge d\bar{z}^j \otimes \partial/\partial z^w$ and summed, then after collecting terms, one obtains exactly

$$2\bar{\partial}_M \xi(t) + [\xi(t), \xi(t)] = 0,$$

and conversely this last equation implies the vanishing of all the individual components given above. Thus the second part of the integrability condition is also as stated, and the theorem is proved. \square

The two conditions in the theorem have different interpretations. The first,

$$\bar{\partial}_M \xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0,$$

requires that for each $t \in \Delta$, the almost complex structure defined by $\xi(t)$ on $M \times \{t\}$ should be integrable. With only this condition, $\xi(t)$ gives a smoothly varying family of complex manifolds. The second condition, that $\xi(t)$ be holomorphic in t , requires that these complex structures should vary holomorphically with $t \in \Delta$.

Some remarks about DGLA. This is a good point to introduce the notion of a *differential graded Lie algebra*. A DGLA consists of

- a \mathbb{Z} -graded vector space $L = \bigoplus_{q \in \mathbb{Z}} L^q$;
- a bilinear map $[-, -]: L \otimes L \rightarrow L$, called the bracket;
- a linear map $d: L \rightarrow L$, called the differential.

These data have to satisfy several axioms. The differential d should be such that

$$d \circ d = 0, \quad d(L^q) \subseteq L^{q+1}, \quad d[a, b] = [da, b] + (-1)^{\deg a} [a, db],$$

while the bracket has to obey the rules

$$[L^q, L^r] \subseteq L^{q+r}, \quad [b, a] + (-1)^{\deg a \deg b} [a, b] = 0,$$

as well as the Jacobi identity

$$(-1)^{\deg a \deg c} [a, [b, c]] + (-1)^{\deg b \deg a} [b, [c, a]] + (-1)^{\deg c \deg b} [c, [a, b]] = 0.$$

The signs in this identity are chosen according to the usual convention for graded objects—interchanging the order of two elements of degree q and r , respectively, produces a sign of $(-1)^{qr}$. This makes formulas easier to remember.

It is not hard to verify that

$$L = \bigoplus_{q \geq 0} A^{0,q}(TM_{1,0}),$$

together with the bracket operation defined above, and $\bar{\partial}$ as the differential, is a DGLA. It is usually called the *Kodaira-Spencer DGLA*.

The equation $da + \frac{1}{2}[a, a] = 0$ for $a \in L^1$ occurring in the theorem is often called the *Maurer-Cartan equation* of the DGLA; in our case, its solutions—as we have seen—correspond to integrable almost complex structures on M that are close to the original complex structure.

The Kodaira-Spencer map. The Kodaira-Spencer map of a family (given by an integrable form $\xi(t)$) also admits of a very simple description in terms of $\xi(t)$. If $v \in T_0 \Delta_{1,0}$ is a tangent vector to Δ at 0, we can form $v \cdot \xi \in A^{0,1}(TM_{1,0})$ by differentiating $\xi(t)$ along v (this includes evaluating at $t = 0$). In local coordinates, if

$$v = \sum_k a^k \frac{\partial}{\partial t^k} \Big|_{t=0} \quad \text{and} \quad \xi(t) = \sum_{i,u} h_i^u(z, t) d\bar{z}^i \otimes \frac{\partial}{\partial z^u},$$

then

$$v \cdot \xi = \sum_k a^k \frac{\partial h_i^u}{\partial t^k}(z, 0) d\bar{z}^i \otimes \frac{\partial}{\partial z^u}.$$

Now $v \cdot \xi$ is actually $\bar{\partial}$ -closed, because $\xi(t)$ is integrable; indeed, we compute

$$\bar{\partial}(v \cdot \xi) = v \cdot (\bar{\partial}_M \xi) = -\frac{1}{2} v \cdot [\xi, \xi] = -[v \cdot \xi, \xi(0)] = 0,$$

using one of the DGLA identities and the fact that $\xi(0) = 0$. Thus $v \cdot \xi$ defines a class in $H^1(M, TM_{1,0})$, and one can check that the map

$$T_0\Delta_{1,0} \rightarrow H^1(M, TM_{1,0}), \quad v \mapsto [v \cdot \xi],$$

is the Kodaira-Spencer map.

3. THE KODAIRA-NIRENBERG-SPENCER THEOREM

One motivation of the work that Kodaira and Spencer did was to find a family of deformations of a given compact complex manifold M in which “every possible small deformation” occurs, meaning that the Kodaira-Spencer map should be surjective. The following theorem solves this problem in a special case; it was conjectured by Kodaira and Spencer, but only proved together with Nirenberg.

Theorem 2. *Let M be a compact complex manifold such that $H^2(M, TM_{1,0}) = 0$. Then there is a small ball $\Delta \subseteq H^1(M, TM_{1,0})$, centered at 0, and a family $\pi: \mathfrak{X} \rightarrow \Delta$ of deformations of M whose Kodaira-Spencer map*

$$H^1(M, TM_{1,0}) \simeq T_0\Delta_{1,0} \rightarrow H^1(M, TM_{1,0})$$

is the identity.

Since we already know that $H^1(M, TM_{1,0})$ parametrizes first-order infinitesimal deformations of M , we can express the content of the theorem differently: If $H^2(M, TM_{1,0}) = 0$, then every first-order deformation of M , represented by some $v \in H^1(M, TM_{1,0})$, can be “integrated.” In other words, there is an actual analytic deformation of M in the direction of v , namely the one obtained from the family in the theorem by restricting to the line segment $\Delta \cap \mathbb{C} \cdot v$.

Of course, to prove the theorem, we should construct a section $\xi(t)$ of the bundle $pr_M^* \mathcal{A}^{0,1}(TM_{1,0})$ over $M \times \Delta$, for a suitable small disk $\Delta \subseteq H^1(M, TM_{1,0})$. This $\xi(t)$ needs to satisfy four conditions:

- (1) $\xi(0) = 0$ (the central fiber should be M);
- (2) $v \cdot \xi = v$ for every $v \in H^1(M, TM_{1,0})$ (the Kodaira-Spencer map should be the identity);
- (3) $\bar{\partial}_M \xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0$ (integrability on each vertical slice $M \times \{t\}$);
- (4) $\bar{\partial}_\Delta \xi(t) = 0$ (second condition for global integrability).

Formal computations. The construction of $\xi(t)$ requires a substantial amount of analysis. The underlying idea, however, is rather simple, and is based on an earlier, unsuccessful method of Kodaira and Spencer. In trying to prove Theorem 2, they attempt a purely *formal* approach to the problem, using power series in $t \in \Delta$. In this approach, all questions of smoothness or convergence are ignored, and a purely formal solution satisfying (3) is sought.

We let $m = h^1(M, TM_{1,0})$, and introduce variables $t = (t_1, \dots, t_m)$. To avoid confusion, we shall always use the following *notational convention*. Capital Greek letters, especially Ξ , will stand for formal objects, either polynomials or formal power series in t , with coefficients in $A^{0,q}(TM_{1,0})$ for some q . Any operation like $\bar{\partial}$ or $[_, _]$ performed on such objects is to be applied termwise. No assumptions about convergence or smoothness are made, and sums and products of series are to be taken formally.

On the other hand, small Greek letters, especially ξ , will be used to denote actual sections of $\mathcal{A}^{0,q}(TM_{1,0})$ over some product $M \times \Delta$; if we want to emphasize the

dependence on t , we write $\xi(t)$, but we might suppress t during computations. If a formal series Ξ does converge for $|t| < \varepsilon$ (in an appropriate sense, to be given below), thus defining a section $\xi(t)$ over $M \times \Delta(\varepsilon)$, we shall still distinguish between the formal object Ξ and the section $\xi(t)$ it gives rise to.

This said, the idea of the formal approach is to construct polynomials Ξ_1, Ξ_2, \dots , each Ξ_s homogeneous of degree s in the variables t , with coefficients in $A^{0,1}(TM_{1,0})$, such that $\Xi = \Xi_1 + \Xi_2 + \dots$ is a formal solution to the problem.

To make sure that condition (2) above is satisfied, we choose $\bar{\partial}$ -closed elements $\beta_1, \dots, \beta_m \in A^{0,1}(TM_{1,0})$ that constitute a basis for $H^1(M, TM_{1,0})$; if we set

$$\Xi_1 = \beta_1 t_1 + \dots + \beta_m t_m,$$

then, at least formally, the Kodaira-Spencer map is the identity. To construct the higher Ξ_s , we set $\Xi^s = \Xi_1 + \dots + \Xi_s$, and note (by looking at homogeneous degrees) that Ξ will satisfy condition (3) above if, and only if,

$$\bar{\partial}_M \Xi^s + \frac{1}{2}[\Xi^s, \Xi^s] \equiv 0 \pmod{t^{s+1}}$$

for all $s \geq 1$.

Now this relation is certainly true for $s = 1$ (since Ξ_1 is $\bar{\partial}_M$ -closed); we may thus proceed by induction, and assume that we have already found a suitable Ξ^s . Then we may write

$$\bar{\partial}_M \Xi^s + \frac{1}{2}[\Xi^s, \Xi^s] \equiv \Omega_{s+1} \pmod{t^{s+2}},$$

where Ω_{s+1} is homogeneous of degree $s + 1$ and has coefficients in $A^{0,2}(TM_{1,0})$. What we have to do is determine Ξ_{s+1} , subject to the requirement that

$$\bar{\partial}_M \Xi^{s+1} + \frac{1}{2}[\Xi^{s+1}, \Xi^{s+1}] \equiv 0 \pmod{t^{s+2}}.$$

If we expand this, and look at degrees, we find that Ξ_{s+1} needs to satisfy

$$\bar{\partial}_M \Xi_{s+1} + \Omega_{s+1} = 0.$$

Since we are assuming that $H^2(M, TM_{1,0}) = 0$, this equation will have a solution provided that $\bar{\partial}_M \Omega_{s+1} = 0$. But, using the DGLA identities, we easily compute that

$$\begin{aligned} \bar{\partial}_M \Omega_{s+1} &\equiv \bar{\partial}_M \left(\bar{\partial}_M \Xi^s + \frac{1}{2}[\Xi^s, \Xi^s] \right) = [\bar{\partial}_M \Xi^s, \Xi^s] \\ &\equiv [\Omega_{s+1}, \Xi^s] - \frac{1}{2}[[\Xi^s, \Xi^s], \Xi^s] = [\Omega_{s+1}, \Xi^s] \equiv 0 \pmod{t^{s+2}}, \end{aligned}$$

which of course implies that $\bar{\partial}_M \Omega_{s+1} = 0$ (because the degree of Ω_{s+1} is only $s + 1$). Thus Ξ_{s+1} , and hence a suitable Ξ^{s+1} , can be found, and the inductive argument works.

Of course, the problem with this approach is that it yields only a formal solution. One could try to show that the formal solution (i.e., the power series) converges, but this will not work in general because there are too many choices in the construction—at every stage, there is more than one solution to the equation $\bar{\partial}_M \Xi_{s+1} + \Omega_{s+1} = 0$. Because it is not clear which of these should be chosen to achieve convergence, not much can be done with the formal approach. Indeed, Kodaira writes in his book that he and Spencer were not able to prove the convergence of any such series, until Nirenberg joined them and suggested the use of more difficult analysis.

Obstructions. The reader will have observed the appearance of “obstructions” when constructing the power series; at each stage, there is a $\bar{\partial}_M$ -closed element Ω_{s+1} , and Ξ_{s+1} has to be found such that $\Omega_{s+1} = -\bar{\partial}_M \Xi_{s+1}$. By assuming that $H^2(M, TM_{1,0}) = 0$, we guarantee that these equations always have a solution; without this “blanket” assumption, there might not be a power series solution to the problem because one of the obstructions Ω_{s+1} might fail to be $\bar{\partial}_M$ -exact.

Consider, for instance, the first step of the construction. Here,

$$\Omega_2 = \frac{1}{2}[\Xi_1, \Xi_1];$$

if we take Ξ_1 to represent a certain infinitesimal deformation, say by fixing values for the variables t , then Ω_2 gives us a class in $H^2(M, TM_{1,0})$. If this so-called *primary obstruction* is already nonzero, there cannot be a formal deformation in the direction of Ξ_1 (meaning a power series solution), let alone an actual analytic deformation (a convergent power series).

If, on the other hand, the primary obstruction is zero in $H^2(M, TM_{1,0})$, a suitable Ξ_2 can be found. In the second step of the construction, we then have

$$\Omega_3 = \frac{1}{2}([\Xi_1, \Xi_2] + [\Xi_2, \Xi_1]) = [\Xi_1, \Xi_2],$$

which again presents an obstruction to constructing Ξ_3 , and so on. It has to be said here that the primary obstruction $[\Xi_1, \Xi_1]$ is well-defined, while all higher obstructions depend on previous choices. In particular, it is possible that at some stage of the construction we run into a nontrivial obstruction, but that a sage choice of some earlier Ξ_s would have avoided this situation.

4. A STRATEGY FOR PROVING THE THEOREM

With the formal computations in mind, one possible strategy for proving the theorem is the following. Construct a power series solution, but try to make all the choices in the construction in a good way, to find a special power series solution that will actually converge if $|t|$ is small. The “correct” choices will be determined by Hodge theory (on the bundle $TM_{1,0}$); proving the convergence still requires some analysis, in particular work with estimates. There will be an additional benefit to this method, in that it proves a lot more than just Theorem 2.

Hodge theory. Let us look at the Hodge-theoretic part first. We choose a Hermitian metric on M , i.e., on the holomorphic tangent bundle $TM_{1,0}$, and use it to define inner products on all of the spaces $A^{0,q}(TM_{1,0})$; each element of $A^{0,q}(TM_{1,0})$ is a section of the bundle $\mathcal{A}^{0,q}(TM_{1,0})$, and this bundle inherits a metric from that on M . The inner product between, say, ψ and ψ' will be written simply as (ψ, ψ') . As usual, the condition

$$(\psi, \bar{\partial}\phi) = (\bar{\partial}^*\psi, \phi),$$

with $\psi \in A^{0,q}(TM_{1,0})$ and $\phi \in A^{0,q-1}(TM_{1,0})$, defines an adjoint operator $\bar{\partial}^*$ to $\bar{\partial}$, and thus the Laplacian

$$\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Because \square is an elliptic operator (of second order), Hodge theory provides us with the familiar decomposition

$$A^{0,q}(TM_{1,0}) = \mathcal{H}^{0,q}(TM_{1,0}) \oplus \square A^{0,q}(TM_{1,0})$$

of the space of all forms into harmonic forms and $\bar{\partial}$ and $\bar{\partial}^*$ -exact forms.

Let us derive some consequences of this decomposition. If we apply the identity to itself and use $\square \mathcal{H}^{0,q}(TM_{1,0}) = 0$, we get

$$A^{0,q}(TM_{1,0}) = \mathcal{H}^{0,q}(TM_{1,0}) \oplus \square \square A^{0,q}(TM_{1,0});$$

thus any $\psi \in A^{0,q}(TM_{1,0})$ can actually be written in the form

$$\psi = H\psi + \square\phi,$$

where $H\psi \in \mathcal{H}^{0,q}(TM_{1,0})$ is the harmonic part of ψ , and $\phi \in \square A^{0,q}(TM_{1,0})$. Actually, ϕ itself is also uniquely determined by ψ —if ϕ' is another choice, then $\square(\phi - \phi') = 0$, which implies $\phi - \phi' = 0$ because $\mathcal{H}^{0,q}(TM_{1,0}) \cap \square A^{0,q}(TM_{1,0}) = 0$. We may therefore define the so-called *Green's operator* G as $G\psi = \phi$, and then have a unique decomposition

$$\psi = H\psi + \square G\psi$$

with $G\psi \in \square A^{0,q}(TM_{1,0})$, and $H\psi$ harmonic.

Even though this is not apparent, G and $\bar{\partial}$ commute with each other. First, $\bar{\partial}$ and \square obviously commute,

$$\bar{\partial}\square = \bar{\partial}(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = \bar{\partial}\bar{\partial}^*\bar{\partial} = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\bar{\partial} = \square\bar{\partial}.$$

From the decomposition for ψ , we get

$$\bar{\partial}\psi = \bar{\partial}H\psi + \bar{\partial}\square G\psi = \square\bar{\partial}G\psi;$$

but $\bar{\partial}G\psi$ is also in the image of \square (since $G\psi$ is, and \square and $\bar{\partial}$ commute), and thus the uniqueness of the decomposition for $\bar{\partial}\psi$ implies that

$$G\bar{\partial}\psi = \bar{\partial}G\psi.$$

Solving $\bar{\partial}$ -equations. Now here is how Hodge theory helps with choosing the power series. At each stage of the construction of the series (when finding the coefficients of Ξ_{s+1} , given those of Ω_{s+1}) we have to solve an equation of the form $\bar{\partial}\psi = \phi$, where ϕ is $\bar{\partial}$ -closed. An obvious necessary condition for the existence of a solution ψ is that the class of ϕ in cohomology be zero; this class is represented by the harmonic part $H\phi$, and so there can only be a solution if $H\phi = 0$. Still, if a solution exists, it is not at all unique.

But now the decomposition above weighs in, for among all possible solutions, there is a unique one that is also $\bar{\partial}^*$ -exact! To see why, suppose for a minute that we had a solution ψ that was $\bar{\partial}^*$ -exact. From the decomposition, we get

$$\psi = H\psi + \square G\psi = H\psi + \bar{\partial}\bar{\partial}^*G\psi + \bar{\partial}^*\bar{\partial}G\psi = H\psi + \bar{\partial}\bar{\partial}^*G\psi + \bar{\partial}^*G\phi,$$

and thus

$$\psi - \bar{\partial}^*G\phi = H\psi + \bar{\partial}\bar{\partial}^*G\psi.$$

The left-hand side is $\bar{\partial}^*$ -exact, while the right-hand side is $\bar{\partial}$ -closed; therefore, both sides have to be equal to zero. From this, we deduce that

$$\psi = \bar{\partial}^*G\phi$$

is uniquely determined by ϕ . Of course, we can check that this really is a solution by computing

$$\bar{\partial}\psi = \bar{\partial}\bar{\partial}^*G\phi = H\phi - \bar{\partial}^*\bar{\partial}G\phi = H\phi - \bar{\partial}^*G\bar{\partial}\phi = 0.$$

The reader should be aware of the following point: When presented with the equation $\bar{\partial}\psi = \phi$, we can always define ψ as $\bar{\partial}^*G\phi$; however, we only get a solution to the equation if $\bar{\partial}\phi = 0$, and if ϕ has no harmonic part.

A good choice of power series. The computation above suggests choosing the power series in the following way. Let β_1, \dots, β_m be a basis of the harmonic space $\mathcal{H}^{0,1}(TM_{1,0})$, and put $\Xi_1 = \beta_1 t_1 + \dots + \beta_m t_m$; with this choice, Ξ_1 is harmonic. As before, we need homogeneous polynomials Ξ_s of degree s , with coefficients in $A^{0,1}(TM_{1,0})$. Since we would like to have

$$\bar{\partial}\Xi = -\frac{1}{2}[\Xi, \Xi],$$

we should try the distinguished solution

$$\Xi = -\frac{1}{2}\bar{\partial}^*G[\Xi, \Xi].$$

Actually, to make sure that Ξ has the correct linear term Ξ_1 , we are going to require that the series $\Xi = \Xi_1 + \Xi_2 + \dots$ satisfy the relation

$$\Xi = \Xi_1 - \frac{1}{2}\bar{\partial}^*G[\Xi, \Xi].$$

By looking at degrees, we find that this is equivalent to

$$\Xi_s + \frac{1}{2}\sum_{i=1}^{s-1}\bar{\partial}^*G[\Xi_i, \Xi_{s-i}] = 0$$

for all $s \geq 2$. Each Ξ_s is therefore determined by all the previous ones, and thus there is a unique formal series Ξ with this property!

The good thing about defining Ξ in this way is that it converges; in due time, we will prove the following result (as before, $\Delta(\varepsilon)$ is a ball around the origin in $H^1(M, TM_{1,0})$ of radius ε).

Lemma 3. *Let $\varepsilon > 0$ be sufficiently small. Then the series Ξ defined above converges (in a sense to be made precise below), and defines a C^∞ -section $\xi(t)$ of the bundle $pr_M^*\mathcal{A}^{0,1}(TM_{1,0})$ over $M \times \Delta(\varepsilon)$. Moreover, $\xi(t)$ is holomorphic in t .*

The bad thing about defining Ξ in this way is that we can no longer be sure that it satisfies the Cartan-Maurer equation $\bar{\partial}\Xi + \frac{1}{2}[\Xi, \Xi] = 0$ that it was supposed to solve. In fact, if the Cartan-Maurer equation is satisfied, then necessarily

$$H[\Xi, \Xi] = -2H(\bar{\partial}\Xi) = 0$$

because $H\bar{\partial} = 0$. This explains the following result, which will also be proved below.

Lemma 4. *Let $\varepsilon > 0$ be sufficiently small, in particular small enough for the previous lemma to apply. Then $\xi(t)$ (and thus Ξ) satisfies the Cartan-Maurer equation $\bar{\partial}_M\xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0$ if, and only if, $H[\xi(t), \xi(t)] = 0$.*

The proofs of both statements, as should be apparent from the occurrence of ε , require estimates. The necessary analysis will be introduced in the next section; proofs of Lemma 3 and Lemma 4 will be given afterwards.

An example, third part. To illustrate the statements made above, let us look at the example of the family of elliptic curves once more. This time, however, we shall begin at the opposite end, by first finding $\xi(t)$ as explained above, and then reconstructing the family from this data.

For the elliptic curve $M = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot i)$, we have $H^1(M, TM_{0,1}) \simeq \mathbb{C}$, with basis the harmonic vector field $d\bar{z} \otimes \partial/\partial z$. We thus use just a single variable $t \in \mathbb{C}$; we also set $\Xi_1 = td\bar{z} \otimes \partial/\partial z$ according to the prescription above. Because M is only one-dimensional, brackets of elements from $A^{0,1}(TM_{1,0})$ are always zero, whence $\Xi = \Xi_1$. This obviously converges for all t , and so we have

$$\xi(t) = td\bar{z} \otimes \frac{\partial}{\partial z}$$

defined on $M \times \mathbb{C}$. This is, not surprisingly, the formula we had obtained before.

We can verify directly that the Cartan-Maurer equation is satisfied; indeed,

$$\bar{\partial}_M \xi(t) + \frac{1}{2}[\xi(t), \xi(t)] \in H^2(M, TM_{0,1}) = 0.$$

Thus if we take t from a small enough disk $\Delta \subseteq \mathbb{C}$ (small enough for $\xi(t)$ to correspond to an almost complex structure), $\xi(t)$ gives rise to a complex structure on $M \times \Delta$, and we have constructed a family $\mathfrak{Y} \rightarrow \Delta$ of deformations of M from scratch.

This family is quite easy to analyze, too. First, the two vector fields

$$\frac{\partial}{\partial \bar{z}} + t \cdot \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial t},$$

span the $(0, 1)$ -part of the tangent bundle, while the $(1, 0)$ -part is spanned by their conjugates

$$\frac{\partial}{\partial z} + \bar{t} \cdot \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial t}.$$

Taking duals, we see that

$$d\bar{z} - \bar{t}dz \quad \text{and} \quad d\bar{t}$$

give a frame for the $(0, 1)$ -part of the cotangent bundle, while

$$dz - td\bar{z} \quad \text{and} \quad dt$$

do the same for the $(1, 0)$ -part.

5. HÖLDER ESTIMATES

To begin with, we need the definition of the *Hölder norms*. Let $k \geq 0$ be a non-negative integer, and let $0 < \alpha < 1$. If f is a complex-valued C^k -function on an open set $U \subseteq \mathbb{R}^{2n}$, we put²

$$\|f\|_{k,\alpha}^U = \sum_{|I| \leq k} \sup_{x \in U} |D^I f(x)| + \sum_{|I|=k} \sup_{x \neq y \in U} \frac{|D^I f(x) - D^I f(y)|}{|x - y|^\alpha}.$$

We use multi-index notation; if $I = (i_1, \dots, i_s)$, then $D^I = \partial/\partial x^{i_1} \dots \partial/\partial x^{i_s}$ is a differential operator of order $s = |I|$. The Hölder norm controls all possible

²If all k -th derivatives of f are Hölder continuous of order α , then the following expression is finite; otherwise, $\|f\|_{k,\alpha}^U = \infty$.

derivatives $D^I f(x)$ up to order k ; it also contains a Lipschitz-type condition on the k -th derivatives. It is a standard fact that the space

$$C^{k,\alpha}(U) = \{ f \in C^k(U) \mid \|f\|_{k,\alpha}^U < \infty \}$$

forms a Banach space with respect to the above norm.

By using local trivializations, we can also define the Hölder norm of a bundle-valued form $\psi \in A^{0,q}(TM_{1,0})$ over the compact complex manifold M . Let \mathcal{U} be a finite open cover of M by coordinate polydisks $U \in \mathcal{U}$. Let local holomorphic coordinates on U be z_U^1, \dots, z_U^n ; these give rise to real coordinates x_U^1, \dots, x_U^{2n} by

$$z_U^j = x_U^{2j-1} + ix_U^{2j}.$$

In each of these sets of local coordinates, write

$$\psi = \sum_{j,v} h_{j,U}^v(x) dz_U^j \otimes \frac{\partial}{\partial z_U^v}$$

with certain functions $h_{j,U}^v(x)$. Then let

$$\|\psi\|_{k,\alpha} = \max_{U \in \mathcal{U}} \max_{j,v} \|h_{j,U}^v\|_{k,\alpha}^U.$$

This definition obviously depends on the choice of \mathcal{U} and the local coordinates; to avoid ambiguities, we shall *choose once and for all* an open cover of M , together with local holomorphic coordinates. With respect to this data, the space of C^k -sections of $\mathcal{A}^{0,q}(TM_{1,0})$ for which the Hölder norm is finite shall be denoted by $A_{k,\alpha}^{0,q}(TM_{1,0})$; as before, it forms a Banach space.

Using the same coordinate systems, we also define the *uniform norm* of a continuous section ψ of the bundle $\mathcal{A}^{0,q}(TM_{1,0})$,

$$\|\psi\|_0 = \max_{U \in \mathcal{U}} \max_{j,v} \sup_{x \in U} |h_{j,U}^v(x)|.$$

Obviously, one has

$$\|\psi\|_0 \leq \|\psi\|_{k,\alpha}$$

whenever $\psi \in A_{k,\alpha}^{0,q}(TM_{1,0})$.

Estimates. In order to prove the convergence of the power series Ξ constructed in Section 4, we will need to have bounds for its homogeneous parts Ξ_s . The necessary estimates for the operators that appear in the construction ($\bar{\partial}^*$, \square , etc.) are collected in this section.

To begin with, there is a positive constant C_0 , such that the inequality

$$(1) \quad (\psi, \phi) \leq C_0 \|\psi\|_0 \|\phi\|_0$$

holds for all continuous sections ψ and ϕ of $\mathcal{A}^{0,q}(TM_{1,0})$. The inequality is easily proved by writing both sides down in local coordinates; the constant C_0 depends on the Hermitian metric for $TM_{0,1}$, as well as on the local coordinates z_U^j that we had chosen above.

There is also a very general Hölder estimate for elliptic operators. While not easy to prove, this is a standard result in the theory of partial differential equations.

Theorem 5. *Let $L = L_d + L_{d-1} + \dots + L_0$ be a pseudo-differential operator of order d , acting on sections of a vector bundle $E \rightarrow M$, and whose principal part L_d*

is elliptic. Then there is a positive constant C , depending only on k , α , and, for each i , on the $C^{k+1+d-i}$ -norm of the coefficients of L_i , such that the estimate

$$\|u\|_{k+d,\alpha} \leq C \cdot \left(\|Lu\|_{k,\alpha} + \|u\|_{0,\alpha} \right)$$

is true for every section u of E .

An example of this situation is given by the Laplacian \square on sections of the bundle $\mathcal{A}^{0,q}(TM_{1,0})$; it is a second-order elliptic operator.

The usefulness of the Hölder norms for our particular convergence problem lies in the following four estimates. There are constants C_1, \dots, C_4 , depending on k , α , and q (and the choices made above), such that for all $\psi, \phi \in A_{k,\alpha}^{0,q}(TM_{1,0})$, we have³

$$(2a) \quad \|\psi\|_{k,\alpha} \leq C_1 \left(\|\square\psi\|_{k-2,\alpha} + \|\psi\|_{0,\alpha} \right)$$

$$(2b) \quad \|\bar{\partial}^*\psi\|_{k,\alpha} \leq C_2 \|\psi\|_{k+1,\alpha}$$

$$(2c) \quad \|[\psi, \phi]\|_{k,\alpha} \leq C_3 \|\psi\|_{k+1,\alpha} \|\phi\|_{k+1,\alpha}$$

$$(2d) \quad \|G\psi\|_{k,\alpha} \leq C_4 \|\psi\|_{k-2,\alpha}.$$

The first inequality is an immediate consequence of Theorem 5, applied to the elliptic operator \square . The second and third inequalities, on the other hand, are quite easy to get from the definition of the Hölder norms; they simply express the fact that when computing $\bar{\partial}^*\psi$ or $[\psi, \phi]$, we have to differentiate once, and therefore a Hölder norm of weight one higher than the original one has to come in.

Here is how to deduce the fourth inequality from the first one. We begin by showing that, for a suitable constant C_5 , one has

$$\|H\psi\|_{l,\alpha} \leq C_5 \|\psi\|_{0,\alpha}$$

whenever $l \geq 2$. This is a standard ‘‘compactness’’ argument. Suppose that no such constant C_5 existed (we shall presently show that this assumption leads to a contradiction); then we would be able to find a sequence ψ_n with

$$\|H\psi_n\|_{l,\alpha} = 1 \quad \text{but} \quad \|\psi_n\|_{0,\alpha} \leq \frac{1}{n}.$$

If we set $\phi_n = H\psi_n$, we thus have $\|\phi_n\|_{l,\alpha} = 1$. The definition of the Hölder norm then implies that in the local coordinates chosen above, all derivatives up to order l of any coefficient of ϕ_n are uniformly bounded and equicontinuous. We may therefore apply the Arzelà-Ascoli theorem, and find a subsequence (still denoted by ϕ_n) that converges in the C^l -norm to some C^l -section ϕ of $\mathcal{A}^{0,q}(TM_{1,0})$; in particular, we get

$$\lim_{n \rightarrow \infty} \|H\psi_n - \phi\|_{0,\alpha} = 0.$$

From this, we can deduce that $\phi \neq 0$. For, applying the estimate (2a) to the section $H\psi_n$, and noting that $\square \circ H = 0$, we get

$$1 = \|H\psi_n\|_{l,\alpha} \leq C_1 \|H\psi_n\|_{0,\alpha}.$$

As $n \rightarrow \infty$, the right-hand side converges to $C_1 \|\phi\|_{0,\alpha}$, and therefore

$$\|\phi\|_{0,\alpha} \geq \frac{1}{C_1} > 0.$$

³In (2a) one should take $k \geq 2$, while (2d) is valid for $k \geq 4$.

On the other hand, we have (using that the inner product is continuous in each factor, because of the estimate (1) above)

$$(\phi, \phi) = \lim_{n \rightarrow \infty} (H\psi_n, H\psi_n) = \lim_{n \rightarrow \infty} (H\psi_n, \psi_n) \leq \|\phi\|_0 \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

because $\|\psi_n\|_0 \leq \|\psi_n\|_{0,\alpha} \leq 1/n$. Thus $\phi = 0$, which is clearly a contradiction. Consequently, there has to be some constant C_5 as claimed.

We now prove (2d), by essentially the same method (assuming, mostly for the sake of convenience, that $k \geq 4$). If we apply the first estimate (2a) to $G\psi$ and use the Hodge decomposition, we obtain

$$\begin{aligned} \|G\psi\|_{k,\alpha} &\leq C_1 \left(\|\square G\psi\|_{k-2,\alpha} + \|G\psi\|_{0,\alpha} \right) \\ &\leq C_1 \left(\|\psi\|_{k-2,\alpha} + \|H\psi\|_{k-2,\alpha} + \|G\psi\|_{0,\alpha} \right) \\ (3) \quad &\leq C_1 \left(\|\psi\|_{k-2,\alpha} + C_5 \|\psi\|_{0,\alpha} + \|G\psi\|_{0,\alpha} \right) \\ &\leq C_1(1 + C_5) \left(\|\psi\|_{k-2,\alpha} + \|G\psi\|_{0,\alpha} \right). \end{aligned}$$

It is thus sufficient to show that for some positive constant C_6 , we have

$$\|G\psi\|_{0,\alpha} \leq C_6 \|\psi\|_{k-2,\alpha}$$

for all ψ . Because $G \circ H = 0$, we have $G\psi = G(\psi - H\psi)$; replacing ψ by $\psi - H\psi$, we may assume that $H\psi = 0$. With this additional assumption, the inequality is readily proved.

Again, suppose no such constant C_6 existed; we could find a sequence ψ_n with $H\psi_n = 0$, satisfying

$$\|G\psi_n\|_{0,\alpha} = 1 \quad \text{but} \quad \|\psi_n\|_{k-2,\alpha} \leq \frac{1}{n}.$$

Setting $\phi_n = G\psi_n$, the inequality (3) from above shows that

$$\|\phi_n\|_{k,\alpha} \leq C_1(1 + C_5) \left(\frac{1}{n} + 1 \right) \leq 2C_1(1 + C_5)$$

is bounded independently of n . Applying the Arzelà-Ascoli theorem one more time, we find a subsequence (which we continue to denote by ϕ_n) that converges in the C^k -norm to some C^k -section ϕ of $\mathcal{A}^{0,q}(TM_{1,0})$. Thus we have

$$\|\phi\|_{0,\alpha} = \lim_{n \rightarrow \infty} \|G\psi_n\|_{0,\alpha} = 1.$$

On the other hand, since \square involves at most second derivatives, $\psi_n = \square G\psi_n$ converges in the C^{k-2} -norm to $\square\phi$; the C^{k-2} -norm of ψ_n is bounded by $\|\psi_n\|_{k-2,\alpha} \leq 1/n$, and so $\square\phi = 0$, which means that ϕ is harmonic. But now

$$(\phi, \phi) = \lim_{n \rightarrow \infty} (\phi_n, \phi) = \lim_{n \rightarrow \infty} (G\psi_n, \phi) = 0$$

because each $G\psi_n$ is orthogonal to the harmonic space. Thus $\phi = 0$, which is clearly in contradiction to the equality $\|\phi\|_{0,\alpha} = 1$ above. It follows that there has to be some constant C_6 as claimed, and this ends the proof of the fourth estimate (2d).

6. PROOF OF LEMMA 5.1

Using the estimates from the previous section, we shall now prove that the power series Ξ is convergent. Ξ has coefficients in $A^{0,1}(TM_{1,0})$, and convergence can only mean absolute convergence with respect to a norm; we shall use the Hölder norm of weight (k, α) , for $k \geq 3$. Here the following notation will be convenient.

Notation. If Ψ is a formal power series (or a formal polynomial) in the variables $t = (t_1, \dots, t_m)$, with coefficients in $A_{k,\alpha}^{0,q}(TM_{1,0})$, say $\Psi = \sum_I \psi_I t^I$, with all $\psi_I \in A_{k,\alpha}^{0,q}(TM_{1,0})$, we shall write $\|\Psi\|_{k,\alpha}$ for the series obtained by taking termwise norms,

$$\|\Psi\|_{k,\alpha} = \sum_I \|\psi_I\|_{k,\alpha} t^I.$$

Thus $\|\Psi\|_{k,\alpha}$ is an element of $\mathbb{R}[[t_1, \dots, t_m]]$.

Secondly, if $a = \sum_I a_I t^I$ and $b = \sum_I b_I t^I$ are two real power series (or polynomials), we shall write

$$a(t) \ll b(t) \quad \text{or} \quad a \ll b$$

to express that each coefficient a_I of a is less or equal to the corresponding coefficient b_I of b .

Estimates. Using the notation just introduced, we can recast the Hölder estimates from the previous section into a very convenient form: With the same constants C_1, \dots, C_4 as before, every two formal power series Ψ and Φ with coefficients in $A_{k,\alpha}^{0,q}(TM_{1,0})$ satisfy the inequalities

$$(4a) \quad \|\Psi\|_{k,\alpha} \ll C_1 \left(\|\square\Psi\|_{k-2,\alpha} + \|\Psi\|_{0,\alpha} \right)$$

$$(4b) \quad \|\bar{\partial}^* \Psi\|_{k,\alpha} \ll C_2 \|\Psi\|_{k+1,\alpha}$$

$$(4c) \quad \|[\Psi, \Phi]\|_{k,\alpha} \ll C_3 \|\Psi\|_{k+1,\alpha} \|\Phi\|_{k+1,\alpha}$$

$$(4d) \quad \|G\Psi\|_{k,\alpha} \ll C_4 \|\Psi\|_{k-2,\alpha}.$$

Each of these is now a statement about a term-by-term inequality of two real power series. The proofs of the first three are quite trivial, and consist in simply applying the old estimates to each term of the series. Here is how to prove the fourth one. Say $\Psi = \sum_I \psi_I t^I$ and $\Phi = \sum_J \phi_J t^J$; then as formal power series,

$$[\Psi, \Phi] = \sum_K \sum_{I+J=K} [\psi_I, \phi_J] t^K,$$

and therefore

$$\|[\Psi, \Phi]\|_{k,\alpha} = \sum_K \sum_{I+J=K} \|[\psi_I, \phi_J]\|_{k,\alpha} t^K.$$

Our original estimate (2d) now gives $\|[\psi_I, \phi_J]\|_{k,\alpha} \leq C_4 \|\psi_I\|_{k+1,\alpha} \|\phi_J\|_{k+1,\alpha}$, and thus we have the term-by-term inequality

$$\|[\Psi, \Phi]\|_{k,\alpha} \ll C_4 \sum_K \sum_{I+J=K} \|\psi_I\|_{k+1,\alpha} \|\phi_J\|_{k+1,\alpha} t^K = C_4 \|\Psi\|_{k+1,\alpha} \|\Phi\|_{k+1,\alpha}.$$

Convergence. If a formal series $\Psi = \sum_s \Psi_s$ with coefficients of class $C^{k,\alpha}$ is such that the real power series $\|\Psi\|_{k,\alpha}$ is convergent for all $|t| < \varepsilon$, then it gives rise to $C^{k,\alpha}$ -section $\psi(t)$ of $pr_M^* \mathcal{A}^{0,q}(TM_{1,0})$ over $M \times \Delta(\varepsilon)$, as follows. Write $\psi_s(t)$ for the section defined by the formal polynomial Ψ_s ; of course, $\psi_s(t)$ is defined for all t , of class $C^{k,\alpha}$ in all variables, and holomorphic in t . The limit

$$\psi(t) = \lim_{S \rightarrow \infty} \sum_{s=1}^S \psi_s(t)$$

now exists in the Banach space $A_{k,\alpha}^{0,q}(TM_{1,0})$; moreover, it exists uniformly in t , as can be seen easily from the convergence of the real power series $\|\Psi\|_{k,\alpha}$. This means that $\psi(t)$ is of class $C^{k,\alpha}$. Moreover, $\psi(t)$ is holomorphic in t , because it is the uniform limit of sections that depend polynomially on t .

Our strategy for proving the convergence of the series Ξ should now be clear—find a real power series A that converges for $|t| < \varepsilon$, and that dominates Ξ termwise,

$$\|\Xi\|_{k,\alpha} \ll A.$$

Then Ξ itself will converge to a $C^{k,\alpha}$ -section of $pr_M^* \mathcal{A}^{0,q}(TM_{1,0})$ on $M \times \Delta(\varepsilon)$.

Proof of convergence in the Hölder norm. We finally prove the convergence of the power series Ξ . Fix $k \geq 3$ and $0 < \alpha < 1$. We shall construct a suitable power series

$$A = \sum_{s=1}^{\infty} A_s \in \mathbb{R}[[t_1, \dots, t_m]]$$

where each A_s is a homogeneous polynomial of degree s , such that

$$\|\Xi\|_{k,\alpha} \ll A.$$

To see how to choose A , we use the Hölder estimates. From the construction of Ξ , we have

$$\Xi_s = \frac{1}{2} \sum_{i=1}^{s-1} \bar{\partial}^* G[\Xi_i, \Xi_{s-i}];$$

to estimate the Hölder norm of Ξ_s , we first look at the individual terms in the sum. Applying (4b) and (4d), we find that

$$\|\bar{\partial}^* G[\Xi_i, \Xi_{s-i}]\|_{k,\alpha} \leq C_2 \|G[\Xi_i, \Xi_{s-i}]\|_{k+1,\alpha} \leq C_2 C_4 \|[\Xi_i, \Xi_{s-i}]\|_{k-1,\alpha}.$$

At the same time, the estimate in (4c) shows that

$$\|[\Xi_i, \Xi_{s-i}]\|_{k-1,\alpha} \leq C_3 \|\Xi_i\|_{k,\alpha} \|\Xi_{s-i}\|_{k,\alpha}.$$

We thus get an estimate for the Hölder norm of Ξ_s as

$$\|\Xi_s\|_{k,\alpha} \leq \frac{1}{2} \sum_{i=1}^{s-1} \|\bar{\partial}^* G[\Xi_i, \Xi_{s-i}]\|_{k,\alpha} \leq C \sum_{i=1}^{s-1} \|\Xi_i\|_{k,\alpha} \|\Xi_{s-i}\|_{k,\alpha}$$

for $C = C_2 C_3 C_4 / 2$. Thus if $\|\Xi_i\|_{k,\alpha} \ll A_i$ for all $i < s$, then to guarantee that $\|\Xi_s\|_{k,\alpha} \leq A_s$, we should have

$$C \sum_{i=1}^{s-1} A_i A_{s-i} \ll A_s.$$

This says that the power series A should satisfy the relation

$$C \cdot A^2 \ll A.$$

For this inequality, a series of the form

$$A = \frac{u}{v} \sum_{s=1}^{\infty} \frac{v^s}{s^2} (t_1 + \cdots + t_m)^s$$

works; here u and v are two positive constants to be chosen later. Indeed, A satisfies

$$A^2 \ll \frac{16u}{v} \cdot A.$$

Proof. Clearly $A = uv^{-1}B(v(t_1 + \cdots + t_m))$, where $B(x) = \sum_{s=1}^{\infty} x^s/s^2$ is a power series in one variable. We have

$$B(x)^2 = \sum_{i,j=1}^{\infty} \frac{x^{i+j}}{i^2 j^2} = \sum_{s=2}^{\infty} \sum_{i=1}^{s-1} \frac{1}{i^2 (s-i)^2} x^s,$$

and the coefficient of x^s can be estimated as

$$\begin{aligned} \sum_{i=1}^{s-1} \frac{1}{i^2 (s-i)^2} &\leq \frac{1}{(s/2)^4} + 2 \sum_{i < s/2} \frac{1}{i^2 (s-i)^2} \leq \frac{1}{(s/2)^4} + \frac{2}{(s/2)^2} \sum_{i < s/2} \frac{1}{i^2} \\ &\leq \frac{8}{s^2} \sum_{i < s/2+1} \frac{1}{i^2} \leq \frac{8}{s^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{8}{s^2} \frac{\pi^2}{6} \leq \frac{16}{s^2}. \end{aligned}$$

Thus we see that $B(x)^2 \ll 16B(x)$, and this implies the inequality for A . \square

Since $\Xi_1 = \beta_1 t_1 + \cdots + \beta_m t_m$, while $A_1 = u(t_1 + \cdots + t_m)$, we should let

$$u = \max_i \|\beta_i\|_{k,\alpha}$$

to insure that $\|\Xi_1\|_{k,\alpha} \ll A_1$. Secondly, we need to have $16u/v = 1/C$, and thus we put

$$v = 16u \cdot C.$$

With these choices, $\|\Xi\|_{k,\alpha} \ll A$ is true. Now the series for $B(x)$ has radius of convergence 1; thus A converges uniformly for

$$|t_1 + \cdots + t_m| < \frac{1}{v},$$

and thus for $|t| < \varepsilon_k = 1/(\sqrt{m} \cdot v)$. (Since the constant C really depends on k , the same is true for the quantity ε_k .) The series Ξ therefore converges absolutely and uniformly on the same domain, and thus defines a $C^{k,\alpha}$ -section $\xi(t)$ of $pr_M^* \mathcal{A}^{0,1}(TM_{1,0})$ over $M \times \Delta(\varepsilon_k)$.

Smoothness of $\xi(t)$. Unfortunately, the argument above is not sufficient to prove smoothness of $\xi(t)$, because it only guarantees that $\xi(t)$ is $C^{k,\alpha}$ for $|t| < \varepsilon_k$, whereas, most unfortunately, ε_k tends to zero as $k \rightarrow \infty$. A different argument is needed. What rescues us this time is again the fact that \square is an elliptic operator.

From

$$\Xi = \Xi_1 - \frac{1}{2} \bar{\partial}^* G[\Xi, \Xi]$$

and the fact that Ξ_1 is harmonic, we obtain $\bar{\partial}^*\Xi = 0$. Thus

$$\square\Xi = \bar{\partial}^*\bar{\partial}\Xi = -\frac{1}{2}\bar{\partial}^*\bar{\partial}\bar{\partial}^*G[\Xi, \Xi] = -\frac{1}{2}\bar{\partial}^*\square G[\Xi, \Xi].$$

Now $\square G[\Xi, \Xi] = [\Xi, \Xi] - H[\Xi, \Xi]$, and since $\bar{\partial}^*H = 0$, we find

$$\square\Xi = -\frac{1}{2}\bar{\partial}^*[\Xi, \Xi].$$

Since this equation involves at most second derivatives, it is also true for $\xi(t)$, provided $|t| < \varepsilon_3$ (we had assumed $k \geq 3$ above), and so

$$\square_M\xi(t) = -\frac{1}{2}\bar{\partial}^*[\xi(t), \xi(t)].$$

On the other hand, $\xi(t)$ is holomorphic in t , being the limit of a power series, and therefore $\xi(t)$ is a solution to the partial differential equation

$$\left(-\sum_i \frac{\partial^2}{\partial t_i \partial \bar{t}_i} + \square_M\right)\xi(t) + \frac{1}{2}\bar{\partial}^*[\xi(t), \xi(t)] = 0$$

on $M \times \Delta(\varepsilon)$ for small enough $\varepsilon > 0$.

The second-order operator

$$E = -\sum_i \frac{\partial^2}{\partial t_i \partial \bar{t}_i} + \square_M$$

occurring in this PDE is elliptic, and even though the equation is not linear, this elliptic term is strong enough to allow us to prove the smoothness of $\xi(t)$. The reasoning goes like this. Write the equation in the form

$$E\xi(t) = -\frac{1}{2}\bar{\partial}^*[\xi(t), \xi(t)],$$

and consider the right-hand side in local coordinates, where

$$\xi(t) = \sum_{i,u} h_i^u(z, t) d\bar{z}^i \otimes \frac{\partial}{\partial z^u}.$$

Since $\bar{\partial}^*$ and $[_, _]$ both involve one differentiation, the terms on the right-hand side that depend on $\xi(t)$ are of the form

$$h_i^u \cdot \frac{\partial^2 h_j^v}{\partial z^a \partial \bar{z}^b}, \quad h_i^u \cdot \frac{\partial h_j^v}{\partial z^a}, \quad h_i^u \cdot \frac{\partial h_j^v}{\partial \bar{z}^b}.$$

If we take all terms that involve second derivatives of $\xi(t)$, and combine them with the second-order operator E on the left-hand side, we thus arrive at an equation that may be written

$$L\xi(t) = R\xi(t),$$

where L is a second-order operator of the form

$$L = E + D_\xi,$$

with D_ξ of second order, having coefficients that depend linearly on $\xi(t)$. Also, R involves at most first derivatives of $\xi(t)$. Now, since $\xi(0) = 0$, we may choose $\varepsilon > 0$ so small that the contribution of D_ξ to L is negligible compared to that of E ; then L is still elliptic. Fix such a choice of ε , and consider $\xi(t)$ over $M \times \Delta(\varepsilon)$ from now on.

We can then show smoothness of $\xi(t)$ by induction. Suppose we have already established that $\xi(t)$ is of class $C^{k,\alpha}$. Then $R\xi(t)$ is of class $C^{k-1,\alpha}$, and the coefficients of L are at least C^k . We can therefore apply the general estimate, Theorem 5, to obtain that

$$\|\xi(t)\|_{k+1,\alpha} \leq C \cdot \left(\|L\xi(t)\|_{k-1,\alpha} + \|\xi(t)\|_{0,\alpha} \right) = C \cdot \left(\|R\xi(t)\|_{k-1,\alpha} + \|\xi(t)\|_{0,\alpha} \right)$$

is bounded. Because of this, one can show (using approximation by smooth sections) that $\xi(t)$ is actually of class $C^{k+1,\alpha}$. By induction, $\xi(t)$ is thus of class C^∞ . Lemma 5.1 is proved.

7. PROOF OF LEMMA 5.2

Recall that we chose a basis $\beta_1, \dots, \beta_m \in \mathcal{H}^{0,1}(TM_{1,0})$ of the space of harmonic forms, and then defined $\Xi = \sum_{s=1}^\infty \Xi_s$ by the recursive conditions

$$\Xi_1 = \beta_1 t_1 + \dots + \beta_m t_m$$

and

$$\Xi_s + \frac{1}{2} \sum_{i=1}^{s-1} \bar{\partial}^* G[\Xi_i, \Xi_{s-i}] = 0.$$

This was the unique formal solution to the equation

$$\Xi = \Xi_1 - \frac{1}{2} \bar{\partial}^* G[\Xi, \Xi].$$

We also showed that this formal series Ξ converges absolutely and uniformly in every Hölder norm, provided $|t| < \varepsilon$, and defines a smooth section $\xi = \xi(t)$ of the bundle $pr_M^* \mathcal{A}^{0,1}(TM_{1,0})$ on $M \times \Delta(\varepsilon)$.

We had already convinced ourselves that the condition $H[\xi, \xi]$ was necessary for the Cartan-Maurer equation to hold. It remains to show the sufficiency of the condition. Thus consider some ξ with $H[\xi, \xi] = 0$. We set

$$\psi = \bar{\partial}\xi + \frac{1}{2}[\xi, \xi],$$

and try to show that $\psi = 0$.

Since $\xi = \xi_1 - \frac{1}{2} \bar{\partial}^* G[\xi, \xi]$ by construction, and since ξ_1 is harmonic, we get

$$\bar{\partial}\xi = -\frac{1}{2} \bar{\partial} \bar{\partial}^* G[\xi, \xi],$$

and thus

$$\psi = -\frac{1}{2} \bar{\partial} \bar{\partial}^* G[\xi, \xi] + \frac{1}{2}[\xi, \xi] = \frac{1}{2} \bar{\partial}^* \bar{\partial} G[\xi, \xi] = \frac{1}{2} \bar{\partial}^* G \bar{\partial}[\xi, \xi],$$

using the decomposition

$$[\xi, \xi] = H[\xi, \xi] + \square G[\xi, \xi] = \bar{\partial}^* \bar{\partial} G[\xi, \xi] + \bar{\partial} \bar{\partial}^* G[\xi, \xi].$$

We can compute $\bar{\partial}[\xi, \xi]$ from the DGLA identities as

$$\bar{\partial}[\xi, \xi] = 2[\bar{\partial}\xi, \xi] = 2[\psi, \xi] - [[\xi, \xi], \xi] = 2[\psi, \xi];$$

by combining the last and third-to-last equations, we finally obtain

$$\psi = \bar{\partial}^* G[\psi, \xi].$$

We now apply the Hölder estimates (whose constants are, of course, independent of t) to get an estimate for the norm of ψ ,

$$\|\psi\|_{k,\alpha} = \|\bar{\partial}^* G[\psi, \xi]\|_{k,\alpha} \leq C_2 C_3 C_4 \|\psi\|_{k,\alpha} \|\xi\|_{k,\alpha}.$$

Here comes the crucial observation. By construction, $\xi(0) = 0$; if we choose t close enough to zero, the quantity $C_2 C_3 C_4 \|\xi\|_{k,\alpha}$ can be made smaller than 1. But then the inequality implies $\|\psi\|_{k,\alpha} = 0$, hence $\psi = 0$. Thus the Cartan-Maurer equation is indeed satisfied, at least when $|t| < \varepsilon$ and $\varepsilon > 0$ is sufficiently small.

8. CONCLUSIONS

All the work having been done in the previous two sections, we are now in a position to draw several interesting conclusions for the problem of deforming complex structures.

Let $\Delta = \Delta(\varepsilon)$ be a ball in $H^1(M, TM_{1,0})$, centered at 0, small enough for the conclusions of Lemma 3 and Lemma 4 to apply. We know that the series $\bar{\Xi}$ converges on $M \times \Delta$, and that it defines a C^∞ -section $\xi(t)$ of the bundle $pr_M^* \mathcal{A}^{0,1}(TM_{1,0})$, holomorphic in the variable $t \in \Delta$.

First of all, we finish the proof of Theorem 2. Since $\xi(t)$ has no constant part, and since the linear part $\xi_1(t)$ was specially chosen as $\xi_1(t) = \beta_1 t_1 + \dots + \beta_m t_m$, we see that $\xi(t)$ satisfies conditions (1), (2) and (4), given after Theorem 2. By Lemma 4, it also satisfies (3), i.e.,

$$\bar{\partial}_M \xi(t) + \frac{1}{2} [\xi(t), \xi(t)] = 0,$$

if, and only if, $H[\xi(t), \xi(t)] = 0$. But

$$H[\xi(t), \xi(t)] \in \mathcal{H}^{0,2}(TM_{1,0}) \simeq H^2(M, TM_{0,1}),$$

and this is zero by assumption. Thus the Maurer-Cartan equation is always satisfied, (3) holds, and Theorem 2 is proved.