STABLE APPROXIMATIONS OF CERTAIN VECTOR BUNDLES

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1. Introduction

Let X be a fixed smooth compact complex surface, and $(x_1, \dots, x_k)$ a multiset of k points on X, in other words, k points on X that are not required to be distinct. Let $\mathscr{I}$ be the ideal sheaf corresponding to the multiset, and assume that $\mathscr{E}_0$ is a holomorphic vector bundle of rank two on X, fitting into the short exact sequence

$$0 \longrightarrow \mathscr{O} \xrightarrow{\ s\ } \mathscr{E}_0 \xrightarrow{\ s^*\ } \mathscr{I} \longrightarrow 0.$$

As usual, s is the distinguished nonzero global section of $\mathscr{E}_0$. The determinant line bundle of $\mathscr{E}_0$ is trivial, while the second Chern class is equal to the integer k.

If X is in particular an algebraic surface, and H a fixed ample line bundle, then we can talk about slope-stability with respect to H. The bundle $\mathscr{E}_0$ is only slope-semistable, of slope 0, but there will be stable bundles among small deformations of $\mathscr{E}_0$. By a theorem of Donaldson, these stable approximations admit (essentially unique) ASD metrics; this note will answer the question what happens to the curvature of these ASD metrics as the approximations get closer to $\mathscr{E}_0$. The following result says that the curvature becomes more and more concentrated at the points $x_1, \dots, x_k$.

Theorem (from *The Geometry of Four-Manifolds*, p. 230)**.** *Let $\mathscr{E}_t$ be a family of holomorphic vector bundles, parametrized by $t \in \mathbb{C}$, such that $\mathscr{E}_t$ is slope-stable for $t \neq 0$, and with $\mathscr{E}_0$ as above. Let A_t be ASD-connections corresponding to $\mathscr{E}_t$ for $t \neq 0$. Then the sequence $[A_t]$ converges weakly to the ideal connection $\big([\theta], (x_1, \dots, x_k)\big)$ as $t \to 0$.*

The remaining sections of the introductions will explain the terminology used in the theorem, as well as the basic point of view towards gauge theory that Donaldson and Kronheimer adopt in their book.

Gauge theory. Donaldson and Kronheimer do not consider a fixed holomorphic bundle and different Hermitian metrics on it. Instead, they fix an underlying topological vector bundle E of rank two, with $c_1(E) = 0$ and $c_2(E) = k$, and a Hermitian metric on it, and then consider different connections on E. Put another way, they fix an SU(2)-bundle E of second Chern class k, which means a bundle associated to a principal SU(2)-bundle on X by the standard representation of SU(2) on $\mathbb{C}^2$. Any SU(2)-connection A on E (represented by the covariant derivative ∇_A) whose curvature is of type (1,1) then makes E into a holomorphic vector bundle $\mathscr{E}_A$, with del-bar operator $\bar\partial_A$. Denote the space of all such connections by $\mathcal{A}^{(1,1)}$.

The complex gauge group $\mathcal{G}^c$ (of complex bundle automorphisms of E) acts on $\mathcal{A}^{(1,1)}$ in the following way. Since the Hermitian metric is fixed, each del-bar operator determines a unique compatible (1,1)-connection; from the obvious action of $\mathcal{G}^c$ on del-bar operators (by pull-back), we therefore get an action on the space of (1,1)-connections. This action extends the action of the usual gauge group $\mathcal{G}$ (of bundle automorphisms preserving the SU(2)-structure). Two connections in $\mathcal{A}^{(1,1)}$ give isomorphic holomorphic structures on E exactly when they are related by a complex gauge transformation.

The following point of view towards complex gauge transformations is often useful. Say $A \in \mathcal{A}^{(1,1)}$ is a connection, and $B = g(A)$ its image under a complex gauge transformation g. The bundle $\mathrm{End}(E) = E^* \otimes E$ admits the induced connection $A * B$, and with respect to this connection, g is a holomorphic section,

$$\bar{\partial}_{A*B} g = 0.$$

In those terms, Donaldson's theorem on the existence of ASD metrics can be restated as follows: If E admits a stable holomorphic structure, then up to gauge equivalence by $\mathcal{G}$, there is a unique ASD connection on E.

Moduli spaces and compactness. In the following, fix a principal SU(2)-bundles $P_k \to X$ with second Chern class k for each nonnegative integer k. There is a moduli space of (gauge equivalence classes of) ASD connections on P_k, denoted here by M_k; if A is an ASD connection, we shall write $[A]$ for the corresponding point in M_k. The space M_0 contains the special class $[\theta]$ of the trivial connection.

To compactify these moduli spaces, the following notion is used. An ideal ASD connection of second Chern class k is a pair

$$\big([A], (x_1, \dots, x_l)\big)$$

consisting of a point $[A] \in M_{k-l}$ and a multiset $(x_1, \dots, x_l)$ of degree l of points of X. The curvature density of an ideal connection is the measure

$$|F(A)|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}$$

on X. A sequence $[A_n]$ in M_k in said to converge weakly to the ideal connection $\big([A], (x_1, \dots, x_l)\big)$ if

- the curvature densities converge as measures, in other words, for every continuous function f on X,
$$\int_X f|F(A_n)|^2 d\mu \to \int_X f|F(A)|^2 d\mu + 8\pi^2 \sum_r f(x_r);$$
- there are bundle maps $\rho_n \colon P_{k-l}\big|_U \to P_k\big|_U$ such that the connections $\rho_n^*(A_n)$ converge in C^∞ to A on compact subsets of $U = X \setminus \{x_1, \dots, x_l\}$.

(As for the second condition, it should be noted that each ρ_n gives rise to a map r_n of associated vector bundles over U. Because any map of principal bundles is an isomorphism, this map r_n is then a unitary bundle isomorphism.)

With the above notion of convergence, the space IM_k of ideal connections of second Chern class k,

$$IM_k = M_k \cup M_{k-1} \times X \cup M_{k-2} \times \mathrm{Sym}^2(X) \cup \cdots,$$

is a (metrizable) Hausdorff space. Let $\bar{M}_k$ be the closure of M_k in IM_k. Uhlenbeck's results on compactness imply that $\bar{M}_k$ is compact.

2. Proof of the Theorem, Outline

As said above, we fix an underlying SU(2)-bundle E; the family of holomorphic bundles $\mathscr{E}_t$ can then be represented by a family $B_t \in \mathcal{A}^{(1,1)}$ of connections on E, where B_t defines the holomorphic structure $\mathscr{E}_t$. To make the argument work, we have to require the connections B_t to converge uniformly in C^∞ to B_0 as $t \to 0$.

To begin the proof, we consider the family of ASD-connections A_t on E. Each A_t gives a point $[A_t]$ in the moduli space M_k, and the theorem is claiming that, as $t \to 0$, these points converge to the point $\bigl([\theta], (x_1, \dots, x_k)\bigr)$. Because of the compactnesss of the space $\bar{M}_k$, it suffices to show that whenever we have a sequence $t_n \to 0$ for which

$$A_n = A_{t_n} \to \bigl([A'], (y_1, \dots, y_l)\bigr),$$

the connection A' is gauge equivalent to the trivial one, and the multiset $(y_1, \dots, y_l)$ is equal to $(x_1, \dots, x_k)$. This will now be proved in several steps. As for notation, we let E' be the SU(2)-bundle of Chern class $(k-l)$ on which the ASD connection A' lives, and $\mathscr{E}'$ the holomorphic bundle structure that A' defines on E'.

Since the proof itself is surprisingly long, a short outline might not be out of place. The argument can be divided quite naturally into two parts. The first part consists in showing that both the bundle E' and the connection A' are trivial. This is accomplished by constructing a nontrivial holomorphic map $h\colon \mathscr{E}_0 \to \mathscr{E}'$, using the fact that both B_{t_n} and A_{t_n} define the same holomorphic structure on the bundle E. The special structure of $\mathscr{E}_0$ can then be used to show that $\mathscr{E}'$ has to be the trivial bundle, essentially by arguments about slope. The fact that E' is trivial automatically gives $k = l$, too.

The second part is topological, and consists in verifying that the two multisets are the same. For each point $z \in \bigl\{x_1, \dots, x_k, y_1, \dots, y_l\bigr\}$, the multiplicity with which it occurs in either multiset is interpreted as the degree of a certain map, and a topological argument, using special trivializations of the bundle E away from the points, then shows the equality of these degrees.

3. Proof of the Theorem, Part 1

In this section, we shall show that E' is the trivial bundle and that $[A'] = [\theta]$. It will be proved below that our convergence assumptions imply the existence of a *nontrivial* holomorphic bundle map $h\colon \mathscr{E}_0 \to \mathscr{E}'$. Because both determinant line bundles, we can then consider the dual map $h^*\colon \mathscr{E}' \to \mathscr{E}_0$, which is also nonzero.

By the easy direction of Donaldson's theorem, the ASD connection A' is either irreducible, in which case $\mathscr{E}'$ is a stable bundle (of slope zero), or reducible, and then $\mathscr{E}' \simeq \mathscr{L} \oplus \mathscr{L}^{-1}$ with a line bundle $\mathscr{L}$ of slope zero.

The first case, that $\mathscr{E}'$ is stable, is actually impossible. To see this, compose h^* with the map $\mathscr{E}_0 \to \mathscr{I}$; the resulting map, from a stable bundle to a torsion-free sheaf of the same slope, must be zero. Therefore, h^* factors as $\mathscr{E}' \to \mathscr{O} \to \mathscr{E}_0$, but for the same reason, the first map in this factorization has to be zero, and this contradicts the fact that h^* is nontrivial.

In the second case, we use the same reasoning. First, we get a map $\mathscr{L} \oplus \mathscr{L}^{-1} \to \mathscr{I} \to \mathscr{O}$. If it is nonzero, then either $\mathscr{L}$ or its dual have a nontrivial global section, which has to be constant since the slope of $\mathscr{L}$ is zero. But then $\mathscr{L} = \mathscr{O}$, hence $\mathscr{E}' = \mathscr{O} \oplus \mathscr{O}$. If the map is zero, then we get a nontrivial map $\mathscr{L} \oplus \mathscr{L}^{-1} \to \mathscr{O}$, which again means that $\mathscr{L}$ and $\mathscr{E}'$ are trivial.

Since, up to gauge equivalence under $\mathcal{G}$, the only ASD connection on the trivial bundle is the trivial one, we are done with this part of the argument.

Existence of a map $h\colon \mathscr{E}_0 \to \mathscr{E}'$. We shall now demonstrate that there exists a nontrivial holomorphic map $h\colon \mathscr{E}_0 \to \mathscr{E}'$. The argument uses the convergence assumptions made above, together with some elliptic estimates.

Let V be the open set $X \setminus \{y_1, \dots, y_l\}$. By our assumption that the sequence $A_n = A_{t_n}$ of ASD connections on the bundle E converge to A' on the bundle E', there are (topological) bundle maps

$$r_n\colon E'\Big|_V \to E\Big|_V$$

such that $A'_n = r_n^*(A_n)$ converges to A' uniformly in C^∞ on compact subsets of V. Of course, each r_n is a unitary isomorphism, since it is induced by a map of principal SU(2)-bundles.

On the bundle E, the ASD connection A_n and the original connection $B_n = B_{t_n}$ define the same complex structure; thus there is a complex gauge transformation $g_n \in \mathcal{G}^c$ with $g_n(A_n) = B_n$. We view g_n as a bundle automorphism of E satisfying

$$\bar{\partial}_{A_n * B_n} g_n = 0.$$

We define $h_n\colon E\Big|_V \to E'\Big|_V$ by $h_n = r_n^{-1} g_n / \|g_n\|_{L^2}$; then

$$|g_n| = |h_n| \cdot \|g_n\|^2$$

because r_n was unitary; h_n also satisfies the equation

$$\bar{\partial}_{B_n * A'_n} h_n = 0.$$

The ellipticity of this equation will be used to find a limit h for this sequence.

Fix temporarily a compact subset K of V. For each value of n, the equation $\bar{\partial}_{B_n * A'_n} h_n = 0$ is elliptic; thus the usual bootstrapping arguments give an estimate for the L^2_3-norm of h_n by some quantity depending only on K, the connections B_n and A'_n, and the L^2-norm of h_n (all norms with respect to the bundle $\mathrm{End}(E)$). If we now allow n to vary, then $B_n \to B_0$ and $A'_n \to A'$ uniformly in C^∞ on K; on the other hand, the L^2-norm of h_n on K is bounded independently of n by construction. We therefore obtain a uniform bound on the L^2_3-norm of h_n, independent of n.

The usual diagonalization argument therefore allows us to choose a subsequence (which we continue to denote by h_n) that converges weakly in L^2_3 on compact subsets to some limit h. But then h satisfies the elliptic equation

$$\bar{\partial}_{B_0 * A'} h = 0,$$

hence is a holomorphic bundle map from $\mathscr{E}_0$ to $\mathscr{E}'$ defined over V. By Hartog's theorem, h extends over the missing points to a global holomorphic bundle map.

Nontriviality of the map h. To complete the argument above, we have to show that $h \neq 0$. This is done by estimating the L^2-norm of h_n, finding a fixed compact subset $K \subset V$ with

$$\int_K |h_n|^2 d\mu \geq \frac{1}{2}$$

for all n. Since $h_n \to h$ strongly in L^2 on K, this inequality is enough to ensure $h \neq 0$.

The argument requires, first of all, the following Weitzenböck formula. The proof is straightforward from the Kähler identities.

Lemma. *Let A be an arbitrary unitary (1,1)-connection on a bundle E over a Kähler surface X, and denote by ∇_A the covariant derivative, by $\bar\partial_A$ the del-bar operator corresponding to A, and by $\hat F_A = \Lambda F_A$ the Hermite-Einstein form of the curvature. If $*$-superscripts denote adjoint operators, then*

$$\bar\partial_A^*\bar\partial_A = \frac{1}{2}\nabla_A^*\nabla_A + i\hat F_A.$$

To get the necessary estimates, consider for a minute the following general situation. Let g be a complex gauge transformation of E, and A and $B = g(A)$ two connections in $\mathcal{A}^{(1,1)}$. If we set

$$\tau = \operatorname{tr}(g^*g),$$

then the L^2-norm of g, viewed as a holomorphic section of $\operatorname{End}(E)$, is simply the integral of this function τ over X. If $\nabla = \nabla_{A*B}$ is the covariant derivative on $\operatorname{End}(E)$, then

$$\begin{aligned}\Delta\tau &= \operatorname{tr}\big(\nabla^*\nabla(g^*g)\big) = \operatorname{tr}\big((\nabla^*\nabla g^*)g + g^*(\nabla^*\nabla g) - 2(\nabla g^*)\cdot(\nabla g)\big)\\ &\le \operatorname{tr}\big((\nabla^*\nabla g^*)g + g^*(\nabla^*\nabla g)\big)\end{aligned}$$

Now use the Weitzenböck formula on the bundle $\operatorname{End}(E)$; for the holomorphic section g, it reads

$$\nabla^*\nabla g = -2i\hat F_{A*B}(g) = 2i\big(g\hat F_A - \hat F_B g\big),$$

and for the dual section g^*, it reads

$$\nabla^*\nabla g^* = 2i\big(\hat F_A g^* - g^*\hat F_B\big).$$

Substitute into the formula above to obtain

$$\Delta\tau \le 2i\operatorname{tr}\big(\hat F_A g^* g - g^*\hat F_B g - g^*\hat F_B g + g^* g\hat F_A\big).$$

Together with the definition of τ, this gives

$$\Delta\tau \le 4\tau\cdot\big(\sup|\hat F_A| + \sup|\hat F_B|\big).$$

If we apply this inequality to our case, using $g = g_n$, $A = A_n$ and $B = B_n$, then $\hat F_{A_n}$ is zero (because A_n is ASD and has first Chern class zero), and $\hat F_{B_n} \to \hat F_{B_0}$. Thus the curvature terms are uniformly bounded, and we have

$$\Delta\tau_n \le C\tau_n$$

for some constant C that is independent of n.

To conclude the argument, one applies the following lemma.

Lemma. *If a non-negative function τ on X satisfies $\Delta\tau \le C\tau$, then there is a constant C'', depending only on C and X, such that for any r-ball $B(r)$ in X,*

$$\int_{B(r)}\tau d\mu \le C'' r^3\int_X \tau d\mu.$$

Proof. Multiply the inequality $\Delta\tau \le C\tau$ on both sides by τ and integrate to get

$$\int_X |\nabla\tau|^2 d\mu \le C\int_X \tau^2 d\mu.$$

With this estimate on the gradient of τ, the four-dimensional Sobolev inequality shows that

$$\int_X \tau^4 d\mu \le C'\Big(\int_X \tau^2 d\mu\Big)^2$$

for some constant C' depending on X and C. On the other hand, Hölder's inequality (with exponents 3 and 3/2) gives

$$\int_X \tau^2 d\mu \leq \left(\int_X \tau d\mu\right)^{2/3} \left(\int_X \tau^4 d\mu\right)^{1/3},$$

and when this is combined with the first inequality, we obtain

$$\int_X \tau^2 d\mu \leq C' \left(\int_X \tau d\mu\right)^2.$$

Finally, substitute this back into the first inequality to get

$$\int_X \tau^4 d\mu \leq (C')^3 \left(\int_X \tau d\mu\right)^4.$$

Now if $B(r)$ is any r-ball in X, then

$$\int_{B(r)} \tau d\mu \leq \left(\int_{B(r)} \tau^4 d\mu\right)^{1/4} \left(\int_{B(r)} 1 d\mu\right)^{3/4} \leq (C')^{3/4} \bigl(\operatorname{vol} B(r)\bigr)^{3/4} \int_X \tau d\mu.$$

Since $\operatorname{vol} B(r)$ is $O(r^4)$, this gives the result. □

In our situation, we have a uniform estimate $\Delta \tau_n \leq C \tau_n$; on each r-ball $B(r)$, we thus get

$$\int_{B(r)} |g_n|^2 d\mu \leq C'' \|g_n\|_{L^2}.$$

from the lemma. This tells us that the contribution to the L^2-norm of g_n, and hence of h_n, from an r-ball around each point y_i is $O(r^3)$, independently of n. We can therefore select a compact set $K \subset V$ such that

$$\int_K |h_n|^2 d\mu \geq \frac{1}{2}$$

for all n, and because $h_n \to h$ in L^2 on K, the limit h cannot be zero.

4. Proof of the Theorem, Part 2

At this point, we know that the bundle E' and the connection A' are both trivial. Because the second Chern class of the limit connection $\bigl([\theta], (y_1, \ldots, y_l)\bigr)$ is still k, we get that $l = k$. It remains to show that the two multisets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ also agree. For this, an essentially topological argument will be used.

Consider the situation on the open set $U = X \setminus \bigl\{x_1, \ldots, x_k\bigr\}$. There, the section s of the bundle $\mathscr{E}_0$ gives a trivial subbundle $\mathscr{O} \subset \mathscr{E}_0$; by choosing a C^∞-complement, we find a trivialization of $\mathscr{E}_0$ over U.

In this trivialization, the del-bar operator $\bar{\partial}_{B_0}$ may be written as

$$\bar{\partial}_{B_0} = \bar{\partial} + \begin{pmatrix} \phi & \beta \\ 0 & -\phi \end{pmatrix},$$

if $\bar{\partial} + \phi$ is the del-bar operator on the trivial subbundle. With respect to the same trivialization, write $\bar{\partial}_{B_t}$ as

$$\bar{\partial}_{B_t} = \bar{\partial} + \begin{pmatrix} \phi + \varepsilon_1(t) & \beta + \varepsilon_2(t) \\ \varepsilon_3(t) & -(\phi + \varepsilon_1(t)) \end{pmatrix};$$

as $t \to 0$, each $\varepsilon_i(t) \to 0$.

To allow comparison with the connections A_t, we shall now find a complex gauge transformation g_t on X such that $g_t(B_t)$ converges to the trivial connection over U. First, a complex gauge transformation on $\mathscr{O}$ allows us to eliminate ϕ. Then, apply a transformation of the form

$$\begin{pmatrix} \lambda(t) & 0 \\ 0 & \lambda(t)^{-1} \end{pmatrix},$$

under which $\bar\partial_{B_t}$ changes into

$$\bar\partial + \begin{pmatrix} \varepsilon_1(t) & \lambda(t)^2(\beta + \varepsilon_2(t)) \\ \lambda(t)^{-2}\varepsilon_3(t) & -\varepsilon_1(t) \end{pmatrix}.$$

A choice of $\lambda(t) = \|\varepsilon_3(t)\|^{1/4}$, for example, will have the desired effect. The reader should note that this complex gauge transformation, which is a priori defined only on the open set U, extends to all of X by Hartog's theorem.

The situation can therefore be stated as follows. Firstly, there is a trivialization τ of E on $U = X \setminus \{x_1, \ldots, x_k\}$, and connections $B'_n = B'_{t_n}$ on E over X that represent the holomorphic structures $\mathscr{E}_{t_n}$ and converge to the product connection on U. Secondly, because of the convergence of A_n, there is a trivialization σ of E on $V = X \setminus \{y_1, \ldots, y_k\}$, such that the connections A'_n (which represent the same holomorphic structures $\mathscr{E}_{t_n}$) converge to the product connection in σ over V, and

$$|F_{A'_n}|^2 \to 8\pi^2 \sum_j \delta_{y_j}.$$

Let g_n be the complex gauge transformation of E on X for which $g_n(B'_n) = A'_n$. On the intersection $U \cap V$, we can represent g_n by a matrix-valued function with respect to the trivializations σ and τ. A convergence argument similar to the one given above allows us to assume that $g_n \to g$. By Hartog's theorem, the limit g extends to a matrix-valued function on all of X, hence is constant. As before, g can be shown to be nontrivial.

Now comes the topological part. Observe that the group $\mathrm{GL}(2, \mathbb{C})$ is homotopy equivalent to $\mathrm{SU}(2)$, and therefore to S^3. Any map from a three-sphere into $\mathrm{GL}(2, \mathbb{C})$ is therefore characterized by its degree, which is an integer. To each point x_i, we can thus associate the degree of the trivialization τ over a small three-sphere around x_i, relative to a fixed trivialization of E in a neighborhood of x_i. This degree is of course simply the multiplicity of the zero of s at x_i.

To each y_i, we can similarly associate the degree of σ over a small sphere, relative to a fixed trivialization of E near y_i. If this degree is, say, d_i, then by Chern-Weil theory we have for any function f on X and any n that

$$\frac{1}{8\pi^2} \int_X f \,\mathrm{tr}\, F_{A_n}^2 = \sum_i d_i f(y_i).$$

Now each A_n is an ASD connection, and so $\mathrm{tr}\, F_{A_n}^2 = |F_{A_n}|^2 d\mu$; the formula above then becomes

$$\frac{1}{8\pi^2} \int_X f |F_{A_n}|^2 d\mu = \sum_i d_i f(y_i).$$

But now the left-hand side converges by assumption to $\sum_i f(y_i)$, and therefore d_i is simplify the multiplicity of y_i in the multiset $(y_1, \ldots, y_k)$.

For each point z in $\{x_1, \dots, x_k, y_1, \dots, y_k\}$, the difference of the multiplicities with which z occurs in the two multisets is thus the degree of g_n over a small three-sphere around z. But since we know that $g_n \to g$, we can use the following lemma to conclude that this degree is zero, hence that the two multisets are the same.

Lemma 1. *Let g be a non-zero 2×2 complex matrix, and let N_r be the intersection of an r-ball around g with the open subset $\mathrm{GL}(2, \mathbb{C})$ of invertible matrices. Then $H_3(N_r, \mathbb{Z}) = 0$ for small values of r, hence any map from S^3 to N_r has degree zero.*

Proof. Since $g \neq 0$, the hypersurface defined by the vanishing of det is smooth at g; N_r is thus the complement of a smooth hypersurface inside a small ball, hence homotopy-equivalent to an S^1. □

Because of this lemma, each of the maps g_n, for n sufficiently large, has degree zero over a small sphere around z, and the same is therefore true for g.