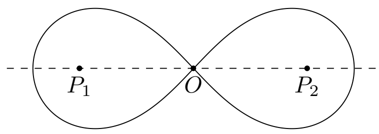
Lecture 1 (January 28)

Introduction. Our topic this semester is abelian varieties. As you probably know, abelian varieties are the higher-dimensional generalization of elliptic curves: smooth projective varieties that have the structure of an abelian group, with the group operations given by algebraic morphisms. During the first half of the semester, we will cover the basic theory, both from the analytic (= complex manifolds) and algebraic (= projective algebraic varieties) point of view. Our main source will be Mumford's book *Abelian Varieties*. After that, I plan to talk about derived categories and the Fourier transform, and about Deligne's theorem on absolute Hodge classes. I will try to provide notes for each lecture.

The lemniscate. Let's start with a bit of historical material, in order to understand where elliptic curves come from. (If you are interested in learning more about this, I recommend the article "The arithmetic-geometric mean of Gauss" by David Cox.) The length of a circular arc is easily computed with the help of trigonometric functions (and their inverses). But trying to compute the arc length of other curves such as ellipses leads to more complicated integrals, and the study of these integrals eventually led to the birth of elliptic curves. One particularly nice example is the *lemniscate*. It is defined as the set of points for which the product of the distances to two given points P_1 and P_2 (called the "foci") is constant.



In polar coordinates (r, θ), the equation of the lemniscate is $r^2 = a^2 \cos(2\theta)$, where $2a$ is the diameter of the lemniscate. The arc length of the lemniscate was first computed by the Bernoullis at the end of the 17-th century. We can easily derive their formula. If we write

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

and use r as the parameter, then $r \in [0, a]$ gives us exactly one quarter of the lemniscate. Therefore the length of the entire lemniscate is

$$L(a) = 4 \int_0^a \sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2}\, dr.$$

Taking derivatives of our parametrization, we get

$$\frac{dx}{dr} = \cos\theta - r\sin\theta \frac{d\theta}{dr} \quad \text{and} \quad \frac{dy}{dr} = \sin\theta + r\cos\theta\frac{d\theta}{dr},$$

and so the expression inside the square root is

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2.$$

From the equation $r^2 = a^2\cos(2\theta)$, we obtain

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{r^2}{a^4\sin^2(2\theta)} = \frac{r^2}{a^4(1 - r^4/a^4)} = \frac{r^2}{a^4 - r^4},$$

and after substituting this into the integral and simplifying, we arrive at

$$L(a) = 4\int_0^a \sqrt{\frac{a^4}{a^4 - r^4}}\, dr = 4a\int_0^1 \frac{dt}{\sqrt{1 - t^4}}.$$

Note that the integral looks a bit similar to

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \arcsin(1) = \frac{\pi}{2}.$$

Probably for that reason, Gauss introduced the notation

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\varpi}{2},$$

because the symbol ϖ (LaTeX code `\varpi`) is a cursive variant of the letter pi. Gauss came across this integral in his study of the arithmetic-geometric mean. For two positive real numbers $a, b > 0$, the arithmetic-geometric mean $M(a, b)$ is the common limit of the two sequences a_n, b_n, defined recursively by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

The two sequences converge very rapidly, and Gauss arrived at the identity

$$M\bigl(\sqrt{2}, 1\bigr) = \frac{\pi}{\varpi}$$

by computing both sides to 11 digits (by hand). This identity can be used to compute ϖ efficiently.

Let's now consider the arc length of the lemniscate

$$\int_0^x \frac{dt}{\sqrt{1-t^4}}$$

as a function of $x \in [-1, 1]$. It is obviously increasing, and takes values in the interval $[-\varpi/2, \varpi/2]$. The inverse function

$$\mathrm{sl}\colon [-\varpi/2, \varpi/2] \to [-1, 1]$$

is called the *lemniscate sine*; its defining property is that

$$\int_0^{\mathrm{sl}\, x} \frac{dt}{\sqrt{1-t^4}} = x.$$

The reason for the name is the obvious analogy with the arc sine function

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

and its inverse. We have $\mathrm{sl}(0) = 0$, $\mathrm{sl}(\varpi/2) = 1$, and $\mathrm{sl}(-\varpi/2) = -1$. Just like the sine function, the lemniscate also satisfies an addition formula. The precise result is due to Euler, I believe, but other people had already found similar addition formulas for the arc lengths of other curves (such as $y = x^3$ or ellipses).

Proposition 1.1. *Suppose that x, y, z are related by the fact that*

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^z \frac{dt}{\sqrt{1-t^4}}.$$

Then x, y, z also satisfy the following algebraic equation:

$$z = \frac{y\sqrt{1-x^4} + x\sqrt{1-y^4}}{1 + x^2 y^2}$$

The interesting point is that the arc length function is *transcendental* (just like the trigonometric functions), but the three values in the formula above are nevertheless related by an *algebraic* equation.

Proof. This is very similar to the proof of the addition formula for sine. Let's think of $z = z(x, y)$ as a function of the two variables x and y; this makes sense when x and y are not too large, because the arc length function has an inverse. Each level set of z is a curve, and by differentiating the relation between x, y, z, we see that

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}} = 0$$

along this curve. Choose a local parametrization $x = x(t)$ and $y = y(t)$ for the curve, in such a way that

$$\frac{dx}{dt} = \sqrt{1-x^4} \quad \text{and} \quad \frac{dy}{dt} = -\sqrt{1-y^4}.$$

For simplicity, let's use a dot to denote the derivative with respect to t. Then

$$\dot{x}^2 = \left(\frac{dx}{dt}\right)^2 = 1 - x^4,$$

and therefore $2\dot{x}\ddot{x} = -4x^3\dot{x}$ or $\ddot{x} = -2x^3$. For the same reason, $\ddot{y} = -2y^3$. Now the trick, which is hard to guess unless you know the proof of the addition formula for sine, is to compute

$$\frac{d}{dt}(y\dot{x} - x\dot{y}) = y\ddot{x} - x\ddot{y} = 2xy(y^2 - x^2).$$

From the formulas for the first derivatives, we also have

$$(y\dot{x} - x\dot{y})(y\dot{x} + x\dot{y}) = y^2\dot{x}^2 - x^2\dot{y}^2 = (y^2 - x^2)(1 + x^2y^2).$$

After dividing both lines, we obtain

$$\frac{d}{dt}\log(y\dot{x} - x\dot{y}) = \frac{\frac{d}{dt}(y\dot{x} - x\dot{y})}{y\dot{x} - x\dot{y}} = \frac{2xy(y\dot{x} + x\dot{y})}{1 + x^2y^2} = \frac{d}{dt}\log(1 + x^2y^2).$$

After integration, this becomes

$$C(1 + x^2y^2) = y\dot{x} - x\dot{y} = y\sqrt{1-x^4} + x\sqrt{1-y^4}$$

for some constant C, and by setting $y = 0$ and $x = z$, we find that $C = z$. This gives the desired algebraic relation between x, y, z. $\square$

If we rewrite the addition formula in terms of $\operatorname{sl} x$, it becomes

$$\operatorname{sl}(x+y) = \frac{\operatorname{sl} y\sqrt{1-\operatorname{sl}^4 x} + \operatorname{sl} x\sqrt{1-\operatorname{sl}^4 y}}{1 + \operatorname{sl}^2 x \operatorname{sl}^2 y}. \tag{1.2}$$

Remembering that $\operatorname{sl}(\varpi/2) = 1$, we deduce that

$$\operatorname{sl}(x + \varpi/2) = \frac{\sqrt{1-\operatorname{sl}^4 x}}{1+\operatorname{sl}^2 x} = \sqrt{\frac{1-\operatorname{sl}^2 x}{1+\operatorname{sl}^2 x}},$$

at least for those values of x where both sides are defined. We can now try to extend the domain of definition. To eliminate the (potentially ambiguous) square root, we rewrite the formula above as

$$\operatorname{sl}^2(x + \varpi/2) = \frac{1-\operatorname{sl}^2 x}{1+\operatorname{sl}^2 x}. \tag{1.3}$$

Applying the formula to itself, we get $\operatorname{sl}^2(x+\varpi) = \operatorname{sl}^2(x)$, and therefore $\operatorname{sl}(x+\varpi) = \pm\operatorname{sl} x$. As $\operatorname{sl}(\varpi/2) = 1$ and $\operatorname{sl}(-\varpi/2) = -1$, we have to choose the minus sign if we want things to be consistent, and so

$$\operatorname{sl}(x+\varpi) = -\operatorname{sl} x.$$

This allows us to extend the lemniscate sine to a function $\operatorname{sl}\colon \mathbb{R} \to [-1, 1]$ that is periodic with period 2ϖ.

Gauss was the first person to consider the lemniscate sine as a function of a *complex* variable; this was taking place around 1800. The integral

$$\int_0^z \frac{dt}{\sqrt{1-t^4}}$$

makes sense for complex numbers $z \in \mathbb{C}$ with $|z| < 1$, by using the standard branch of the square root function. It is again invertible, at least in a neighborhood of the origin, and we denote the inverse function by the same symbol sl. The substitution $t \mapsto it$ proves the formula

$$\int_0^{iz} \frac{dt}{\sqrt{1-t^4}} = i \int_0^z \frac{dt}{\sqrt{1-t^4}},$$

and so we have $\operatorname{sl}(iz) = i \operatorname{sl}(z)$. The addition formula in (1.2) shows that

$$\operatorname{sl}(x+iy) = \frac{\operatorname{sl} x \sqrt{1-\operatorname{sl}^4 y} + i \operatorname{sl} y \sqrt{1-\operatorname{sl}^4 x}}{1-\operatorname{sl}^2 x \operatorname{sl}^2 y}$$

for $x, y \in [-\varpi/2, \varpi/2]$, and so our complex-valued lemniscate sine is defined on the square of side length ϖ centered at the origin. From this formula, we can see that $\operatorname{sl} z$ has a pole when $\operatorname{sl} x = \pm 1$ and $\operatorname{sl} y = \pm 1$, hence at the points $(\pm 1 \pm i)/2$. It also has a unique zero at the point $z = 0$. For the same reason as before, the function $\operatorname{sl} z$ satisfies the two identities

$$\operatorname{sl}(z+\varpi) = -\operatorname{sl}(z) \quad \text{and} \quad \operatorname{sl}(z+i\varpi) = -\operatorname{sl}(z),$$

and this allows us to extend $\operatorname{sl} z$ to a well-defined meromorphic function on the entire complex plane. It has simple zeros at the point $\mathbb{Z}\varpi + \mathbb{Z}i\varpi$ and simple poles at the points $\varpi(1+i)/2 + \mathbb{Z}\varpi + \mathbb{Z}i\varpi$. Note that $\operatorname{sl} z$ has two independent periods, namely 2ϖ and $(1+i)\varpi$. Unlike the usual trigonometric function, the lemniscate sine is doubly periodic. Such functions are also called *elliptic functions*. Gauss did not publish any of his results, and it took almost 30 years until Abel and Jacobi developed the theory of elliptic functions.

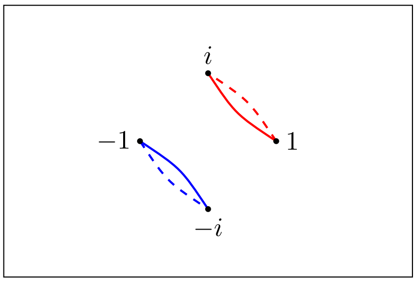
Geometric interpretation. We can interpret the above results about the integral

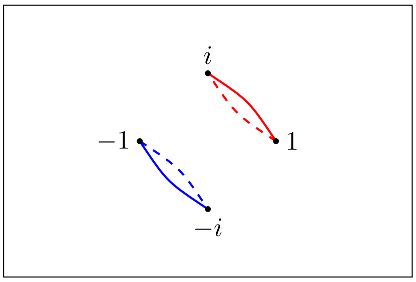
$$\int \frac{dt}{\sqrt{1-t^4}}$$

in terms of compact Riemann surfaces. Since the square-root function $\sqrt{1-x^4}$ has two branches, we introduce a new variable y such that $y^2 = 1 - x^4$, and then consider the one-form dx/y instead of $dt/\sqrt{1-t^4}$. This rule actually defines a double covering of $\mathbb{P}^1$. To see why, let's use the coordinate $u = 1/x$ on the chart at infinity, and define $v^2 = 1 - u^4$. If we glue the two according to the rule $v = -u^2 y$, then we get a compact Riemann surface C together with a two-to-one map $C \to \mathbb{P}^1$ that is branched at the four points ± 1 and $\pm i$. Moreover, dx/y is a well-defined one-form on C because

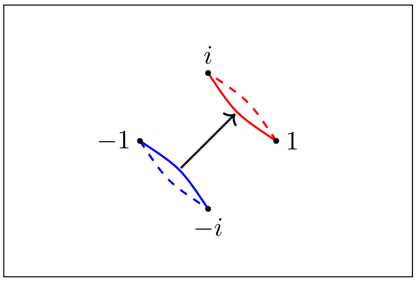
$$\frac{dx}{y} = \frac{-du/u^2}{-v/u^2} = \frac{du}{v}.$$

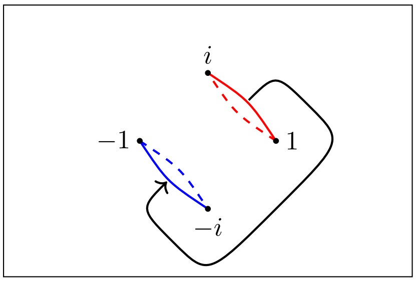
We can visualize the double covering as follows. Take two copies of the complex plane (or the Riemann sphere), make branch cuts from the point 1 to the point i and from the point -1 to the point $-i$, and then glue the two copies together as indicated in the picture below. This produces a surface with one handle, which means that the Riemann surface C has genus 1, hence is a torus.





We can integrate dx/y along paths in C to obtain a multi-valued holomorphic function on C. It is multi-valued because C is not simply connected. The ambiguity in the values is given by the integrals of dx/y along the two basic closed loops in C. The first loop goes around the torus.





By moving the contour of integration in both copies of the plane, one can see that the integral of dx/y over this loop is equal to

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^4}} - \int_{1}^{-1} \frac{dt}{\sqrt{1-t^4}} = 2\varpi.$$

Similarly, the loop that goes around the neck of the torus leads to the integral $(1+i)\varpi$. The inverse of this multi-valued function will therefore be doubly periodic, with periods 2ϖ and $(1+i)\varpi$.

Elliptic curves. Let's now consider elliptic functions in general. An elliptic function is holomorphic (actually, meromorphic) function on $\mathbb{C}$ with two linearly independent periods. If we call the two periods γ_1 and γ_2, then the subgroup $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a discrete subgroup of $\mathbb{C}$ of rank 2. The quotient $\mathbb{C}/\Gamma$ is compact, and so Γ is called a lattice. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic function with periods in Γ, meaning that $f(z+\gamma) = f(z)$ for every $\gamma \in \Gamma$. Because $\mathbb{C}/\Gamma$ is compact, f must be bounded, hence constant. In order to get interesting elliptic functions, we have to allow poles.

The most basic elliptic function is the *Weierstrass $\wp$-function.* The symbol $\wp$ (LaTeX code `\wp`) is the old handwritten German p, but for simplicity, I will use the regular letter P instead. Consider the infinite series

$$P(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

It converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Gamma$; the reason is that, for z in a compact set, each term is bounded by a constant times $1/|\gamma|^3$, and it is easy to see that the series

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{|\gamma|^3}$$

is convergent. Therefore $P(z)$ is a well-defined meromorphic function with a double pole at each point of Γ; it is clearly even because $P(-z) = P(z)$. To show that it is Γ-periodic, we consider the derivative

$$P'(z) = \sum_{\gamma \in \Gamma} \frac{-2}{(z-\gamma)^3}.$$

The series again converges absolutely and uniformly on compact subsets, and $P'(z)$ is visibly Γ-periodic. This means that

$$P(z+\gamma) = P(z) + C(\gamma)$$

for some constant $C(\gamma) \in \mathbb{C}$. By putting $z = -\gamma/2$ and remembering that $P(z)$ is even, we get $C(\gamma) = 0$, and so $P(z)$ is itself Γ-peridic.

Let's compute the Laurent series around $z = 0$. Here we use the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{and} \quad \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

From the definition of $P(z)$, we get

$$\begin{aligned} P(z) &= \frac{1}{z^2} + \sum_{\gamma \neq 0} \frac{1}{\gamma^2} \left(\frac{1}{(1-z/\gamma)^2} - 1 \right) = \frac{1}{z^2} + \sum_{\gamma \neq 0} \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\gamma^{n+2}} \\ &= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) G_{n+2} z^n, \end{aligned}$$

where the constants G_n, depending on the lattice Γ, are defined by the formula

$$G_n = \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^n}$$

for $n \geq 3$. By symmetry, G_n can only be nonzero for even values of n; therefore

$$P(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n} \tag{1.4}$$

is the Laurent expansion of $P(z)$ around the point $z = 0$.

The next lemma shows that the Weierstrass ℘-function is related to cubic curves.

Lemma 1.5. *The ℘-function satisfies the differential equation*

$$P'(z)^2 = 4P(z)^3 - 60G_4 P(z) - 140G_6.$$

Proof. From the Laurent series in (1.4), we get

$$\begin{aligned} P(z) &= z^{-2} + 3G_4 z^2 + 5G_6 z^4 + \cdots \\ P'(z) &= -2z^{-3} + 6G_4 z + 20G_6 z^3 + \cdots \\ P'(z)^2 &= 4z^{-6} - 24G_4 z^{-2} - 80G_6 + \cdots \\ 4P(z)^3 &= 4z^{-6} + 36G_4 z^{-2} + 60G_6 + \cdots \end{aligned}$$

Consequently, the function $P'(z)^2 - 4P(z)^3 + 60G_4 P(z) + 140G_6$ has no pole at $z = 0$, and because it is doubly periodic, it is therefore holomorphic, hence constant. The constant value is the value at $z = 0$, which is 0. □

If we set $g_2 = 60G_4$ and $g_3 = 140G_6$, then the differential equation takes the form $P'(z)^2 = 4P(z)^3 - g_2 P(z) - g_3$. Now consider the holomorphic mapping

$$\mathbb{C} \setminus \Gamma \to \mathbb{C}^2, \quad z \mapsto \big(P(z), P'(z)\big).$$

Its image is contained in the cubic curve with equation $y^2 = 4x^3 - g_2 x - g_3$. Let us denote by $E = \mathbb{C}/\Gamma$ the quotient, which is a compact Riemann surface of genus

1. Let $0 \in E$ be the image of the origin in $\mathbb{C}$. We get an induced holomorphic mapping

$$E \setminus \{0\} \to \mathbb{C}^2, \quad z + \Gamma \mapsto \bigl(P(z), P'(z)\bigr).$$

Since $P(z)$ has a pole of order 2 at $z = 0$, whereas $P'(z)$ has a pole of order 3, this extends to a holomorphic mapping

$$E \to \mathbb{P}^2$$

by sending the point $0 \in E$ to the point $[0,1,0] \in \mathbb{P}^2$. The image is now contained inside the projective cubic curve with equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$$

Lecture 2 (January 30)

Elliptic curves. Last time, we introduced the Weierstrass $\wp$-function

$$P(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right),$$

where $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a lattice in $\mathbb{C}$. We showed that it is meromorphic and Γ-periodic, and that it satisfies the differential equation

$$P'(z)^2 = 4P(z)^3 - g_2 P(z) - g_3,$$

where g_2, g_3 are certain constants that depend on Γ. We are really interested in the compact Riemann surface $E = \mathbb{C}/\Gamma$, which is topologically a torus. Using the differential equation, we concluded that the image of the holomorphic mapping

$$h \colon E \to \mathbb{P}^2, \quad z + \Gamma \mapsto [P(z), P'(z), 1],$$

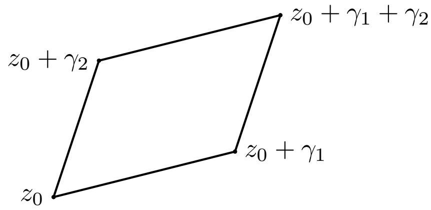
is contained in the cubic curve C with equation $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$. Now we are going to argue that h is an isomorphism between E and this cubic curve.

For that, we need two basic facts about doubly periodic meromorphic functions. Let f be a meromorphic function on $\mathbb{C}$ that is Γ-periodic. We may also view f as a holomorphic mapping from E to $\mathbb{P}^1$. If f has a zero or pole at a point $x \in E$, we let $\operatorname{ord}_x(f)$ denote the order of the zero (with a plus sign) or the order of the pole (with a minus sign); if x is neither a zero nor a pole, we set $\operatorname{ord}_x(f) = 0$. The fact that f is Γ-periodic constrains the number and location of the zeros and poles, in the following way:

(1) We have $\displaystyle\sum_{x \in E} \operatorname{ord}_x(f) = 0$.

(2) We have $\displaystyle\sum_{x \in E} \operatorname{ord}_x(f) \cdot x = 0$ as points in E.

Remember that we can add and subtract points in $E = \mathbb{C}/\Gamma$, because the quotient is an abelian group. Both formulas are consequences of the residue theorem. Let's quickly look at how this works. Consider the parallelogram D spanned by the two basis vectors $\gamma_1, \gamma_2 \in \Gamma$, with one corner at a point $z_0 \in \mathbb{C}$, and choose z_0 such that the boundary ∂D of the parallogram does not pass through any zeros or poles of f.



At each zero or pole of f, the meromorphic function f'/f has a simple pole with residue equal to the order of the zero or pole. The residue theorem therefore gives

$$\sum_{z \in D} \operatorname{ord}_z(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz.$$

The integral on the right-hand side is equal to zero because f is Γ-periodic. Since D is a fundamental domain for the action of Γ on $\mathbb{C}$, the left-hand side is equal to $\sum_{x \in E} \operatorname{ord}_x(f)$, and so we get the formula in (1). The formula in (2) is proved in a similar manner. Again by the residue theorem, we have

$$\sum_{z \in D} \operatorname{ord}_z(f) \cdot z = \frac{1}{2\pi i} \int_{\partial D} \frac{z f'(z)}{f(z)}\, dz.$$

This time, the integral on the right-hand side evaluates to an element in $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$, and so we get

$$\sum_{z \in D} \operatorname{ord}_z(f) \cdot z \in \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2,$$

which translates to the formula in (2).

We can now prove that $E = \mathbb{C}/\Gamma$ is isomorphic to the cubic curve C.

Proposition 2.1. *The mapping $h\colon E \to \mathbb{P}^2$ induces an isomorphism between E and the cubic curve C.*

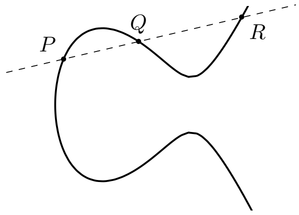
Proof. Let's first show that h is surjective. The point $[0, 1, 0]$ is the image of $0 \in E$, so let's consider points of the form $[a, b, 1]$ with $b^2 = 4a^3 - g_2 a - g_3$. We need to find $z_0 \in \mathbb{C}$ such that $P(z) = a$ and $P'(z) = b$. The function $P(z) - a$ is Γ-periodic and meromorphic, and has (up to translation by by Γ) a unique pole of order 2. According to the discussion above, it must therefore have exactly two zeros; since $P(z)$ is even, these will be of the form $\pm z_0$ for some $z_0 \in \mathbb{C}$. The differential equation for $P(z)$ then gives $P'(z_0)^2 = 4P(z_0)^3 - g_2 P(z_0) - g_3 = b^2$, and therefore $P'(z_0) = \pm b$. If $P(z_0) = b$, we can take $z = z_0$; otherwise, we take $z = -z_0$.

Now let's prove that h is injective. It is easy to see that the points in Γ are the only points mapping to $[0, 1, 0]$, so it is enough to consider two points $z_1, z_2 \in \mathbb{C}$ such that $P(z_1) = P(z_2)$ and $P'(z_1) = P'(z_2)$. We need to argue that $z_1 = z_2$. As before, $P(z) - P(z_1)$ must have exactly two zeros, which are z_1 and $-z_1$, and so either $z_2 \equiv z_1 \mod \Gamma$ and we are done, or $z_2 \equiv -z_1 \mod \Gamma$. In the second case, we get $P'(z_1) = P'(z_2) = -P'(z_1)$, and so $P'(z_1) = 0$. But we will see in a moment that the only zeros of $P'(z)$ are the translates of $\gamma_1/2$, $\gamma_2/2$, and $(\gamma_1 + \gamma_2)/2$, and no two of these differ by an element of Γ.

Next, we argue that the cubic curve C is nonsingular, and therefore a compact Riemann surface. This amounts to saying that all three roots of the cubic polynomial $4x^3 - g_2 x - g_3$ are distinct. Because h is surjective, these roots are of the form $P(z)$, where $z \in \mathbb{C}$ is any point such that $P'(z) = 0$. Now $P'(z)$ has (up to translation by Γ) a unique pole of order 3, and so it must have exactly 3 zeros. Because $P'(z)$ is odd, each of $\gamma_1/2$, $\gamma_2/2$, and $(\gamma_1 + \gamma_2)/2$ is a zero, and so these are all the zeros. They are different modulo Γ, and so our cubic polynomial has three distinct roots.

Now $h\colon E \to C$ is a bijective holomorphic mapping between two complex manifolds, and therefore biholomorphic (by the inverse function theorem). This proves that $E = \mathbb{C}/\Gamma$ is isomorphic to the cubic curve C. $\square$

Let's also briefly discuss the group structure. As you know, the points of a nonsingular cubic curve form a group; three points P, Q, R on the cubic are collinear (in $\mathbb{P}^2$) if and only if they satisfy $P + Q + R = 0$ as elements of the group.



In fact, the isomorphism $u\colon E \to C$ also respects the group structure, because of the following proposition.

Proposition 2.2. *Let $u, v, w \in \mathbb{C}$ be three distinct points. Then $u + v + w \in \Gamma$ if and only if the points $h(u)$, $h(v)$, and $h(w)$ are collinear in $\mathbb{P}^2$.*

Proof. I promised to put the proof in the notes, so here it is. Because h is bijective, we only need to prove one implication. Let's focus on the case where $u, v, w \notin \Gamma$. The fact that the three points are collinear then says that

$$\det \begin{pmatrix} P(u) & P(v) & P(w) \\ P'(u) & P'(v) & P'(w) \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

This means that there are constants $A, B, C \in \mathbb{C}$ such that $AP(z)+BP'(z)+C = 0$ for $z \in \{u, v, w\}$. Because this function has a unique pole of order 3, these must be the only zeros (up to translation by Γ). In particular, they are simple zeros, and so the second consequence of the residue theorem tells us that $u + v + w \in \Gamma$. A similar argument works when one of the three points belongs to the lattice Γ. $\square$

Exercise 2.1. The lemniscate sine is related to the Weierstrass $\wp$-function, but perhaps not quite in the way one would expect. Prove the formula

$$\frac{\varpi^2}{\mathrm{sl}^2(\varpi z)} = P(z),$$

where $P(z)$ is the Weierstrass $\wp$-function for the lattice $\mathbb{Z}+\mathbb{Z}i$ of Gaussian integers. (*Hint:* Look at the first few terms in the Laurent series.) Can you find the equation of the cubic curve for this lattice?

Abelian varieties. Let's now start looking at abelian varieties, from the point of view of complex geometry. Consider a compact and connected complex Lie group X. This means that X is a complex manifold, say of dimension n, that is compact and connected; it also means that the group operations

$$X \times X \to X, \quad (x, y) \mapsto x \cdot y, \qquad X \to X, \quad x \mapsto x^{-1},$$

are *holomorphic* mappings. Let $e \in X$ denote the identity element. We are going to prove that X is commutative, and that it has the form V/Λ, where V is an n-dimensional complex vector space, and $\Lambda \subseteq V$ is a discrete subgroup of rank $2n$.

First, we need to review a few basic facts about 1-parameter subgroups. Let $V = T_eX$ denote the tangent space to X at the identity element $e \in X$; this is an n-dimensional complex vector space. The result we need is that for every vector $v \in V$, there is a unique holomorphic mapping

$$\phi_v\colon \mathbb{C} \to X$$

that is a group homomorphism and whose differential

$$(d\phi_v)_e\colon \mathbb{C} = T_0\mathbb{C} \to V = T_eX$$

maps $1 \in \mathbb{C}$ to the given vector $v \in V$. This is a consequence of the existence and uniqueness result for solutions to ordinary differential equations. (Briefly, the tangent vector $v \in T_eX$ can be extended in a unique way to a holomorphic tangent vector field $\tilde{v}$ on all of X, using the group structure. Then ϕ_v solves the initial value problem $\phi'(t) = \tilde{v}_{\phi(t)}$, $\phi(0) = e$. The solution is holomorphic by Cauchy's theorem, which is a nice but fairly elementary result.) We need three additional facts:

(1) The mapping
$$\phi\colon \mathbb{C} \times V \to X, \quad (t, v) \mapsto \phi_v(t),$$
is holomorphic (because the solution to a holomorphic initial value problem depends holomorphically on the initial data).

(2) Let us define the *exponential mapping* by the formula
$$\exp\colon V \to X, \quad \exp(v) = \phi_v(1).$$
This is a holomorphic mapping by (1). The uniqueness of ϕ_v implies that $\phi_v(st) = \phi_{sv}(t)$ for every $s, t \in \mathbb{C}$; therefore
$$\phi_v(t) = \exp(tv)$$
for $t \in \mathbb{C}$ and $v \in V$. By construction, the differential
$$(d\exp)_e\colon V = T_0 V \to V = T_e X$$
is the identity mapping. By the (holomorphic) inverse function theorem, exp is therefore a biholomorphic isomorphism between a neighborhood of $0 \in V$ and a neighborhood of $e \in X$.

(3) Suppose that $h\colon X_1 \to X_2$ is both a holomorphic mapping and a group homomorphism. Then one has
$$h\bigl(\exp_{X_1}(v)\bigr) = \exp_{X_2}\bigl((dh)_e(v)\bigr)$$
for every $v \in V$. The reason is that the composition
$$h \circ \phi_v\colon \mathbb{C} \to X_2$$
is a holomorphic group homomorphism with
$$d(h \circ \phi_v)_e(1) = (dh)_e \circ (d\phi_v)_e(1) = (dh)_e(v).$$
The result we want therefore follows from the uniqueness statement.

We can now prove that the group structure on X is commutative.

Lemma 2.3. *Every compact connected complex Lie group is abelian.*

Proof. For $x \in X$, consider the conjugation mapping
$$C_x\colon X \to X, \quad C_x(y) = xyx^{-1};$$
it is cleary biholomorphic and an automorphism of the group X. The differential
$$(dC_x)_e\colon V \to V$$
is therefore an automorphism of $V = T_e X$. This gives us a holomorphic mapping
$$X \to \mathrm{GL}(V), \quad x \mapsto (dC_x)_e.$$
Because $\mathrm{GL}(V)$ sits inside the vector space $\mathrm{End}(V) \cong \mathbb{C}^{n^2}$, and X is compact and connected, this mapping must be constant; the constant value is
$$(dC_x)_e = (dC_e)_e = \mathrm{id}_V.$$
We can now apply (3) from above and conclude that
$$C_x\bigl(\exp(v)\bigr) = \exp\bigl((dC_x)_e(v)\bigr) = \exp(v)$$
for every $x \in X$ and every $v \in V$. This says that the image $\exp(V)$ of the exponential mapping lies in the center of the group X. It also generates X as a group (because $\exp(V)$ contains a neighorhood of $e \in X$ and X is compact and connected); it follows that X is commutative. $\square$

There are of course many compact real Lie groups that are not commutative; the magic comes from the fact that the group operations need to be holomorphic. Next, let's prove that X is isomorphic to the quotient of V by a discrete subgroup.

Lemma 2.4. *The exponential mapping* $\exp\colon V \to X$ *is a surjective group homomorphism. Its kernel* $\Lambda = \ker(\exp)$ *is a lattice in* V*, and* $X \cong V/\Lambda$*.*

Proof. For any two vectors $v, w \in V$, consider the holomorphic mapping

$$\mathbb{C} \to X, \quad t \mapsto \exp(tv)\exp(tw).$$

Because X is commutative, this is a group homomorphism; its differential takes $1 \in \mathbb{C}$ to the vector $v + w$. By uniqueness, it follows that

$$\exp(tv)\exp(tw) = \exp\bigl(t(v+w)\bigr).$$

Setting $t = 1$, we conclude that $\exp\colon V \to X$ is a group homomorphism. We already know that $\exp(V)$ generates X as a group; now $\exp(V)$ is also a subgroup, and so $\exp(V) = X$. The kernel $\Lambda = \ker(\exp)$ is a discrete subgroup of V (because exp is bijective in a neighborhood of $0 \in V$). The quotient V/Λ is isomorphic to X, hence compact; this means that Λ is a lattice in V. $\square$

A bit of terminology. A discrete subgroup Λ of a complex vector space V is called a *lattice* if the quotient V/Λ is compact. It is easy to see that Λ must then be isomorphic, as a group, to $\mathbb{Z}^{2n}$, where $n = \dim V$. The quotient V/Λ is called a *compact complex torus*. So the result above is saying that every compact and connected complex Lie group is a compact complex torus.

From now on, we are going to use additive notation

$$X \times X \to X, \quad (x,y) \mapsto x + y, \qquad X \to X, \quad x \mapsto -x,$$

for the group operations; the identity element is always $0 \in X$. By choosing a basis for $\Lambda \cong \mathbb{Z}^{2n}$, we see that

$$X_{\mathbb{R}} \cong (\mathbb{R}/\mathbb{Z})^{2n} \cong (\mathbb{S}^1)^{2n}$$

as real Lie groups. We will consider the problem of how to keep track of the different possible complex manifold structures on this later on.

Corollary 2.5. *As a group, X is divisible, and for every $m \in \mathbb{Z}$, we have*

$$X[m] = \bigl\{\, x \in X \bigm| m \cdot x = 0 \,\bigr\} \cong (\mathbb{Z}/m\mathbb{Z})^{2n}.$$

Proof. This is clear from the fact that $X_{\mathbb{R}} \cong (\mathbb{R}/\mathbb{Z})^{2n}$. $\square$

Lecture 3 (February 4)

Cohomology of compact complex tori. Let $X = V/\Gamma$ be a compact complex torus of dimension n. This means that V is an n-dimensional complex vector space, and $\Gamma \subseteq V$ is a lattice of rank $2n$. Let's see how to describe the cohomology of X in terms of V and Γ. Observe that Γ generates the underlying $\mathbb{R}$-vector space

$$V_{\mathbb{R}} \cong \Gamma \otimes_{\mathbb{Z}} \mathbb{R},$$

because $\Gamma \cong \mathbb{R}^{2n}$ and $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$. Over the complex numbers, we have

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{C} \cong V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \bar{V},$$

where $\bar{V}$ denotes the conjugate vector space: the underlying abelian group is still $(V, +)$, but the complex numbers act via $z \cdot v = \bar{z}v$. This is true for the complexification of any complex vector space. Indeed, let $J \in \operatorname{End}_{\mathbb{R}}(V_{\mathbb{R}})$ be the endomorphism $J(v) = iv$; then $J^2 = -\operatorname{id}$. The complexification

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E_i(J) \oplus E_{-i}(J)$$

decomposes into the $\pm i$-eigenspaces of J, and the two maps

$$V \to E_i(J), \quad v \mapsto v \otimes 1 - Jv \otimes i$$
$$\bar{V} \to E_{-i}(J), \quad v \mapsto v \otimes 1 + Jv \otimes i$$

are isomorphisms of $\mathbb{C}$-vector spaces.

As complex vector spaces, $V \cong T_0 X$ is isomorphic to the holomorphic tangent space at the point $0 \in X$. Since X is a group, the holomorphic tangent bundle is trivial; this means that we have a natural isomorphism

$$T_X \cong \mathscr{O}_X \otimes_{\mathbb{C}} V.$$

Dually, we get $\Omega_X^1 \cong \mathscr{O}_X \otimes_{\mathbb{C}} V^*$, and therefore

$$\Omega_X^p \cong \mathscr{O}_X \otimes_{\mathbb{C}} \bigwedge^p V^*.$$

We can also describe the lattice Γ intrinsically:

$$\Gamma \cong \pi_1(X, 0) \cong H_1(X, \mathbb{Z}),$$

where an element $\gamma \in \Gamma$ corresponds to the homotopy class (or homology class) of the closed loop $[0,1] \to X$, $t \mapsto t \cdot \gamma + \Gamma$. According to the universal coefficients theorem, we then have

$$H^1(X, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}\bigl(H_1(X, \mathbb{Z}), \mathbb{Z}\bigr) \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) = \Gamma^*.$$

The entire integral cohomology is equally easy to describe.

Lemma 3.1. *We have* $H^k(X, \mathbb{Z}) \cong \bigwedge^k H^1(X, \mathbb{Z}) \cong \bigwedge^k \Gamma^*$.

Proof. The cup product gives us a natural map

$$\bigwedge^k H^1(X, \mathbb{Z}) \to H^k(X, \mathbb{Z}), \quad \gamma_1 \wedge \cdots \wedge \gamma_k \mapsto \gamma_1 \cup \cdots \cup \gamma_k.$$

Since $X \cong (\mathbb{S}^1)^{2n}$ as smooth manifolds, the Künneth formula implies that the map is an isomorphism. $\square$

We can also describe the de Rham cohomology and the Dolbeault cohomology, by relating V and $\bar{V}$ to differential forms on X. Choose a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ be the dual basis; we view these linear functions as a holomorphic system of coordinates on $V \cong \mathbb{C}^n$. Their differentials are invariant under translation by Γ, and so they give us well-defined 1-forms

$$dz_1, \dots, dz_n \in A^{1,0}(X), \quad d\bar{z}_1, \dots, d\bar{z}_n \in A^{0,1}(X)$$

on $X = V/\Gamma$. Every smooth form $\alpha \in A^{p,q}(X)$ of type (p,q) can then be written as

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} \, dz_I \wedge d\bar{z}_J,$$

with coefficients $\alpha_{I,J} \in A^0(X)$ that are smooth fuctions on X.

Lemma 3.2. *The (p,q)-forms of the shape*

$$\sum_{|I|=p, |J|=q} c_{I,J} \, dz_I \wedge d\bar{z}_J$$

with $c_{I,J} \in \mathbb{C}$ give a basis for the Dolbeault cohomology group $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Proof. This is a consequence of the Hodge theorem that we proved last semester. Choose a hermitian inner product h on V. It determines a hermitian metric on V and on $X = V/\Gamma$, whose associated $(1,1)$-form is

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h(v_j, v_k) dz_j \wedge d\bar{z}_k.$$

This is obviously closed, and so the metric is Kähler. By the Hodge theorem, every de Rham (and Dolbeault) cohomology class contains a unique harmonic representative. But the harmonic forms for this metric are exactly the forms with constant coefficients, as in the statement of the lemma. (Mumford's book contains a more elementary proof, using Fourier series.) $\square$

We can also say this without choosing coordinates. In degree 1, the isomorphism $V^* \cong H^{1,0}(X)$ sends a linear functional $f\colon V \to \mathbb{C}$ to the holomorphic 1-form df; the isomorphism $\bar{V}^* \cong H^{0,1}(X)$ sends a conjugate-linear functional $f\colon V \to \mathbb{C}$ to the anti-holomorphic 1-form df. In higher degrees, we have

$$H^{p,q}(X) \cong \bigwedge^p V^* \otimes \bigwedge^q \bar{V}^*,$$

by taking wedge products.

The above description of integral cohomology (in terms of Γ) and de Rham cohomology (in terms of V and $\bar{V}$) are compatible in the following way: the diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathbb{C}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathscr{O}_X) \arrow[d, "\cong"] \\
\operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C}) \arrow[r] & \operatorname{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C})
\end{tikzcd}$$

is commutative. The second arrow in the bottom row is the projection

$$\operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C},\mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(V \oplus \bar{V},\mathbb{C}) \to \operatorname{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C}).$$

The commutativity of the diagram requires a little bit of checking that I will skip.

Holomorphic line bundles. Our next goal is to describe all holomorphic line bundles on $X = V/\Gamma$, in a way that is suitable for determining their spaces of sections and deciding which line bundles are ample. In particular, this will tell us which compact complex tori can be embedded into projective space.

One way to describe holomorphic line bundles is via the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \xrightarrow{e^{2\pi i(-)}} \mathscr{O}_X^\times \longrightarrow 0.$$

The long exact sequence in cohomology reads

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^\times) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X).$$

Let $\operatorname{Pic}(X)$ denote the set of isomorphism classes of holomorphic line bundles; this is a group under tensor product. We have $\operatorname{Pic}(X) \cong H^1(X,\mathscr{O}_X^\times)$, and so we get a short exact sequence

$$0 \longrightarrow \operatorname{Pic}^0(X) \longrightarrow \operatorname{Pic}(X) \xrightarrow{c_1} \ker\bigl(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)\bigr) \longrightarrow 0,$$

where $\operatorname{Pic}^0(X) \cong H^1(X,\mathscr{O}_X)/H^1(X,\mathbb{Z})$ is the set of isomorphism classes of topologically trivial holomorphic line bundles. The element $c_1(L) \in H^2(X,\mathbb{Z})$ is the first Chern class of the holomorphic line bundle $L \in \operatorname{Pic}(X)$.

Our starting point is the fact that on $V \cong \mathbb{C}^n$, all holomorphic line bundles are trivial. Let $q\colon V \to X$ be the quotient map. Given $L \in \operatorname{Pic}(X)$, the pullback

$$q^*L \cong V \times \mathbb{C}$$

is trivial. The group Γ acts on q^*L in a way that is compatible with the translation action on V. We can write this action in the form

$$\gamma \cdot (v, z) = \big(v + \gamma, e_\gamma(v) \cdot z\big),$$

where $e_\gamma \in \Gamma(V, \mathscr{O}_V^\times)$ is a nowhere vanishing holomorphic function on V. If we set $H^\times = \Gamma(V, \mathscr{O}_V^\times)$, we can write this more concisely as $e_\gamma \in H^\times$. The group Γ acts on $H^\times$ by translation, according to the rule

$$(\gamma \cdot e)(v) = e(v + \gamma).$$

Obviously, for $\gamma, \delta \in \Gamma$, we have

$$(\gamma + \delta) \cdot (v, z) = \gamma \cdot \delta \cdot (v, z),$$

and this translates into the cocycle condition

$$e_{\gamma+\delta}(v) = e_\gamma(v + \delta) \cdot e_\delta(v). \tag{3.3}$$

If we change the trivialization of q^*L by multiplying pointwise by a nowhere vanishing holomorphic function $g \in H^\times$, then our cocycle changes to

$$e'_\gamma(v) = e_\gamma(v) \cdot g(v + \gamma)/g(v). \tag{3.4}$$

If you know the definition of group cohomology, you may recognize that these two conditions are describing the first group cohomology $H^1(\Gamma, H^\times)$.

Group cohomology. Let's put our discussion of line bundles on hold for a moment and briefly review group cohomology. Let G be a group, and let M be a G-module; this means that M is an abelian group with a left action by G, or in other words, a left module over the group algebra $\mathbb{Z}G$. The subspace of G-invariants

$$M^G = \big\{\, m \in M \;\big|\; gm = m \text{ for all } g \in G \,\big\} = \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$$

is a left-exact functor on G-modules, and group cohomology is the derived functors:

$$H^i(G, M) = \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$$

In practice, one uses a specific resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module to compute group cohomology. We therefore define $H^i(G, M)$ as the i-th cohomology of the following complex. For each $p \in \mathbb{N}$, set

$$C^p = C^p(G, M) = \{\text{functions } f \colon G^p \to M\},$$

and define the differential $d \colon C^p \to C^{p+1}$ by the formula

$$\begin{aligned}(df)(g_0, \dots, g_p) = g_0 \cdot f(g_1, \dots, g_p) + \sum_{i=0}^{p-1} (-1)^{i+1} f(g_0, \dots, g_i g_{i+1}, \dots, g_p) \\ + (-1)^{p+1} f(g_0, \dots, g_{p-1}).\end{aligned}$$

One checks that $d \circ d = 0$, and so this is indeed a complex.

Example 3.5. We have $C^0 = M$, and therefore

$$H^0(G, M) = \big\{\, m \in M \;\big|\; gm = m \text{ for all } g \in G \,\big\} = M^G$$

is the space of G-invariants, as it should be. Let's also compute $H^1(G, M)$. Now a 1-chain is just a function $f \colon G \to M$, and because

$$(df)(g, h) = gf(h) - f(gh) + f(g),$$

the cocycle condition $df = 0$ translates into the identity

$$f(gh) = gf(h) + f(g), \qquad \text{for all } g, h \in G.$$

It follows that

$$H^1(G, M) = \frac{\big\{\, f \colon G \to M \;\big|\; f(gh) = gf(h) + f(g) \,\big\}}{\big\{\, g \mapsto gm - m \;\big|\; m \in M \,\big\}}.$$

In the discussion above, $H^\times = \Gamma(V, \mathscr{O}_V^\times)$ is a Γ-module, but with the group structure written multiplicatively. Taking this into account, the conditions in (3.3) and (3.4) are therefore exactly describing $H^1(\Gamma, H^\times)$.

There are two other useful facts. The first is that a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of G-modules gives rise to long exact sequence in group cohomology (as usual for the functors Ext^i). The second is that group cohomology can be used to compute sheaf cohomology. Suppose that $\mathscr{F}$ is a sheaf on $X = V/\Gamma$. Assuming that the pullback sheaf $q^*\mathscr{F}$ has no higher cohomology, one has

$$H^i(X, \mathscr{F}) \cong H^i\big(\Gamma, H^0(V, q^*\mathscr{F})\big),$$

where $H^0(V, q^*\mathscr{F})$ is a Γ-module. This says, for example, that

$$H^1(X, \mathscr{O}_X^\times) \cong H^1(\Gamma, H^\times),$$

as suggested by the discussion above.

Holomorphic line bundles, continued. We return to our study of holomorphic line bundles. From $L \in \operatorname{Pic}(X)$, we get a cohomology class in $H^1(\Gamma, H^\times)$, represented by the cocycle e_γ from (3.3). Conversely, a cocycle determines a holomorphic line bundle by letting Γ act on $V \times \mathbb{C}$ according to the rule

$$\gamma \cdot (v, z) = \big(v + \gamma, e_\gamma(v) \cdot z\big).$$

The quotient $(V \times \mathbb{C})/\Gamma \to V/\Gamma$ is then a holomorphic line bundle on $X = V/\Gamma$. So all we need is nice description of these cocycles.

Let's start by describing the possible first Chern classes

$$c_1(L) \in \ker\big(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)\big).$$

We know that

$$H^2(X, \mathbb{Z}) \cong \bigwedge^2 H^1(X, \mathbb{Z}) \cong \bigwedge^2 \Gamma^*,$$

and so each cohomology class is represented uniquely by an alternating form

$$E \colon \Gamma \times \Gamma \to \mathbb{Z}.$$

When is such a class in the kernel of $H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)$? We can extend E uniquely to an alternating bilinear form on

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{C} \cong V \oplus \bar{V},$$

and since $H^2(X, \mathscr{O}_X) \cong \bigwedge \bar{V}^*$, this extension needs to be trivial on $\bar{V} \times \bar{V}$. This translates into the condition that

$$E\big(v \otimes 1 + Jv \otimes i, w \otimes 1 + Jw \otimes i\big) = 0$$

for $v, w \in V$. Expanding and looking at the real part, we deduce that

$$E(Jv, Jw) = E(v, w) \qquad \text{for all } v, w \in V. \tag{3.6}$$

It is easy to see that this condition is equivalent to the existence of a hermitian bilinear form

$$H \colon V \times V \to \mathbb{C}$$

such that $E = \operatorname{Im} H$. Indeed, H must be given by the formula

$$H(v, w) = E(Jv, w) + iE(v, w),$$

and the condition in (3.6) ensures that H is hermitian symmetric. We can summarize this in the following lemma.

Lemma 3.7. *An alternating bilinear form $E\colon \Gamma\times\Gamma \to \mathbb{Z}$ represents the first Chern class of a holomorphic line bundle on X iff there is a hermitian form $H\colon V\times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$.*

Equivalently, we can start from the hermitian form $H\colon V \times V \to \mathbb{C}$, and then the condition is that $E = \operatorname{Im} H$ needs to take integer values on the subset $\Gamma \times \Gamma$.

Note. If this seems too abstract, here is a more concrete way of thinking about the lemma. Let's start from a hermitian form $H\colon V \times V \to \mathbb{C}$. Choose a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ be the dual basis. Setting $h_{j,k} = H(v_j, v_k)$, we get a closed $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k}\, dz_j \wedge d\bar{z}_k \in A^{1,1}(X),$$

and the fact that H is hermitian ensures that $\omega \in A^2(X,\mathbb{R})$. In order for ω to be the first Chern class of a holomorphic line bundle, the cohomology class $[\omega] \in H^2(X,\mathbb{R})$ needs to be in the image of $H^2(X,\mathbb{Z})$, which means that the integral of ω over every homology class in $H_2(X,\mathbb{Z})$ should be an integer. A basis for $H_2(X,\mathbb{Z})$ is given by the images of the maps

$$c_{\gamma,\delta}\colon [0,1]^2 \to X, \quad (s,t) \mapsto s\gamma + t\delta + \Gamma$$

for $\gamma, \delta \in \Gamma$. Writing $\gamma = \sum_j \gamma_j v_j$ and $\delta = \sum_j \delta_j v_j$, the integral of ω over the image of the map $c_{\gamma,\delta}$ is then

$$\int_{[0,1]^2} c_{\gamma,\delta}^* \omega = \int_0^1 \int_0^1 \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} (\gamma_j ds + \delta_j dt) \wedge \big(\bar{\gamma}_k ds + \bar{\delta}_k dt\big) = \operatorname{Im} H(\gamma,\delta).$$

So the condition is precisely that $E = \operatorname{Im} H$ should take integer values on $\Gamma \times \Gamma$.

Now let's compute the first Chern class from the cocycle e_γ in (3.3). Setting $H = \Gamma(V, \mathscr{O}_V)$, we have a short exact sequence of Γ-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \xrightarrow{e^{2\pi i(-)}} H^\times \longrightarrow 0,$$

and therefore a long exact sequence in cohomology. The connecting homomorphism $\delta\colon H^1(\Gamma, H^\times) \to H^2(\Gamma,\mathbb{Z})$ fits into a commutative diagram

$$\begin{tikzcd}
H^1(\Gamma, H^\times) \arrow[r, "\delta"] \arrow[d, "\cong"] & H^2(\Gamma,\mathbb{Z}) \arrow[d, "\cong"] \arrow[dr, bend left, "\cong"] & \\
H^1(X, \mathscr{O}_X^\times) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r, "\cong"] & \bigwedge^2 \Gamma^*.
\end{tikzcd}$$

Our cocycle $e = \{e_\gamma\}$ is an element in $C^1(\Gamma, H^\times)$. To compute its image under the connecting homomorphism, we need to lift it to $f = \{f_\gamma\} \in C^1(\Gamma, H)$, and then apply the differential $d\colon C^1(\Gamma, H) \to C^2(\Gamma, H)$. So we write

$$e_\gamma(v) = e^{2\pi i f_\gamma(v)}$$

with $f_\gamma \in H$, and then

$$F(\gamma,\delta) = (df)(\gamma,\delta) = f_\delta(v+\gamma) - f_{\gamma+\delta}(v) + f_\gamma(v) \in \mathbb{Z}.$$

Under the isomorphism $H^2(\Gamma,\mathbb{Z}) \cong \bigwedge^2 \Gamma^*$, this 2-cocycle then goes to the alternating form $E\colon \Gamma\times\Gamma \to \mathbb{Z}$ given by the formula

$$E(\gamma,\delta) = F(\gamma,\delta) - F(\delta,\gamma) = \big(f_\delta(v+\gamma) - f_\delta(v)\big) - \big(f_\gamma(v+\delta) - f_\gamma(v)\big). \tag{3.8}$$

Since this is the first Chern class of a line bundle, we have $E = \operatorname{Im} H$ for a hermitian form $H\colon V \times V \to \mathbb{C}$.

The Appell-Humbert theorem. We can now solve our problem in the following way. Fix a hermitian form $H\colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. Let's describe all line bundles $L \in \operatorname{Pic}(X)$ such that $c_1(L) \in H^2(X, \mathbb{Z})$ is represented by E. There are two cases.

The first (and easier) case is when $H = 0$. Here we are looking for line bundles $L \in \operatorname{Pic}(X)$ with $c_1(L) = 0$. Recall that

$$\operatorname{Pic}^0(X) \cong H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathscr{O}_X^\times).$$

From Hodge theory, we have an isomorphism of $\mathbb{R}$-vector spaces

$$H^1(X, \mathbb{R}) \to H^1(X, \mathscr{O}_X)$$

and therefore $\operatorname{Pic}^0(X) \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \cong H^1(X, \mathbb{R}/\mathbb{Z})$. Since $x \mapsto e^{2\pi i x}$ maps $\mathbb{R}/\mathbb{Z}$ isomorphically to the circle group $U(1) = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$, the image of $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ in $\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$ is therefore isomorphic to

$$\operatorname{Hom}_{\mathbb{Z}}\bigl(H_1(X, \mathbb{Z}), U(1)\bigr) \cong \operatorname{Hom}_{\mathbb{Z}}\bigl(\Gamma, U(1)\bigr).$$

In terms of cocycles, this means that every group homomorphism

$$\alpha\colon \Gamma \to U(1)$$

gives us a constant cocycle $e_\gamma(v) = \alpha(\gamma)$; this obviously satisfies the cocycle condition in (3.3). So we get

$$\operatorname{Hom}_{\mathbb{Z}}\bigl(\Gamma, U(1)\bigr) \cong \operatorname{Pic}^0(X), \quad \alpha \mapsto \{e_\gamma = \alpha(\gamma)\}.$$

The general case is when $H \neq 0$. Here the best possible choice of cocycle is

$$e_\gamma(v) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \cdot \alpha(\gamma), \tag{3.9}$$

where $\alpha\colon \Gamma \to U(1)$. This needs to satisfy the cocycle condition in (3.3), and so

$$e^{\pi H(v,\gamma+\delta) + \frac{\pi}{2} H(\gamma+\delta,\gamma+\delta)} \alpha(\gamma+\delta) = e^{\pi H(v+\delta,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) e^{\pi H(v,\delta) + \frac{\pi}{2} H(\delta,\delta)} \alpha(\delta).$$

After cancelling common factors and remembering that $E = \operatorname{Im} H$, this turns into

$$\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta) \cdot e^{i\pi E(\gamma,\delta)}. \tag{3.10}$$

So $\alpha\colon \Gamma \to U(1)$ is no longer a group homomorphism, but it is not off by very much because $e^{i\pi E(\gamma,\delta)} = \pm 1$.

We also need to make sure that the first Chern class is represented by $E = \operatorname{Im} H$. Going back to (3.8), the condition is that

$$\bigl(f_\delta(v+\gamma) - f_\delta(v)\bigr) - \bigl(f_\gamma(v+\delta) - f_\gamma(v)\bigr) = E(\gamma, \delta).$$

For e_γ as in (3.9), the lifting is

$$f_\gamma = \frac{H(v,\gamma)}{2i} + \frac{H(\gamma,\gamma)}{4i} + \frac{1}{2\pi i} \log \alpha(\gamma),$$

which is of course only determined up to $\mathbb{Z}$ (because of the logarithm). After plugging this into the formula above, we get

$$\frac{H(\gamma,\delta) - H(\delta,\gamma)}{2i} = E(\gamma, \delta),$$

which is correct because $E = \operatorname{Im} H$. (In fact, (3.9) is determined uniquely if we look for a lifting f_γ that is affine linear in v and satisfies the equation above.)

Definition 3.11. Let $H\colon V \times V \to \mathbb{C}$ be a hermitian form such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. For any $\alpha\colon \Gamma \to U(1)$ such that (3.10) holds, we define the holomorphic line bundle

$$L(H, \alpha) = (V \times \mathbb{C})/\Gamma$$

over $X = V/\Gamma$, where the Γ-action is given by

$$\gamma \cdot (v, z) = \Big(v + \gamma,\, e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot z \Big).$$

Then (H, α) is called the *Appell-Humbert datum* for the line bundle $L(H, \alpha)$.

The main result (that we have almost proved at this point) is that

$$\operatorname{Pic}(X) = \big\{\, L(H,\alpha) \bigm| (H,\alpha) \text{ is an Appell-Humbert datum} \,\big\}$$

describes all holomorphic line bundles on X. More on this next time.

Lecture 4 (February 6)

The Appell-Humbert theorem. Last time, we described all holomorphic line bundles on a compact complex torus $X = V/\Gamma$. There were two pieces of data:

(1) A hermitian form $H \colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. Let $\operatorname{Herm}_{\mathbb{Z}}(V, \Gamma)$ denote the set of all such.

(2) A mapping $\alpha \colon \Gamma \to U(1)$ such that

$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta) e^{i\pi E(\gamma,\delta)} \qquad \text{for all } \gamma, \delta \in \Gamma.$$

We call such a pair (H, α) an *Appell-Humbert datum.* Let $\operatorname{AH}(V, \Gamma)$ be the set of Appell-Humbert data. To each $(H, \alpha) \in \operatorname{AH}(V, \Gamma)$, we associated a holomorphic line bundle $L(H, \alpha)$ on X, defined as the quotient of $V \times \mathbb{C}$ by the Γ-action

$$\gamma \cdot (v, z) = \big(v + \gamma, e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot z \big).$$

We now get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}\big(\Gamma, U(1)\big) \arrow[r] \arrow[d] & \operatorname{AH}(V,\Gamma) \arrow[r] \arrow[d, "L"] & \operatorname{Herm}_{\mathbb{Z}}(V,\Gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Pic}^0(X) \arrow[r] & \operatorname{Pic}(X) \arrow[r] & \ker\big(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)\big) \arrow[r] & 0
\end{tikzcd}$$

The first arrow in the first line sends a homomorphism α to the pair $(0, \alpha)$, and the second arrow sends an Appell-Humbert datum (H, α) to the hermitian form H. The vertical arrow in the middle sends (H, α) to the associated line bundle $L(H, \alpha)$. We could not quite state the main result last time, so here it is.

Theorem 4.1 (Appell-Humbert)**.** *The mapping* $L \colon \operatorname{AH}(V, \Gamma) \to \operatorname{Pic}(X)$ *is an isomorphism of abelian groups.*

Proof. The group operation in $\operatorname{AH}(V, \Gamma)$ is given by the rule

$$(H_1, \alpha_1) \cdot (H_2, \alpha_2) = (H_1 + H_2, \alpha_1 \alpha_2).$$

This is compatible with the group structures on $\operatorname{Hom}_{\mathbb{Z}}(\Gamma, U(1))$ and on $\operatorname{Herm}_{\mathbb{Z}}(V, \Gamma)$. Now if two line bundles are represented by cocycles, in the way we introduced last time, then their tensor product is represented by the pointwise product of the two cocycles. Together with the explicit formula for $L(H, \alpha)$, this shows that

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \cong L(H_1 + H_2, \alpha_1 \alpha_2),$$

and so L is indeed a group homomorphism. We showed last time that the first and third vertical arrow in the diagram are isomorphisms; by the five lemma, the arrow in the middle is also an isomorphism. □

Example 4.2. Let's look at the case of elliptic curves. Here $V = \mathbb{C}$, with coordinate z, and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$ is a point in the upper halfplane. The pairing $E = \operatorname{Im} H$ is determined by the integer $m = E(\tau, 1)$, which is the first Chern class of the line bundle. As always, we extend E $\mathbb{R}$-linearly; then

$$m = E(\operatorname{Re}\tau + i \operatorname{Im}\tau, 1) = \operatorname{Re}\tau E(1,1) + \operatorname{Im}\tau E(i, 1),$$

and therefore $E(i,1) = m/\operatorname{Im}\tau$. The hermitian pairing H is then determined by

$$H(1,1) = E(i,1) + iE(1,1) = \frac{m}{\operatorname{Im}\tau}.$$

So the quantity $\operatorname{Im}\tau$ shows up in the Appell-Humbert description of line bundles.

Global sections. Next, we are going to compute the space of global sections of $L(H,\alpha)$, and determine under what conditions $L(H,\alpha)$ is ample. Along the way, we'll prove the following interesting fact: If L is a holomorphic line bundle on a compact complex torus X, and if $H^0(X,L) \neq 0$, then there is a surjective holomorphic group homomorphism $q\colon X \to Y$ to another compact complex torus Y, and an *ample* line bundle M on Y, such that $L \cong q^*M$.

Consider a line bundle of the form $L(H,\alpha)$. From the description as $V \times \mathbb{C}/\Gamma$, we see that a global section of $L(H,\alpha)$ is the same thing as a holomorphic function $\theta\colon V \to \mathbb{C}$ with the property that

$$\theta(v+\gamma) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot \theta(v) \tag{4.3}$$

for every $\gamma \in \Gamma$. Such functions are called *theta functions* for the pair (H,α). We will see in a moment how these are related to the classical theta function.

It turns out that the existence or non-existence of sections depends on the properties of the hermitian form H. There are three cases:

Case 1. The hermitian form H is degenerate. Recall that $E = \operatorname{Im} H$ is integral on $\Gamma \times \Gamma$. Consider the null space

$$\begin{aligned} V_0 &= \left\{ v \in V \;\middle|\; H(v,w) = 0 \text{ for all } w \in V \right\} \\ &= \left\{ v \in V \;\middle|\; E(v,\gamma) = 0 \text{ for all } \gamma \in \Gamma \right\}. \end{aligned}$$

The first line shows that V_0 is a complex subspace of V, and the second line shows that $\Gamma_0 = V_0 \cap \Gamma$ is again a lattice in V_0. Define $V_1 = V/V_0$ and $\Gamma_1 = \Gamma/\Gamma_0$; then $X_1 = V_1/\Gamma_1$ is again a compact complex torus. Because V_0 is the nullspace, H descends to a nondegenerate hermitian form H_1 on V_1.

For $\gamma \in \Gamma_0$, the transformation rule in (4.3) gives

$$\theta(v+\gamma) = \alpha(\gamma)\theta(v),$$

and since $|\alpha(\gamma)| = 1$, this shows that θ is bounded on each coset $v + V_0$. By Liouville's theorem, θ is constant, and so there is a holomorphic function

$$\theta_1\colon V_1 \to \mathbb{C}$$

such that $\theta(v) = \theta_1(v + V_0)$. It then follows that $\alpha(\gamma) = 1$ for $\gamma \in \Gamma_0$, and so there is a function $\alpha_1\colon \Gamma_1 \to U(1)$ with the property that $\alpha(\gamma) = \alpha_1(\gamma + \Gamma_0)$. If we let $q\colon X \to X_1$ denote the quotient mapping, this means that $L(H,\alpha) \cong q^*L(H_1,\alpha_1)$ is the pullback of a holomorphic line bundle from the smaller torus X_1. Without loss of generality, we therefore need to consider only the case when H is nondegenerate.

Case 2. There is a nonzero vector $w \in V$ such that $H(w,w) < 0$. We are going to show that this forces $\theta = 0$. In order to use the transformation rule in (4.3), we pick a compact subset $K \subseteq V$ such that $V = K + \Gamma$. For every $t \in \mathbb{C}$, we can then write $tw = k_t + \gamma_t$, with $k_t \in K$ and $\gamma_t \in \Gamma$. Now fix a point $v \in V$ and consider the restriction of θ to the complex line $v + tw$. We have

$$|\theta(v+tw)| = |\theta(v + k_t + \gamma_t)| = |e^{\pi H(v+k_t,\gamma_t) + \frac{\pi}{2} H(\gamma_t,\gamma_t)}| \cdot |\theta(v+k_t)|$$

If we rewrite the exponent in terms of w, we get

$$\begin{aligned} \pi H(v+k_t,\gamma_t) + \frac{\pi}{2} H(\gamma_t,\gamma_t) &= \pi H(v+k_t, tw-k_t) + \frac{\pi}{2} H(tw-k_t, tw-k_t) \\ &= \pi H(w,w)|t|^2 + O(|t|), \end{aligned}$$

because $v \in V$ is fixed and $k_t \in K$ lies in a compact subset. As $H(w,w) < 0$, this expression goes to $-\infty$ when $|t| \to \infty$. Because the function $\theta(v+tw)$ is holomorphic in t, it follows that $\theta(v+tw) = 0$; but then $\theta(v) = 0$, and so $\theta = 0$. Under the assumption that H is nondegenerate, $L(H,\alpha)$ can therefore have nontrivial sections only when H is positive definite.

Case 3. The hermitian form H is positive definite. If we pick a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ denote the dual basis, then the first Chern class of $L(H,\alpha)$ is represented by the closed $(1,1)$-form

$$\frac{i}{2} \sum_{j,k=1}^{n} H(v_j, v_k) dz_j \wedge d\bar{z}_k.$$

This is now a *positive* form, which means that the line bundle $L(H,\alpha)$ is positive (in Kodaira's sense). According to the Kodaira embedding theorem, a sufficiently large power of $L(H,\alpha)$ will therefore embed X into projective space. (Borrowing a piece of terminology from algebraic geometry, we may say that $L(H,\alpha)$ is an *ample* line bundle.) So we have proved the following criterion for X to be projective.

Theorem 4.4. *A compact complex torus $X = V/\Gamma$ is projective iff there exists a positive definite hermitian form $H \colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$.*

The discussion in Case 1 also shows that if $L = L(H,\alpha)$ is a holomorphic line bundle on X such that $H^0(X,L) \neq 0$, then there is surjective holomorphic group homomorphism $q \colon X \to X_1$ to a (possibly smaller) compact complex torus X_1, and an ample line bundle $L_1 = L(H_1, \alpha_1)$, such that $L \cong q^* L_1$. Unlike in other parts of algebraic geometry, the existence of sections is therefore very closely related to ampleness.

Now let us actually determine the space of global sections of $L(H,\alpha)$, under the assumption that H is positive definite. This will also allow us to figure out exactly what power of $L(H,\alpha)$ we need to get an embedding into projective space. The proof is a bit tricky, so let's think about the classical case first.

Example 4.5. Consider $V = \mathbb{C}$ and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, with $\tau \in \mathbb{H}$. For simplicity, let's take $E(\tau, 1) = 1$, and $\alpha(1) = \alpha(\tau) = 1$; these two values determine α uniquely. We already computed that

$$H(1,1) = \frac{1}{\operatorname{Im} \tau},$$

and so H is positive definite. A theta function for (H,α) is an entire function $\theta \colon \mathbb{C} \to \mathbb{C}$ that satisfies the two functional equations

$$\begin{aligned} \theta(z+1) &= e^{\pi H(z,1) + \frac{\pi}{2} H(1,1)} \cdot \theta(z) = e^{\frac{\pi}{2}(2z+1)/\operatorname{Im}\tau} \cdot \theta(z) \\ \theta(z+\tau) &= e^{\pi H(z,\tau) + \frac{\pi}{2} H(\tau,\tau} \cdot \theta(z) = e^{\frac{\pi}{2}(2z\bar{\tau} + |\tau|^2)/\operatorname{Im}\tau} \cdot \theta(z). \end{aligned}$$

Now the (very classical) idea is to make θ periodic, meaning invariant under the substitution $z \mapsto z+1$, and then to use Fourier series. We can achieve this by completing the square: consider the new entire function

$$\vartheta(z) = e^{-\frac{\pi}{2} z^2 / \operatorname{Im}\tau} \cdot \theta(z).$$

The first functional equation then gives $\vartheta(z+1) = \vartheta(z)$, and so we can expand $\vartheta(z)$ into a Fourier series of the form

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n z},$$

with coefficients $c_n \in \mathbb{C}$. (This is actually quite elementary: using the holomorphic mapping $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^{2\pi i z}$, descend ϑ to a holomorphic function on $\mathbb{C}^*$; the Fourier series is then just the Laurent series of this holomorphic function.)

After simplifying, the second functional equation reads

$$\vartheta(z+\tau) = e^{-i\pi\tau - 2\pi i z}\vartheta(z).$$

If we substitute the Fourier series into this equation, we get

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n\tau} e^{2\pi i n z} = e^{-i\pi\tau} \sum_{n\in\mathbb{Z}} c_n e^{2\pi i(n-1)z},$$

and after comparing coefficients, we arrive at the identity

$$c_{n+1} = c_n e^{i\pi(2n+1)\tau}.$$

This shows that all the Fourier coefficients c_n are uniquely determined by c_0. If we set $c_0 = 1$, we get $c_n = e^{i\pi n^2\tau}$, and

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau + 2\pi i n z}$$

is exactly the classical *Jacobi theta function.* The series converges absolutely and uniformly on compact subsets; in fact,

$$|\vartheta(z)| \leq \sum_{n\in\mathbb{Z}} |e^{i\pi n^2\tau+2\pi i n z}| = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \operatorname{Im}\tau} e^{2\pi n \operatorname{Re} z}$$

converges very rapidly on any strip of the form $|\operatorname{Re} z| \leq C$. The conclusion is that the line bundle $L(H,\alpha)$ has a unique holomorphic section, which looks like

$$\theta(z) = e^{\frac{\pi}{2}z^2/\operatorname{Im}\tau} \cdot \vartheta(z),$$

where ϑ is Jacobi's theta function.

Now we carry out the same kind of computation in general. Let us fix a positive definite hermitian form $H\colon V\times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma\times\Gamma$. After choosing a basis for $\Gamma \cong \mathbb{Z}^{2n}$, we can represent E as a $2n\times 2n$-matrix with integer entries; let's denote the determinant of this matrix by $\det E$.

Theorem 4.6. *We have* $\dim H^0\big(X, L(H,\alpha)\big) = \sqrt{\det E}$.

We divide the proof into six steps. The general idea is the same as in the example. We find a subgroup $\Gamma' \subseteq \Gamma$ of rank n, and complete the square in order to make θ invariant under translation by this sublattice. We then study the coefficients in the Fourier series in order to determine all possible theta functions for (H,α).

Step 1. We find a subgroup $\Gamma' \subseteq \Gamma$ of rank n on which E is trivial. We can turn the pairing E into a group homomorphism

$$E\colon \Gamma \to \Gamma^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}), \quad \gamma \mapsto E(\gamma,-).$$

This is injective (because E is nondegenerate over $\mathbb{R}$ and Γ is torsion-free), and the image has index equal to $\det E$. Now suppose that Γ' is any subgroup of Γ such that $E|_{\Gamma'\times\Gamma'} = 0$. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma' \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "E"] & \Gamma/\Gamma' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\Gamma/\Gamma')^* \arrow[r] & \Gamma^* \arrow[r] & (\Gamma')^* \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(\Gamma/\Gamma',\mathbb{Z})
\end{tikzcd}$$

with exact rows; the Ext-group on the bottom is nonzero exactly when Γ/Γ' has torsion. If $\gamma\in\Gamma$ is in the kernel of $\Gamma/\Gamma' \to (\Gamma')^*$, then $E(\gamma,\delta) = 0$ for every $\delta\in\Gamma'$, and so $\Gamma' + \mathbb{Z}\gamma$ is a bigger subgroup on which E is identically zero. If we take Γ' to be maximal with this property, then

$$\Gamma/\Gamma' \to (\Gamma')^*$$

must therefore be injective; consequently, Γ/Γ' is torsion-free, and $\operatorname{rk}\Gamma = 2\operatorname{rk}\Gamma'$, which gives $\operatorname{rk}\Gamma' = n$. Let m be the index of Γ/Γ' in $(\Gamma')^*$. Because the first vertical arrow in the diagram is the dual of the third one, it follows that

$$\det E = \bigl(\Gamma^* : \Gamma\bigr) = \bigl((\Gamma')^* : \Gamma/\Gamma'\bigr)^2 = m^2,$$

or equivalently, $m = \sqrt{\det E}$. So we can restate the theorem as

$$\dim H^0\bigl(X, L(H,\alpha)\bigr) = m.$$

The subgroup Γ' will play the role that $\mathbb{Z} \subseteq \mathbb{Z} + \mathbb{Z}\tau$ played in the example.

Step 2. Now suppose that $\theta\colon V \to \mathbb{C}$ is a theta function for (H,α), with

$$\theta(v+\gamma) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) \cdot \theta(v)$$

We want to make θ invariant under translation by Γ', but in order to "complete the square", we need to turn our hermitian form H into a quadratic form. Let

$$W_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma' \subseteq V_{\mathbb{R}}$$

be the $\mathbb{R}$-vector space spanned by Γ'. We have $\dim_{\mathbb{R}} W_{\mathbb{R}} = n$, and because Γ/Γ' is torsion free, we also have $W_{\mathbb{R}} \cap \Gamma = \Gamma'$. Recall that $J \in \operatorname{End}(V_{\mathbb{R}})$ is the endomorphism $J(v) = iv$. The hermitian form H is related to E by the formula $H(v,w) = E(Jv,w) + iE(v,w)$, and so H is identically zero on $W_{\mathbb{R}} \cap J(W_{\mathbb{R}})$. Because H is positive definite, we get $W_{\mathbb{R}} \cap J(W_{\mathbb{R}}) = 0$, and therefore

$$V = W_{\mathbb{R}} \oplus J(W_{\mathbb{R}})$$

for dimension reasons. This shows that $V = \mathbb{C} \otimes_{\mathbb{R}} W_{\mathbb{R}} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$. Let $p\colon V \to W_{\mathbb{R}}$ and $q\colon V \to W_{\mathbb{R}}$ be the two projections; then

$$v = p(v) + Jq(v) \qquad \text{for any } v \in V.$$

Now consider the restriction $H|_{W_{\mathbb{R}} \times W_{\mathbb{R}}}$. Because $E = \operatorname{Im} H$, this is an $\mathbb{R}$-valued symmetric bilinear form; let $B\colon V \times V \to \mathbb{C}$ be the unique $\mathbb{C}$-valued symmetric bilinear form such that $B|_{W_{\mathbb{R}} \times W_{\mathbb{R}}} = H|_{W_{\mathbb{R}} \times W_{\mathbb{R}}}$. This will play the role that the quadratic function $z^2/\operatorname{Im}\tau$ played in the example.

We also need to deal with the factor $\alpha(\gamma)$ that was not there in the example. For $\gamma, \delta \in \Gamma'$, we have $E(\gamma,\delta) = 0$, and therefore $\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)$. By choosing a basis for $\Gamma' \cong \mathbb{Z}^n$, we can find a homomorphism

$$\lambda\colon \Gamma' \to \mathbb{R}$$

with the property that $\alpha(\gamma) = e^{2\pi i \lambda(\gamma)}$ for $\gamma \in \Gamma'$. Since $V = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$, this extends uniquely to a $\mathbb{C}$-linear mapping $\lambda\colon V \to \mathbb{C}$.

Step 3. As in the example, we now consider the new holomorphic function

$$\vartheta\colon V \to \mathbb{C}, \quad \vartheta(v) = e^{-\frac{\pi}{2} B(v,v)} e^{-2\pi i \lambda(v)} \cdot \theta(v).$$

A brief computation shows that this satisfies the functional equation

$$\vartheta(v+\gamma) = e^{-2\pi i \lambda(\gamma)} \alpha(\gamma) \cdot e^{\pi\bigl(H(v,\gamma) - B(v,\gamma)\bigr) + \frac{\pi}{2}\bigl(H(\gamma,\gamma) - B(\gamma,\gamma)\bigr)} \cdot \vartheta(v). \tag{4.7}$$

When $\gamma \in \Gamma'$, both factors are trivial, and so ϑ is invariant under translation by Γ'. We can therefore expand it into a Fourier series

$$\vartheta(v) = \sum_{\chi \in \operatorname{Hom}_{\mathbb{Z}}(\Gamma',\mathbb{Z})} c_\chi e^{2\pi i \chi(v)}.$$

The Fourier coefficients $c_\chi \in \mathbb{C}$ are indexed by homomorphisms $\chi\colon \Gamma' \to \mathbb{Z}$. Note that each χ extends uniquely to a $\mathbb{C}$-linear mapping $\chi\colon V \to \mathbb{C}$, which is how we define the $\chi(v)$ in the exponent.

Lecture 5 (February 11)

Our first task is to finish the proof of Theorem 4.6 from last time. In class, I reviewed the notation and the first half of the argument; look at the notes from last time before reading on.

Step 4. Let's see what the functional equation in (4.7) tells about the Fourier coefficients of ϑ. For that, we need to rewrite the terms with $H-B$ in a more manageable way. Each $\gamma \in \Gamma$ determines a homomorphism

$$\hat{\gamma}\colon \Gamma' \to \mathbb{Z}, \quad \hat{\gamma}(\delta) = E(\gamma, \delta).$$

As we observed during Step 1 of the proof, the mapping

$$\Gamma/\Gamma' \to (\Gamma')^*, \quad \gamma + \Gamma' \mapsto \hat{\gamma},$$

is injective, and the image has index $m = \sqrt{\det E}$. Now if $\gamma \in \Gamma$ and $\delta \in \Gamma'$, then

$$H(\delta,\gamma) - B(\delta,\gamma) = \overline{H(\gamma,\delta)} - B(\gamma,\delta) = \overline{H(\gamma,\delta)} - H(\gamma,\delta) = -2iE(\gamma,\delta),$$

because $B|_{W_\mathbb{R}\times W_\mathbb{R}} = H|_{W_\mathbb{R}\times W_\mathbb{R}}$ and both B and H are $\mathbb{C}$-linear in their first argument. Consequently,

$$H(\delta,\gamma) - B(\delta,\gamma) = -2i\hat{\gamma}(\delta),$$

and because $V = \mathbb{C} \otimes_\mathbb{Z} \Gamma'$ and everything is $\mathbb{C}$-linear, we get

$$H(v,\gamma) - B(v,\gamma) = -2i\hat{\gamma}(v) \qquad \text{for all } v \in V.$$

This allows us to rewrite (4.7) as

$$\vartheta(v+\gamma) = e^{-2\pi i\lambda(\gamma)}\alpha(\gamma) \cdot e^{-2\pi i\hat{\gamma}(v) - i\pi\hat{\gamma}(\gamma)} \cdot \vartheta(v).$$

If we now substitute the Fourier expansion for ϑ into this identity, we get

$$\sum_\chi c_\chi e^{2\pi i\chi(\gamma)} e^{2\pi i\chi(v)} = e^{-2\pi i\lambda(\gamma)}\alpha(\gamma) \cdot e^{-i\pi\hat{\gamma}(\gamma)} \sum_\chi c_\chi e^{2\pi i(\chi(v) - \hat{\gamma}(v))}.$$

Comparing coefficients on both sides, we find that

$$c_{\chi+\hat{\gamma}} = e^{2\pi i\lambda(\gamma)}\alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i\chi(\gamma)} \cdot c_\chi. \tag{5.1}$$

This shows that all the Fourier coefficients are uniquely determined once we know the values on each coset of Γ/Γ' inside $(\Gamma')^*$. Since the index of this subgroup is m, we conclude that there are at most m linearly independent solutions, and therefore

$$\dim H^0\big(X, L(H,\alpha)\big) \le m.$$

Step 5. It remains to prove that we get exactly m linearly inpedendent theta functions. For that, we have to prove that each time we have a solution to (5.1), the corresponding Fourier series actually converges. Let's fix a homomorphism $\chi_0 \in (\Gamma')^*$, and consider its coset in $(\Gamma')^*$. We set $c_{\chi_0} = 1$, and $c_\chi = 0$ unless $\chi = \chi_0 + \hat{\gamma}$ for some $\gamma \in \Gamma$. Solving the equations in (5.1) above, we find that

$$c_{\chi_0+\hat{\gamma}} = e^{2\pi i\lambda(\gamma)}\alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i\chi_0(\gamma)}.$$

The Fourier series with these coefficients is

$$\sum_{\hat{\gamma}} e^{2\pi i\lambda(\gamma)}\alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i\chi_0(\gamma)} e^{2\pi i\chi_0(v) + 2\pi i\hat{\gamma}(v)}.$$

Note that each term only depends on $\hat{\gamma}$, as indicated, because all the factors where γ appears are equal to 1 when $\gamma \in \Gamma'$. Anyway, the Fourier series is clearly dominated, in absolute value, by the series

$$\sum_{\hat{\gamma}} e^{-\pi \operatorname{Im} \hat{\gamma}(\gamma)} e^{-2\pi \operatorname{Im} \chi_0(\gamma)} e^{-2\pi \operatorname{Im} \chi_0(v) - 2\pi \operatorname{Im} \hat{\gamma}(v)}.$$

We will prove in a moment that that $\operatorname{Im} \hat{\gamma}(\gamma) = H\big(q(\gamma), q(\gamma)\big)$, where $q \colon V \to W_{\mathbb{R}}$ is the projection. As long as v stays in a compact subset, the exponent in the exponential therefore looks like

$$-\pi H\big(q(\gamma), q(\gamma)\big) + O\big(\lVert\gamma\rVert\big),$$

where $\lVert - \rVert$ is any inner product on V. Because H is positive definite, and q embeds Γ/Γ' as a lattice into $W_{\mathbb{R}}$, the quadratic term is negative definite, and as in the case of the Jacobi theta function, this ensures that the series converges. Our Fourier series is therefore absolutely and uniformly convergent on compact subsets, and so each of the m linearly independent choices of Fourier coefficients gives rise to a theta function for (H, α).

Step 6. It remains to prove that

$$\operatorname{Im} \hat{\gamma}(\gamma) = H\big(q(\gamma), q(\gamma)\big).$$

Recall that $p \colon V \to W_{\mathbb{R}}$ and $q \colon V \to W_{\mathbb{R}}$ are the two projections, so $v = p(v) + Jq(v)$. We showed earlier that

$$H(v, \gamma) - B(v, \gamma) = -2i\hat{\gamma}(v).$$

Plugging in $v = \gamma$ gives

$$\operatorname{Im} \hat{\gamma}(\gamma) = \operatorname{Re} \frac{H(\gamma, \gamma) - B(\gamma, \gamma)}{2}.$$

Because H is hermitian and $J(v) = iv$, we have

$$H(\gamma, \gamma) = H\big(p(v), p(v)\big) + H\big(q(v), q(v)\big) - iH\big(p(v), q(v)\big) + iH\big(q(v), p(v)\big).$$

At the same time, B is bilinear, and equal to H on $W_{\mathbb{R}} \times W_{\mathbb{R}}$, and so

$$\begin{aligned} B(\gamma, \gamma) &= B\big(p(v), p(v)\big) - B\big(q(v), q(v)\big) + iB\big(p(v), q(v)\big) + iB\big(q(v), p(v)\big) \\ &= H\big(p(v), p(v)\big) - H\big(q(v), q(v)\big) + iH\big(p(v), q(v)\big) + iH\big(q(v), p(v)\big). \end{aligned}$$

Taking the difference, we obtain

$$\frac{H(\gamma, \gamma) - B(\gamma, \gamma)}{2} = H\big(q(v), q(v)\big) - iH\big(p(v), q(v)\big),$$

and the real part of the right-hand side is obviously $H\big(q(v), q(v)\big)$.

Riemann-Roch theorem. We can also express Theorem 4.6 in a more cohomological way, as follows. Let (H, α) be Appell-Humbert data, with H positive definite; then the line bundle $L = L(H, \alpha)$ is ample. Since the canonical bundle of X is trivial, the Kodaira vanishing theorem shows that $H^i(X, L) = 0$ for $i > 0$. Therefore the Euler characteristic of L is equal to

$$\chi(X, L) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, L) = \dim H^0(X, L).$$

Our computation for the dimension of the space of sections, together with Corollary 5.3 below, gives

$$\chi(X, L) = \sqrt{\det E} = \frac{1}{n!} c_1(L)^n.$$

Because the tangent bundle T_X is trivial, this is exactly the formula one gets from Grothendieck's Riemann-Roch theorem.

Some matrix calculations. Let $E\colon \Gamma\times\Gamma\to\mathbb{Z}$ be an alternating bilinear form, such that the induced group homomorphism

$$E\colon \Gamma\to \operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}),\quad \gamma\mapsto E(\gamma,-),$$

is injective. For the sake of completeness, I am including proofs for the assertions about E that we used in the previous two sections. The key technical point is the following lemma.

Lemma 5.2. *There is a basis $e_1,\dots,e_{2n}\in\Gamma$ such that the $2n\times 2n$-matrix with entries $E(e_i,e_j)$ has the form*

$$\begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

for positive integers $m_1\mid m_2\mid\cdots\mid m_n$. In particular,

$$\det E=(m_1\cdots m_n)^2$$

is always the square of an integer.

Proof. Choose two vectors $e_1,e_2\in\Gamma$ such that $m_1=E(e_1,e_2)$ is the smallest possible positive integer among the values of E. For any $\gamma\in\Gamma$, we have

$$\begin{aligned} E(\gamma-ae_1-be_2,e_1)&=E(\gamma,e_1)+bm_1,\\ E(\gamma-ae_1-be_2,e_2)&=E(\gamma,e_2)-am_2. \end{aligned}$$

By minimality of m_1, both integers $E(\gamma,e_1)$ and $E(\gamma,e_2)$ must be divisible by m_1, and so we can uniquely choose $a,b\in\mathbb{Z}$ such that $\gamma-ae_1-be_2$ becomes orthogonal to e_1 and e_2. This means that $\Gamma=\mathbb{Z}e_1\oplus\mathbb{Z}e_2\oplus\Gamma'$, where Γ' is the subgroup

$$\Gamma'=\left\{\,\gamma\in\Gamma \;\middle|\; E(\gamma,e_1)=E(\gamma,e_2)=0\,\right\}.$$

Again by minimality of m_1, all values of E on Γ' must be divisible by m_1. The result we want now follows by induction on the rank of Γ. ∎

One consequence is that the image of the homomorphism $E\colon\Gamma\to\operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$ has index equal to $\det E$. The reason is that the image of

$$\begin{pmatrix} 0 & m\\ -m & 0\end{pmatrix}\colon \mathbb{Z}^2\to\mathbb{Z}^2$$

is the subgroup $m\mathbb{Z}^2$, which clearly has index m^2. We used this fact during the proof of Theorem 4.6. Another consequence is the following description of $\det E$ in terms of intersection numbers.

Corollary 5.3. *Set $L=L(H,\alpha)$, with H positive definite. Then*

$$\sqrt{\det E}=\frac{1}{n!}c_1(L)^n.$$

Proof. Choose a basis $e_1,\dots,e_{2n}\in\Gamma$ as in the lemma, and let $e_1^*,\cdots,e_{2n}^*\in\Gamma^*$ be the dual basis. As elements of $H^2(X,\mathbb{Z})\cong\bigwedge^2\Gamma^*$, we then have

$$c_1(L)=\sum_{j<k}E(e_j,e_k)\,e_j^*\wedge e_k^*=\sum_{i=1}^n m_i\,e_{2i-1}^*\wedge e_{2i}^*,$$

where $L=L(H,\alpha)$. Therefore

$$\frac{1}{n!}c_1(L)^n=m_1\cdots m_n\cdot e_1^*\wedge\cdots\wedge e_{2n}^*,$$

and as elements of $H^{2n}(X,\mathbb{Z}) \cong \bigwedge^{2n} \Gamma^*$, this gives

$$\frac{1}{n!}c_1(L)^n = m_1 \cdots m_n = \sqrt{\det E}. \qquad \square$$

Some terminology. An *abelian variety* is by definition a compact complex torus $X = V/\Gamma$ that can be embedded into projective space. According to Theorem 4.4, this is equivalent to the existence of a positive definite hermitian form $H \colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. If that is the case, then any line bundle of the form $L(H,\alpha)$ is ample; for historical reasons, such a line bundle is called a *polarization*. If we choose a basis for Γ as in Lemma 5.2, such that

$$E = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix},$$

then the n-tuple of integers $(m_1, m_2, \dots, m_n)$ with $m_1 \mid m_2 \mid \cdots \mid m_n$ is called the *type* of the polarization. A polarization is called *principal* if $m_1 = \cdots = m_n = 1$; this is equivalent to saying that the homomorphism

$$E \colon \Gamma \to \Gamma^*, \quad \gamma \mapsto E(\gamma, -),$$

is an isomorphism. (In that case, E is also said to be *unimodular*.)

Exercise 5.1. If $m_1 \geq 2$, show that $L(H,\alpha)$ is the m_1-th tensor power of some other holomorphic line bundle.

Jacobians. Let C be a compact Riemann surface of genus $g \geq 1$. The most important example of a principally polarized abelian variety is the Jacobian

$$J(C) = \operatorname{Pic}^0(C) \cong H^1(C, \mathscr{O}_C)/H^1(C,\mathbb{Z}).$$

Let's verify that this is the case. The starting point is the Hodge decomposition

$$H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C) \cong H^0(C, \Omega_C^1) \oplus H^1(C, \mathscr{O}_C).$$

The mapping $H^1(C,\mathbb{R}) \to H^1(C,\mathscr{O}_C)$ is an isomorphism of $\mathbb{R}$-vector spaces: if $\alpha \in H^1(C,\mathbb{R})$, the in the Hodge decomposition $\alpha = \alpha^{1,0} + \alpha^{0,1}$, one has $\alpha^{1,0} = \overline{\alpha^{0,1}}$, and so $\alpha^{0,1} = 0$ implies $\alpha = 0$. It follows that the composition

$$H^1(C,\mathbb{Z}) \to H^1(C,\mathbb{R}) \to H^1(C,\mathscr{O}_C)$$

embeds $\Gamma = H^1(C,\mathbb{Z})$ as a lattice into $V = H^1(C,\mathscr{O}_C)$, and so the quotient is a compact complex torus of dimension g.

To show that it is an abelian variety, we need to find a positive definite hermitian form H such that $E = \operatorname{Im} H$ is integral. Consider the alternating pairing

$$E \colon H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z}, \quad E(\gamma,\delta) = [C] \cap (\gamma \cup \delta).$$

We have $H^1(C,\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}\bigl(H_1(C,\mathbb{Z}),\mathbb{Z}\bigr)$, and by Poincaré duality, the mapping

$$H^1(C,\mathbb{Z}) \to H_1(C,\mathbb{Z}), \quad \gamma \mapsto [C] \cap \gamma,$$

is an isomorphism. Therefore E is unimodular.

Using the embedding $H^1(C,\mathbb{Z}) \hookrightarrow H^1(C,\mathbb{C})$, we can view each element γ as a de Rham cohomology class. As such, we have

$$E(\gamma,\delta) = \int_C \gamma \wedge \delta = \int_C \gamma^{0,1} \wedge \overline{\delta^{0,1}} + \int_C \overline{\gamma^{0,1}} \wedge \delta^{0,1} = 2 \operatorname{Re} \int_C \gamma^{0,1} \wedge \overline{\delta^{0,1}}$$

The Hodge-Riemann bilinear relations show that the hermitian form

$$H \colon H^{0,1}(C) \times H^{0,1}(C) \to \mathbb{C}, \quad H(\gamma^{0,1}, \delta^{0,1}) = -2i \int_C \gamma^{0,1} \wedge \overline{\delta^{0,1}}$$

is positive definite. (There is again nothing deep here: locally, $\gamma^{0,1}$ looks like $f d\bar{z}$ for some function f, and therefore

$$-i\gamma^{0,1} \wedge \overline{\gamma^{0,1}} = -i|f|^2 d\bar{z} \wedge dz = 2|f|^2 \, dx \wedge dy \geq 0;$$

therefore the integral is nonnegative, and vanishes iff $\gamma^{0,1} = 0$.)

The computation above tells us that

$$E(\gamma, \delta) = -\operatorname{Im} H(\gamma^{0,1}, \delta^{0,1}),$$

and so we should redefine the pairing E as

$$E \colon H^1(C, \mathbb{Z}) \times H^1(C, \mathbb{Z}) \to \mathbb{Z}, \quad E(\gamma, \delta) = -[C] \cap (\gamma \cup \delta)$$

in order for it to be the first Chern class of an ample line bundle. Since E is unimodular, the Jacobian $J(C)$ is therefore a principally polarized abelian variety.

Morphisms. Let $X_1 = V_1/\Gamma_1$ and $X_2 = V_2/\Gamma_2$ be two compact complex tori. The following simple lemma shows that, up to translation, every holomorphic mapping from X_1 to X_2 is a group homomorphism.

Lemma 5.4. *Let $f \colon X_1 \to X_2$ be a holomorphic mapping between two compact complex tori. Then f is the composition of a group homomorphism and a translation.*

Proof. If $f(0) = y$, we can compose f with the holomorphic mapping

$$X_2 \to X_2, \quad x \mapsto x - y,$$

and arrange that $f(0) = 0$. So it suffices to prove that if $f(0) = 0$, then f is a group homomorphism. Because $V_1 \to X_1$ and $V_2 \to X_2$ are the universal covering spaces, f lifts uniquely to a holomorphic mapping $\tilde{f} \colon V_1 \to V_2$ with $\tilde{f}(0) = 0$, as in the following diagram:

$$\begin{tikzcd} V_1 \arrow[r, "\tilde{f}"] \arrow[d] & V_2 \arrow[d] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

For every $\gamma \in \Gamma_1$, we must have

$$\tilde{f}(v + \gamma) - \tilde{f}(v) \in \Gamma_2,$$

and after differentiating this formula, we see that all the first derivatives of $\tilde{f}$ are holomorphic and doubly periodic, hence constant. As $\tilde{f}(0) = 0$, this implies that $\tilde{f}$ is a linear map; but then f is clearly a group homomorphism as well. $\square$

Lecture 6 (February 13)

Translations. Our next goal is to prove a more precise version of the Kodaira embedding theorem for abelian varieties. In preparation for that, we first investigate how the group structure on a compact complex torus interacts with holomorphic line bundles.

Let $X = V/\Gamma$ be a compact complex torus. For every point $a \in X$, we have the translation automorphism

$$t_a \colon X \to X, \quad t_a(x) = a + x.$$

It is biholomorphic, with inverse t_{-a}. If we choose a vector $w \in V$ such that $q(w) = a$, where $q \colon V \to X$ is the quotient map, then t_a is induced by the linear translation $v \mapsto v + w$.

Let's consider the pullback $t_a^* L$, where L is a holomorphic line bundle on X. Write $L = L(H, \alpha)$, where (H, α) is a Appell-Humbert datum. Choose a vector

$v_a \in V$ such that $q(v_a) = a$, where $q\colon V \to X$ is the quotient map. Then L is represented by the cocycle

$$\gamma \mapsto e_\gamma(v) = e^{\pi H(v,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma)$$

and therefore $t_a^* L$ is represented by the cocycle

$$\gamma \mapsto e^{\pi H(v+w,\gamma) + \frac{\pi}{2} H(\gamma,\gamma)} \alpha(\gamma) = e^{\pi H(w,\gamma)} \cdot e_\gamma(v).$$

Therefore the tensor product $t_a^* L \otimes L^{-1}$ is represented by the constant cocycle

$$\gamma \mapsto e^{\pi H(w,\gamma)},$$

and is therefore an element of $\operatorname{Pic}^0(X)$. After modifying it by a coboundary

$$e^{\pi H(w,\gamma)} \cdot \frac{e^{-\pi H(v+\gamma,w)}}{e^{-\pi H(v,w)}} = e^{\pi H(w,\gamma) - \pi H(\gamma,w)} = e^{2\pi i E(w,\gamma)},$$

it becomes an Appell-Humbert datum for a unique line bundle in $\operatorname{Pic}^0(X)$, because $\gamma \mapsto e^{2\pi i E(w,\gamma)}$ is a group homomorphism from Γ to the circle group $U(1)$.

Example 6.1. If $c_1(L) = 0$, then we have $H = 0$, and therefore $t_a^* L \cong L$. So any holomorphic line bundle in $\operatorname{Pic}^0(X)$ is *translation-invariant.*

We see from these simple formulas that a holomorphic line bundle L determines a holomorphic group homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X), \quad a \mapsto t_a^* L \otimes L^{-1}. \tag{6.2}$$

It is holomorphic because the cocycle $e^{\pi H(w,\gamma)}$ depends holomorphically on $w \in V$; and it is a group homomorphism because the cocycle is linear in w. Note that when $w \in \Gamma$, the cocycle $e^{2\pi i E(w,\gamma)}$ is trivial because $E(\Gamma \times \Gamma) \subseteq \mathbb{Z}$.

Lemma 6.3. *If the line bundle L is ample, the group homomorphism ϕ_L is surjective, and its kernel is a subgroup of X isomorphic to Γ^*/Γ. In particular,* $\ker \phi_L$ *is a finite abelian group of order* $(\dim H^0(X,L))^2$*.*

Proof. If we again write $L = L(H,\alpha)$, then L is ample exactly when H is positive definite (and $E = \operatorname{Im} H$ is nondegenerate). This means that

$$V \to \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}), \quad w \mapsto H(w,-),$$

is an isomorphism of complex vector spaces. According to the discussion above, the image of ϕ_L therefore contains every line bundle in $\operatorname{Pic}^0(X)$ that can be represented by a cocycle of the form $e^{f(\gamma)}$, where $f\colon \bar{V} \to \mathbb{C}$ is $\mathbb{C}$-linear. But we have $\operatorname{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X,\mathbb{Z})$ and $H^1(X,\mathscr{O}_X) \cong \operatorname{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C})$, and so this gives all line bundles in $\operatorname{Pic}^0(X)$.

Let's compute the kernel. We have seen that $\phi_L(a)$ is represented by Appell-Humbert datum $(0, \gamma \mapsto e^{2\pi i E(w,\gamma)})$, and so it is trivial exactly when $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \Gamma$. Now E is nondegenerate, and so the map

$$V_{\mathbb{R}} \to \operatorname{Hom}_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R}), \quad w \mapsto E(w,-),$$

is an isomorphism of $\mathbb{R}$-vector spaces. Under this isomorphism, the subgroup $\Gamma^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$ corresponds exactly to those $w \in V_{\mathbb{R}}$ such that $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \Gamma$; the reason is that $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma$. Therefore

$$\ker \phi_L = \bigl\{ w \in V \bigm| E(w,\gamma) \in \mathbb{Z} \text{ for every } \gamma \in \Gamma \bigr\} / \Gamma \cong \Gamma^*/\Gamma.$$

As we saw during the proof of Theorem 4.6, this is a group of order

$$\det E = \bigl(\dim H^0(X,L)\bigr)^2,$$

and so the proof is complete. ∎

Example 6.4. When L is a principal polarization ($\det E = 1$), the group Γ^*/Γ is trivial; in that case, our homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is an isomorphism of abelian varieties. Later on, when we treat abelian varieties using algebraic methods, we are going to use this kind of result in order to *define* the Picard variety $\operatorname{Pic}^0(X)$.

The fact that ϕ_L is a group homomorphism means that

$$t_{a+b}^* L \otimes L^{-1} \cong t_a^* L \otimes L^{-1} \otimes t_b^* L \otimes L^{-1}.$$

If we clean this up a bit, it becomes

$$t_{a+b}^* L \otimes L \cong t_a^* L \otimes t_b^* L$$

for any two points $a, b \in X$. This result is known as the "theorem of the square".

The Lefschetz theorem. We are now going to prove a sharp version of the Kodaira embedding theorem.

Theorem 6.5 (Lefschetz)**.** *Let $L = L(H, \alpha)$ be a holomorphic line bundle such that the hermitian form H is positive definite.*

(a) The line bundle L^2 is base-point free, and its global sections give a holomorphic mapping

$$\varphi_2 \colon X \to \mathbb{P}\big(H^0(X, L^2)\big).$$

(b) The line bundle L^3 is very ample, and its global sections give an embedding

$$\varphi_3 \colon X \to \mathbb{P}\big(H^0(X, L^3)\big).$$

The numbers 2 and 3 are exactly as in the case of elliptic curves: any elliptic curve has a 2:1 map to $\mathbb{P}^1$, and can be embedded into $\mathbb{P}^2$ as a cubic curve. In general, by Corollary 5.3, we have

$$\dim H^0(X, L^k) = \frac{1}{n!} c_1(L^k)^n = k^n \dim H^0(X, L),$$

and so the projective spaces in question are fairly big once n gets larger.

Let's start by proving (a). According to Theorem 4.6, we have

$$\dim H^0(X, L) = \sqrt{\det E} \geq 1$$

because H is positive definite. Let $s_0 \in H^0(X, L)$ be any nontrivial section. The idea is to use translations in order to generate additional sections of L^2. Recall from above that

$$t_a^* L \otimes t_{-a}^* L \cong L^2$$

for any $a \in X$. This shows that $t_a^* s_0 \otimes t_{-a}^* s_0$ is a global section of L^2. The proof of (a) is now very easy. To show that L^2 is base-point free, we need to find, at any given point $x \in X$, a global section of L^2 that does not vanish at x. For that, we only have to choose $a \in X$ so that the two points $x \pm a$ do not lie on the zero locus of s_0; then ${t_a}^* s_0 \otimes t_{-a}^* s_0$ does the job.

It remains to prove (b). The argument that I gave in class was incomplete – as Spencer pointed out, I did not really prove that φ_3 is injective. So I am going to deviate from what I said in class, and use the notes to present Mumford's argument. Before doing that, let's briefly review a bit of general theory. Suppose that X is a compact complex manifold, and L a holomorphic line bundle that is base-point free. If we set $d = \dim H^0(X, L) - 1$, and choose a basis $s_0, \dots, s_d \in H^0(X, L)$, then we get a holomorphic mapping

$$\varphi \colon X \to \mathbb{P}^d, \quad x \mapsto \big(s_0(x), s_1(x), \dots, s_d(x)\big).$$

It is proper because X is compact. To show that φ is an embedding, we have to prove two things:

(1) φ is injective. By compactness, this ensures that φ is a homeomorphism between X and $\varphi(X)$.
(2) φ is an immersion. Concretely, this means that for every $x \in X$, the map on tangent spaces $d\varphi_x \colon T_x X \to T_{\varphi(x)}\mathbb{P}^d$ is injective. This ensures that $\varphi(X)$ is a complex manifold and φ is biholomorphic.

Proof that φ_3 is injective. Let's now prove (1) for the line bundle L^3. Recall that global sections of $L = L(H, \alpha)$ are theta functions for (H, α); these are holomorphic functions $\theta \colon V \to \mathbb{C}$ that satisfy the functional equation

$$\theta(v + \gamma) = e^{\pi H(v,\gamma)+\frac{\pi}{2}H(\gamma,\gamma)}\alpha(\gamma) \cdot \theta(v). \tag{6.6}$$

For any two vectors $u, w \in V$, the product

$$\theta(v - u)\theta(v - w)\theta(v + u + w)$$

is a theta function for $(3H, \alpha^3)$, and therefore a global section of L^3. Suppose that there are two points $x_1, x_2 \in X$ with $\varphi_3(x_1) = \varphi_3(x_2)$. If we lift $x_1, x_2 \in X$ to vectors $v_1, v_2 \in V$, then it follows that there is a constant $C \neq 0$ such that

$$\phi(v_1) = C\phi(v_2)$$

for every theta function ϕ for the Appell-Humbert datum $(3H, \alpha^3)$. In particular, for every pair of vectors $v, w \in V$, we will have

$$\theta(v_1 - v)\theta(v_1 - w)\theta(v_1 + v + w) = C\theta(v_2 - v)\theta(v_2 - w)\theta(v_2 + v + w) \tag{6.7}$$

for all theta function for (H, α). We are going to deduce from this condition that $v_2 - v_1 \in \Gamma$, and hence that $x_1 = x_2$.

Consider (6.7) as a function of $v \in V$. To eliminate the constant C, we take logarithmic derivatives. Let $\omega = (d\theta)/\theta$, which is a meromorphic 1-form on V. After differentiating (6.7), we obtain

$$\omega(v_1 + v + w) - \omega(v_1 - v) = \omega(v_2 + v + w) - \omega(v_2 - v),$$

and so the meromorphic 1-form $\omega(v_2+v)-\omega(v_1+v)$ is invariant under translation by arbitrary elements of V, hence constant. We can therefore write it as $df(v)$, where $f \colon V \to \mathbb{C}$ is $\mathbb{C}$-linear. Since $\omega(v_2 + v) - \omega(v_1 + v)$ is the logarithmic derivative of $\theta(v_2 + v)/\theta(v_1 + v)$, it follows that there is a constant $A \in \mathbb{C}$ such that

$$\theta(v + v_2) = Ae^{f(v)}\theta(v + v_1)$$

for every $v \in V$. Set $w = v_2 - v_1$, and replace v by $v - v_1$ to put this into the form

$$\theta(v + w) = Be^{f(v)}\theta(v),$$

where $B \in \mathbb{C}$ is some other constant.

If we now substitute into the functional equation in (6.6) and cancel terms that appear on both sides, we get $e^{\pi H(w,\gamma)} = e^{f(\gamma)}$ for every $\gamma \in \Gamma$. This means that

$$\pi H(w, \gamma) - f(\gamma) \in 2\pi i \cdot \mathbb{Z}.$$

Recalling that $E = \operatorname{Im} H$, we have

$$\pi H(w, \gamma) - f(\gamma) = \pi H(\gamma, w) - f(\gamma) + 2\pi i E(w, \gamma) \in 2\pi i \cdot \mathbb{Z},$$

and so $\pi H(\gamma, w) - f(\gamma) \in i \cdot \mathbb{R}$. Because it is also $\mathbb{C}$-linear in the first argument, it follows that

$$\pi H(v, w) = f(v) \qquad \text{for every } v \in V. \tag{6.8}$$

We conclude that $E(w,\gamma) \in \mathbb{Z}$ for every $\gamma \in \mathbb{Z}$, and so our vector $w = v_2 - v_1$ belongs to the larger lattice

$$\hat{\Gamma} = \{ v \in V \mid E(v,\gamma) \in \mathbb{Z} \text{ for every } \gamma \in \mathbb{Z} \}.$$

Recall that $\hat{\Gamma} \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$, and that $\hat{\Gamma}/\Gamma$ is a finite group of order $\det E$. This already shows that some integer multiple of w lies in Γ.

We are going to finish the proof of (1) by showing that $w \in \Gamma$. Observe that θ is actually a theta function for the larger lattice $\Gamma' = \Gamma + \mathbb{Z}w$. The reason is that, because of (6.8), we have

$$\theta(v+w) = Be^{\pi H(v,w)}\theta(v) = Be^{-\frac{\pi}{2}H(w,w)} \cdot e^{\pi H(v,w)+\frac{\pi}{2}H(w,w)}\theta(v).$$

Because an integer multiple of w lies in Γ, the constant $Be^{-\frac{\pi}{2}H(w,w)}$ must be of absolute value 1, and so we can extend $\alpha\colon \Gamma \to U(1)$ uniquely to $\alpha'\colon \Gamma' \to U(1)$ by requiring that $\alpha'(w) = Be^{-\frac{\pi}{2}H(w,w)}$ and $\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)e^{i\pi E(\gamma,\delta)}$ for all $\gamma, \delta \in \Gamma'$. With this choice, every theta function θ for the pair (H,α) and the lattice Γ is then also a theta function for the pair (H,α') and the bigger lattice Γ'.

The dimension of the space of theta functions for (H,α) and Γ is, according to Theorem 4.6, equal to the square root of the order of the group Γ^*/Γ. If $\Gamma' \neq \Gamma$, then this is strictly larger than the order of the group Γ'^*/Γ', and so for dimension reasons, it is not possible for every theta function for Γ to also be a theta function for Γ'. The conclusion is that $\Gamma' = \Gamma$, and hence that $w \in \Gamma$. This proves that φ_3 is injective.

Proof that φ_3 is an immersion. Next, we prove (2) for φ_3. Suppose there is a point $x_0 \in X$ and a tangent vector $\xi \in T_{x_0}X$ that is mapped to zero under the differential of φ_3. Choose a basis $v_1, \dots, v_n \in V$ and let $z_1, \dots, z_n \in V^*$ be the dual basis; as usual, we view $z_1, \dots, z_n$ as coordinates on V, and hence as local coordinates on X. Write $\xi = \sum_{j=1}^n c_j \partial/\partial_j$. Choose a lifting of $x_0 \in X$ to a vector $v_0 \in V$. After computing the derivatives in an affine coordinate chart on projective space, we find that there is a constant $c_0 \in \mathbb{C}$ such that

$$\sum_{j=1}^n c_j \frac{\partial \phi}{\partial z_j}(v_0) = c_0 \phi(v_0)$$

for every theta function ϕ for the pair $(3H, \alpha^3)$. As before, we apply this to functions of the form $\phi(v) = \theta(v-u)\theta(v-w)\theta(v+u+w)$ with $u, w \in V$, where θ is any theta function for the pair (H,α). For given θ, consider the meromorphic function

$$f = \theta^{-1} \sum_{j=1}^n c_j \frac{\partial \theta}{\partial z_j}.$$

After substituting into the relation above, we get

$$f(v_0 - u) + f(v_0 - w) + f(v_0 + u + w) = c_0$$

for all $u, w \in V$. By the usual argument with first derivatives, it follows that $f(v) = \ell(v) + f(0)$ for a linear functional $\ell\colon V \to \mathbb{C}$.

Define $c = \sum_{j=1}^n c_j v_j \in V$. We compute that

$$\frac{d}{dt}\theta(v+tc) = \sum_{j=1}^n c_j \frac{\partial \theta}{\partial z_j}(v+tc) = \big(t\ell(c) + f(v)\big) \cdot \theta(v+tc).$$

After integration, this leads to the identity

$$\theta(v+tc) = e^{\frac{1}{2}t^2\ell(c) + tf(v)}\theta(v)$$

for every $v \in V$ and every $t \in \mathbb{C}$. If we now plug this into the functional equation in (6.6) and cancel terms that appear on both sides, we find that

$$e^{\pi H(tc,\gamma)} = e^{\frac{1}{2}t^2\ell(c)+tf(v)}.$$

By varying $v \in V$, we conclude that $f = 0$, and hence that $\ell = 0$. By varying $t \in \mathbb{C}$, it follows that $H(c,\gamma) = 0$ for every $\gamma \in \Gamma$. Because H is nondegenerate, this finally gives $c = 0$. We conclude that $\xi = 0$, and hence that φ_3 is indeed an immersion.

Lecture 7 (February 18)

Principally polarized abelian varieties. Let $X = V/\Gamma$ be a compact complex torus. Recall from Lecture 5 that a polarization is a positive definite hermitian form $H\colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. (In Lecture 5, I said incorrectly that a polarization was an ample line bundle; instead, the polarization is just the first Chern class of an ample line bundle.) According to Lemma 5.2, we can always find a basis for Γ such that

$$E = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $m_1 \mid m_2 \mid \cdots \mid m_n$, and therefore $\det E = (m_1 \cdots m_n)^2$. The polarization is principal if $m_1 = \cdots = m_n = 1$. In that case, the homomorphism

$$E\colon \Gamma \to \Gamma^*, \quad \gamma \mapsto E(\gamma, -),$$

is an isomorphism. For any choice of $\alpha\colon \Gamma \to U(1)$ with the property that

$$\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)e^{i\pi E(\gamma,\delta)},$$

the line bundle $L = L(H,\alpha)$ is then ample, and $\dim H^0(X,L) = 1$. The divisor of the (essentially unique) nontrivial section of L is called a *theta divisor*.

Note that the principal polarization does not uniquely determine an ample line bundle. Since $e^{i\pi E(\gamma,\delta)} = \pm 1$, we can cut down on the number of choices by requiring that $\alpha(\Gamma) \subseteq \{\pm 1\}$. But even then, there are still 2^{2n} possible choices for α, and there is no way to pick a canonical one. On the other hand, the line bundle $L^2 = L(2H,1)$ is uniquely determined by H, because $e^{2\pi i E(\gamma,\delta)} = 1$, and so $\alpha \equiv 1$ works for $2H$.

When H is a principal polarization, the holomorphic group homomorphism

$$\phi_L\colon X \to \operatorname{Pic}^0(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

is also an isomorphism. This means that every line bundle with trivial first Chern class can be written in the form $t_x^* L \otimes L^{-1}$ for a unique point $x \in X$.

Let's now determine all possible principally polarized abelian varieties in a given dimension. It is customary to call this dimension g, as in the case of Jacobians (where g is the genus of the compact Riemann surface in question).

Example 7.1. As a warm-up, let's do the case $g = 1$. Any elliptic curve can be written in the form $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$ is a point in the upper halfplane (so $\operatorname{Im} \tau > 0$). The principal polarization is $E(\tau, 1) = 1$, and then $H(1,1) = 1/\operatorname{Im}\tau$. We can choose a different basis for the lattice, say of the form

$$a\tau + b \quad \text{and} \quad c\tau + d,$$

for integers $a, b, c, d \in \mathbb{Z}$, subject to the condition that

$$1 = E(a\tau + b, c\tau + d) = ad - bc.$$

In terms of matrices, this is saying that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

If we again use the second basis vector in the lattice as the basis for the vector space, then the new lattice is $\mathbb{Z} + \mathbb{Z}\tau'$, where

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

This is still a point on the upper halfplane $\mathbb{H}$, because

$$\operatorname{Im} \tau' = \frac{\operatorname{Im} \tau}{|c\tau + d|^2} > 0.$$

The isomorphism between the two elliptic curves is then

$$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau' \to \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, \quad z \mapsto (c\tau + d)z.$$

To summarize, every elliptic curve can be written as $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ for $\tau \in \mathbb{H}$, and two such curves are isomorphic if and only if τ and τ' belong to the same orbit of the group $\mathrm{SL}_2(\mathbb{Z})$. In that sense, the moduli space of elliptic curves is the quotient $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This statement needs to be taken with some care, though, because $\mathrm{SL}_2(\mathbb{Z})$ does not act freely: for example, the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

acts trivially on $\mathbb{H}$, but on the level of the elliptic curves, it acts nontrivially (as the automorphism $z \mapsto -z$). There are also special points, for example $\tau = i$ or $\tau = \frac{1}{2}(-1 + \sqrt{-3})$, whose stabilizer is even larger (and where the elliptic curve has additional automorphisms).

Let's now describe all principally polarized abelian varieties of dimension $g \geq 1$. We start from $X = V/\Gamma$, where $\dim V = g$ and $\operatorname{rk} \Gamma = 2g$. The principal polarization is $H \colon V \times V \to \mathbb{C}$, and $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. Using Lemma 5.2, but changing the order of the basis elements, we can find a basis $e_1, \dots, e_{2g} \in \Gamma$ such that E becomes the block matrix (of size $2g \times 2g$)

$$E = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right]$$

where I_g is the identity matrix of size $g \times g$. We can now use the lattice elements $e_{g+1}, \dots, e_{2g}$ as a basis for the $\mathbb{C}$-vector space $\mathbb{C}$, and write the other g lattice elements $e_1, \dots, e_g$ in terms of that basis as

$$e_k = \sum_{j=1}^{g} \Omega_{j,k} e_{g+j}$$

for certain complex numbers $\Omega_{j,k} \in \mathbb{C}$. With this convention, we have $V = \mathbb{C}^n$, and the lattice takes the form $\Gamma = \mathbb{Z}^n + \Omega\mathbb{Z}^n$; in other words, the lattice is spanned by the columns of the following block matrix (of size $g \times 2g$):

$$\left[\begin{array}{c|c} \Omega & I_g \end{array}\right]$$

The positive definite hermitian form H is then represented by the $g\times g$-matrix with entries $H_{j,k} = H(e_{g+j}, e_{g+k}) \in \mathbb{R}$ (because $E = \operatorname{Im} H$ vanishes on these vectors). This matrix is symmetric and positive definite. From

$$1 = E(e_k, e_{g+k}) = \operatorname{Im} H(e_k, e_{g+k}) = \sum_{j=1}^{g} \operatorname{Im}\Omega_{j,k}\cdot H_{j,k}, = \sum_{j=1}^{g} \operatorname{Im}\Omega_{j,k}\cdot H_{k,j},$$

we see that H is the inverse matrix to $\operatorname{Im}\Omega$. Therefore $\operatorname{Im}\Omega$ must be positive definite, and the polarization is represented, in the basis $e_{g+1},\dots,e_{2g} \in V$, by the matrix $(\operatorname{Im}\Omega)^{-1}$.

For a similar reason, the matrix Ω is also symmetric. Indeed,

$$H(e_j, e_k) = \sum_{p,q=1}^{g} \Omega_{p,j}\overline{\Omega}_{q,k} H(e_{g+p}, e_{g+q}) = \sum_{p,q=1}^{g} \Omega_{p,j}\overline{\Omega}_{q,k} H_{p,q}$$

is also real (for $1 \leq j,k \leq g$), and therefore

$$0 = \sum_{p,q=1}^{g} \left(\operatorname{Im}\Omega_{p,j}\operatorname{Re}\Omega_{q,k} - \operatorname{Re}\Omega_{p,j}\operatorname{Im}\Omega_{q,k}\right) H_{p,q} = \operatorname{Re}\Omega_{j,k} - \operatorname{Re}\Omega_{k,j},$$

remembering that $\operatorname{Im}\Omega$ and H are inverse matrices. Therefore $\Omega^t = \Omega$.

The analogue of the upper halfplane is the so-called *Siegel space*

$$\mathcal{H}_g = \left\{\, \Omega \in \operatorname{Mat}_{g\times g}(\mathbb{C}) \;\middle|\; \Omega^t = \Omega, \text{ and } \operatorname{Im}\Omega \text{ is positive definite} \,\right\}.$$

Every principally polarized abelian variety can be written in the form

$$\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g),$$

where $\Omega \in \mathcal{H}_g$. In the standard basis on $\mathbb{C}^g$, the polarization is represented by the positive definite matrix $(\operatorname{Im}\Omega)^{-1}$.

What happens when we choose a different basis for the lattice? Suppose that $e_1',\dots,e_{2g}' \in \Gamma$ is another basis, still with the property that $E(e_j', e_{g+j}') = 1$. We can represent the change of basis by the block matrix (of size $2g\times 2g$)

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with $A,B,C,D \in \operatorname{Mat}_{g\times g}(\mathbb{Z})$, and the condition that $E\colon \Gamma\times\Gamma \to \mathbb{Z}$ has the same shape as before translates into the matrix equation

$$\left[\begin{array}{c|c} A^t & C^t \\ \hline B^t & D^t \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right].$$

In other words, $M \in \operatorname{Sp}_g(\mathbb{Z})$ is an element of the symplectic group. A brief computation shows that the matrix Ω gets transformed into the new matrix

$$\Omega' = (C\Omega + D)^{-1}(A\Omega + B).$$

So the parameter space (or moduli space) for principally polarized abelian varieties of dimension g is the quotient space

$$\mathcal{A}_g = \mathcal{H}_g/\operatorname{Sp}_g(\mathbb{Z}),$$

with the same caveats as before. This has dimension $g(g+1)/2$.

Subtori and isogenies. We'll end the complex-analytic treatment of abelian varieties by a quick look at the structure of abelian varieties. Let $X = V/\Gamma$ be a compact complex torus.

A *subtorus* is a connected (closed, hence compact) complex subgroup. As we saw in Lecture 2, such a subgroup again has the form $X' = V'/\Gamma'$, and so $V' \subseteq V$ is a complex subspace, and $\Gamma' = V' \cap \Gamma$ should be a lattice in V'. We can think of a subtorus either as a discrete subgroup of Γ of some even rank $2k$, whose span is a complex subspace of dimension k; or as a complex subspace of dimension k that intersects Γ in a discrete subgroup of rank $2k$.

Example 7.2. Any holomorphic mapping $f \colon X \to Y$ from a compact complex torus to a compact complex torus that satisfies $f(0) = 0$ is a group homomorphism. So the connected component of $\ker f$ is a subtorus.

If $X' \subseteq X$ is a subtorus, then the quotient X/X' is again a compact complex torus; to see this, write the quotient as

$$X/X' \cong \frac{V/V'}{\Gamma/\Gamma'}.$$

A compact complex torus X is called *simple* if the only subtori are $\{0\}$ and X. Elliptic curves are simple (for dimension reasons); in fact, if we choose a random lattice in V, then the resulting compact complex torus will be simple.

Now we would like to prove that every abelian variety can be decomposed into simple abelian varieties. Here "decomposed" could mean "written as a product", but that doesn't quite work, so we have to settle for something a bit weaker. The relevant definition is the following.

Definition 7.3. A group homomorphism $f \colon X \to Y$ from a compact complex torus X to a compact complex torus Y is called an *isogeny* if f is surjective and $\ker f$ is a finite group. In that case, we say that X and Y are *isogenous.*

Consider an isogeny $f \colon X_1 \to X_2$. The induced map $\tilde{f} \colon V_1 \to V_2$ must be an isomorphism, and $\tilde{f}(\Gamma_1) \subseteq \Gamma_2$. Then $\ker f \cong \tilde{f}^{-1}(\Gamma_2)/\Gamma_1$, and so this must be a finite group. Equivalently, we can identify the two vector spaces using $\tilde{f}$, and after that identification, our isogeny has the form

$$V/\Gamma_1 \to V/\Gamma_2,$$

where $\Gamma_1 \subseteq \Gamma_2$ is a subgroup of finite index. This shows that being isogenous really is an equivalence relation. Indeed, if $\Gamma_1 \subseteq \Gamma_2$ is a subgroup of finite index, then Γ_2 will have finite index in $\Gamma_{1,m} = \frac{1}{m}\Gamma_1$ for some integer $m \geq 1$. Because the mapping

$$V/\Gamma_{1,m} \to V/\Gamma_1, \quad v \mapsto mv,$$

is an isomorphism, this gives us the desired isogeny

$$V/\Gamma_2 \to V/\Gamma_{1,m} \to V/\Gamma_1.$$

So working up to isogeny basically means replacing the lattice Γ by the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$.

Example 7.4. For any nonzero integer $m \in \mathbb{Z}$, the homomorphism

$$[m] \colon X \to X, \quad x \mapsto mx,$$

is an isogeny. The kernel is the set $X[m]$ of points of order m in the group X; as we saw in Lecture 2, this is a group with $m^{2 \dim X}$ elements.

Example 7.5. If X is an abelian variety, and L an ample line bundle, then the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X), \quad x \mapsto t_x^* L \otimes L^{-1},$$

is an isogeny: indeed, the kernel is a finite group of order $\dim H^0(X, L)^2$.

One has the following simple structure theorem for abelian varieties. It is known as the *Poincaré complete irreducibility theorem.*

Theorem 7.6. *Every abelian variety is isogenous to a product of simple abelian varieties, and the factors are unique up to isogeny.*

Unlike other structure theorems in geometry, this one is completely elementary. We first prove the following lemma about simple tori.

Lemma 7.7. *Let X and Y be simple compact complex tori. Then any holomorphic group homomorphism $f\colon X \to Y$ is either constant or an isogeny.*

Proof. Consider a holomorphic group homomorphism $f\colon X \to Y$. The image $\operatorname{im} f$ is a subtorus of Y, and because Y is simple, we either have $\operatorname{im} f = \{0\}$ or $\operatorname{im} f = Y$. In the first case, f is constant. In the second case, the connected component of $\ker f$ is a subtorus of X, and because X is simple and f is not constant, this connected component must be trivial. But then f is surjective with finite kernel, and so it is an isogeny. $\square$

We can now prove the theorem.

Proof. This is exactly the same as the prime factorization of integers. Let's first prove existence. By induction on $\dim X$, we only need to show that if X contains a nontrivial subtorus X', then X is isogenous to $X' \times X''$ for some other subtorus X''. Write $X = V/\Gamma$ and $X' = V'/\Gamma'$, with $\Gamma' = V' \cap \Gamma$. Choose a polarization $H\colon V \times V \to \mathbb{C}$ and set $E = \operatorname{Im} H$ as usual. The orthogonal complement

$$V'' = \left\{ v \in V \;\middle|\; H(v, v') = 0 \text{ for every } v' \in V' \right\}$$

satisfies $V = V' \oplus V''$ because H is positive definite. Moreover, we have

$$\Gamma'' = V'' \cap \Gamma = \left\{ \gamma \in \Gamma \;\middle|\; E(\gamma, \gamma') = 0 \text{ for every } \gamma' \in \Gamma' \right\}$$

because $V' = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma'$. Since E is nondegenerate, $\Gamma' \oplus \Gamma'' \subseteq \Gamma$ is a subgroup of finite index. Therefore Γ'' is a lattice in V''; the quotient $X'' = V''/\Gamma''$ is a compact complex torus; and the induced mapping

$$X' \oplus X'' \to X$$

is an isogeny.

Now let's prove uniqueness. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be two collections of simple abelian varieties, and suppose that we have an isogeny

$$f\colon X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, consider the induced homomorphism $f_{j,i}\colon X_i \to Y_j$. By Lemma 7.7, it is either constant or an isogeny. After rearranging the order of the factors, we may assume that $X_1, \dots, X_p$ and $Y_1, \dots, Y_q$ are isogenous to each other (and therefore of the same dimension), but not isogenous to any of the other factors. If we view our isogeny

$$f\colon (X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_m) \to (Y_1 \times \cdots \times Y_q) \times (Y_{q+1} \times \cdots \times Y_n)$$

as a 2×2-matrix, it has the form

$$f = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix},$$

where $g\colon X_1 \times \cdots \times X_p \to Y_1 \times \cdots \times Y_q$ and $h\colon X_{p+1} \times \cdots \times X_m \to Y_{q+1} \times \cdots \times Y_n$ are homomorphisms. Because f is surjective with finite kernel, both g and h must be surjective with finite kernel. For dimension reasons, we get $p = q$. But $h\colon X_{p+1} \times \cdots \times X_m \to Y_{q+1} \times \cdots \times Y_n$ is still an isogeny, and so we can finish the proof by induction on the number of factors. $\square$

Lecture 8 (February 20)

Abelian varieties. We are now going to look at abelian varieties from the point of view of algebraic geometry. Let k be an algebraically closed field; the theory can be developed in that generality, but some of the results are going to be a bit different when $\operatorname{char} k \neq 0$.

Definition 8.1. An *abelian variety* is a complete variety X (over the field k) that has the structure of a group, such that the group operations

$$m\colon X \times X \to X, \quad m(x,y) = xy, \qquad i\colon X \to X, \quad i(x) = x^{-1},$$

are morphisms (= regular maps).

Nonsingular cubic curves in $\mathbb{P}^2$ (in characteristic different from 2 and 3) are an example: the group law on the points of a nonsingular cubic can be described by morphisms. We can get higher-dimensional abelian varieties by taking products; other examples are less easy to come by.

We are going to show later that every abelian variety is projective; but in the definition, we only assume that X is complete (or, in scheme language, proper over $\operatorname{Spec} k$). We are mostly going to work with varieties, and not with schemes, so all the points of X are closed points. Generally speaking, we want to prove the same kind of results that we proved in the complex-analytic setting: the structure of X as a group; line bundles and their global sections; maps to projective space; etc.

Let's start with a few basic observations. First, X is always nonsingular. By definition, X is a variety, so it is reduced and irreducible. The set of nonsingular points is therefore Zariski-open and dense in X. Now X, being a group, is homogeneous, and so the existence of one nonsingular point implies that all points are nonsingular. More precisely, for any $x \in X$, consider the translation morphism

$$t_x\colon X \to X, \quad t_x(y) = m(x,y).$$

This is an automorphism (with inverse $t_{i(x)}$). Choose a nonsingular point $x_0 \in X$, and let $x \in X$ be an arbitrary point. Then translation by $m(x, i(x_0))$ takes the point x_0 to the point x, and since x_0 is nonsingular, x must also be nonsingular.

Second, let's prove that X is an abelian group. We will give two proofs for this; you should remember the technique, because it is very useful for studying group actions on algebraic varieties.

Lemma 8.2. *The group operation on an abelian variety is commutative.*

Proof. As in the complex case, we look at the conjugation morphism

$$C_x\colon X \to X, \quad C_x(y) = xyx^{-1}.$$

This is an automorphism, with inverse $C_{x^{-1}}$. It takes the identity element $e \in X$ to itself, and so it acts (by pullback of regular functions) on the local ring $\mathscr{O}_{X,e}$. The idea is to show that this action is trivial, by proving that it is trivial modulo larger and larger powers of the maximal ideal $\mathfrak{m}_e$. Because $e \in X$ is a nonsingular point, the quotient $\mathfrak{m}_e/\mathfrak{m}_e^2$ is a k-vector space of dimension $n = \dim X$; by Nakayama's lemma, we have $\mathfrak{m}_e = (f_1, \dots, f_n)$ for a system of parameters $f_1, \dots, f_n \in \mathscr{O}_{X,e}$. Now the automorphism

$$C_x^*\colon \mathscr{O}_{X,e} \to \mathscr{O}_{X,e}$$

preserves the maximal ideal $\mathfrak{m}_e$, and so for each $\ell \in \mathbb{N}$, it induces an automorphism

$$C_x^*\colon \mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1} \to \mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}.$$

Thinking of the elements in the quotient as polynomials of degree $\leq \ell$ in n-variables, we see that the quotient on the right-hand side is a finite-dimensional k-vector space of dimension $\binom{n+\ell}{\ell}$. So we get a function

$$f\colon X \to \mathrm{End}_k\bigl(\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}\bigr)$$

that sends a point $x \in X$ to the endomorphism C_x^* modulo $\mathfrak{m}_e^{\ell+1}$, viewed as an element of the k-vector space on the right-hand side. It is not hard to see that c is a morphism of algebraic varieties. Indeed, the mapping

$$C\colon X \times X \to X, \quad C(x,y) = C_x(y) = xyx^{-1},$$

is a morphism (by the definition of abelian varieties). Choose affine open neighborhoods V, W of the point $e \in X$, and U of the point $x \in X$, such that $C(U \times V) \subseteq W$. Then pullback of regular functions gives a morphism of k-algebras

$$C^*\colon k[W] \to k[U \times V] \cong k[U] \otimes_k k[V],$$

where $k[U] = \Gamma(U, \mathscr{O}_X)$ is the k-algebra of regular functions on U. Since $C(x,e) = e$, this induces a morphism of k-algebras

$$\mathscr{O}_{X,e} \to k[U] \otimes_k \mathscr{O}_{X,e},$$

and from this, it is easy to see that if we view $f|_U\colon U \to \mathrm{End}_k\bigl(\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}\bigr)$ as a matrix of size $\binom{n+\ell}{\ell}$, then the entries are regular functions on U. This means that f is a morphism of algebraic varieties.

The rest of the proof is easy. By assumption, X is complete, and so the morphism f must be constant; because $f(e) = \mathrm{id}$, it follows that C_x^* acts as the identity on $\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}$. By Krull's intersection theorem, we have

$$\bigcap_{\ell \in \mathbb{N}} \mathfrak{m}_e^{\ell+1} = (0),$$

and so it follows that C_x^* is the identity on $\mathscr{O}_{X,e}$. Therefore C_x acts as the identity on a Zariski-open neighorhood of $e \in X$, and because X is a variety, C_x is the identity everywhere. But then $C_x(y) = y$, and this means that X is commutative. □

From now on, we are going to write the group operation on an abelian variety additively: so $m(x,y) = x + y$ and $i(x) = -x$, and the identity element is $0 \in X$.

As in the complex case, we can describe the tangent and cotangent bundles of an abelian variety. Let $T = T_{X,0}$ be the Zariski tangent space at $0 \in X$; if we set $\Omega_0 = \mathfrak{m}_0/\mathfrak{m}_0^2$, then $T = \mathrm{Hom}_k(\Omega_0, k)$, and both are k-vector spaces of dimension $\dim X$. For every $x \in X$, translation induces an isomorphism

$$t_{-x}^*\colon \mathfrak{m}_0/\mathfrak{m}_0^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2,$$

and so a cotangent vector $\theta \in \Omega_0$ defines an algebraic 1-form ω_θ by the rule $(\omega_\theta)_x = t_{-x}^*(\theta)$. As before, one can check on affines that ω_θ is a global section of the sheaf of Kähler differentials $\Omega_{X/k}^1$, and that this procedure defines a morphism of sheaves

$$\Omega_0 \otimes_k \mathscr{O}_X \to \Omega_{X/k}^1.$$

By construction, it is an isomorphism on fibers, meaning after tensoring by $\mathscr{O}_{X,x}/\mathfrak{m}_x$; by Nakayama's lemma, it is therefore an isomorphism of sheaves. After dualizing, we find that

$$T_X \cong T \otimes_k \mathscr{O}_X,$$

and so the tangent bundle of X is trivial. Similarly, we can take wedge powers to get

$$\Omega_{X/k}^p \cong \bigwedge^p \Omega_0 \otimes_k \mathscr{O}_X.$$

On global sections, this gives

$$H^0\bigl(X, \Omega_{X/k}^p\bigr) \cong \bigwedge^p \Omega_0,$$

because $H^0(X, \mathscr{O}_X) = k$ by completeness of X. All global algebraic p-forms on X are therefore translation invariant, exactly as on compact complex tori.

A fourth result, with a similar infinitesimal proof, is that the group of points of an abelian variety is divisible, provided we avoid the characteristic of the field k.

Lemma 8.3. *As long as n is not divisible by* $\mathrm{char}(k)$*, the homomorphism*

$$n_X \colon X \to X, \quad x \mapsto n \cdot x,$$

is surjective.

Proof. The morphism $m \colon X \times X \to X$ induces a k-linear mapping

$$dm \colon T_{X \times X,(0,0)} \to T_{X,0}$$

on tangent spaces. Set $T = T_{X,0}$. The tangent space to $X \times X$ at the point $(0, 0)$ is isomorphic to $T \oplus T$, with the two copies given by the images of $T_{X,0}$ under the two inclusions $i_1 \colon X \to X \times X$, $i_1(x) = (x, 0)$, and $i_2 \colon X \to X \times X$, $i_2(x) = (0, x)$. Because $m \circ i_1 = m \circ i_2 = \mathrm{id}$, it follows that

$$dm \colon T \oplus T \to T$$

is just the sum map $(t_1, t_2) \mapsto t_1 + t_2$. From this, it is easy to see that

$$dn_X \colon T \to T$$

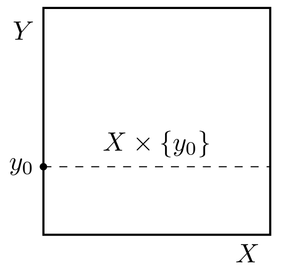
is multiplication by the integer n. Therefore dn_X is an isomorphism if n is not divisible by $\mathrm{char}(k)$. For dimension reasons, this means that n_X must be surjective: otherwise, the dimension of the image would be strictly less that $\dim X$, and so all fibers of n_X would have dimension ≥ 1. But if the fiber through the point $0 \in X$ has positive dimension, we can find a tangent vector $t \in T$ such that $dn_X(t) = 0$, and this contradicts the fact that dn_X is an isomorphism. $\square$

The proof shows more: because n_X is a homomorphism, the differential dn_X is actually an isomorphism at every point of X, and so $n_X \colon X \to X$ is finite étale. (In the case of compact complex tori, multiplication by n was a finite covering space.) We will later compute the degree of n_X, but this is more involved than on compact complex tori.

The rigidity theorem and its consequences. In order to go further, we need the following somewhat technical result, called the *rigidity theorem*. It is one of the important properties of complete varieties.

Theorem 8.4. *Let X be a complete variety over k, let Y, Z be varieties, and let $f \colon X \times Y \to Z$ be a morphism. Suppose that there is a point $y_0 \in Y$ such that $f(X \times \{y_0\})$ is a single point $z_0 \in Z$. Then $f = g \circ p_2$ for a morphism $g \colon Y \to Z$.*

This is saying that if one of the slices $X \times \{y_0\}$ is contracted to a point, then all slides $X \times \{y\}$ are contracted to a point (and $g(y)$ is that point).



Proof. Choose a point $x_0 \in X$ and define $g\colon Y \to Z$ by the formula $g(y) = f(x_0, y)$. Let $p_2\colon X \times Y \to Y$ be the second projection. In order to prove that $f = g \circ p_2$, it is enough to show that this holds on a Zariski-open set containing $X \times \{y_0\}$; the reason is that $X \times Y$ is irreducible. Choose an affine open set $U \subseteq Z$ such that $z_0 \in U$. The idea is to show that all nearby slices $X \times \{y\}$ also map into U.

The complement $Z \setminus U$ is a closed subset of Z. Because X is complete, the morphism $p_2\colon X \times Y \to Y$ is proper, which means that the image of any closed subset is closed. For that reason,

$$W = p_2\bigl(f^{-1}(Z \setminus U)\bigr) \subseteq Y$$

is a closed subset of Y. It does not contain the point y_0, because f maps $X \times \{y_0\}$ to the point $z_0 \in U$, and so $V = Y \setminus W$ is a Zariski-open set containing y_0. By construction, we have $f(X \times \{y\}) \subseteq U$ for every $y \in V$. Because U is affine and X is complete, f is therefore constant on $X \times \{y\}$. This shows that we have $f(x, y) = f(x_0, y) = g(y)$ for every $y \in V$. The identity $f = g \circ p_2$ therefore holds on the open set $X \times V$, as required. □

This has several useful consequences for abelian varieties.

Corollary 8.5. *Every morphism between two algebraic varieties is a group homomorphism composed with a translation.*

Proof. Let $f\colon X \to Y$ be a morphism from an abelian variety to an abelian variety. After composing f with the translation $t_{-f(e)}\colon Y \to Y$, we may assume that $f(e) = e$. We then claim that f must be a group homomorphism. To see that this is true, consider the morphism

$$F\colon X \times X \to Y, \quad F(x, y) = f(xy) - f(x) - f(y).$$

We have $F(x, e) = F(e, x) = e$, and so F contracts both $X \times \{e\}$ and $\{e\} \times X$. By the rigidity theorem, we must have $F(x, y) = e$ for all $x, y \in X$, and so f is a group homomorphism. □

We can also give another proof for the fact that X is commutative.

Corollary 8.6. *The group structure on an abelian variety is commutative.*

Proof. For the sake of clarity, let's briefly revert to multiplicative notation. Consider the morphism $i\colon X \to X$, $i(x) = x^{-1}$. It satisfies $i(0) = 0$, and so it must be a group homomorphism (by the previous corollary). This gives

$$y^{-1}x^{-1} = i(xy) = i(x)i(y) = x^{-1}y^{-1},$$

which obviously implies that the group operation is commutative. □

The last result for today is another special property of abelian varieties. If S and T are varieties, we can describe morphisms into the product $S \times T$ (which, in scheme-theoretic language, would be the fiber product over $\operatorname{Spec} k$). Indeed, the universal property says that a morphism $X \to S \times T$ is the same thing as a pair of morphisms $X \to S$ and $X \to T$ (all over k, of course); in other words, we have an isomorphism of sets

$$\operatorname{Hom}(X, S \times T) \cong \operatorname{Hom}(X, S) \times \operatorname{Hom}(X, T).$$

For abelian varieties, there is a similar result for maps *from* a product. Suppose that S and T are *complete* varieties, and that each comes with a choice of base point $s_0 \in S$ and $t_0 \in T$. We'll write (S, s_0) for the variety together with the point. Now suppose we have two morphisms $f\colon S \to X$ and $g\colon T \to X$ such that $f(s_0) = g(t_0) = 0$. The composition

$$S \times T \xrightarrow{f \times g} X \times X \xrightarrow{m} X$$

gives us a morphism $h\colon S \times T \to X$ with $h(s_0, t_0) = 0$. More concretely, we have

$$h\colon S \times T \to X, \quad h(s,t) = f(s) + g(t).$$

From h, we can of course recover f and g because $f(s) = h(s, t_0)$ and $g(t) = h(s_0, t)$. This shows that the function

$$\begin{aligned} \operatorname{Hom}\bigl((S, s_0), (X, 0)\bigr) \times \operatorname{Hom}\bigl((T, t_0), (X, 0)\bigr) &\to \operatorname{Hom}\bigl((S \times T, s_0 \times t_0), (X, 0)\bigr) \\ (f, g) &\mapsto m \circ (f \times g), \end{aligned}$$

is injective. It is also surjective: Given $h\colon S \times T \to X$ with $h(s_0, t_0) = 0$, we define $f(s) = h(s, t_0)$ and $g(t) = h(s_0, t)$, and then $h(s,t) = f(s) + g(t)$ by the rigidity theorem. (The difference $h(s,t) - f(s) - g(t)$ again contracts both $S \times \{t_0\}$ and $\{s_0\} \times T$, and so it must be constant.)

So, in somewhat more fancy language, the functor

$$(S, s_0) \mapsto \operatorname{Hom}\bigl((S, s_0), (X, 0)\bigr),$$

from the category of complete varieties with base point to the category of sets takes products to products.

Lecture 9 (February 25)

Cohomology and base change. Our next goal is to study line bundles and their cohomology on abelian varieties. In the complex case, we saw that line bundles come in big families – because we can always tensor by line bundles in $\operatorname{Pic}^0(X)$ – and so we need to first understand how cohomology groups of line bundles behave in families. A family of line bundles parametrized by a variety T is of course just a line bundle L on the product $X \times T$, and we are interested how the cohomology of the restrictions $L_t = L|_{X \times \{t\}}$ depends on $t \in T$. The technical tool is cohomology and base change, which Mumford treats very nicely in his book.

Here is the general setting. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. For every point $y \in Y$, we have the fiber

$$X_y = X \times_Y \operatorname{Spec} k(y),$$

which is a scheme over the field field $k(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$. Let's denote by

$$\mathscr{F}_y = \mathscr{F}|_{X_y} = \mathscr{F} \otimes_{\mathscr{O}_Y} k(y)$$

the restriction of $\mathscr{F}$ to the closed subscheme X_y. Cohomology and base change is about the cohomology groups $H^k(X_y, \mathscr{F}_y)$, and how they relate to the higher direct image sheaves $R^k f_* \mathscr{F}$. The key assumption is *flatness*.

Definition 9.1. We say that $\mathscr{F}$ is *flat* over Y if, for every point $x \in X$, the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$ is flat over $\mathscr{O}_{Y,f(x)}$, via the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$. Since $\mathscr{F}$ is quasi-coherent, this is equivalent to saying that for every pair of affine open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$ is flat over $\mathscr{O}_Y(V)$, via the ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$.

Example 9.2. If $A \to B$ is a ring homomorphism, and M is a B-module, then $\tilde{M}$ is a quasi-coherent sheaf on $\operatorname{Spec} B$; it is flat if and only if M is flat as an A-module.

Example 9.3. The second projection $p_2\colon X \times T \to T$ is flat, and any locally free sheaf on $X \times T$ (such as a line bundle) is therefore flat over T.

The geometric part of cohomology and base change is the following theorem by Grothendieck. In class, I just outlined the proof, but I filled in most of the details in the notes.

Theorem 9.4. *Let $f\colon X \to Y$ be a proper morphism between noetherian schemes, with $Y = \operatorname{Spec} A$ affine. Let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then there is a bounded complex $K^\bullet$ of finitely-generated projective A-modules, of the form*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0,$$

such that for every B-algebra A, one has a functorial isomorphism

$$H^p\big(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\big) \cong H^p\big(K^\bullet \otimes_A B\big)$$

for all $p \in \mathbb{Z}$.

Note that $K^\bullet$ is a complex of A-modules, and so its cohomology groups

$$H^p(K^\bullet) = \frac{\ker d^p\colon K^p \to K^{p+1}}{\operatorname{im} d^{p-1}\colon K^{p-1} \to K^p}$$

are again A-modules. The complex $K^\bullet$ gives us a functorial way to describe all the objects we are interested in. For example, if we take $B = A$, we get

$$H^p(K^\bullet) \cong H^p(X, \mathscr{F}),$$

which is the A-module corresponding to the sheaf $R^p f_* \mathscr{F}$. On the other hand, we can take $B = k(y)$, where $y \in Y$ is any closed point; then $X \times_Y \operatorname{Spec} k(y) = X_y$ and $\mathscr{F} \otimes_A k(y) = \mathscr{F}_y$, and so

$$H^p\big(K^\bullet \otimes_A k(y)\big) \cong H^p(X_y, \mathscr{F}_y).$$

So the theorem translates the whole problem of cohomology and base change into understanding how the cohomology groups of a bounde complex of finitely-generated projective A-modules (= locally free sheaves) change from point to point.

Here is an outline of the proof, in four steps.

Step 1. The morphism f is proper, and $\mathscr{F}$ is coherent on X, and so all the higher direct image sheaves $R^p f_* \mathscr{F}$ are coherent on Y. (This theorem is also due to Grothendieck.) Because $Y = \operatorname{Spec} A$ is affine, $R^p f_* \mathscr{F}$ is the quasi-coherent sheaf associated to the A-module $H^p(X, \mathscr{F})$, and the theorem is saying that $H^p(X, \mathscr{F})$ is a finitely-generated A-module. (If you want to see the proof, have a look at `Tag 0203` in the *Stacks Project.*)

Step 2. We can compute the cohomology of quasi-coherent sheaves using Čech cohomology. Because f is proper and Y is affine, we can cover X by finitely many affine open subsets; let $\mathcal{U} = \{U_i\}_{i \in I}$ be the open covering. Let $C^\bullet = C^\bullet(\mathcal{U}, \mathscr{F})$ be the Čech complex, with terms

$$C^p = \bigoplus_{i_0, \dots, i_p} \mathscr{F}\big(U_{i_0} \cap \cdots \cap U_{i_p}\big),$$

and the usual differential. The intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ are affine (because X is separated), and so the flatness of $\mathscr{F}$ implies that each C^p is a flat A-module. Because affine open coverings are acyclic (for quasi-coherent sheaves), the Čech complex computes the sheaf cohomology of $\mathscr{F}$:

$$H^p(X, \mathscr{F}) \cong H^p(C^\bullet)$$

are isomorphic as A-modules. Now suppose that B is any A-algebra. Then $\mathcal{U}_B = \{U_i \times_Y \operatorname{Spec} B\}_{i \in I}$ is an affine open covering of $X \times_Y \operatorname{Spec} B$, and it is easy to deduce from the definition of the fiber product that

$$C^p\big(\mathcal{U}_B, \mathscr{F} \otimes_A B\big) \cong C^p(\mathcal{U}, \mathscr{F}) \otimes_A B$$

as B-modules. This gives us isomorphisms

$$H^p\big(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\big) \cong H^p\big(C^\bullet \otimes_A B\big),$$

which are clearly functorial in B.

Step 3. The complex $C^\bullet$ has almost all the properties we want, except that the A-modules C^p are not finitely-generated. The following lemma allows us to replace $C^\bullet$ by a smaller complex that is finitely-generated.

Lemma 9.5. *Consider a bounded complex of A-modules $C^\bullet$, of the form*

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0,$$

whose cohomology groups $H^p(C^\bullet)$ are finitely-generated A-modules. Then there is a bounded complex of finitely-generated A-modules $K^\bullet$, of the form

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0,$$

and a morphism of complexes $\phi\colon K^\bullet \to C^\bullet$ that induces isomorphisms on cohomology. We can arrange that $K^1, \dots, K^n$ are finitely-generated free A-modules, and if $C^0, \dots, C^n$ are flat A-modules, then K^0 is also flat, hence projective.

One piece of terminology. A morphism of complexes $\phi\colon K^\bullet \to C^\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on cohomology: for every $p \in \mathbb{Z}$, the morphism $\phi\colon H^p(K^\bullet) \to H^p(C^\bullet)$ is an isomorphism.

Proof. This is a basic result in homological algebra, similar to the construction of free resolutions for A-modules. We construct the desired complex step-by-step. For simplicity, set $H^p = H^p(C^\bullet)$, which are finitely-generated A-modules, nonzero only for $p = 0, \dots, n$. Let's denote the differentials in the complex $C^\bullet$ by $\delta^p\colon C^p \to C^{p+1}$. To begin with, H^n is finitely-generated, and so we can choose a finitely-generated free A-module K^n and a surjection $K^n \to H^n$. Because $H^n = C^n / \operatorname{im} \delta^{n-1}$, we can lift this to a morphism $\phi^n\colon K^n \to C^n$. We now have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_0^n \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d, "\phi^n"] & H^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{im} \delta^{n-1} \arrow[r] & C^n \arrow[r] & H^n \arrow[r] & 0
\end{tikzcd}$$

where we define $K_0^n = (\phi^n)^{-1}(\operatorname{im} \delta^{n-1})$ as the preimage of $\operatorname{im} \delta^{n-1}$. Note that K_0^n is again a finitely-generated A-module (because K^n and H^n are); we can therefore choose a finitely-generated A-module K^{n-1} and a surjection $K^{n-1} \to K_0^n$. Because $\operatorname{im} \delta^{n-1} = C^{n-1} / \ker \delta^{n-1}$, we can again lift the morphism from K^{n-1} to $\operatorname{im} \delta^{n-1}$ to a morphism $\phi^{n-1}\colon K^{n-1} \to C^{n-1}$, giving us another commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0^{n-1} \arrow[r] \arrow[d] & K^{n-1} \arrow[r] \arrow[d, "\phi^{n-1}"] & K_0^n \arrow[r] \arrow[d, "\phi^n"] & 0 \\
0 \arrow[r] & \ker \delta^{n-1} \arrow[r] & C^{n-1} \arrow[r, "\delta^{n-1}"] & \operatorname{im} \delta^{n-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows; of course, $K_0^{n-1} = (\phi^{n-1})^{-1}(\ker \delta^{n-1})$. Define $d^{n-1}\colon K^{n-1} \to K^n$ as the composition $K^{n-1} \to K_0^n \to K^n$; then $\ker d^{n-1} = K_0^{n-1}$ and $K^n / \operatorname{im} d^{n-1} \cong H^n$, and so ϕ^n induces an isomorphism between the n-th cohomology of our (partial) complex $K^\bullet$ and the n-th cohomology of $C^\bullet$.

The composition $K_0^{n-1} \to \ker \delta^{n-1} \to H^{n-1}$ may not be surjective, but because H^{n-1} is finitely-generated, we can add a finitely-generated free A-module to both K_0^{n-1} and K^{n-1} (and let d^{n-1} act on it as zero); this makes sure that $K_0^{n-1} \to H^{n-1}$ is surjective. After this change, the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1^{n-1} \arrow[r] \arrow[d] & \ker d^{n-1} \arrow[r] \arrow[d, "\phi^{n-1}"] & H^{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{im} \delta^{n-2} \arrow[r] & \ker \delta^{n-1} \arrow[r] & H^{n-1} \arrow[r] & 0
\end{tikzcd}$$

is exact, where $K_1^{n-1} = (\phi^{n-1})^{-1}(\operatorname{im}\delta^{n-2})$. Since K_1^{n-1} is finitely-generated, we can map a finitely-generated free A-module K^{n-2} onto it, and so on. In other words, we keep repeating the whole procedure n times: for each $p = 1, \dots, n$, we get a morphism $\phi^p \colon K^p \to C^p$ from a finitely-generated free A-module, and a differential $d^p \colon K^p \to K^{p+1}$, such that the induced morphism from the p-th cohomology of $K^\bullet$ to the p-th cohomology of $C^\bullet$ is an isomorphism. In the final step of the construction, for $p = 0$, we need to define

$$K^0 = \left\{ (x, y) \in C^0 \times K_0^1 \;\middle|\; \delta^0(x) = \phi^1(y) \right\}$$

in order for the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\delta^0 & \longrightarrow & K^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & \\
0 & \longrightarrow & \ker\delta^0 & \longrightarrow & C^0 & \xrightarrow{\delta^0} & \operatorname{im}\delta^0 & \longrightarrow & 0
\end{array}$$

to be exact. Because $\ker\delta^0 = H^0$ is finitely-generated, the A-module K^0 will be finitely-generated, but not necessarily free.

It remains to show that if $C^0, \dots, C^n$ are flat A-modules, then K^0 is also flat; flat and finitely-generated implies projective, so K^0 will then be a projective A-module, as claimed. For that, we consider the mapping cone complex $L^\bullet$ for the morphism $\phi \colon K^\bullet \to C^\bullet$. This is the complex with terms

$$L^p = K^{p+1} \oplus C^p$$

and with differential

$$d \colon L^p \to L^{p+1}, \quad d(x, y) = \big(-dx, \delta(y) - \phi(x)\big).$$

It is easy to see that this fits into a short exact sequence of complexes

$$0 \to C^\bullet \to L^\bullet \to K^{\bullet+1} \to 0,$$

where the usual homological algebra convention is that the differential in the complex $K^{\bullet+1}$ is $-\delta$. The long exact sequence in cohomology reads

$$\cdots \longrightarrow H^{p-1}(L^\bullet) \longrightarrow H^p(K^\bullet) \xrightarrow{\phi} H^p(C^\bullet) \longrightarrow H^{p+1}(L^\bullet) \longrightarrow \cdots$$

and because ϕ is quasi-isomorphism, we get $H^p(L^\bullet) = 0$ for all $p \in \mathbb{Z}$, and so the complex $L^\bullet$ is exact. All the terms L^p are flat A-modules, with the possible exception of $L^{-1} = K^0$. From this and exactness, it follows readily that L^{-1} is also flat; as we said earlier, this means that K^0 is actually a projective A-module. □

In fact, we can do a little bit better. Suppose we are interested in the local behavior near a point $y_0 \in Y$. We can localize at y_0, meaning replace A by the local ring $\mathscr{O}_{Y,y_0}$. Now each time we need to choose a finitely-generated free A-module in the construction above, we can choose a *minimal* one, using Nakayama's lemma. Indeed, suppose that M is a finitely-generated A-module, where $(A, \mathfrak{m})$ is a local ring. Then $M/\mathfrak{m}M$ is a finite-dimensional vector space over $A/\mathfrak{m}$, and if we choose elements $m_1, \dots, m_n \in M$ whose images in $M/\mathfrak{m}M$ form a basis, then $m_1, \dots, m_n$ generate M by Nakayama's lemma. This gives us a surjection $A^n \to M$ with n minimal. If we use this device at each step, then K^0 will also be a free A-module (because every projective module over a local ring is free), and the differentials d^p in the complex $K^\bullet$ will have the property that $\operatorname{im} d^p \subseteq \mathfrak{m}K^{p+1}$. In other words, the complex $K^\bullet$ will be a *minimal complex*, in the following sense.

Definition 9.6. A complex of free A-modules $K^\bullet$ over a local ring $(A, \mathfrak{m})$ is *minimal* if $\operatorname{im} d^p \subseteq \mathfrak{m}K^{p+1}$ for every $p \in \mathbb{Z}$, or equivalently, if the tensor product $K^\bullet \otimes_A A/\mathfrak{m}$ has trivial differentials.

The complex $K^\bullet$ will actually make sense on some affine open set $\operatorname{Spec} A'$ containing the point y_0; at the cost of replacing $\operatorname{Spec} A$ by this smaller open set, we can therefore always achieve that the complex $K^\bullet \otimes k(y_0)$ has trivial differentials at a given point $y_0 \in Y$.

Step 4. It remains to show that we have

$$H^p\bigl(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B\bigr) \cong H^p\bigl(K^\bullet \otimes_A B\bigr)$$

for every A-algebra B. Since this holds for the Čech complex $C^\bullet$ by construction, we can apply the following general lemma.

Lemma 9.7. *Suppose that $\phi \colon K^\bullet \to C^\bullet$ is a quasi-isomorphism. If $C^0, \dots, C^n$ and $K^0, \dots, K^n$ are flat A-modules, then*

$$\phi \otimes_A B \colon K^\bullet \otimes_A B \to C^\bullet \otimes_A B$$

is also a quasi-isomorphism.

Proof. Consider again the mapping cone complex $L^\bullet$. The argument we gave earlier shows that ϕ is a quasi-isomorphism if and only if $L^\bullet$ is exact. Because every $L^p = K^{p+1} \oplus C^p$ is flat, the tensor product $L^\bullet \otimes_A B$ is still exact. But this is the mapping cone complex of $\phi \otimes_A B$, and so $\phi \otimes_A B$ is a quasi-isomorphism. □

Consequences of Grothendieck's theorem. The rest of the theory is basically just linear algebra. Let's first investigate the dimensions of the fiberwise cohomology groups.

Corollary 9.8. *Let $f \colon X \to Y$ be a proper morphism, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y.*

(a) The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is upper semicontinuous on Y.

(b) The Euler characteristic function

$$y \mapsto \chi(\mathscr{F}_y) = \sum_{p \in \mathbb{Z}} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is locally constant on Y.

Proof. The problem is local on Y, and so we may assume that $Y = \operatorname{Spec} A$ is affine, and that we have a bounded complex $K^\bullet$ of finitely-generated free A-modules as in the theorem. (This works because projective A-modules are locally free.) So each differential $d^p \colon K^p \to K^{p+1}$ is now just a matrix with entries in the ring A. For every point $y \in Y$, we have

$$\begin{aligned} \dim H^p(X_y, \mathscr{F}_y) &= \dim H^p\bigl(K^\bullet \otimes_A k(y)\bigr) \\ &= \dim \ker d^p \otimes_A k(y) - \dim \operatorname{im} d^{p-1} \otimes_A k(y) \\ &= \dim K^p \otimes_A k(y) - \dim \operatorname{im} d^p \otimes_A k(y) - \dim \operatorname{im} d^{p-1} \otimes_A k(y). \end{aligned}$$

Taking the alternating sum over $p \in \mathbb{Z}$, we get

$$\sum_{p \in \mathbb{Z}} (-1)^p \dim H^p(X_y, \mathscr{F}_y) = \sum_{p \in \mathbb{Z}} (-1)^p \dim K^p \otimes_A k(y)$$

which is independent of y because each K^p is a free A-module. This gives (b).

For (a), we need to prove that each set

$$\bigl\{\, y \in Y \bigm| \dim H^p(X_y, \mathscr{F}_y) \geq \ell \,\bigr\}$$

is the set of closed points of a closed subscheme of Y. By the computation above, it suffices to show that the same is true for the sets

$$\bigl\{\, y \in Y \bigm| \dim \operatorname{im} d^p \otimes_A k(y) \leq \ell \,\bigr\}.$$

But this set is defined by all the minors of size $(\ell+1)$ of the matrix representing the differential d^p, and so it is a closed subscheme. $\square$

The next corollary is the actual base change theorem. It says that if the dimensions of the cohomology groups $H^p(X_y, \mathscr{F}_y)$ are constant, then they fit together into a locally free sheaf, namely $R^p f_* \mathscr{F}$.

Corollary 9.9. *Let $f\colon X \to Y$ be a proper morphism, with Y reduced. Let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then the following two conditions are equivalent:*

(a) The function $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is constant.

(b) The coherent sheaf $R^p f_* \mathscr{F}$ is locally free, and the base change morphism

$$R^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

is an isomorphism for every $y \in Y$.

If this happens, then the base change morphism

$$R^{p-1} f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^{p-1}(X_y, \mathscr{F}_y)$$

in the next lower degree is also an isomorphism.

Proof. The problem is again local on Y, and so we may assume that $Y = \operatorname{Spec} A$ is affine, and that we have a complex $K^\bullet$ as in the theorem. To simplify the notation, let us set $K^\bullet(y) = K^\bullet \otimes_A k(y)$ and $d^p(y) = d^p \otimes_A k(y)$. This time, though, we also choose a point $y_0 \in Y$, and arrange that the complex $K^\bullet$ is minimal at y_0, in the sense that $K^\bullet(y_0)$ has trivial differentials. Obviously, this means that

$$H^p(X_{y_0}, \mathscr{F}_{y_0}) \cong H^p\big(K^\bullet(y_0)\big) \cong K^p(y_0).$$

Now suppose that $\dim H^p(X_y, \mathscr{F}_y)$ is constant, and therefore equal to $\dim K^p(y_0) = \dim K^p(y)$. Because this is the cohomology of

$$K^{p-1}(y) \xrightarrow{d^{p-1}(y)} K^p(y) \xrightarrow{d^p(y)} K^{p+1}(y),$$

we must have $d^{p-1}(y) = d^p(y) = 0$ for every $y \in Y$, which means that the entries of the matrices for d^{p-1} and d^p vanish at every point $y \in Y$. Because Y is reduced, this means that $d^{p-1} = d^p = 0$. But then

$$H^p(X, \mathscr{F}) = H^p(K^\bullet) = K^p$$

is a free A-module. The associated coherent sheaf is $R^p f_* \mathscr{F}$, which is therefore locally free. It is clear from this that the base change morphism is an isomorphism.

Now let's see where the (somewhat unexpected) additional assertion comes from. We have $d^{p-1} = 0$, and so $R^{p-1} f_* \mathscr{F}$ is the coherent sheaf associated to

$$H^{p-1}(X, \mathscr{F}) = H^{p-1}(K^\bullet) = K^{p-1} / \operatorname{im} d^{p-2}.$$

In other words, we have an exact sequence

$$K^{p-2} \xrightarrow{d^{p-2}} K^{p-1} \longrightarrow H^{p-1}(X, \mathscr{F}) \longrightarrow 0$$

Because tensor product is right exact, we can tensor with $k(y)$ and

$$K^{p-2}(y) \xrightarrow{d^{p-2}(y)} K^{p-1}(y) \longrightarrow H^{p-1}(X, \mathscr{F}) \otimes_A k(y) \longrightarrow 0$$

is still exact. This gives the desired isomorphism between $H^{p-1}(X, \mathscr{F}) \otimes_A k(y)$ and $H^{p-1}\big(K^\bullet(y)\big)$. $\square$

This has many nice consequences. For example, suppose that $H^{p+1}(X_y, \mathscr{F}_y) = 0$ for every $y \in Y$; this will happen for example if p is the maximum of the fiber dimensions $\dim X_y$. Then the base change morphism

$$R^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

is an isomorphism for every $y \in Y$. For that reason, base change always holds for the cohomology groups in the largest possible degree. Similarly, suppose that we have the vanishing $H^p(X_y, \mathscr{F}_y) = 0$ for every $y \in Y$ and every $p \geq p_0$. By repeatedly applying Corollary 9.9, we conclude that $R^p f_* \mathscr{F} = 0$ for all $p \geq p_0$. In this way, we can turn a fiberwise vanishing statement into the vanishing of the higher direct image sheaves, which will be useful if we are, for example, doing computations with the Leray spectral sequence.

The seesaw theorem. We now apply the base change theorem to the case of line bundles. Let X be a complete variety, T an arbitrary variety, and suppose we have a line bundle L on the product $X \times T$. For every $t \in T$, denote by L_t the restriction of L to $X \times \{t\}$. In the notation from above, we are working with the second projection $p_2 \colon X \times T \to T$, which is clearly proper and flat.

Theorem 9.10. *Under these assumptions, the set*

$$T_1 = \left\{ t \in T \;\middle|\; L_t \textit{ is trivial on } X \times \{t\} \right\}$$

is closed in T, and there is a line bundle M on T_1 such that

$$L|_{X \times T_1} \cong p_2^* M.$$

Proof. We observe that a line bundle L on a proper variety X is trivial if and only if $H^0(X, L) \neq 0$ and $H^0(X, L^{-1}) \neq 0$. The reason is that a nonzero section $s \in H^0(X, L)$ gives a nonzero morphism $s \colon \mathscr{O}_X \to L$, and a nonzero section $t \in H^0(X, L^{-1})$ gives a nonzero morphism $t \colon L \to \mathscr{O}_X$. Their composition $t \circ s$ is a nonzero morphism from $\mathscr{O}_X$ to itself, hence a nonzero constant by properness. After multiplying by the inverse of this constant, we can assume that $t \circ s = 1$. But then $s \colon \mathscr{O}_X \to L$ is an isomorphism with inverse $t \colon L \to \mathscr{O}_X$.

This observation proves that

$$T_1 = \left\{ t \in T \;\middle|\; \dim H^0(X \times \{t\}, L_t) \geq 1 \text{ and } \dim H^0(X \times \{t\}, L_t^{-1}) \geq 1 \right\}.$$

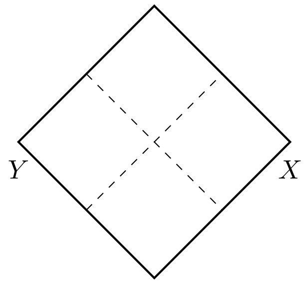
By Corollary 9.8, this is a closed subset of T. To prove the other half, we can replace T by T_1 and assume without loss of generality that L_t is trivial for every $t \in T$. Then $\dim H^0(X \times \{t\}, L_t) = 1$ is constant, and so the direct image sheaf

$$M = (p_2)_* L$$

is a locally free sheaf of rank 1, hence a line bundle. By construction, the induced morphism $p_2^* M \to L$ is an isomorphism on every fiber $X \times \{t\}$ (because L_t is trivial), and therefore an isomorphism on $X \times T$. □

Lecture 10 (February 27)

Last time, someone asked where the name "seesaw theorem" comes from. In one of the explanatory paragraphs in his collected works, André Weil writes that he introduced the name in a course on abelian varieties that he taught at the University of Chicago in 1954/55. Unfortunately, he does not explain why the theorem made him think of a seesaw. Ravi Vakil (in *The Rising Sea*) says that he has no idea why it is called the seesaw theorem. Herbert Lange (in his book *Complex Abelian Varieties*) says that it is "called the seesaw theorem for obvious reasons". Perhaps the reason is that if we draw $X \times Y$ like this



then the two slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ look like the two opposite positions of a seesaw. But your guess is as good as mine.

Anyway, here is a useful corollary.

Corollary 10.1. *Let L be a line bundle on $X \times Y$, where X, Y are varieties, and X is complete. If $L|_{X\times\{y\}}$ is trivial for every $y \in Y$, and if $L|_{\{x_0\}\times Y}$ is trivial for some point $x_0 \in X$, then L is trivial.*

Proof. By the seesaw theorem, we have $L \cong p_2^* M$ for a line bundle M on Y; now restrict to $\{x_0\} \times Y$ to conclude that M is trivial. □

The theorem of the cube. Our main topic today is the "theorem of the cube", which is a result about line bundles on $X \times Y \times Z$. It is the crucial ingredient in proving results about line bundles on abelian varieties. Here is the statement.

Theorem 10.2. *Let L be a line bundle on $X \times Y \times Z$, where X, Y, Z are varieties, and X and Y are complete. Suppose that there are points $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ such that the three line bundles*

$$L|_{\{x_0\}\times Y\times Z}, \quad L|_{X\times\{y_0\}\times Z}, \quad L|_{X\times Y\times\{z_0\}}$$

are trivial. Then L is trivial.

Note that this only works for three or more factors: a line bundle on $X \times Y$ can be trivial on $\{x_0\} \times Y$ and on $X \times \{y_0\}$ without being trivial. We can get some intuition for the statement from the case of complex manifolds. If $\mathrm{Pic}(X)$ denotes the group of holomorphic line bundles, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

where $\mathrm{Pic}^0(X)$ means line bundles with trivial first Chern class. Now consider a holomorphic line bundle L on $X \times Y \times Z$, say with X, Y, Z connected. By the Künneth formula, we have

$$\begin{aligned} H^2(X \times Y \times Z, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}) \otimes H^1(Y, \mathbb{Z}) \oplus \\ H^1(X, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z}) \oplus H^1(Y, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}). \end{aligned}$$

Each summand involves at most two factors of the product, because we are looking at H^2. If the restriction of L to all three slices $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ is trivial, it follows from this that $c_1(L) = 0$. Because $\mathrm{Pic}^0(X) \cong H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$, we also get from the Künneth formula that

$$\mathrm{Pic}^0(X \times Y \times Z) \cong \mathrm{Pic}^0(X) \times \mathrm{Pic}^0(Y) \times \mathrm{Pic}^0(Z),$$

and so a line bundle $L \in \mathrm{Pic}^0(X \times Y \times Z)$ that is trivial on all three slices is trivial.

Before giving the proof, let's first deduce the following nice corollary.

Corollary 10.3. *If X and Y are complete varieties, then every line bundle on $X \times Y \times Z$ is isomorphic to a line bundle of the form*

$$p_{12}^* L_{12} \otimes p_{13}^* L_{13} \otimes p_{23}^* L_{23},$$

where L_{12}, L_{13}, L_{23} are line bundles on the three double products.

Proof. Choose three points $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Let M_1 denote the restriction of L^{-1} to $X \times \{y_0\} \times \{z_0\}$, and define M_2 and M_3 similarly. After replacing L by the tensor product

$$L \otimes p_1^* M_1 \otimes p_2^* M_2 \otimes p_3^* M_3,$$

we can assume without loss of generality that L is trivial on those three subvarieties. Now suppose that L_{12} is a line bundle on $X \times Y$ that is trivial on $\{x_0\} \times Y$ and on $X \times \{y_0\}$, and similarly for L_{13} and L_{23}. The condition that

$$M = L^{-1} \otimes p_{12}^* L_{12} \otimes p_{13}^* L_{13} \otimes p_{23}^* L_{23}$$

should be trivial on $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ then uniquely determines L_{12}, L_{13}, and L_{23}. For example, we have

$$M|_{X \times Y \times \{z_0\}} \cong L^{-1}|_{X \times Y \times \{z_0\}} \otimes L_{12},$$

because L_{13} is trivial on $X \times \{z_0\}$ and L_{23} is trivial on $Y \times \{z_0\}$; therefore we can set $L_{12} = L|_{X \times Y \times \{z_0\}}$. With these choices, M is trivial on all three slices. The theorem of the cube implies that M is trivial, and this gives the desired result. □

Proof of the theorem. Let L be a line bundle on $X \times Y \times Z$ such that

$$L|_{\{x_0\} \times Y \times Z}, \quad L|_{X \times \{y_0\} \times Z}, \quad L|_{X \times Y \times \{z_0\}}$$

are trivial. We want to prove that L itself must be trivial. The proof will hopefully make it clear why we need X and Y to be complete.

Step 1. To get started, we observe that it is enough to prove that $L|_{\{x\} \times Y \times \{z\}}$ is trivial for every $(x, z) \in X \times Z$. This is because of the seesaw theorem: Y is complete, and if L is trivial on every fiber of $p_{13} \colon X \times Y \times Z \to X \times Z$, it is the pullback of a line bundle from $Y \times Z$; but that line bundle must be trivial because we are assuming that L is trivial on $\{x_0\} \times Y \times Z$.

Step 2. This observation allows us to reduce the problem to the case where X is a nonsingular curve. Let $x \in X$ be an arbitrary point. Choose a complete irreducible curve $C \subseteq X$ that passes through the two points x_0 and x. (Such a curve clearly exists when X is projective; and by Chow's lemma, any complete variety admits a surjective map from a projective variety.) Let $f \colon \tilde{C} \to C$ be the normalization; then $\tilde{C}$ is nonsingular and irreducible. Consider the pullback $M = (f \times \mathrm{id} \times \mathrm{id})^* L$ of the line bundle along the morphism

$$f \times \mathrm{id} \times \mathrm{id} \colon \tilde{C} \times Y \times Z \to X \times Y \times Z.$$

It still satisfies the assumptions in the theorem of the cube, but now on the product $\tilde{C} \times Y \times Z$. If we can show that M is trivial on every subvariety of the form $\{c\} \times Y \times \{z\}$, then M is trivial; and because x is in the image of f, this then implies that L is trivial on $\{x\} \times Y \times \{z\}$. So if we can prove the theorem of the cube when $\dim X = 1$ and X is nonsingular, then it will hold in general.

Remark. We don't actually need Chow's lemma here. For fixed $z \in Z$, the set of points $x \in X$ such that L is trivial on $\{x\} \times Y \times \{z\}$ is closed (by Theorem 9.10), and so it is enough to prove this for all x in an affine open neighborhood of the point x_0. But any two points in an affine variety can clearly be connected by an irreducible curve.

Step 3. From now on, we assume that X is an complete, irreducible, and nonsingular curve. By the same argument as in Step 1, it is enough to prove that the line bundle

$$L_{(y,z)} = L|_{X \times \{y\} \times \{z\}}$$

is trivial for every $(y, z) \in Y \times Z$; in fact, we can even replace Z by a dense open subset, because the set of all such points is closed in $Y \times Z$ by the seesaw theorem.

Let ω_X be the canonical line bundle on the curve X, and let $g = \dim H^0(X, \omega_X)$ be the genus of the curve. We can choose g points $P_1, \dots, P_g \in C$ such that the divisor $D = P_1 + \dots + P_g$ satisfies $\dim H^0\big(X, \omega_X(-D)\big) = 0$: take a nontrivial section of ω_X and pick the first point P_1 such that the section does not vanish at P_1; then $\dim H^0\big(X, \omega_X(-P_1)\big) = g - 1$; and so on. By Serre duality, we get

$$\dim H^1\big(X, \mathscr{O}_X(D)\big) = \dim H^0\big(X, \omega_X(-D)\big) = 0.$$

We now adjust the line bundle L as follows. Let $p_1 \colon X \times Y \times Z \to X$ be the first projection, and define

$$L' = L \otimes p_1^* \mathscr{O}_X(D).$$

As before, we set $L'_{(y,z)} = L'|_{X \times \{y\} \times \{z\}}$; evidently,

$$L'_{(y,z)} \cong L_{(y,z)} \otimes \mathscr{O}_X(D). \tag{10.4}$$

Because L is trivial on $X \times Y \times \{z_0\}$, we get $L'_{(y,z_0)} \cong \mathscr{O}_X(D)$ for all $y \in Y$; consequently, the first cohomology $H^1\big(X, L'_{(y,z_0)}\big) = 0$.

By Corollary 9.8, the set

$$F = \big\{ (y, z) \in Y \times Z \bigm| \dim H^1\big(X, L'_{(y,z)}\big) \geq 1 \big\}$$

is closed in $Y \times Z$. Because Y is proper, the image $p_2(F) \subseteq Z$ is also closed. We have just seen that it does not contain the point z_0. We can therefore find an open set $Z' \subseteq Z$ containing the point z_0, such that $p_2(F) \cap Z_0 = \emptyset$. This means concretely that

$$H^1\big(X, L'_{(y,z)}\big) = 0$$

for every $(y, z) \in Y \times Z'$. After replacing Z by the dense open subset Z', we can assume that this holds for every $(y, z) \in Y \times Z$.

Step 4. We can use this to compute the space of global sections. By Corollary 9.8, the Euler characteristic is constant, and so

$$\begin{aligned} \dim H^0\big(X, L'_{(y,z)}\big) = \chi\big(L'_{(y,z)}\big) = \chi\big(L'_{(y,z_0)}\big) = \chi\big(X, \mathscr{O}_X(D)\big) \\ = \deg D - g + 1 = 1 \end{aligned}$$

by the Riemann-Roch theorem. Every line bundle $L'_{(y,z)}$ therefore has (up to scaling) a unique nontrivial global section, and so it determines a unique effective divisor on X (of degree $g = \deg D$). As we move $(y, z) \in Y \times Z$, these divisors are going to sweep out a divisor $\tilde{D}$ on $X \times Y \times Z$.

To construct $\tilde{D}$ rigorously, we can argue as follows. First, $\dim H^0\big(X, L'_{(y,z)}\big) = 1$ is constant, and so Corollary 9.9 implies that the pushforward $(p_{23})_* L'$ is a line bundle on $Y \times Z$. On any open set $U \subseteq Y \times Z$ where this line bundle is trivial, we can choose a nowhere vanishing section $s_U \in H^0\big(U, (p_{23})_* L'\big)$. By the definition of the pushforward, it comes from a section $\tilde{s}_U \in H^0\big(X \times U, L'\big)$, and we let $\tilde{D}_U$ be the divisor of $\tilde{s}_U$. If $V \subseteq Y \times Z$ is another open set of this type, then s_U and s_V differ from each other by an element of $H^0(U \cap V, \mathscr{O}_X^\times)$, and so $\tilde{D}_U$ and $\tilde{D}_V$ agree on $X \times (U \cap V)$. Consequently, there is a well-defined divisor $\tilde{D}$ such that $\tilde{D}|_U = \tilde{D}_U$. It is clear from the construction that $\tilde{D}|_{X \times \{y\} \times \{z\}}$ is the divisor of the unique nontrivial section of $L'_{(y,z)}$.

Step 5. We'll complete the proof by showing that $\tilde{D} = p_1^*(D)$. Observe that

$$\tilde{D}|_{X\times\{y\}\times\{z_0\}} = D \quad \text{and} \quad \tilde{D}|_{X\times\{y_0\}\times\{z\}} = D$$

for every $y \in Y$ and every $z \in Z$; the reason is that $L'_{(y,z_0)} \cong L'_{(y_0,z)} \cong \mathscr{O}_X(D)$. So if we take a point $P \in X$ with $P \neq P_j$ for $j = 1, \dots, g$, then the divisor

$$\tilde{D}_P = \tilde{D}|_{\{P\}\times Y\times Z}$$

does not intersect the two closed subsets $\{P\} \times Y \times \{z_0\}$ and $\{P\} \times \{y_0\} \times Z$. The projection $p_2(\tilde{D}_P) \subseteq Z$ is a closed subset (because Y is complete); because it does not contain the point z_0, it must be a proper closed subset. For dimension reasons, this implies that the divisor $\tilde{D}_P$ is supported on a finite union of closed subsets of the form $\{P\} \times Y \times T_j$, where $T_j \subseteq Z$ has codimension one. But $\tilde{D}_P$ also does not intersect $\{P\} \times \{y_0\} \times Z$, and this is now only possible if $\tilde{D}_P$ is empty.

Step 6. The conclusion is that $\tilde{D}$ does not intersect the set $\{P\} \times Y \times Z$, and being a divisor, it must therefore be of the form

$$\tilde{D} = \sum_{j=1}^{g} c_j \cdot \{P_j\} \times Y \times Z$$

for certain integers $c_1, \dots, c_g \in \mathbb{N}$. But $\tilde{D}|_{X\times\{y\}\times\{z_0\}} = D$, and so $c_1 = \cdots = c_g = 0$, or equivalently, $\tilde{D} = p_1^*(D)$. This gives $L'_{(y,z)} \cong \mathscr{O}_X(D)$. If we now go back to (10.4), we find that

$$L_{(y,z)} \cong L'_{(y,z)} \otimes \mathscr{O}_X(-D) \cong \mathscr{O}_X,$$

and so L is indeed trivial on all subvarieties of the form $X \times \{y\} \times \{z\}$. As we said above, this is enough to conclude that L is trivial on $X \times Y \times Z$.

Line bundles on abelian varieties. The theorem of the cube has many nice consequences for line bundles on abelian varieties. Let X be an abelian variety, and let L be a line bundle on X. The group operation is $m \colon X \times X \to X$, and by extension, we also write

$$m \colon X \times X \times X \to X, \quad m(x, y, z) = x + y + z.$$

Denote by $p_{i,j} \colon X \times X \times X \to X \times X$ the projections, and set

$$m_{i,j} \colon X \times X \times X \to X, \quad m_{i,j} = m \circ p_{i,j}.$$

Consider the line bundle

$$M = m^*L \otimes m_{1,2}^*L^{-1} \otimes m_{1,3}^*L^{-1} \otimes m_{1,3}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L.$$

Because $0 \in X$ is the neutral element, it is easy to see that M is trivial on all three slices $\{0\} \times X \times X$, $X \times \{0\} \times X$, and $X \times X \times \{0\}$. By the theorem of the cube, M is trivial on $X \times X \times X$, and therefore

$$m^*L \cong m_{1,2}^*L \otimes m_{1,3}^*L \otimes m_{1,3}^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \otimes p_3^*L^{-1}. \tag{10.5}$$

If we now have three morphisms $f, g, h \colon T \to X$ from some other variety T, we can pull back this identity along the mapping $(f, g, h) \colon T \to X \times X \times X$; this proves the following result.

Corollary 10.6. *Let $f, g, h \colon T \to X$ be three morphisms to an abelian variety. For any line bundle L on X, one has*

$$(f + g + h)^*L \cong (f + g)^*L \otimes (f + h)^*L \otimes (g + h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

Lecture 11 (March 4)

We continue our study of line bundles on abelian varieties, based on the theorem of the cube. At the end of the previous class, we proved that if f, g, h are three morphisms from an arbitrary variety T to an abelian variety X, and if L is any line bundle on X, then

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}. \tag{11.1}$$

As a first application of this formula, we have the so-called "theorem of the square"; over the complex numbers, we already proved this back in Lecture 6.

Corollary 11.2. *Let L be a line bundle on an abelian variety, and $x, y \in X$ any two points. Then $t_{x+y}^*L \otimes L^{-1} \cong t_x^*L \otimes t_y^*L$.*

Proof. Let $f\colon \to X$ be the constant map $f \equiv x$, let $g\colon X \to X$ be the constant map $g \equiv y$, and let $h = \mathrm{id}$ be the identity. Then $f + h = t_x$, $g + h = t_y$, and $f + g + h = t_{x+y}$, and we get the desired isomorphism by applying (11.1). □

As in Lecture 6, the theorem of the square has the following interpretation. Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of line bundles on X; this is an abelian group under tensor product. Any line bundle L on X determines a function

$$\phi_L\colon X \to \mathrm{Pic}(X), \quad \phi_L(x) = t_x^*L \otimes L^{-1}.$$

The theorem of the square shows that $\phi_L(x+y) = \phi_L(x) \otimes \phi_L(y)$, and so ϕ_L is a group homomorphism. Moreover, any line bundle of the form $t_x^*L \otimes L^{-1}$ is translation-invariant, because

$$t_y^*\big(t_x^*L \otimes L^{-1}\big) = t_{x+y}^*L \otimes t_y^*L^{-1} \cong t_x^*L \otimes L^{-1}.$$

Later on, we are going to show that the set of translation-invariant line bundles is itself an abelian variety, denoted $\mathrm{Pic}^0(X)$, and that $\phi_L\colon X \to \mathrm{Pic}^0(X)$ is a morphism of abelian varieties.

Example 11.3. In terms of divisors, the theorem of the cube becomes a result about linear equivalence: for any divisor D, one has

$$t_{x+y}^*D + D \equiv t_x^*D + t_y^*D,$$

where $\equiv$ means linear equivalence. In particular, we always have

$$t_x^*D + t_{-x}^*D \equiv 2D,$$

just as in the complex case.

A second application concerns the homomorphisms

$$n_X\colon X \to X, \quad x \mapsto n \cdot x$$

and how they affect line bundles.

Corollary 11.4. *Let $n \in \mathbb{Z}$. For any line bundle L on X, one has*

$$n_X^*L \cong L^{n(n-1)/2} \otimes (-1)_X^*L^{n(n-1)/2}.$$

Proof. Take $f = (n+1)_X$, $g = 1_X$, and $h = (-1)_X$. Then $f + g + h = (n+1)_X$, $f + g = (n+2)_X$, $f + h = n_X$, and $g + h \equiv 0$, and so (11.1) gives

$$(n+1)_X^*L \cong (n+2)_X^*L \otimes n_X^*L \otimes (n+1)_X^*L^{-1} \otimes L^{-1} \otimes (-1)_X^*L^{-1}.$$

We can put this into the nicer-looking form

$$(n+2)_X^*L \otimes (n+1)_X^*L^{-2} \otimes n_X^*L \cong L \otimes (-1)_X^*L,$$

and then we recognize this as the "second difference" of the function $\mathbb{Z} \to \operatorname{Pic}(X)$, $n \mapsto n_X^* L$. Recall that if $f \colon \mathbb{Z} \to G$ is a function from the integers into an abelian group, f has degree ≤ 1 iff the first difference

$$f(n+1) - f(n) = a$$

is constant, equal to some $a \in G$; in that case, $f(n) = n \cdot a + b$, where $b = f(0)$. Similarly, f has degree ≤ 2 if the second difference

$$f(n+2) - 2f(n+1) + f(n) = a$$

is constant, and in that case, $f(n) = \binom{n}{2}a + \binom{n}{1}b + \binom{n}{0}c$ for some $b, c \in G$. Applied to our situation, this gives

$$n_X^* L \cong \bigl(L \otimes (-1)_X^* L\bigr)^{n(n-1)/2} \otimes M_1^n \otimes M_2,$$

for certain line bundles M_1, M_2, and by taking $n = 0$ and $n = 1$, one finds that $M_1 \cong L$ and $M_2 \cong \mathscr{O}_X$. Therefore

$$n_X^* L \cong \bigl(L \otimes (-1)_X^* L\bigr)^{n(n-1)/2} \otimes L^n,$$

which simplifies to the formula we wanted. □

Example 11.5. A line bundle L is called *symmetric* if $L \cong (-1)_X^* L$; this happens for example if $L = \mathscr{O}_X(D)$ for a divisor D that is invariant under the involution $x \mapsto -x$. When L is symmetric, one has

$$n_X^* L \cong L^{n^2}.$$

Similarly, L is called *anti-symmetric* if $L^{-1} \cong (-1)_X^* L$; in that case,

$$n_X^* L \cong L^n.$$

Those are the two extreme cases. Of course, for any L, the tensor product $L \otimes (-1)_X^* L$ will be symmetric, and $L^{-1} \otimes (-1)_X^* L$ will be anti-symmetric.

The homomorphism ϕ_L and ampleness. Let's look at the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

in more detail. In the complex case, we proved that if $L = L(H, \alpha)$, with H positive definite, then ϕ_L has a finite kernel, of order $(\dim H^0(X, L))^2$. In general, ϕ_L gives us a useful way for detecting whether or not L is ample.

Definition 11.6. For a line bundle L on an abelian variety X, we define

$$K(L) = \ker \phi_L = \bigl\{\, x \in X \bigm| t_x^* L \cong L \,\bigr\}$$

which is a subgroup of the abelian group X.

In fact, $K(L)$ is also closed in the Zariski topology. To see why, consider the line bundle $m^* L \otimes p_2^* L^{-1}$ on the product $X \times X$, where $m \colon X \times X \to X$ is the group operation. For $x \in X$, we have

$$m^* L \otimes p_2^* L^{-1}|_{\{x\} \times X} \cong t_x^* L \otimes L^{-1},$$

and so the subgroup $K(L)$ can also be written in the form

$$K(L) = \bigl\{\, x \in X \bigm| m^* L \otimes p_2^* L^{-1} \text{ is trivial on } \{x\} \times X \,\bigr\}.$$

By the seesaw theorem (in Theorem 9.10), this is a closed subset of X.

Now let D be an effective divisor on X, and consider the line bundle $L = \mathscr{O}_X(D)$; in other words, we are assuming that $H^0(X, L) \neq 0$.

Theorem 11.7. *The following four conditions are equivalent:*

(a) L is ample.
(b) $K(L)$ is a finite group.

(c) *The group* $H = \{ x \in X \mid t_x^* D = D \}$ *is finite.*
(d) *The linear system* $|2D|$ *has no base points, and defines a* finite *morphism to projective space.*

Note that in (c), $t_x^* D = D$ means equality as divisors, so every irreducible component of D needs to be invariant under translation by x. The most interesting implication is that finiteness of $K(L)$ implies ampleness of L; but also note that (d) is very similar to the Lefschetz theorem (in Theorem 6.5).

Proof. Clearly $H \subseteq K(L)$, and so (b) trivially implies (c). It is also not hard to see that (d) implies (a). Indeed, the morphism $\phi_{|2D|} \colon X \to \mathbb{P}^N$ has the property that $\phi_{|2D|}^* \mathscr{O}_{\mathbb{P}^N}(1) \cong L^2$. Now the pullback of an ample line bundle by a finite morphism remains ample, and so L must be ample. (This fact is a substitute for the complex-analytic description of ampleness in terms of positive metrics.)

Let's show that (a) implies (b). We know that $K(L)$ is a closed subgroup, and so the connected component containing the point $0 \in K(L)$ is an abelian variety $Y \subseteq X$. To prove that $K(L)$ is finite, we need to show that $\dim Y = 0$. By construction, we have $t_y^* L \cong L$ for every $y \in Y$. Now consider the restriction $L_Y = L|_Y$. This is an ample line bundle on the abelian variety Y, with the property that $t_y^* L_Y \cong L_Y$ for every $y \in Y$. By the seesaw theorem (applied to the line bundle $m^* L_Y \otimes p_2^* L_Y^{-1}$ on $Y \times Y$), it follows that $m^* L_Y \otimes p_1^* L_Y^{-1} \otimes p_2^* L_Y^{-1}$ is trivial, and hence that

$$m^* L_Y \cong p_1^* L_Y \otimes p_2^* L_Y.$$

If we now pull back this identity along the mapping $Y \to Y \times Y$, $y \mapsto (y, -y)$, we get

$$\mathscr{O}_Y \cong L_Y \otimes (-1)_Y^* L_Y.$$

But both line bundles on the right-hand side are ample, and an ample line bundle on a complete variety Y can only be trivial if $\dim Y = 0$. Therefore $K(L)$ must be finite.

The most interesting implication is from (c) to (d). We already know that $|2D|$ has no base points: the reason is that

$$t_x^* D + t_{-x}^* D \equiv 2D,$$

and so for any $y \in X$, we only need to choose $x \in X$ such that $y \pm x \notin \operatorname{Supp} D$ to get a divisor linearly equivalent to $2D$ that does not pass through the point y. So we always have a morphism

$$\phi = \phi_{|2D|} \colon X \to \mathbb{P}^N,$$

where $\mathbb{P}^N$ is really the projectivization of the k-vector space $H^(X, L^2)$. We need to show that ϕ is a finite morphism. Because X is proper, ϕ is proper, and so it suffices to prove that ϕ has finite fibers. Let's argue by contradiction and assume that ϕ does not have finite fibers. Then there is an irreducible proper curve $C \subseteq X$ such that $\phi(C)$ is a point. Because the divisors in $|2D|$ correspond to hyperplanes in $\mathbb{P}^N$, and because a hyperplane either passes through a given point or is disjoint from it, we find that every divisor in $|2D|$ either contains the curve C, or is disjoint from it. In particular, for every $x \in X$, the divisor $t_x^* D + t_{-x}^* D$ either contains C, or is disjoint from C. Because C cannot be contained in all translates of D for obvious reasons, we can certainly find a point $x \in X$ such that C is disjoint from the divisor $t_x^* D$.

Now write $t_x^* D = m_1 D_1 + \cdots + m_k D_k$ as a sum of irreducible divisors. The lemma below implies that each D_j is invariant under all translations of the form $t_{x_2 - x_1}$ with $x_1, x_2 \in C$. But this clearly contradicts the finiteness of H, and so the morphism ϕ_L must have been finite after all. $\square$

Lemma 11.8. *Let E be an irreducible divisor on an abelian variety. If there is an irreducible curve C such that $E \cap C = \emptyset$, then $t^*_{x_1 - x_2} E = E$ for all $x_1, x_2 \in C$.*

Proof. Consider the line bundle $L = \mathscr{O}_X(E)$. Because C is disjoint from E, the restriction $L|_C$ is trivial, and therefore has degree 0. Because the degree is constant in families, the restriction of $t^*_x L$ to C will have degree 0 for every $x \in X$. (To prove this rigorously, we can pull back to the normalization and use the Riemann-Roch theorem to express the degree in terms of the Euler characteristic; we know from Corollary 9.8 that the Euler characteristic is constant in families.) This implies that if the curve $t_x(C)$ intersects E, then it must be contained in E (because a line bundle of degree 0 with a nontrivial section is trivial).

Now let $x_1, x_2 \in C$ and $y \in E$. Then the curve $t_{y-x_2}(C)$ intersects E in the point y, and so $t_{y-x_2}(C) \subseteq E$; therefore $y + x_1 - x_2 \in E$ for every $y \in E$, which says exactly that $t^*_{x_1 - x_2} E = E$. □

The theorem shows that on abelian varieties, ampleness of a line bundle can be detected on curves. A very neat corollary of the theorem is that abelian varieties are always projective.

Corollary 11.9. *Every abelian variety is projective.*

Proof. Let $U \subseteq X$ be an affine open set containing the point $0 \in X$. Because X is complete and nonsingular, the complement $X \setminus U$ is a union of irreducible divisors $D_1, \dots, D_r$ (because regular functions on nonsingular varieties extend over subvarieties of codimension ≥ 2). Set $D = D_1 + \dots + D_r$. The subgroup

$$H = \left\{ x \in X \;\middle|\; t^*_x D = D \right\}$$

is closed in X, and translation by any $x \in H$ preserves $U = X \setminus D$. Because $0 \in U$, this shows that $H \subseteq U$. But now H is complete and U is affine, and so H must be finite. Theorem 11.7 implies that $\mathscr{O}_X(D)$ is ample. □

Torsion points. As in the complex case, we can also prove that X is always a divisible group.

Corollary 11.10. *The group X is divisible, and $X_n = \left\{ x \in X \;\middle|\; n \cdot x = 0 \right\}$ is finite.*

Proof. For divisibility, we only need to prove that the homomorphism $n_X \colon X \to X$ is surjective for every $n \neq 0$. For dimension reasons, it is enough to prove that $\ker n_X$ is finite. Let L be an ample line bundle (which exists because X is projective). Then

$$n^*_X L \cong L^{n(n+1)/2} \otimes (-1)^*_X L^{n(n-1)/2},$$

and the line bundle on the right-hand side is again ample. Since an ample line bundle cannot be trivial on a complete variety of positive dimension, we find that $\ker(n_X)$ must be 0-dimensional, and therefore finite. □

In the complex case, the fact that $X \cong (\mathbb{R}/\mathbb{Z})^{2g}$ made it easy to compute the kernel of n_X. We can prove somewhat similar results in general, except when the characteristic $p = \operatorname{char}(k)$ divides n.

Proposition 11.11. *Let $n \in \mathbb{Z}$ be an integer.*

(a) The degree of n_X is equal to n^{2g}, where $g = \dim X$.
(b) n_X is separable iff $p \nmid n$.
(c) If $p \nmid n$, then $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.
(d) There is an integer $r \in \{0, 1, \dots, g\}$ such that $X_{p^e} \cong (\mathbb{Z}/p^e\mathbb{Z})^r$.

Suppose that $f: X \to Y$ is a surjective morphism between two n-dimensional varieties. The extension of function fields

$$k(Y) \subseteq k(X)$$

is finite algebraic, and we define $\deg f = \big(k(X) : k(Y)\big)$. When f is separable, meaning when the field extension is separable, the number of points in the fiber $f^{-1}(y)$ is equal to $\deg f$ for most $y \in Y$. (More precisely, there is a nonempty Zariski-open subset of Y where this is true.) When f is inseparable, we define the *separable degree* of f as the separable degree of the field extension $k(Y) \subseteq k(X)$; then the number of points in the general fiber is equal to the separable degree.

Recall from the complex case that an *isogeny* $f: X \to Y$ is a surjective homomorphism between two abelian varieties whose kernel is finite. The typical examples are the homomorphisms $n_X : X \to X$ with $n \neq 0$. In the case of an isogeny, all fibers have the same number of points; therefore the number of points in $X_n = \ker(n_X)$ is equal to the separable degree of n_X. We'll compute this degree next time.

Lecture 12 (March 6)

Let's first restate the result from last time. We were looking at the homomorphism $n_X : X \to X$, $x \mapsto nx$, and its kernel

$$X_n = \{\, x \in X \mid nx = 0 \,\},$$

which is the subgroup of n-torsion points on X.

Proposition 12.1. *Set $g = \dim X$ and $p = \operatorname{char}(k)$.*

- *(a) We have $\deg n_X = n^{2g}$.*
- *(b) If $p \nmid n$, then n_X is separable and $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.*
- *(c) If $p \mid n$, then n_X is not separable and there is an integer $r \in \{0, 1, \dots, g\}$ such that $X_{p^e} \cong (\mathbb{Z}/p^e\mathbb{Z})^r$.*

Recall that a homomorphism $f: X \to Y$ between abelian varieties is called an *isogeny* if it is surjective with finite kernel (and therefore $\dim X = \dim Y$). We define the degree $\deg f$ as the degree of the field extension $f^* : k(Y) \to k(X)$. We say that f is a *separable isogeny* if the field extension is separable; this is always the case in characteristic zero, or when $\deg f$ is not a multiple of p. In that case, the number of elements in the subgroup $\ker f$ is equal to $\deg f$. If the field extension is not separable, we can let $L \subseteq k(X)$ be the subfield of all elements that are separable over $k(Y)$; the field extension $L \subseteq k(X)$ is purely inseparable. In general, the number of elements in $\ker f$ is only equal to the *separable degree* $\deg_s(f) = \big(L : k(Y)\big)$. Lastly, we need a basic fact from intersection theory: if $D_1, \dots, D_g$ are Cartier divisors on Y, then their pullbacks $f^*D_1, \dots, f^*D_g$ are Cartier divisors on X, and we have the equality of intersection numbers

$$(f^*D_1 \cdots f^*D_g)_X = \deg f \cdot (D_1 \cdots D_g)_Y.$$

Proof of the proposition. For (a), we pick an ample and symmetric divisor D; this means that $(-1)_X^* D \equiv D$. We showed last time that $n_X^* D \equiv n^2 D$. Now the formula from above gives

$$\deg n_X (D \cdots D)_X = \big(n_X^* D \cdots n_X^* D\big)_X = n^{2g} (D \cdots D)_X,$$

and so $\deg n_X = n^{2g}$. This part is the same as in the complex case. For (b), suppose that $p \nmid n$. The degree of n_X is then not divisible by p, and so n_X is separable, and the number of elements in $X_n = \ker(n_X)$ is therefore n^{2g}. From this, we see that X_n is a finite abelian group; the order of every element divides n; and for every divisor $m \mid n$, the number of elements whose order divides m is exactly m^{2g}. Looking at the classification of finite abelian groups, this is only possible if $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

For (c), let's now assume that $p \mid n$. Let $T_{X,0}$ be the tangent space at the zero element, and Ω_0 the dual k-vector space. We showed in Lecture 8 that the differential

$$dn_X : T_{X,0} \to T_{X,0}$$

is multiplication by n, hence trivial if $p \mid n$. Because $\Omega^1_{X/k} \cong \Omega_0 \otimes_k \mathscr{O}_X$, it follows that $n_X^* : \Omega^1_{X/k} \to \Omega^1_{X/k}$ is trivial (as a morphism of sheaves). So if $f \in k(X)$ is a rational function, then f is regular on some open subset U, and so $df \in H^0(U, \Omega^1_{X/k})$. But then

$$0 = n_X^*(df) = d\big(n_X^* f\big),$$

and because we are in characteristic p (and k is algebraically closed), we must have $n_X^* f = g^p$ for some other rational function $g \in k(X)$. Therefore the field extension

$$n_X^* : k(X) \to k(X)$$

actually factors through the subfield $k(X)^p$, and so it is not separable. This means that X_n has fewer than n^{2g} elements.

Now consider $p_X : X \to X$. We sort of convinced ourselves in class that the (purely inseparable) field extension $k(X)^p \subseteq k(X)$ has degree at least p^g, because the transcendence degree of $k(X)$ is equal to $\dim X = g$. This means that the separable degree of $p_X^* : k(X) \to k(X)$ must be equal to p^r for some $0 \leq r \leq g$. Therefore X_p is a finite abelian group with p^r elements in which every element has order p; clearly $X_p \cong (\mathbb{Z}/p\mathbb{Z})^r$. Because X_n is divisible, it is easy to deduce by induction on $e \geq 1$ that $X_{p^e} \cong (\mathbb{Z}/p^e\mathbb{Z})^r$. □

Example 12.2. Elliptic curves over a field of characteristic p are a good example. By the general result above, the group X_p is either $\mathbb{Z}/p\mathbb{Z}$ or trivial. In the case when X_p is trivial, the elliptic curve is called *supersingular*.

We can always realize an elliptic curve as a nonsingular cubic curve in $\mathbb{P}^2$, defined by a cubic polynomial $f(x, y, z)$. If $p \neq 2, 3$, so that we can complete the square and the cube, we can put this polynomial into Weierstrass form

$$y^2 z = x^3 + axz^2 + bz^3,$$

for constants $a, b \in k$; or into Legendre form

$$y^2 z = x(x - z)(x - \lambda z)$$

for a constant $\lambda \in k$. (In both cases, the polynomial on the right-hand side must not have any repeated roots; so for example $\lambda \neq 0, 1$.) Two such cubic curves are isomorphic (as abstract curves), if and only if there is an automorphism of $\mathbb{P}^2$ that takes one to the other, if and only if they have the same j-*invariant*; this is

$$j(A, B) = 1728 \frac{4A^3}{4A^3 + 27B^3}$$

for curves in Weierstrass form, and

$$j(\lambda) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}$$

for curves in Legendre form.

One can show that a nonsingular cubic curve is supersingular iff the coefficient of $(xyz)^{p-1}$ in the polynomial $f(x, y, z)^{p-1}$ vanishes. That allows us to give some concrete examples. (Note that actually computing the subgroup of p-torsion points by hand is very difficult: the geometric description of the group law is simple, but the formulas are not so simple.) For instance, consider the curve

$$y^2 = x^3 + 1.$$

Here $(y^2z - x^3 - z^3)^{p-1}$ only contains terms of the form

$$(y^2z)^a(x^3)^b(z^3)^c$$

with $a+b+c = p-1$. To get $(xyz)^{p-1}$, we need $p-1 = 2a$ and $a = 3b$, so $p = 6b+1$. (And in that case, the coefficient is the product of two factorials that are not divisible by p.) So this curve is supersingular exactly when $p \equiv 1 \mod 6$.

How common are supersingular curves? Since the number of elements in X_p is equal to the separable degree, X is supersingular exactly when the field extension $p_X^* \colon k(X) \to k(X)$ is purely inseparable. Assume again that X is defined by a cubic polynomial $f(x,y,z)$. Define a new cubic polynomial $f_p(x,y,z)$ by the rule

$$f(x,y,z)^p = f_p(x^p, y^p, z^p);$$

in other words, all the coefficients of f get raised to the p-th power. We then have the Frobenius morphism

$$F \colon V(f) \to V(f_p), \quad F(x,y,z) = (x^p, y^p, z^p),$$

which is purely inseparable of degree p. By general theory, p_X purely inseparable of degree p^2 implies that $p_X = F^2$. In particular, the cubic curve defined by $f(x,y,z)$ must be isomorphic to the cubic curve defined by $f_{p^2}(x,y,z)$. For curves in Legendre form, for example, this means that

$$j(\lambda) = j(\lambda^{p^2}) = \big(j(\lambda)\big)^{p^2},$$

which is saying that $j(\lambda)$ lies in the subfield with p^2 elements. (Remember that k is algebraically closed.) This shows that there are rather few supersingular curves.

Quotients by finite groups. Our next goal is to construct $\operatorname{Pic}^0(X)$ as an abelian variety. The general idea is that $\phi_L \colon X \to \operatorname{Pic}^0(X)$ is surjective when L is ample, and so $\operatorname{Pic}^0(X)$ should be the quotient of X by the finite subgroup $K(L)$. Before we can do that, we have to review very quickly a few results about such quotients.

Let X be a variety, and let G be a finite group of automorphisms of X. The main technical assumption is that for all points $x \in X$, the orbit $Gx = \{\, gx \mid g \in G \,\}$ should be contained in some affine open subset of X. This is true for example when X is quasi-projective: take a projective completion, and remove a hyperplane section not containing any point of Gx.

Theorem 12.3. *Under these assumptions, there is a morphism $\pi \colon X \to Y$ to a variety Y, such that $Y = X/G$ as topological spaces, and such that the morphism $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$ induces an isomorphism between $\mathscr{O}_Y$ and the subsheaf $(\pi_* \mathscr{O}_X)^G$ of G-invariant functions. The morphism π is finite, surjective, and separable; if G acts freely, then π is étale.*

We denote Y by the symbol X/G and call it the quotient of X by G. It has the following universal property: if $f \colon X \to Z$ is any morphis such that $f \circ g = f$ for all $g \in G$, then f factors uniquely through a morphism $h \colon Y \to Z$. The construction of the quotient is straightforward. The statement about orbits implies that we can cover X by affine open subsets that are invariant under the G-action. If $U = \operatorname{Spec} A$ is such an affine open, we define the quotient as the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$, where $A^G \subseteq A$ is the subring of G-invariant functions. One shows that this has the universal property; for that reason, the individual quotients U/G then glue together into a variety Y with the desired properties.

We can also describe coherent sheaves on $Y = X/G$. Suppose that $\mathscr{F}$ is a coherent $\mathscr{O}_Y$-module. The pullback $\pi \mathscr{F}$ is a coherent $\mathscr{O}_X$-module, and for every $g \in G$, we have $\pi \circ g = \pi$, and therefore $g^* \pi^* \mathscr{F} \cong \pi^* \mathscr{F}$. We say that a coherent $\mathscr{O}_X$-module $\mathscr{G}$ is *G-equivariant* if we have a collection of isomorphisms

$$\phi_g \colon g^* \mathscr{G} \to \mathscr{G}$$

that are compatible with composition, in the sense that the diagram

$$\begin{tikzcd} h^*g^*\mathscr{G} \arrow[r, "h^*\phi_g"] \arrow[d, equal] & h^*\mathscr{G} \arrow[d, "\phi_h"] \\ (gh)^*\mathscr{G} \arrow[r, "\phi_{gh}"] & \mathscr{G} \end{tikzcd}$$

is commutative. In that case, G acts on the direct image sheaf $\pi_*\mathscr{G}$, and the subsheaf $(\pi_*\mathscr{G})^G$ of G-invariants is a coherent $\mathscr{O}_Y$-module.

Proposition 12.4. *Suppose that G acts freely on X. The functors $\mathscr{F} \mapsto \pi^*\mathscr{F}$ and $\mathscr{G} \mapsto (\pi_*\mathscr{G})^G$ define an equivalence between the category of coherent $\mathscr{O}_Y$-modules and the category of G-equivariant coherent $\mathscr{O}_X$-modules.*

For the study of abelian varieties, line bundles are of particular interest. When the group G is abelian, these are closely related to characters. For a finite abelian group G, we are going to write

$$\hat{G} = \operatorname{Hom}(G, k^\times)$$

for the group of characters of G with values in the field k. Suppose that X is complete and that G acts freely on X. Let L be a line bundle on Y whose pullback π^*L is trivial. We get a G-equivariant structure on $\mathscr{O}_X$, namely a collection of isomorphisms $\phi_g \colon \mathscr{O}_X \to \mathscr{O}_X$, such that $\phi_{gh} = \phi_h \circ h^*\phi_g$. Because X is complete, each ϕ_g is multiplication by a nonzero constant $\alpha(g) \in k^\times$, and the compatibility condition means exactly that $\alpha \colon G \to k^\times$ is a character. Conversely, given such a character, we can recover the line bundle L as the subsheaf of G-invariants in $\pi_*\mathscr{O}_X$ (with the G-action depending on the character, of course); concretely,

$$L \cong \bigl\{ f \in \pi_*\mathscr{O}_X \bigm| g(f) = \alpha(g) \cdot f \text{ for all } g \in G \bigr\}.$$

These considerations prove the following proposition.

Proposition 12.5. *Suppose that G acts freely on a complete variety X. For every character $\alpha \in \hat{G}$, consider the subsheaf*

$$L_\alpha = \bigl\{ f \in \pi_*\mathscr{O}_X \bigm| g(f) = \alpha(g) \cdot f \textit{ for all } g \in G \bigr\}.$$

Then L_α is a line bundle on X/G, and we have $L_\alpha \otimes L_\beta \cong L_{\alpha+\beta}$. Moreover,

$$\hat{G} \cong \ker\bigl(\pi^* \colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)\bigr)$$

are isomorphic groups.

Specializing further, suppose that G is a finite abelian group, whose order is not divisible by the characteristic $p = \operatorname{char}(k)$. In that case, every finite-dimensional representation of G on a k-vector space is a direct sum of characters. Indeed, every finite-dimensional representation is completely reducible, because for any given G-invariant subspace, we can write down a G-invariant complement (by an explicit formula whose denominator $|G|$ is invertible in the field k). Furthermore, every irreducible representation is 1-dimensional (because G is abelian), hence is given by a character. For exactly the same reason, the G-action on $\pi_*\mathscr{O}_X$ decomposes into a direct sum of line bundles, and so we get a decomposition

$$\pi_*\mathscr{O}_X \cong \bigoplus_{\alpha \in \hat{G}} L_\alpha.$$

Recall here that $\hat{G}$ and G have the same number of elements; because $\pi \colon X \to Y$ is separable, this number is just the degree of π. Because of the projection formula

$$\pi_*\pi^*\mathscr{F} \cong \mathscr{F} \otimes_{\mathscr{O}_Y} \pi_*\mathscr{O}_X,$$

it then follows that $\mathscr{F}$ is isomorphic to a direct summand in $\pi_*\pi^*\mathscr{F}$.

We can apply the results above to the case of abelian varieties.

Corollary 12.6. *Let X be an abelian variety. There is a one-to-one correspondence between finite subgroups $K \subseteq X$ and (isomorphism classes of) separable isogenies $f\colon X \to Y$. The correspondence sends $f\colon X \to Y$ to the finite subgroup* $\ker f$*; and it sends K to the quotient $\pi\colon X \to Y$.*

Here two isogenies $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ are isomorphic if there is an isomorphism $g\colon Y_1 \to Y_2$ such that $g \circ f_1 = f_2$.

Proof. A finite subgroup $K \subseteq X$ acts freely on X by translations, and so the quotient X/K is a nonsingular complete variety, and $\pi\colon X \to X/K$ is finite, surjective, and separable. Because K is a subgroup, X/K has the structure of a group. It is in fact an abelian variety. Indeed, the product $(X/K) \times (X/K)$ is isomorphic to $(X \times X)/(K \times K)$, and by the universal property of quotients, the group action $m\colon X \times X \to X$ descends to $n\colon (X/K) \times (X/K) \to X/K$:

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "\pi\times\pi"] & X \arrow[d, "\pi"] \\ (X/K) \times (X/K) \arrow[r, "n"] & X/K \end{tikzcd}$$

It follows that $\pi\colon X \to X/K$ is a separable isogeny, and clearly $K = \ker \pi$.

Conversely, given a separable isogeny $f\colon X \to Y$, we let $K = \ker f$, and define $\pi\colon X \to X/K$ as the quotient. By the universal property of quotients, we get the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ X/K \arrow[ur, bend right, "g"'] & \end{tikzcd}$$

Both X/K and Y are nonsingular, and g is finite and bijective, and therefore an isomorphism. This proves that the two operations are inverse to each other. $\square$

This result also shows that there is a sort of duality between X and line bundles on X, in the following sense. Consider a separable isogeny $f\colon X \to Y$, of degree prime to $p = \operatorname{char}(k)$. By the corollary, we have $Y \cong X/K$, where $K = \ker f$. Now Proposition 12.5 shows that

$$\hat{K} = \operatorname{Hom}(K, k^\times) \cong \ker\bigl(f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)\bigr).$$

So the kernel of $f\colon X \to Y$ and the kernel of $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ have the same number of elements, and in fact, are "dual" to each other in the sense that one group is the group of characters on the other group.

Lecture 13 (March 11)

Translation-invariant line bundles. Let X be an abelian variety. Over the complex numbers, $\operatorname{Pic}^0(X)$ is the space of holomorphic line bundles with trivial first Chern class; this is again an abelian variety of the same dimension. Our goal today is to construct this abelian variety over any field of characteristic zero. We showed in Lecture 6 that all line bundles in $\operatorname{Pic}^0(X)$ are translation-invariant, in the sense that $t_x^* L \cong L$ for every $x \in X$. We use this property as the definition over other fields (where we don't have a good theory of first Chern classes in cohomology).

Definition 13.1. If X is an abelian variety, we define

$$\operatorname{Pic}^0(X) = \bigl\{\, L \in \operatorname{Pic}(X) \bigm| t_x^* L \cong L \text{ for all } x \in X \,\bigr\},$$

the group of (isomorphism classes of) translation-invariant line bundles.

In terms of the group homomorphism

$$\phi_L \colon X \to \operatorname{Pic}(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

the subgroup $\operatorname{Pic}^0(X) \subseteq \operatorname{Pic}(X)$ consists of all those line bundles for which $\phi_L \equiv 0$. By the theorem of the square, we have

$$t_y^* \phi_L(x) = t_{x+y}^* L \otimes t_y^* L^{-1} \cong t_x^* L \otimes L^{-1}$$

and so $\phi_L(x) \in \operatorname{Pic}^0(X)$. Therefore

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

takes values in the subgroup $\operatorname{Pic}^0(X)$. We are going to construct an abelian variety $\hat{X}$ that is isomorphic to $\operatorname{Pic}^0(X)$ as a group (in a functorial way).

We begin with series of observations about translation-invariant line bundles.

Observation 1. We have $L \in \operatorname{Pic}^0(X)$ iff $m^* L \cong p_1^* L \otimes p_2^* L$ on $X \times X$. This is a consequence of the seesaw theorem. Indeed, the restriction of the line bundle $m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ to the slice $X \times \{x\}$ is isomorphic to $t_x^* L \otimes L^{-1}$, and therefore trivial when $L \in \operatorname{Pic}^0(X)$. Because the line bundle is also trivial on $\{0\} \times X$, it must be trivial on $X \times X$ by Theorem 9.10.

Observation 2. If $f, g \colon S \to X$ are two morphisms from a variety (or scheme) S, then $(f+g)^* L \cong f^* L \otimes g^* L$. This follows from Observation 1 by pulling back along the morphism $(f, g) \colon S \to X \times X$.

Observation 3. Let $n_X \colon X \to X$ be the morphism $n_X(X) = n \cdot x$. By induction, the previous observation implies that $n_X^* L \cong L^n$. In particular, $(-1)_X^* L \cong L^{-1}$, and so L is anti-symmetric.

Observation 4. For every $L \in \operatorname{Pic}(X)$, we have $n_X^* L \otimes L^{-n^2} \in \operatorname{Pic}^0(X)$. By rewriting the identity in Corollary 11.4, we get

$$n_X^* L \otimes L^{-n^2} \cong \bigl(L \otimes (-1)_X^* L^{-1}\bigr)^{(n-n^2)/2},$$

and so it is enough to prove that $L \otimes (-1)_X^* L^{-1} \in \operatorname{Pic}^0(X)$. We compute

$$\begin{aligned} t_y^* \bigl(L \otimes (-1)_X^* L^{-1}\bigr) &\cong t_y^* L \otimes (-1)_X^* t_{-y}^* L^{-1} \\ &\cong t_y^* L \otimes (-1)_X^* \bigl(t_{-y}^* L^{-1} \otimes L\bigr) \otimes (-1)_X^* L^{-1} \\ &\cong t_y^* L \otimes \bigl(t_{-y}^* L \otimes L^{-1}\bigr) \otimes (-1)_X^* L^{-1} \\ &\cong L^2 \otimes L^{-1} \otimes (-1)_X^* L^{-1} \cong L \otimes (-1)_X^* L^{-1}, \end{aligned}$$

where we used the fact that $t_{-y}^* L \otimes L^{-1} \in \operatorname{Pic}^0(X)$ (and Observation 2) to go from the second to the third line; and the identity $t_y^* L \otimes t_{-y}^* L \cong L^2$ from the theorem of the square to go from the third to the fourth line.

Observation 5. If $L \in \operatorname{Pic}(X)$ has finite order, then $L \in \operatorname{Pic}^0(X)$. Indeed, if L^n is trivial for some $n \geq 1$, then one has

$$0 = \phi_{L^n}(x) = n\phi_L(x) = \phi_L(nx)$$

for every $x \in X$, and because X is divisible, this implies that $\phi_L \equiv 0$ and hence that $L \in \operatorname{Pic}^0(X)$.

Observation 6. Let S be a variety, and let L be a line bundle on $X \times S$; as usual, we think of this as a family of line bundles $L_s = L|_{X \times \{s\}}$ on X, parametrized by the variety S. Then for any two points $s_0, s_1 \in S$, one has $L_{s_1} \otimes L_{s_0}^{-1} \in \operatorname{Pic}^0(X)$. What this means is that the connected components of $\operatorname{Pic}(X)$ are copies of $\operatorname{Pic}^0(X)$, in the sense that an irreducible (hence connected) family of line bundles can only change L_{s_0} by line bundles in $\operatorname{Pic}^0(X)$.

Proof. After replacing L by $L \otimes p_1^* L_{s_0}^{-1}$, we may assume that L_{s_0} is trivial; then the claim is that $L_s \in \operatorname{Pic}^0(X)$ for all $s \in S$. The restriction of L to $\{0\} \times S$ is a line bundle on S, hence locally trivial; after replacing S by an open subset, we may therefore assume in addition that $L|_{\{0\}\times S}$ is trivial. In order to show that $L_s \in \operatorname{Pic}^0(X)$, it is enough to prove that $m^* L_s \otimes p_1^* L_s^{-1} \otimes p_2^* L_s^{-1}$ is trivial. To do that, we go to the product $X \times X \times S$, and consider the line bundle

$$M = \mu^* L \otimes p_{12}^* L^{-1} \otimes p_{13}^* L^{-1},$$

where $\mu \colon X \times X \times S \to X \times S$ is the morphism $\mu(x, y, s) = (x + y, s)$. The assumptions on L imply that M is trivial on $X \times X \times \{s_0\}$, on $\{0\} \times X \times S$, and on $X \times \{0\} \times S$. The theorem of the cube implies that M is trivial, and this gives the result we want after restricting to $X \times X \times \{s\}$. □

Observation 7. If $L \in \operatorname{Pic}^0(X)$ is nontrivial, then $H^i(X, L) = 0$ for every $i \in \mathbb{Z}$.

Proof. We prove this by induction on $i \geq 0$. Suppose that $s \in H^0(X, L)$ is a nontrivial global section. Then $(-1)_X^* s$ is a nontrivial global section of $(-1)_X^* L \cong L^{-1}$, and so $s \otimes (-1)_X^* s$ is a nontrivial global section of $L \otimes L^{-1} \cong \mathscr{O}_X$, hence a nonzero constant (because X is complete). But then the original section s cannot vanish anywhere, and so L is trivial, contrary to our initial assumption.

For $i > 0$, consider the composition

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rr, bend left, "\mathrm{id}"] & X \times X \arrow[r, "m"] & X \end{tikzcd}$$

where $j(x) = (x, 0)$ and $m(x, y) = x + y$. It gives us a factorization

$$\begin{tikzcd} H^i(X, L) \arrow[r, "m^*"] \arrow[rr, bend left, "\mathrm{id}"] & H^i(X \times X, m^* L) \arrow[r, "j^*"] & H^i(X, L). \end{tikzcd}$$

From Observation 1, we know that $m^* L \cong p_1^* L \otimes p_2^* L$, and so

$$H^i(X \times X, m^* L) \cong \bigoplus_{p+q=i} H^p(X, L) \otimes H^q(X, L)$$

by the Künneth formula. But now all summands are trivial (by induction), and so $H^i(X \times X, m^* L) = 0$; the above factorization then gives $H^i(X, L) = 0$ as well. □

Observation 8. If L is an ample line bundle, the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is surjective. This is the key result for describing $\operatorname{Pic}^0(X)$.

Proof. Fix a translation-invariant line bundle $M \in \operatorname{Pic}^0(X)$. We need to find a point $x \in X$ such that $M \cong t_x^* L \otimes L^{-1}$. Suppose to the contrary that no such point exists. We'll derive a contradiction by looking at the line bundle

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* (L^{-1} \otimes M^{-1})$$

on the product $X \times X$. We have

$$K|_{\{x\}\times X} \cong t_x^* L \otimes L^{-1} \otimes M^{-1},$$

and because $t_x^* L \otimes L^{-1}$ is not isomorphic to M, this line bundle is nontrivial, and therefore has no cohomology (by the previous observation). According to Corollary 9.9, applied to the first projection $p_1 \colon X \times X \to X$, it follows that

$R^i(p_1)_*K = 0$ for every $i \in \mathbb{Z}$. By the Leray spectral sequence (or an exercise in Hartshorne), we now get

$$H^i(X \times X, K) = 0$$

for all $i \in \mathbb{Z}$.

Now let's consider the second projection $p_2 \colon X \times X \to X$. Here we have

$$K|_{X \times \{x\}} \cong t_x^* L \otimes L^{-1},$$

which is trivial exactly when x belongs to the subgroup $K(L) = \ker \phi_L$. Since L is ample, $K(L)$ is a finite group by Theorem 11.7. Therefore $K|_{X \times \{x\}}$ has no cohomology except when $x \in K(L)$. Another application of base change shows that the support of the coherent sheaves $R^q(p_2)_*K$ is contained in $K(L)$, and so $H^p(X, R^q(p_2)_*K) = 0$ for $p \geq 1$ for dimension reasons. The Leray spectral sequence therefore degenerates and gives us isomorphisms

$$0 = H^i(X \times X, K) \cong H^0\big(X, R^i(p_2)_*K\big).$$

It follows that $R^i(p_2)_*K = 0$, and hence (by Corollary 9.9) that $K_{X \times \{x\}}$ has no cohomology for every $x \in X$. But this is absurd because this bundle is isomorphic to $\mathscr{O}_X$ when $x = 0$, and $H^0(X, \mathscr{O}_X) = k$. □

If we take L to be an ample line bundle – which exists because X is projective (by Corollary 11.9) – then the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

is surjective, and its kernel is the finite subgroup $K(L)$. As a group, $\operatorname{Pic}^0(X)$ is therefore isomorphic to the quotient $X/K(L)$.

Example 13.2. Suppose that $\dim X = 1$, so that X is an elliptic curve, with zero element $x_0 \in X$. The line bundle $L = \mathscr{O}_X(x_0)$ is ample, and the homomorphism

$$\phi_L \colon X \to \operatorname{Pic}^0(X)$$

takes a point $x \in X$ to the line bundle $\mathscr{O}_X(x - x_0)$ corresponding to the divisor $x - x_0$; it is well-known that this is an isomorphism.

Construction of the dual abelian variety. According to the results from last time, the quotient $\hat{X} = X/K(L)$ is actually an abelian variety. So we get an isomorphism of groups $\hat{X} \cong \operatorname{Pic}^0(X)$. The abelian variety $\hat{X}$ should therefore be a "moduli space" for translation-invariant line bundles on X. What extra structure do we need to make that statement precise?

(A) We need a "universal" line bundle P on the product $X \times \hat{X}$. For every point $\alpha \in \hat{X}$, we want the line bundle

$$P_\alpha = P|_{X \times \{\alpha\}}$$

to represent the element of $\operatorname{Pic}^0(X)$ corresponding to α under the isomorphism $\hat{X} \cong \operatorname{Pic}^0(X)$. If we impose the additional condition that $P|_{\{0\} \times X}$ is trivial, then P is determined up to isomorphism (by the seesaw theorem). This line bundle is called the *Poincaré bundle.*

(B) All families of line bundles in $\operatorname{Pic}^0(X)$ should come from P, in the following sense. Suppose that S is a normal variety (for technical reasons), and that K is a line bundle on $X \times S$ such that

$$K_s = K|_{X \times \{s\}} \in \operatorname{Pic}^0(X)$$

for every $s \in S$, and such that $K|_{\{0\} \times X}$ is trivial. We then get a function

$$f \colon S \to \hat{X}$$

by sending a point $s \in S$ to the unique point $f(s) \in \hat{X}$ such that $K_s \cong P_{f(s)}$. (There is a unique point because $\hat{X} \cong \mathrm{Pic}^0(X)$ as groups.) Then we want the function f to be a morphism of varieties, and $K \cong (\mathrm{id} \times f)^* P$.

The two conditions actually determine the pair $(\hat{X}, P)$ up to isomorphism. The reason is that if we have another pair (Y, Q) with the same properties, then (B), applied to the line bundle Q on $X \times Y$, gives us a unique morphism

$$f \colon Y \to \hat{X}$$

such that $(\mathrm{id} \times f)^* P \cong Q$. For the same reason, (B) applied to the line bundle P on $X \times \hat{X}$ gives us a unique morphism

$$g \colon \hat{X} \to Y$$

such that $(\mathrm{id} \times g)^* Q \cong P$. Uniqueness then implies that $f \circ g = \mathrm{id}_{\hat{X}}$ and $g \circ f = \mathrm{id}_Y$, and so Y is isomorphic to $\hat{X}$, and the pullback of Q is isomorphic to P.

Remark. The properties above make $\hat{X}$ a so-called "fine" moduli space. This way of describing moduli spaces – where families of objects parametrized by S are in one-to-one correspondence with morphisms from S into the moduli space – is due to Grothendieck. The fact that this determines the moduli space up to isomorphism is then basically Yoneda's lemma: a scheme (or variety) is uniquely determined by knowing all morphisms from other schemes (or varieties) into it.

Now let's actually construct the dual abelian variety $\hat{X}$. As explained above, we choose an ample line bundle L on the abelian variety X, and then define

$$\hat{X} = X/K(L)$$

as the quotient by the finite subgroup $K(L) = \ker \phi_L$. Let $\pi \colon X \to \hat{X}$ be the quotient map; this is a surjective homomorphism with finite kernel, hence an isogeny. The mapping $\phi_L \colon X \to \mathrm{Pic}^0(X)$ then induces an isomorphism of groups $\hat{X} \cong \mathrm{Pic}^0(X)$.

Next, we construct the Poincaré bundle P on $X \times \hat{X}$. If we set

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1},$$

then the Poincaré bundle must satisfy

$$(\mathrm{id} \times \pi)^* P \cong K.$$

This is dictated by (B), applied to the line bundle K on the product $X \times X$: we have $K_x \cong t_x^* L \otimes L^{-1} = \phi_L(x)$, and this exactly corresponds to the point $\pi(x)$ under our isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$. So the question becomes whether there is a line bundle P on $X \times \hat{X}$ such that $(\mathrm{id} \times \pi)^* P \cong K$. Now

$$\mathrm{id} \times \pi \colon X \times X \to X \times \hat{X}$$

is an isogeny with kernel $\{0\} \times K(L)$, and so according to Proposition 12.4 from last time, all we need is to lift the translation action by the finite group $\{0\} \times K(L)$ on $X \times X$ to an action on the line bundle K.

So let's take a point $a \in K(L)$ and compute:

$$\begin{aligned} t_{(0,a)}^* K &\cong t_{(0,a)}^* m^* M \otimes t_{(0,a)}^* p_1^* L^{-1} \otimes t_{(0,a)}^* p_2^* L^{-1} \\ &\cong m^* t_a^* L \otimes p_1^* L^{-1} \otimes p_2^* t_a^* L^{-1} \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} = K, \end{aligned}$$

because $t_a^* L \cong L$, due to the fact that $a \in K(L)$. This means that we can choose a collection of isomorphisms

$$\phi_a \colon t_{(0,a)}^* K \to K.$$

Each ϕ_a is of course only unique up to a nonzero constant. In order for K to be equivariant, we need $\phi_a \circ \phi_b = \phi_{a+b}$, and so we need to make the right choice of ϕ_a. This can be done as follows. Observe that

$$\begin{aligned} K|_{\{0\}\times X} &\cong m^* L|_{\{0\}\times X} \otimes p_1^* L^{-1}|_{\{0\}\times X} \otimes p_2^* L^{-1}|_{\{0\}\times X} \\ &\cong L \otimes \left(\mathscr{O}_X \otimes L^{-1}|_0\right) \otimes L \cong \mathscr{O}_X \otimes L^{-1}|_0 \end{aligned}$$

is a trivial line bundle with fiber the 1-dimensional k-vector space $L^{-1}|_0$. We can normalize each ϕ_a by requiring that it acts trivially (meaning, as the identity) on the fiber of this line bundle. This uniquely determines ϕ_a, and the uniqueness also gives $\phi_{a+b} = \phi_a \circ \phi_b$. So we get a line bundle P on $X \times \hat{X}$, unique up to isomorphism, such that

$$(\mathrm{id} \times \pi)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \tag{13.3}$$

It is easy to see that (A) holds: write a given point $\alpha \in \hat{X}$ as $\alpha = \pi(x)$ for some $x \in X$, and observe that

$$P_\alpha = P|_{X\times\{\alpha\}} \cong (\mathrm{id} \times \pi)^* P|_{X\times\{x\}} \cong t_x^* L \otimes L^{-1},$$

which is correct because α go to $\phi_L(x)$ under our isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$.

It remains to check (B), and here we are going to use the fact that k has characteristic 0. Suppose that S is a normal variety, and that K is a line bundle on $X \times S$ with the property that

$$K_s = K|_{X\times\{s\}} \in \mathrm{Pic}^0(X)$$

and such that $K|_{\{0\}\times X}$ is trivial. We need to construct a morphism $f\colon S \to \hat{X}$ such that $K_s \cong P_{f(s)}$ for every $s \in S$. We'll do this by constructing the graph of f inside $S \times \hat{X}$. To that end, consider the line bundle

$$E = p_{12}^* K \otimes p_{13}^*(P^{-1})$$

on the product $X \times S \times \hat{X}$. For a pair $(s, \alpha) \in S \times \hat{X}$, we have

$$E|_{X\times\{s\}\times\{\alpha\}} \cong K_s \otimes P_\alpha^{-1},$$

and we want $\alpha = f(s)$ exactly when this line bundle is trivial. So let

$$\Gamma = \left\{\, (s,\alpha) \in S \times \hat{X} \;\middle|\; E \text{ is trivial on } X \times \{s\} \times \{\alpha\} \,\right\}.$$

According to Theorem 9.10, this is a closed subset of $S \times \hat{X}$. Because $K_s \in \mathrm{Pic}^0(X)$, and $\hat{X} \cong \mathrm{Pic}^0(X)$, for every $s \in S$, there is a unique point $\alpha \in \hat{X}$ such that $(s, \alpha) \in \Gamma$, and so the first projection $p_1\colon \Gamma \to S$ is bijective. Now Γ is a reduced variety, and S is a normal variety, and because we are in characteristic zero, it follows that p_1 is birational. Because S is normal, p_1 is then an isomorphism (by Zariski's main theorem). This shows that Γ is the graph of a morphism $f\colon S \to \hat{X}$. By the seesaw theorem, the restriction of E to $X \times \Gamma$ is trivial; pulling back along the morphism $X \times S \to X \times S \times \hat{X}$, $(x, s) \mapsto (x, s, f(s))$, we then get

$$K \cong (\mathrm{id} \times f)^* P$$

as desired.

Lecture 14 (March13)

Properties of the dual abelian variety. Last time, we constructed the dual abelian variety $\hat{X}$ and the Poincaré bundle P on $X \times \hat{X}$. For a point $\alpha \in \hat{X}$, we introduced the notation

$$P_\alpha = P|_{X\times\{a\}} \in \mathrm{Pic}^0(X);$$

this is the line bundle corresponding to α under the isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$. In class, I first went over the proof of the universal property again. During the proof, we used the fact that the field k has characteristic zero; the general case needs a bit more work.

We then looked at a few basic properties of the construction. First, let L be any line bundle on the abelian variety X, and consider the homomorphism

$$\phi_L \colon X \to \mathrm{Pic}^0(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1}.$$

This is in fact a morphism of abelian varieties; more precisely, under our isomorphism $\hat{X} \cong \mathrm{Pic}^0(X)$, the homomorphism ϕ_L comes from a morphism $f \colon X \to \hat{X}$. For the proof, consider the line bundle

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

on the product $X \times X$. We have

$$K|_{X \times \{x\}} \cong t_x^* L \otimes L^{-1} \quad \text{and} \quad K|_{\{0\} \times X} \cong \mathscr{O}_X,$$

and so we can apply the universal property (which we called (B) last time). This gives us a unique morphism $f \colon X \to \hat{X}$ such that $K \cong (\mathrm{id} \times f)^* P$. Restricting to $X \times \{x\}$, we get $P_{f(x)} \cong t_x^* L \otimes L^{-1} = \phi_L(x)$, and so f does indeed realize ϕ_L. Note that f is a group homomorphism (because ϕ_L is).

The next result says that the dual abelian variety is really a functor on the category of abelian varieties. Recall that a morphism of abelian varieties is a morphism that is also a group homomorphism. We showed that any morphism $f \colon X \to Y$ with $f(0) = 0$ is a homomorphism.

Proposition 14.1. *Let $f \colon X \to Y$ be a morphism of abelian varieties. Then the pullback homomorphism $f^* \colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ defines a morphism $\hat{f} \colon \hat{Y} \to \hat{X}$.*

Proof. Let's write P_X for the Poincaré bundle on $X \times \hat{X}$, and P_Y for the one on $Y \times \hat{Y}$. On $X \times \hat{Y}$, consider the line bundle $(f \times \mathrm{id})^* P_Y$. Its restriction to $\{0\} \times \hat{Y}$ is trivial because $f(0) = 0$; the restrictions to $X \times \{\alpha\}$ are in $\mathrm{Pic}^0(X)$ by Observation 6 from last time (because this holds when $\alpha = 0$). By the universal property for $\hat{X}$, there is thus a unique morphism $\hat{f} \colon \hat{Y} \to \hat{X}$ such that

$$(f \times \mathrm{id})^* P_Y \cong (\mathrm{id} \times \hat{f})^* P_X. \tag{14.2}$$

Here is a diagram of the two morphisms:

$$\begin{array}{ccc} X \times \hat{Y} & \xrightarrow{f \times \mathrm{id}} & Y \times \hat{Y} \\ \downarrow{\scriptstyle \mathrm{id} \times \hat{f}} & & \\ X \times \hat{X} & & \end{array}$$

If we restrict the isomorphism to $X \times \{\alpha\}$, we obtain

$$P_{X, \hat{f}(\alpha)} \cong f^* P_{Y,\alpha},$$

which is saying that the morphism $\hat{f}$ realizes the pullback f^* on line bundles. □

We can say a bit more in the case of isogenies.

Proposition 14.3. *Let $f \colon X \to Y$ be an isogeny. Then $\hat{f} \colon \hat{Y} \to \hat{X}$ is also an isogeny, and $\ker f$ and $\ker \hat{f}$ are dual abelian groups, in the sense that*

$$\ker \hat{f} \cong \mathrm{Hom}\big(\ker f, k^\times\big).$$

Proof. We showed at the end of Lecture 12 that

$$\ker\bigl(f^* \colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)\bigr) \cong \operatorname{Hom}\bigl(\ker f, k^\times\bigr)$$

is true for separable isogenies (and all isogenies are separable because we are assuming that k has characteristic zero). So it suffices to show that if f^*L is trivial for a line bundle $L \in \operatorname{Pic}(Y)$, then $L \in \operatorname{Pic}^0(Y)$. This implies that $\ker \hat{f}$ is dual to $\ker f$, hence finite, and then $\hat{f}$ must be an isogeny for dimension reasons. The proof is very easy: $\ker f^*$ is a finite group (because it is dual to the finite group $\ker f$), and so L has finite order; but we showed that any line bundle of finite order is in $\operatorname{Pic}^0(Y)$. $\square$

Example 14.4. The isogeny $n_X \colon X \to X$ has the property that $\hat{n}_X \colon \hat{X} \to \hat{X}$ is equal to $n_{\hat{X}}$. This follows from the identity $n_X^* L \cong L^n$ for $L \in \operatorname{Pic}^0(X)$ that we proved last time.

Example 14.5. Over the complex numbers, we can write an abelian variety as $X = V/\Gamma$, where V is a g-dimensional complex vector space, and Γ is a lattice of rank $2g$. The dual abelian variety is

$$\operatorname{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z}).$$

Now $H_1(X, \mathbb{Z}) \cong \Gamma$, and therefore

$$H^1(X, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$$

is the lattice dual to Γ. We also have

$$H^1(X, \mathscr{O}_X) \cong \operatorname{Hom}_{\mathbb{C}}\bigl(\bar{V}, \mathbb{C}\bigr),$$

with a conjugate-linear functional $f \colon V \to \mathbb{C}$ mapping to the translation-invariant $(0,1)$-form df. The embedding of the dual lattice works by extending a homomorphism $\varphi \colon \Gamma \to \mathbb{Z}$ uniquely to a linear functional $\varphi_{\mathbb{C}} \colon \Gamma \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$, and then projecting to the second summand in

$$\operatorname{Hom}_{\mathbb{C}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(V \oplus \bar{V}, \mathbb{C}) \cong \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \oplus \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}).$$

This explains the reason for calling $\operatorname{Pic}^0(X)$ the "dual" abelian variety.

Symmetric description of the dual abelian variety. While this is not clear from our construction of $\hat{X}$ (as a quotient of X), the two abelian varieties X and $\hat{X}$ really play the same role. To make this precise, we make the following definition.

Definition 14.6. A *divisorial correspondence* between two abelian varieties X and Y is a line bundle Q on $X \times Y$ such that $Q|_{\{0\} \times Y}$ and $Q|_{X \times \{0\}}$ are trivial.

We could realize Q by a divisor on $X \times Y$, which would then be a divisorial correspondence in the proper sense, but it is much better to work with line bundles. By Observation 6 from last time, we have

$$Q|_{\{x\} \times Y} \in \operatorname{Pic}^0(Y) \quad \text{and} \quad Q|_{X \times \{y\}} \in \operatorname{Pic}^0(X)$$

for every $x \in X$ and every $y \in Y$.

Proposition 14.7. *Let X and Y be abelian varieties of the same dimension, and let Q be a divisorial correspondence between X and Y. Then the following two conditions are equivalent:*

(a) $Q|_{\{x\} \times Y}$ trivial implies that $x = 0$.
(b) $Q|_{X \times \{y\}}$ trivial implies that $y = 0$.

If either of these conditions is satisfied, then $X \cong \hat{Y}$ and $Y \cong \hat{X}$, and Q is isomorphic to the pullback of both Poincaré bundles P_X and P_Y.

Proof. We only need to prove that (a) implies (b); the converse follows by interchanging X and Y. Let's first consider Q as a family of line bundles on Y. By the universal property of the dual abelian variety, we get a unique morphism $f\colon X \to \hat{Y}$ such that $Q \cong (f \times \mathrm{id})^* s^* P_Y$, where $s\colon Y \times \hat{Y} \to \hat{Y} \times Y$ is the morphism $s(y, \eta) = (\eta, y)$ that swaps the two factors. But (a) tells us that

$$P_{Y,f(x)} \cong Q|_{\{x\}\times Y}$$

is trivial only when $x = 0$, and so $\ker f = \{0\}$. Therefore f is injective, hence bijective (because $\dim X = \dim Y$), hence an isomorphism (because $\mathrm{char}(k) = 0$).

We can also view Q as a family of line bundles on X, and so we also get a unique morphism $g\colon Y \to \hat{X}$ such that $Q \cong (\mathrm{id} \times g)^* P_X$.

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id} \times g"] \arrow[d, "f \times \mathrm{id}"] & X \times \hat{X} \\ \hat{Y} \times Y & \end{tikzcd}$$

In order to prove (b), we need to show that g is injective. Let $K \subseteq \ker g$ be any finite subgroup of g; we shall argue that $K = \{0\}$, which is enough to conclude that g is injective. Because $K \subseteq \ker g$, we get a factorization

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend left, "g"] & Z \arrow[r, "\tilde{g}"] & \hat{X}, \end{tikzcd}$$

where $Z = Y/K$ is the quotient. If we set $L = (\mathrm{id} \times \tilde{g})^* P_X$, which is a line bundle on $X \times Z$, then $Q \cong (\mathrm{id} \times \pi)^* L$. Viewing L as a family of line bundles on Z, we get a third morphism $h\colon X \to \hat{Z}$, with the property that $L \cong (h \times \mathrm{id})^* s^* P_Z$. Let $\hat{\pi}\colon \hat{Z} \to \hat{Y}$ be the morphism dual to $\pi\colon Y \to Z$. According to (14.2), we have

$$(\pi \times \mathrm{id})^* P_Z \cong (\mathrm{id} \times \hat{\pi})^* P_Y.$$

If we combine this with the formulas for Q and L, we get

$$\begin{aligned} Q &\cong (\mathrm{id} \times \pi)^* (h \times \mathrm{id})^* s^* P_Z \cong (h \times \mathrm{id})^* s^* (\pi \times \mathrm{id})^* P_Z \\ &\cong (h \times \mathrm{id})^* s^* (\mathrm{id} \times \hat{\pi})^* P_Y \cong (h \times \mathrm{id})^* (\hat{\pi} \times \mathrm{id})^* s^* P_Y \\ &\cong ((\hat{\pi} \circ h) \times \mathrm{id})^* s^* P_Y. \end{aligned}$$

But Q is also isomorphic to $(f \times \mathrm{id})^* s^* P_Y$, and so the uniqueness of the morphism (in the universal property of the dual abelian variety) implies that $f = \hat{\pi} \circ h$. In other words, we found a factorization

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend left, "f"] & \hat{Z} \arrow[r, "\hat{\pi}"] & \hat{Y}. \end{tikzcd}$$

Now f is an isomorphism by (a), and so h must be injective. For dimension reasons, h is then an isomorphism, and so $\hat{\pi}$ is an isomorphism as well. By Proposition 14.3, the kernel of $\hat{\pi}$ is dual to $K = \ker \pi$. Therefore K is trivial, and so $g\colon Y \to \hat{X}$ is injective, as claimed. This proves (b). Along the way, we have shown that

$$f\colon X \to \hat{Y} \quad \text{and} \quad g\colon Y \to \hat{X}$$

are isomorphisms, and that $(\mathrm{id} \times g)^* P_X \cong Q \cong (f \times \mathrm{id})^* s^* P_Y$. □

We can apply this to the Poincaré bundle P_X on the product $X \times \hat{X}$; this is a divisorial correspondence, and $P_\alpha = P_X|_{X\times\{\alpha\}}$ is trivial only when $\alpha = 0$. The proposition then tells us that the dual abelian variety of $\hat{X}$ is isomorphic to the original abelian variety X, and that the Poincaré bundle $P_{\hat{X}}$ is isomorphic to $s^* P_X$, where $s\colon X \times \hat{X} \to \hat{X} \times X$ again swaps the two factors.

Positive characteristic and schemes. In the construction of the dual abelian variety, we had to assume that k has characteristic zero to prove the universal property. Ultimately, it comes down to the fact that when we have a line bundle L on $X \times S$, we are treating the locus in S such that L_s is trivial as a set, instead of as a scheme. (This applies in particular to the subgroup $K(L)$ inside X.) That is also the reason for the (unsatisfying) assumption that the parameter space S in the universal property needs to be normal. To fix these problems, we first need to revisit the seesaw theorem and make it work for schemes.

Proposition 14.8. *Let X be a complete variety, S a scheme, and L a line bundle on $X \times S$. There is a unique closed subscheme $S_0 \subseteq S$ such that:*

(a) $L|_{X \times S_0} \cong p_2^* L_0$ for a line bundle L_0 on S_0.
(b) If $f\colon T \to S$ is a morphism of schemes such that $(\mathrm{id} \times f)^* L \cong p_2^* K$ for a line bundle K on T, then f factors through S_0.

Proof. The proof is basically the same as that of Theorem 9.10, we just need to pay a little bit more attention to the details. For a closed point $s \in S(k)$, let's put as usual $L_s = L|_{X \times \{s\}}$. We already know that the set of $s \in S(k)$ such that L_s is trivial is closed in the Zariski topology. All we need to do is to put a natural scheme structure on this set. The problem being local, we may fix a point $s \in S(k)$ such that L_s is trivial, and then replace S by an affine open neighborhood $\operatorname{Spec} A$ of the point s. According to Theorem 9.4, we can find a bounded complex

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely-generated free A-modules – we can make them free by shrinking S, if necessary – such that for every B-algebra A, one has

$$H^p\big(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, L \otimes_A B\big) \cong H^p(K^\bullet \otimes_A B).$$

We may further assume that the complex is minimal at the point s; if we let $\mathfrak{m} \subseteq A$ denote the maximal ideal corresponding to $s \in S(k)$, then this means that the complex $K^\bullet \otimes_A A/\mathfrak{m}$ has trivial differentials. Because this complex computes the cohomology of $L \otimes_A A/\mathfrak{m} \cong L_s$, and because L_s is trivial, we get $H^0(X, L_s) \cong k$, and so K^0 must have rank one, hence $K^0 \cong A$. Likewise, $K^1 \cong A^r$ for some $r \geq 1$, and the differential $d\colon K^0 \to K^1$ is therefore represented by r elements $f_1, \dots, f_r \in A$. For the time being, let $I = (f_1, \dots, f_r) \subseteq A$ be the ideal generated by these elements. Taking $B = A/I$, we get

$$H^0\big(X \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I), L \otimes_A A/I\big) \cong H^0(K^\bullet \otimes_A A/I) \cong A/I, \tag{14.9}$$

because $d\colon K^0 \otimes_A A/I \to K^1 \otimes_A A/I$ is of course trivial by construction. So the restriction of L to the closed subscheme $X \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$ has a nontrivial global section. In fact, we get a line bundle L_0 on $\operatorname{Spec}(A/I)$, corresponding to the free A/I-module in (14.9), and the global section is really a morphism from $p_2^* L_0$ to the restriction of L.

As in Theorem 9.10, we now repeat this procedure for the line bundle L^{-1}; this gives us several additional elements $g_1, \dots, g_p \in A$, which we add to the ideal I. The desired closed subscheme is then $S_0 = \operatorname{Spec}(A/I)$. The reason is that both L and L^{-1} have a nontrivial global section on $X \times_S S_0$ (and so L_s is trivial for every closed point of S_0). The argument above gives us a line bundle L_0 on S_0, and an isomorphism $p_2^* L_0 \cong L|_{X \times S_0}$. This proves (a).

For (b), we may assume (by uniqueness) that $T = \operatorname{Spec} B$ is affine and that the line bundle K is trivial. The morphism $f\colon T \to S$ is given by a morphism of

k-algebras $\varphi\colon A \to B$, and to show that f factors through S_0, we need to prove that $I \subseteq \ker \varphi$. Because $(\mathrm{id} \times f)^* L \cong p_2^* K$, we get

$$B \cong H^0\big(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, L \otimes_A B\big) \cong H^0(K^\bullet \otimes_A B),$$

and because $K^0 \cong A$, this is only possible if the differential $d\colon K^0 \otimes_A B \to K^1 \otimes_A B$ is zero. But this means exactly that $\varphi(f_1) = \cdots = \varphi(f_r) = 0$. $\square$

As before, this improved version of the seesaw theorem implies the theorem of the cube for schemes.

Corollary 14.10. *Let L be a line bundle on $X \times Y \times S$, where X, Y are complete varieties, and S is a scheme. Suppose that there are points $x_0 \in X$, $y_0 \in Y$, and $s_0 \in S$ such that the three line bundles*

$$L|_{\{x_0\} \times Y \times S}, \quad L|_{X \times \{y_0\} \times S}, \quad L|_{X \times Y \times \{s_0\}}$$

are trivial. Then L is trivial.

With this result in hand, we can now construct the dual abelian variety in general. Let L be an ample line bundle on X. We proved that

$$\phi_L\colon X \to \operatorname{Pic}^0(X)$$

is surjective, and that its kernel $K(L)$ is a finite group. The dual abelian variety should therefore still be the quotient of X by this subgroup, in a suitable sense.

We first observe that the closed subgroup $K(L) \subseteq X$ has a natural scheme structure on it. Indeed, if we take the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

on $X \times X$, and consider the first copy of X as the parameter space, then Proposition 14.8 shows that there is a unique closed subscheme $X_0 \subseteq X$ such that

$$L|_{X_0 \times X} \cong p_1^* L_0$$

for some line bundle L_0 on X_0. Because $M|_{\{0\} \times X}$ is trivial, L_0 must be trivial, and so $X_0 \subseteq X$ is the maximal closed subscheme of X such that $L|_{X_0 \times X}$ is trivial. The set of closed points of X_0 is of course our subgroup $K(L)$, and so this puts a scheme structure on $K(L)$. From now on, we are going to denote this subscheme by the same symbol $K(L)$. We'll show next time that the group operation $m\colon X \times X \to X$ restricts to a morphism $K(L) \times K(L) \to K(L)$, and this makes $K(L)$ into a "group scheme". We can then define the dual abelian variety as

$$\hat{X} = X/K(L),$$

but where we now take the scheme structure on $K(L)$ into account when taking the quotient. (In characteristic zero, every group scheme is reduced; but in positive characteristic, $K(L)$ might be nonreduced, and then the quotient is different.)

Lecture 15 (March 25)

Group schemes. This was the first lecture after spring break, so I briefly reviewed what we had done before the break. Let X be an abelian variety over an algebraically closed field k. We are interested in the group

$$\operatorname{Pic}^0(X) = \big\{\, L \in \operatorname{Pic}(X) \,\big|\, t_x^* L \cong L \text{ for every } x \in X \,\big\}$$

of translation-invariant line bundles on X, and in particular, in constructing an abelian variety $\hat{X}$, the so-called "dual" abelian variety, that is isomorphic to $\operatorname{Pic}^0(X)$ as a group. Let L be an ample line bundle. We showed that

$$\phi_L\colon X \to \operatorname{Pic}^0(X), \quad \phi_L(x) = t_x^* L \otimes L^{-1},$$

is a surjective homomorphism, and that $K(L) = \ker \phi_L$ is a finite subgroup of X. In characteristic zero, we then defined the dual abelian variety as the quotient

$$\hat{X} = X/K(L).$$

We also constructed a universal line bundle P on $X \times \hat{X}$, called the Poincaré bundle, with the property that

$$(\mathrm{id} \times \pi)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

The pair $(\hat{X}, P)$ serves as a "moduli space" for families of line bundles in $\mathrm{Pic}^0(X)$, but we were only able to prove this for families parametrized by normal varieties.

To construct the dual abelian variety in all characteristics, we need to take into account that $K(L)$ is not just a finite set, but that it has a natural scheme structure. This follows from the scheme version of the seesaw theorem. Indeed, by Proposition 14.8, there is a maximal closed subscheme $X_0 \subseteq X$ with the property that the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

is trivial on $X \times X_0$; the set of closed points of X_0 is exactly our subgroup $K(L)$, and this endows $K(L)$ with a scheme structure. In fact, $K(L)$ is an example of a "group scheme": a group-object in the category of schemes (of finite type over k).

Definition 15.1. A *group scheme* is a scheme G (of finite type over the field k) with a closed point $e \in G(k)$ and two morphisms

$$m \colon G \times G \to G \quad \text{and} \quad i \colon G \to G,$$

subject to the following conditions:

(1) m is associative, meaning that the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

is commutative.

(2) e is the unit element, meaning that the diagram

$$\begin{tikzcd} G \arrow[dr, "(\mathrm{id},e)"'] \arrow[drr, bend left, "\mathrm{id}"] & & \\ & G \times G \arrow[r, "m"] & G \\ G \arrow[ur, "(e,\mathrm{id})"] \arrow[urr, bend right, "\mathrm{id}"'] & & \end{tikzcd}$$

is commutative; here we view e as a morphism $e \colon \mathrm{Spec}\, k \to G$.

(3) i is the inverse, meaning that the diagram

$$\begin{tikzcd} & G \times G \arrow[dr, bend left, "m"] & \\ G \arrow[ur, bend left, "(\mathrm{id},i)"] \arrow[r] \arrow[dr, bend right, "(i,\mathrm{id})"'] & \mathrm{Spec}\, k \arrow[r, "e"] & G \\ & G \times G \arrow[ur, bend right, "m"'] & \end{tikzcd}$$

is commutative.

If G is a group scheme, then for every scheme S (of finite type over k), the set of G-valued points $G(S) = \mathrm{Hom}_{\mathrm{Spec}\, k}(S, G)$ becomes a group; conversely, if $G(S)$ is a group in a way that is functorial in S, then G has the structure of a group scheme.

Example 15.2. An abelian variety is obviously a group scheme; by definition, every abelian variety is reduced and irreducible. If we define

$$X_n = \ker\bigl(n_X \colon X \to X\bigr)$$

as the kernel of the morphism $x \mapsto nx$, then X_n is a closed subscheme of length $\deg n_X = n^{2\dim X}$. We saw earlier that it has $n^{2\dim X}$ points when n is not a multiple of the characteristic $\operatorname{char}(k)$; but for example X_p always has at most $p^{\dim X}$ many points, and must therefore be nonreduced.

Example 15.3. For $n \geq 1$, the n-th roots of unity form a group scheme

$$\mu_n = \operatorname{Spec} k[x]/(x^n - 1).$$

The group operation is given by the morphism of k-algebras

$$k[x]/(x^n - 1) \to k[y,z]/(y^n - 1, z^n - 1), \quad x \mapsto yz.$$

When the field k has characteristic p, the group scheme μ_p is nonreduced and only has a single closed point, because

$$k[x]/(x^p - 1) = k[x]/(x-1)^p.$$

Example 15.4. In characteristic p, the Frobenius morphism

$$F \colon k[x] \to k[x], \quad F(x) = x^p,$$

is a ring homomorphism. The fiber over the origin is the group scheme

$$\operatorname{Spec} k[x]/(x^p),$$

which is again nonreduced with a single closed point. (The group operation is now $x \mapsto y + z$, in the same notation as in the previous example.)

The examples show that, in characteristic p, group schemes can have a nontrivial (meaning nonreduced) scheme structure. In characteristic zero, this does not happen, because of the following theorem.

Theorem 15.5. *Every group scheme over a field of characteristic* 0 *is nonsingular.*

The proof has two steps. First, one shows that the sheaf of Kähler differentials $\Omega^1_{G/k}$ on a group scheme G is always locally free. Recall that, according to one construction of the Kähler differentials, $\Omega^1_{G/k} \cong \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of the diagonal $\Delta \colon G \to G \times G$. Because G is a group scheme, we can describe the diagonal in terms of the group operations. Let

$$s = m \circ (\mathrm{id}, i) \colon G \times G \to G$$

be the morphism that acts on closed points as $s(x,y) = xy^{-1}$. One can show that

$$\begin{array}{ccc} G & \xrightarrow{\Delta} & G \times G \\ \big\downarrow & & \big\downarrow{\scriptstyle s} \\ \operatorname{Spec} k & \xrightarrow{e} & G \end{array}$$

is a Cartesian diagram, and therefore

$$\Omega^1_{G/k} \cong \mathcal{I}/\mathcal{I}^2 \cong s^*\bigl(\mathfrak{m}_e/\mathfrak{m}_e^2\bigr) \cong \mathscr{O}_G \otimes_k \mathfrak{m}_e/\mathfrak{m}_e^2,$$

where $\mathfrak{m}_e$ is the ideal sheaf of the closed point $e \in G(k)$. This proves that the sheaf of Kähler differentials is locally free of rank equal to the dimension of the k-vector space $\mathfrak{m}_e/\mathfrak{m}_e^2$. This much is true independently of the characteristic of the field.

Example 15.6. For $\mu_p = \operatorname{Spec} k[x]/(x^p - 1)$, we have $d(x-1)^p = p(x-1)^{p-1} = 0$, and so the module of Kähler differentials is isomorphic to

$$\Omega^1_{k[x]/k} \otimes_{k[x]} k[x]/(x^p - 1) \cong k[x]/(x^p - 1),$$

hence free of rank one.

The characteristic zero magic happens in the following lemma.

Lemma 15.7. *Let X be a scheme of finite type over a field k of characteristic zero. If the sheaf of Kähler differentials $\Omega^1_{X/k}$ is locally free, then X is nonsingular.*

Proof. I did not present the proof in class, but here it is. Let $x \in X(k)$ be an arbitrary closed point. It is enough to show that the local ring $\mathscr{O}_{X,x}$ is regular. So we may assume that $(A, \mathfrak{m})$ is a local k-algebra with residue field $A/\mathfrak{m} \cong k$, and that the module of Kähler differentials $\Omega^1_{A/k}$ is locally free. We need to show that A is regular, which means that $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. Set $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$, and choose n elements $f_1, \dots, f_n \in \mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis over k. Because $\Omega^1_{A/k} \otimes_A k \cong \mathfrak{m}/\mathfrak{m}^2$, the rank of the free A-module $\Omega^1_{A/k}$ is equal to n, and so we have an isomorphism of A-modules

$$\Omega^1_{A/k} \cong A^n.$$

By the universal property of the Kähler differentials, this gives us n derivations $\delta_1, \dots, \delta_n \in \operatorname{Der}_k(A)$, with the property that $\delta_i(f_j) = 1$ if $i = j$, and 0 otherwise. It follows that $\delta_i(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell-1}$ for all $\ell \geq 1$.

Now both $\dim_k \mathfrak{m}/\mathfrak{m}^2$ and $\dim A$ don't change under completion, and so we may replace A by its completion

$$\hat{A} = \varprojlim_{\ell} A/\mathfrak{m}^\ell.$$

Because $\delta_i(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell-1}$, our derivations extend to $\hat{A}$ as well; we may therefore assume that A is complete to begin with. Because A is complete, we then get a homomorphism of k-algebras

$$\alpha \colon k[[x_1, \dots, x_n]] \to A, \quad \alpha(x_i) = f_i,$$

from the ring of formal power series, and α is easily seen to be surjective. For any $f \in A$, we denote by $f(0) \in k$ its image in $A/\mathfrak{m}$. Because $\operatorname{char}(k) = 0$, we can also define a function

$$\beta \colon A \to k[[x_1, \dots, x_n]], \quad \beta(f) = \sum_{k_1, \dots, k_n} \frac{1}{k_1! \cdots k_n!} \bigl(\delta_1^{k_1} \cdots \delta_n^{k_n} f\bigr)(0),$$

that sends every $f \in A$ to its Taylor series; a short computation proves that β is a ring homomorphism. The composition

$$\beta \circ \alpha \colon k[[x_1, \dots, x_n]] \to k[[x_1, \dots, x_n]]$$

is the identity modulo $(x_1, \dots, x_n)^2$, and is therefore an automorphism; in particular, α must be injective, and so α is an isomorphism. This proves that A is a regular local ring. □

We also need talk briefly about quotients. Suppose that G is a finite (hence affine) group scheme. An action of G on a scheme X is a morphism $G \times X \to X$, subject to the condition that certain diagrams commute. As in Lecture 12, one can define the quotient X/G, under the assumption that the orbit of every closed point is contained in an affine open subset of X. If $U = \operatorname{Spec} A$ is a G-invariant affine open subset, then $G \times U \to U$ corresponds to a morphism of k-algebras

$$\delta \colon A \to \Gamma(G, \mathscr{O}_G) \otimes_k A,$$

and one defines the subalgebra of G-invariant functions as

$$A^G = \{ f \in A \mid \delta(f) = 1 \otimes f \}.$$

One can show that A^G is again a finitely-generated k-algebra, and the quotient U/G is defined as $\operatorname{Spec} A^G$.

The dual abelian variety in general. Let X be an abelian variety, and let L be an ample line bundle on X. Consider the line bundle

$$M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \tag{15.8}$$

on $X \times X$, and let $K(L) \subseteq X$ be the maximal closed subscheme such that the restriction of M to $X \times K(L)$ is trivial; the set of closed points of this subscheme is the group $\ker \phi_L$ that we used earlier. This is actually a group scheme: the group operation is induced by $m \colon X \times X \to X$.

Lemma 15.9. *The group operation $m \colon X \times X \to X$ on the abelian variety restricts to a morphism $m \colon K(L) \times K(L) \to K(L)$.*

Proof. Set $K = K(L)$. We need to show that the composition

$$K \times K \longrightarrow X \times X \xrightarrow{m} X$$

factors through the closed subscheme K. By the universal property in Proposition 14.8, this is equivalent to the pullback line bundle $(\mathrm{id} \times m)^* M$ being trivial on $X \times K \times K$. We are going to use the following notation:

$$\begin{tikzcd}
X & X \times K \arrow[l, "p_1"'] \arrow[r, "m"] & X & \\
X \times K \arrow[u, "m"] & X \times K \times K \arrow[l, "p_{12}"'] \arrow[u, "m \times \mathrm{id}"'] \arrow[r, "\mathrm{id} \times m"] \arrow[d, "p_{23}"] & X \times X \arrow[u, "m"'] \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
& K \times K \arrow[r, "m"] & X &
\end{tikzcd}$$

Because of (15.8), we have

$$(\mathrm{id} \times m)^* M \cong (m \times \mathrm{id})^* m^* L \otimes p_{23}^* m^* L^{-1} \otimes p_1^* L^{-1}.$$

We can rewrite the first factor as

$$(m \times \mathrm{id})^* m^* L \cong (m \times \mathrm{id})^* M \otimes (m \times \mathrm{id})^* p_1^* L \otimes p_3^* L \cong p_{12}^* m^* L \otimes p_3^* L,$$

because M is trivial on $X \times K$ (by definition of K). Similarly, we have

$$p_{12}^* m^* L \cong p_{12}^* M \otimes p_1^* L \otimes p_2^* L \cong p_1^* L \otimes p_2^* L,$$

again because M is trivial on $X \times K$. Combining the three previous lines gives

$$\begin{aligned}
(\mathrm{id} \times m)^* M &\cong p_1^* L \otimes p_2^* L \otimes p_3^* L \otimes p_{23}^* m^* L^{-1} \otimes p_1^* L^{-1} \\
&\cong p_{23}^* \big(m^* L^{-1} \otimes p_1^* L \otimes p_2^* L \big) \cong p_{23}^* M^{-1},
\end{aligned}$$

which is trivial because M is trivial on $K \times K$. This proves the lemma. □

We can now define the dual abelian variety as the quotient

$$\hat{X} = X / K(L)$$

by the finite group scheme $K(L)$. If we let $\pi \colon X \to \hat{X}$ be the quotient morphism, we again get a Poincaré bundle P on the product $X \times \hat{X}$, with the property that

$$(\mathrm{id} \times \pi)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

By construction, the two line bundles

$$P|_{X \times \{0\}} \quad \text{and} \quad P|_{\{0\} \times \hat{X}}$$

are trivial, and the maximal closed subscheme $Z \subseteq \hat{X}$ such that P is trivial on $X \times Z$ is the reduced singleton $Z = \{0\}$. The pair $(\hat{X}, P)$ is now a moduli space for translation-invariant line bundles on X on the category of all schemes (of finite type over k). Indeed, one can prove the following universal property (similar to what we did in Lecture 13, but using Proposition 14.8).

Theorem 15.10. *Given a scheme S, and a line bundle L on $X \times S$ such that*

$$L_s = L|_{X \times \{s\}} \in \operatorname{Pic}^0(X)$$

for every closed point $s \in S(k)$, and such that $L|_{\{0\} \times S}$ is trivial, there is a unique morphism $f\colon S \to \hat{X}$ with the property that $L \cong (\mathrm{id} \times f)^* P$.

Cohomology of the structure sheaf. We showed that nontrivial line bundles in $\operatorname{Pic}^0(X)$ have no cohomology whatsoever. But we still haven't computed the cohomology groups of the trivial line bundle $\mathscr{O}_X$. We are going to do this by computing the cohomology of the Poincaré bundle P at the same time. The result is exactly the same as over the complex numbers.

Theorem 15.11. *Let X be an abelian variety of dimension g.*

(a) We have $\dim_k H^i(X, \mathscr{O}_X) = \binom{g}{i}$.

(b) The cohomology of the Poincaré bundle is

$$H^i(X \times \hat{X}, P) \cong \begin{cases} 0 & \text{if } i \neq g, \\ k & \text{if } i = g. \end{cases}$$

From (a), it follows that the natural map

$$\bigwedge^i H^1(X, \mathscr{O}_X) \to H^i(X, \mathscr{O}_X)$$

is an isomorphism of k-vector spaces. We will carry out the proof of the theorem in six steps; the main ingredient is (as usual) the base change theorem.

Step 1. Let $p_2\colon X \times \hat{X} \to \hat{X}$ be the second projection. For $i \in \mathbb{N}$, define

$$\mathscr{F}_i = R^i (p_2)_* P,$$

which is a coherent sheaf on $\hat{X}$. Because $\dim X = g$, we have $\mathscr{F}_i = 0$ for $i > g$. We are going to prove (b) by computing these higher direct image sheaves. For any closed point $\alpha \in \hat{X}(k)$, we set

$$P_\alpha = P|_{X \times \{\alpha\}} \in \operatorname{Pic}^0(X),$$

and observe that P_α is trivial iff $\alpha = 0$. By Observation 7 in Lecture 13, we therefore have $H^i(X, P_\alpha) = 0$ for all $i \in \mathbb{Z}$ and all $\alpha \neq 0$; the base change theorem (in Corollary 9.9) therefore tells us that $\mathscr{F}_i$ is supported at the closed point $0 \in \hat{X}(k)$.

Step 2. Since $\operatorname{Supp} \mathscr{F}_i = \{0\}$, we have $H^j(\hat{X}, \mathscr{F}_i) = 0$ for $j > 0$, and so we get (from the Leray spectral sequence) that

$$H^i\big(X \times \hat{X}, P\big) \cong H^0\big(\hat{X}, \mathscr{F}_i\big) = 0$$

for $i > g$. From this, one can deduce by Serre duality that the same thing is true for $i < g$, and hence that $\mathscr{F}_i = 0$ for $i \neq g$; this is a nice exericse, but we will prove it in a different way in the next two steps of the argument.

Step 3. Now let's study what happens near the point $0 \in \hat{X}(k)$. We can work over the local ring $A = \mathscr{O}_{\hat{X},0}$; as usual, we denote the maximal ideal by $\mathfrak{m} = \mathfrak{m}_0$. According to Theorem 9.4, we can find a bounded complex

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely-generated free A-modules that "universally" computes the higher direct images $R^i(p_2)_*P$, in the sense that for any A-algebra B, one has

$$H^i\big(K^\bullet \otimes_A B\big) \cong H^i\big(X \times \hat{X}, P \otimes_A B\big).$$

Since we are working over a local ring, we may choose the complex $K^\bullet$ to be minimal, which means that all the differentials $d\colon K^i \to K^{i+1}$ have entries in the maximal ideal $\mathfrak{m}$. From

$$H^i\big(K^\bullet \otimes_A k\big) \cong H^i\big(X, P_0\big) \cong H^i(X, \mathscr{O}_X)$$

and minimality, we see that

$$\operatorname{rk} K^i = \dim_k H^i(X, \mathscr{O}_X).$$

In particular, we have $K^i = 0$ for $i > g$, and so our minimal complex takes the form

$$0 \to K^0 \to K^1 \to \cdots \to K^g \to 0.$$

Let M_i be the finitely-generated A-module corresponding to $\mathscr{F}_i = R^i(p_2)_*P$; these are the cohomology modules of the complex $K^\bullet$.

Step 4. The following simple lemma from commutative algebra now lets us conclude that $M_i = 0$ for $i < g$.

Lemma 15.12. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension g. Let*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

be a bounded complex of finitely-generated free A-modules, such that all cohomology modules $H^i(K^\bullet)$ have finite length. Then $H^i(K^\bullet) = 0$ for $i < g$.

Proof. The statement is trivial for $g = 0$, and so we can argue by induction on $g \geq 0$. Choose an element $f \in \mathfrak{m}$ such that $f \notin \mathfrak{m}^2$; then the quotient ring $\bar{A} = A/Af$ is regular of dimension $g - 1$ (by dimension theory). If we set $\bar{K}^\bullet = K^\bullet \otimes_A \bar{A}$, we get a short exact sequence of complexes

$$0 \longrightarrow K^\bullet \xrightarrow{f} K^\bullet \longrightarrow \bar{K}^\bullet \longrightarrow 0$$

and therefore an exact sequence in cohomology

$$H^i(K^\bullet) \xrightarrow{f} H^i(K^\bullet) \longrightarrow H^i(\bar{K}^\bullet) \longrightarrow H^{i+1}(K^\bullet) \xrightarrow{f} H^{i+1}(K^\bullet).$$

Because all cohomology modules of $K^\bullet$ have finite length, it follows that the cohomology modules of $\bar{K}^\bullet$ also have finite length. By induction, we therefore get $H^i(\bar{K}^\bullet) = 0$ for $i < g - 1$. From the exact sequence, $f\colon H^i(K) \to H^i(K)$ is then injective for $i < g$; but because $H^i(K)$ has finite length, it is annihilated by some power of f, and so $H^i(K) = 0$ for $i < g$. $\square$

If we apply this to our complex, we find that $M_i = 0$ for $i < g$, and hence that M_g is the only nontrivial cohomology module of $K^\bullet$. In other words, $K^\bullet$ is a minimal free resolution of the A-module M_g.

Step 5. Now let's combine this with what we know about the Poincaré bundle (from the construction of $\hat{X}$). We have $H^0(X, \mathscr{O}_X) = k$, and so $K^0 \cong A$; setting $n = \operatorname{rk} K^1$, we also get $K^1 \cong A^n$. The differential

$$d\colon K^0 \to K^1$$

is therefore given by n elements $f_1, \dots, f_n \in \mathfrak{m}$ (by minimality). We showed during the proof of Proposition 14.8 that the maximal closed subscheme of $\hat{X}$ over which P is trivial is defined by the ideal $(f_1, \dots, f_n)$. In our case, this closed subscheme is $\{0\}$, and so we must have $(f_1, \dots, f_n) = \mathfrak{m}$. Consider now the dual complex

$$0 \to (K^n)^* \to \dots \to (K^1)^* \to (K^0)^* \to 0.$$

By the lemma from Step 4, this complex is again exact in all places except at the right end, and there, the cohomology is $A/(f_1, \dots, f_n) = A/\mathfrak{m} = k$. The dual complex is therefore a minimal free resolution of the residue field k.

Step 6. But we know from commutative algebra what the minimal free resolution of $A/\mathfrak{m}$ looks like in a regular local ring: it is the Koszul complex for a regular sequence $x_1, \dots, x_g \in \mathfrak{m}$. The Koszul complex is the tensor product of the g complexes

$$0 \longrightarrow A \xrightarrow{x_i} A \longrightarrow 0$$

and therefore has the shape

$$0 \to A^{\binom{g}{0}} \to A^{\binom{g}{1}} \to A^{\binom{g}{2}} \to \dots \to A^{\binom{g}{g-1}} \to A^{\binom{g}{g}} \to 0.$$

Because minimal free resolutions are unique (up to isomorphism), the dual complex of $K^\bullet$, and hence $K^\bullet$ itself, must be a Koszul complex as well. This gives

$$\dim_k H^i(X, \mathscr{O}_X) = \operatorname{rk} K^i = \binom{g}{i},$$

which proves (a). We also find that $M_g = H^g(K^\bullet) \cong A/\mathfrak{m}$, and so $\mathscr{F}_g$ is the structure sheaf of the closed point $0 \in \hat{X}(k)$. We now get (b) from the computation in Step 2. □

Note. In fact, we have shown that

$$R^i(p_2)_* P \cong \begin{cases} 0 & \text{if } i \neq g, \\ \mathscr{O}_0 & \text{if } i = g. \end{cases} \tag{15.13}$$

This result will be important when we study derived categories of abelian varieties.

Corollary 15.14. *We have* $\dim H^q(X, \Omega^p_{X/k}) = \binom{g}{p}\binom{g}{q}$.

Proof. The Kähler differentials $\Omega^1_{X/k}$ are locally free of rank $g = \dim X$. Therefore $\Omega^p_{X/k} = \bigwedge^p \Omega^1_{X/k}$ is locally free of rank $\binom{g}{p}$, and the formula follows from the theorem. □

Lecture 16 (March 27)

The derived category. Derived categories were introduced to have a better foundation for the theory of derived functors. When we calculate derived functors such as Tor or Ext, we typically find a (locally free, or flat, or injective) resolution of our given module/sheaf, apply the functor in question to each term of the resolution, and then take cohomology. The main idea behind the derived category is to keep not just the cohomology modules/sheaves, but the complexes themselves. Because the same module/sheaf can be resolved in many different ways, keeping the complex only makes sense if we declare different complexes obtained in this way to be isomorphic. This leads to the notion of a *quasi-isomorphism*: a morphism between two complexes that induces isomorphisms on cohomology.

Example 16.1. Consider the case of modules over a ring. Every module M has a (typically infinite) free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

and in the derived category, we want to consider the complex $F_\bullet$ as being isomorphic to M. If $G_\bullet$ is another free resolution of M, then a basic result in homological algebra says that there is a morphism of complexes $f\colon F_\bullet \to G_\bullet$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] & F_1 \arrow[r, "d"] \arrow[d, "f"] & F_0 \arrow[r] \arrow[d, "f"] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$

commute. This morphism is only unique up to homotopy: for any other choice $f'\colon F_\bullet \to G_\bullet$, there are homomorphisms $s\colon F_n \to G_{n+1}$ such that $f' - f = ds + sd$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed] & F_1 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & F_0 \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$

If we want to consider M, $F_\bullet$, and $G_\bullet$ as being isomorphic to each other, the two liftings of $\mathrm{id}\colon M \to M$ should be equal, and so we are forced to consider morphisms of complexes up to homotopy.

Example 16.2. In other cases, say for computing Ext, we might want to replace M by an injective resolution of the form

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots,$$

Now an injective resolution and a free resolution do not have much in common; the only thing we can say is that we have a morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d] & F_1 \arrow[r, "d"] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d"] & I^1 \arrow[r, "d"] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

that is an isomorphism on the level of cohomology—being resolutions of M, both complexes have cohomology only in degree zero. If we want both complexes to be isomorphic as objects of the derived category, we need to make sure that such quasi-isomorphisms have inverses.

Quasi-isomorphisms also arise naturally if we consider resolutions of complexes.

Example 16.3. An injective resolution of a complex $M^\bullet$ of modules is a complex $I^\bullet$ of injective modules, and a morphism of complexes $M^\bullet \to I^\bullet$ that induces isomorphisms on cohomology. This generalizes the usual definition for a single module to complexes.

Unfortunately, not every quasi-isomorphism has an inverse. The following example (in the category of $\mathbb{Z}$-modules) shows one way in which this can happen.

Example 16.4. In the category of $\mathbb{Z}$-modules, the morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism; but it clearly has no inverse, not even up to homotopy, because there are no nontrivial homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.

Let me now explain the classical construction of the derived category. Let $\mathfrak{A}$ be an arbitrary abelian category (such as modules over a ring, or coherent sheaves on a scheme). Depending on what kind of complexes we want to consider, there are several derived categories: the unbounded derived category $\mathrm{D}(\mathfrak{A})$, whose objects are arbitrary complexes of objects in $\mathfrak{A}$; the categories $\mathrm{D}^+(\mathfrak{A})$ and $\mathrm{D}^-(\mathfrak{A})$, whose objects are semi-infinite complexes that are allowed to be infinite in the positive respectively negative direction; and finally the bounded derived category $\mathrm{D}^b(\mathfrak{A})$, whose objects are bounded complexes of objects in $\mathfrak{A}$. All of these categories are constructed in two stages; we explain this in the case of $\mathrm{D}^b(\mathfrak{A})$.

(1) Starting from the category of bounded complexes $\mathrm{K}^b(\mathfrak{A})$, form the so-called homotopy category $\mathrm{H}^b(\mathfrak{A})$. It has exactly the same objects, but the morphisms between two complexes are taken up to homotopy; in other words,

$$\mathrm{Hom}_{\mathrm{H}^b(\mathfrak{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet) \big/ \mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet),$$

where $\mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet)$ denotes the subgroup of those morphisms that are homotopic to zero.

(2) Now form the derived category $\mathrm{D}^b(\mathfrak{A})$ by inverting quasi-isomorphisms; this can be done by a formal construction similar to the passage from $\mathbb{Z}$ to $\mathbb{Q}$. That is to say, in $\mathrm{D}^b(\mathfrak{A})$, a morphism between two complexes $A^\bullet$ and $B^\bullet$ is represented by a fraction f/h, which stands for the diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "h"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $f\colon C^\bullet \to B^\bullet$ is a morphism of complexes and $h\colon C^\bullet \to A^\bullet$ is a quasi-isomorphism. As with ordinary fractions, there is an equivalence relation that we shall not dwell on; it is also not entirely trivial to show that the composition of two morphisms is again a morphism.

In other words, the objects of the derived category are still just complexes; but the set of morphisms between two complexes has become more complicated (especially because a morphism may involve an additional complex).

Example 16.5. For us, the most interesting case is when the abelian category is $\mathrm{Coh}(X)$, the category of coherent sheaves on a scheme X. By applying the above construction, we get the bounded derived category $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$; once again, the objects of this category are just bounded complexes of coherent sheaves. For practical purposes, a broader definition of the derived category is more useful. Inside the unbounded derived category $\mathrm{D}(\mathscr{O}_X)$ of all complexes of sheaves of $\mathscr{O}_X$-modules, consider the full subcategory $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; by definition, a complex

$$\cdots \to \mathscr{F}^{-1} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \cdots$$

belongs to this subcategory if its cohomology sheaves $\mathcal{H}^i(\mathscr{F}^\bullet)$ are coherent, and nonzero for only finitely many values of i. Clearly,

$$\mathrm{D}^b\big(\mathrm{Coh}(X)\big) \subseteq \mathrm{D}^b_{coh}(\mathscr{O}_X),$$

and under some mild assumptions on X, this inclusion is actually an equivalence of categories. The larger category has the advantage of being more flexible: for example, an injective resolution of a coherent sheaf is an object of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ but not of $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$.

Morphisms in the derived category. The definition of the derived category leads to several questions. The first one is whether one can describe the space of morphisms between two complexes in more basic terms. At least in the case of complexes with only one nonzero cohomology object, this is possible.

We first define the following *shift functor*. Given a complex $A^\bullet \in \mathrm{K}(\mathfrak{A})$ and an integer $n \in \mathbb{Z}$, we obtain a new complex $A^\bullet[n]$ by setting

$$A^\bullet[n] = A^{\bullet+n};$$

we also multiply all the differentials in the original complex by the factor $(-1)^n$. (This convention makes it easier to remember certain formulas.) For example, if $A^\bullet$ is the complex

$$\cdots \longrightarrow A^{-1} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \longrightarrow \cdots$$

then $A^\bullet[1]$ is the same complex shifted to the left by one step,

$$\cdots \longrightarrow A^0 \xrightarrow{-d} A^1 \xrightarrow{-d} A^2 \xrightarrow{-d} A^3 \longrightarrow \cdots$$

and with the sign of all differentials changed. This operation passes to the derived category, and defines a collection of functors $[n]\colon D(\mathfrak{A}) \to D(\mathfrak{A})$.

Example 16.6. Morphisms in $\mathrm{D}^b(\mathfrak{A})$ are related to Ext-groups in the sense of Yoneda. (When the abelian category $\mathfrak{A}$ has enough injective objects, these are the same as the derived functors of Hom, computed using an injective resolution.) If A and B are two objects of the abelian category $\mathfrak{A}$, then one has

$$\mathrm{Hom}_{\mathrm{D}^b(\mathfrak{A})}\bigl(A, B[n]\bigr) \simeq \mathrm{Ext}^n(A, B);$$

in particular, this group is trivial for $n < 0$. Let us consider the case $n = 1$. An element of $\mathrm{Ext}^1(A, B)$ is represented by a short exact sequence of the form

$$0 \to B \to E \to A \to 0.$$

Now the morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is obviously a quasi-isomorphism; on the other hand, we have

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

and together, they determine a morphism in $\mathrm{D}^b(\mathfrak{A})$ from A (viewed as a complex in degree 0) to $B[1]$ (viewed as a complex in degree -1).

Exercise 16.1. Show that, conversely, every element of $\mathrm{Hom}_{\mathrm{D}^b(\mathfrak{A})}\bigl(A, B[1]\bigr)$ gives rise to an extension of A by B, and that the two constructions are inverse to each other.

Other models for the derived category. Recall that the objects of the bounded derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ are complexes of sheaves of $\mathscr{O}_X$-modules whose cohomology sheaves are coherent and vanish outside some bounded interval. I already mentioned that, under some mild assumptions on X, this category is equivalent to the much smaller category $\mathrm{D}^b\bigl(\mathrm{Coh}(X)\bigr)$, whose objects are bounded complexes of coherent sheaves on X. There are various other models for the derived category, each based on a certain class of sheaves (such as injective sheaves or flat sheaves). Let me illustrate this principle with the following example.

Example 16.7. Let $\mathrm{Inj}(\mathscr{O}_X)$ denote the (additive, but not abelian) category of injective sheaves of $\mathscr{O}_X$-modules. Every $\mathscr{O}_X$-module has a semi-infinite resolution by injectives; using the Cartan-Eilenberg construction, every semi-infinite complex of $\mathscr{O}_X$-modules is quasi-isomorphic to a semi-infinite complex of injectives. This means that the inclusion

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \subseteq \mathrm{D}^+(\mathscr{O}_X)$$

is an equivalence of categories. By restricting to complexes with bounded and coherent cohomology sheaves, we also obtain an equivalence of categories

$$\mathrm{D}^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The advantage of using injectives is that we do not need to worry about inverses for quasi-isomorphisms. Indeed, suppose that $f\colon I_1^\bullet \to I_2^\bullet$ is a quasi-isomorphism between two complexes of injective $\mathscr{O}_X$-modules. The universal mapping property of injectives implies that there is a morphism of complexes $g\colon I_2^\bullet \to I_1^\bullet$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity. Thus

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq H^+\big(\mathrm{Inj}(\mathscr{O}_X)\big)$$

and, extending our earlier notation in the obvious way, also

$$H^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The same construction works for sheaves of flat $\mathscr{O}_X$-modules; under certain additional assumptions on the scheme X, one can also use locally free sheaves.

In this model for the derived category, the morphisms are much easier to describe. Nevertheless, it is better to work with the category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, because it gives us more flexibility: we can choose injective or flat or locally free resolutions as the occasion demands.

Triangulated categories. The derived category is no longer an abelian category, because the kernel and cokernel of a morphism do not make sense. (This is due to all the additional morphisms that we have introduced when adding inverses for quasi-isomorphisms.) But there is a replacement for short exact sequences, the so-called distinguished triangles, and $D^b(\mathfrak{A})$ is an example of a *triangulated category*.

A triangulated category is given by specifying a class of triangles. The motivation for introducing triangles lies in the mapping cone construction from homological algebra; let us briefly review this construction, and explain in what sense it acts as a substitute for short exact sequences. Given a morphism of complexes $f\colon A^\bullet \to B^\bullet$, the *mapping cone* of f is the complex

$$C_f^\bullet = B^\bullet \oplus A^\bullet[1] = B^\bullet \oplus A^{\bullet+1}$$

with differential $d(b,a) = (db + fa, -da)$. (The terminology comes from the mapping cone in algebraic topology.) Since we defined $A^\bullet[1]$ by changing the sign of all differentials, this makes the sequence of complexes

$$0 \to B^\bullet \to C_f^\bullet \to A^\bullet[1] \to 0$$

short exact. In total, we have a sequence of four morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \to A^\bullet[1], \tag{16.8}$$

and the composition of any two adjacent morphisms is zero *up to homotopy*.

Exercise 16.2. Verify that the composite morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \qquad \text{and} \qquad C_f^\bullet \to A^\bullet[1] \to B^\bullet[1]$$

are both homotopic to zero.

A sequence of four morphisms as in (16.8) is called a *triangle*; this is because we can arrange it into the shape of a triangle, with the convention that the arrow marked [1] really goes from $C_f^\bullet$ to $A^\bullet[1]$:

$$\begin{tikzcd} & A^\bullet \arrow[dr] & \\ C_f^\bullet \arrow[ur, "{[1]}"] & & B^\bullet \arrow[ll] \end{tikzcd}$$

The short exact sequence of complexes gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C_f^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

for the cohomology of the complexes. In order to write down this long exact sequence, all we need is the four morphisms in (16.8). Taking this example as a model, we say that any sequence of four morphisms of complexes

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a *distiguished triangle* if it is isomorphic, in the derived category, to a triangle coming from the mapping cone construction. (In particular, the composition of two adjacent morphisms in the triangle is then actually homotopic to zero.) This definition endows the derived category with the structure of a triangulated category.

Here are two basic properties of distinguished triangles that you should try to verify as an exercise. There are many others, and by abstracting from this example, Verdier arrived at the concept of a triangulated category; since the precise definition is not relevant for our purposes, we shall not dwell on the details.

Exercise 16.3. Suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle. Show that $B^\bullet \to C^\bullet \to A^\bullet[1] \to B^\bullet[1]$ and $C^\bullet[-1] \to A^\bullet \to B^\bullet \to C^\bullet$ are again distinguished triangles. This means that distinguished triangles can be "rotated" in both directions.

Exercise 16.4. Show that a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

in the abelian category $\mathfrak{A}$.

I already mentioned that distinguished triangles are a replacement for short exact sequences; let me elaborate on this point a bit. On the one hand, the prototypical example of a distinguished triangle in (16.8) came from the short exact sequence of the mapping cone. On the other hand, once we look at complexes up to quasi-isomorphism, every short exact sequence of complexes is actually that of a mapping cone (under some conditions on $\mathfrak{A}$). Let me illustrate this claim with the example of modules over a ring.

Example 16.9. Suppose we have a short exact sequence of complexes of R-modules

$$0 \to B^\bullet \to C^\bullet \to A^\bullet[1] \to 0.$$

Up to quasi-isomorphism, we can replace any complex by a free resolution, and so we may assume that $A^\bullet$ is a complex of free R-modules. We can then choose splittings

$$C^n \simeq B^n \oplus A^{n+1}.$$

With respect to this decomposition, the differential $d\colon C^n \to C^{n+1}$ is represented by a matrix

$$\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$$

for some homomorphism $f\colon A^n \to B^n$; the identity $d \circ d = 0$ implies that f defines a morphism of complexes from $A^\bullet$ to $B^\bullet$, and our exact sequence of complexes is the one for the mapping cone of f.

In closing, let me mention one other general fact that is frequently useful. Namely, suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$. Then for every $E^\bullet \in \mathrm{D}^b(\mathfrak{A})$, one gets two long exact sequences of abelian groups

$$\cdots \to \operatorname{Hom}(E^\bullet, A^\bullet) \to \operatorname{Hom}(E^\bullet, B^\bullet) \to \operatorname{Hom}(E^\bullet, C^\bullet) \to \operatorname{Hom}\bigl(E^\bullet, A^\bullet[1]\bigr) \to \cdots$$

$$\cdots \to \operatorname{Hom}\bigl(A^\bullet[1], E^\bullet\bigr) \to \operatorname{Hom}(C^\bullet, E^\bullet) \to \operatorname{Hom}(B^\bullet, E^\bullet) \to \operatorname{Hom}(A^\bullet, E^\bullet) \to \cdots$$

where $\operatorname{Hom}(-,-)$ means the set of morphisms in $\mathrm{D}^b(\mathfrak{A})$.

Derived functors. From now on, we shall concentrate on the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, where X is a scheme. Here is a very useful fact:

Example 16.10. If X is nonsingular and quasi-compact, so that every coherent sheaf on X has a finite resolution by locally free sheaves, then every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is quasi-isomorphic to a bounded complex of locally free sheaves.

Our goal is to define derived functors for the commonly used functors in algebraic geometry, such as $\otimes$, $\mathcal{H}om$, or pushforwards and pullbacks. The original functors are either left or right exact, and in classical homological algebra, the higher derived functors correct the lack of exactness. In the setting of triangulated categories, the relevant definition is the following.

Definition 16.11. An additive functor between two triangulated categories is *exact* if it takes distinguished triangles to distinguished triangles.

If we have an exact functor $F\colon \mathrm{D}^b(\mathfrak{A}) \to \mathrm{D}^b(\mathfrak{B})$ between the derived categories of two abelian categories, we get a long exact sequence in cohomology: if

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$, then

$$F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet) \to F(A^\bullet)[1]$$

is a distinguished triangle in $\mathrm{D}^b(\mathfrak{B})$, and so

$$\cdots \to H^i F(A^\bullet) \to H^i F(B^\bullet) \to H^i F(C^\bullet) \to H^{i+1} F(A^\bullet) \to \cdots$$

is a long exact sequence in the abelian category $\mathfrak{B}$. This explains the terminology.

When defining a derived functor, we have two choices:

(1) Use a definition that works only for certain complexes, such as complexes of injective sheaves or flat sheaves. Then show that the subcategory consisting of such complexes is equivalent to the entire derived category. In this way, we obtain a non-constructive definition of the functor.
(2) Use a definition that works for arbitrary complexes. This may require more effort, but seems better from a mathematical point of view.

Example 16.12. Let $f\colon X \to Y$ be a morphism of schemes, say quasi-compact and separated (in order for f_* to preserve quasi-coherence). We want to define the derived functor $\mathbf{R}f_*\colon \mathrm{D}^+\bigl(\mathrm{QCoh}(X)\bigr) \to \mathrm{D}^+\bigl(\mathrm{QCoh}(Y)\bigr)$. Since we already know that injective sheaves are acyclic, we should obviously define

$$\mathbf{R}f_* I^\bullet = f_* I^\bullet$$

if $I^\bullet$ is a complex of injective sheaves. Since the subcategory $\mathrm{D}^+\bigl(\mathrm{Inj}(X)\bigr)$ is equivalent to $\mathrm{D}^+\bigl(\mathrm{QCoh}(X)\bigr)$, we can choose an inverse functor to the inclusion – this basically amounts to choosing an injective resolution for every complex of quasi-coherent sheaves – and compose the two. In this way, we obtain a functor

$$\mathbf{R}f_* \colon \mathrm{D}^+\big(\mathrm{QCoh}(X)\big) \to \mathrm{D}^+\big(\mathrm{QCoh}(Y)\big).$$

If f is proper, then f_* preserves coherence, and $\mathbf{R}f_*$ restricts to a functor

$$\mathbf{R}f_* \colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y).$$

It remains to verify that $\mathbf{R}f_*$ is an exact functor.

Exercise 16.5. Show that $\mathbf{R}f_*$ takes distinguished triangles to distinguished triangles. (Hint: It is enough to prove this for a triangle of the form

$$I_1^\bullet \to I_2^\bullet \to C_\varphi^\bullet \to I_1^\bullet[1],$$

for $\varphi \colon I_1^\bullet \to I_2^\bullet$ a morphism between two complexes of injective sheaves.)

Example 16.13. If the above definition of $\mathbf{R}f_*$ involves too many choices for your taste, here is another possibility. Flasque sheaves are also acyclic for f_*, and have the advantage that there is a canonical resolution by flasque sheaves, the so-called *Godement resolution.* Given a sheaf of abelian groups $\mathscr{F}$, let $G^0(\mathscr{F})$ denote the sheaf of discontinuous sections: for any open subscheme $U \subseteq X$,

$$G^0(\mathscr{F})(U) = \prod_{x \in U} \mathscr{F}_x.$$

This sheaf is flasque and contains $\mathscr{F}$ as a subsheaf. Now we define $G^1(\mathscr{F})$ by applying the same construction to the cokernel of $\mathscr{F} \hookrightarrow G^0(\mathscr{F})$; in general, we set $G^{n+1}(\mathscr{F}) = G^0\big(G^n(\mathscr{F})/G^{n-1}(\mathscr{F})\big)$. The resulting complex of sheaves

$$0 \to \mathscr{F} \to G^0(\mathscr{F}) \to G^1(\mathscr{F}) \to G^2(\mathscr{F}) \to \cdots$$

is exact; this is the Godement resolution $G^\bullet(\mathscr{F})$. The same construction produces canonical flasque resolutions for complexes of sheaves: apply the construction to each sheaf in the complex to get a double complex, and then take the associated single complex. This allows us to define $\mathbf{R}f_*$ by setting

$$\mathbf{R}f_* F = f_* G^\bullet(F)$$

for any $F \in \mathrm{D}^+(\mathscr{O}_X)$. One can show that $\mathbf{R}f_*\mathscr{F}$ is canonically isomorphic to $f_*\mathscr{F}$ when $\mathscr{F}$ is a flasque sheaf; up to isomorphism, the two constructions of $\mathbf{R}f_*$ are therefore the same.

By one of those methods, one can also define the derived functors $\overset{\mathbf{L}}{\otimes}$, $\mathbf{R}\mathcal{H}om$, $\mathbf{R}\Gamma$, $\mathbf{R}\mathrm{Hom}$, as well as $\mathbf{L}f^*$ for morphisms $f \colon X \to Y$. All of the properties of the underived functors carry over to this setting: for example, $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$. In classical homological algebra, the composition of two functors leads to a spectral sequence (such as the Grothendieck spectral sequence); in the derived category, this simply becomes an identity between two derived functors.

Example 16.14. For two morphisms $f \colon X \to Y$ and $g \colon Y \to Z$, one has $\mathbf{R}g_* \circ \mathbf{R}f_* \simeq \mathbf{R}(g \circ f)_*$. This can be proved by observing that the pushforward of an injective sheaf is again injective: for a complex of injective sheaves,

$$(g \circ f)_* I^\bullet = g_*\big(f_* I^\bullet\big).$$

A special case of this is the formula $\mathbf{R}\Gamma(Y, -) \circ \mathbf{R}f_* \simeq \mathbf{R}\Gamma(X, -)$, which is the derived category version of the Leray spectral sequence.

Example 16.15. Similar reasoning proves the formula $\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om \simeq \mathbf{R}\mathrm{Hom}$.

The big advantage of working in the derived category is that many relations among the underived functors that are true only for locally free sheaves, now hold in general. Technically, this is true on nonsingular varieties, because every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is then quasi-isomorphic to a bounded complex of locally free sheaves.

As a case in point, let us consider the projection formula. The version in Hartshorne says that if $f\colon X \to Y$ is a morphism of schemes, and if $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, then $f_*\big(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}\big) \simeq f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E}$. In the derived category, we have the following generalization.

Proposition 16.16. *Let $f\colon X \to Y$ be a morphism of schemes, with Y nonsingular and quasi-compact. Then one has*

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^*G\big) \simeq \mathbf{R}f_*F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G$$

for every $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and every $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. We may assume without loss of generality that G is a bounded complex of locally free sheaves and that F is a complex of injective sheaves. In that case,

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^*G\big) = f_*\big(F \otimes_{\mathscr{O}_X} f^*G\big),$$

and by the usual projection formula, this is isomorphic to

$$f_*F \otimes_{\mathscr{O}_Y} G = \mathbf{R}f_*F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G. \qquad \square$$

Lecture 17 (April 1)

Grothendieck duality. In additional to the general definitions from last time, we also need two basic tools for actually working with derived categories. The first one is Grothendieck duality. The general theory is fairly complicated, and so we shall only discuss a special case that is sufficient for the purposes of this course.

Let me begin by recalling Serre's duality theorem. It says that if $\mathscr{F}$ is a coherent sheaf on a smooth projective variety X, then

$$\mathrm{Ext}^{n-i}(\mathscr{F}, \omega_X) \simeq \mathrm{Hom}_{\mathbb{C}}\big(H^i(X, \mathscr{F}), \mathbb{C}\big),$$

where $n = \dim X$ and ω_X denotes the canonical bundle of X. We can reformulate this using the derived category. Because of the relationship between Ext-groups and morphisms in the derived category, we have

$$H^i(X, \mathscr{F}) \simeq \mathrm{Ext}^i(\mathscr{O}_X, \mathscr{F}) \simeq \mathrm{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\big(\mathscr{O}_X, \mathscr{F}[i]\big)$$
$$\mathrm{Ext}^{n-i}(\mathscr{F}, \omega_X) \simeq \mathrm{Hom}\big(\mathscr{F}[i], \omega_X[n]\big).$$

Serre duality can therefore be rewritten in the form

$$\mathrm{Hom}\big(F, \omega_X[n]\big) \simeq \mathrm{Hom}\big(\mathrm{Hom}(\mathscr{O}_X, F), \mathbb{C}\big),$$

where $F = \mathscr{F}[i]$. Using suitable resolutions, this can be improved to the following general result in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$.

Theorem 17.1. *Let X be a smooth projective variety, and let F and G be two objects of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. Then one has an isomorphism of $\mathbb{C}$-vector spaces*

$$\mathrm{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\Big(F, G \otimes \omega_X[n]\Big) \simeq \mathrm{Hom}\Big(\mathrm{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}(G, F), \mathbb{C}\Big)$$

that is functorial in F and G.

Grothendieck duality is a relative version of Serre duality, where instead of a single variety, one has a proper morphism $f\colon X \to Y$. In Grothendieck's formulation, duality becomes a statement about certain functors: we have the derived pushforward functor $\mathbf{R}f_*\colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y)$, and the problem is to construct a right adjoint $f^!\colon \mathrm{D}^b_{coh}(\mathscr{O}_Y) \to \mathrm{D}^b_{coh}(\mathscr{O}_X)$, pronounced "$f$-shriek". In other words, we would like to define $f^!$ in such a way that we have functorial isomorphisms

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_Y)}\bigl(\mathbf{R}f_* F, G\bigr) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\bigl(F, f^! G\bigr)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. For arbitrary proper morphisms, the construction requires considerable technical effort; it is explained in Hartshorne's book *Residues and Duality.* (There is also a modern treatment by Amnon Neeman, based on the Brown' representability theorem.) But in the special case that both X and Y are smooth projective, there is a much simpler construction due to Alexei Bondal and Mikhail Kapranov.

Theorem 17.2. *If $f\colon X \to Y$ is a morphism between two smooth projective varieties, then*

$$f^! G = \omega_X[\dim X] \otimes \mathbf{L}f^*\bigl(G \otimes \omega_Y^{-1}[-\dim Y]\bigr)$$

for any $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. This follows very easily from the fact that $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$ – if we use Serre duality to interchange left and right. Fix two objects $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. Applying Serre duality on Y, we get

$$\operatorname{Hom}\bigl(\mathbf{R}f_* F, G \otimes \omega_Y[\dim Y]\bigr) \simeq \operatorname{Hom}\Bigl(\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr), \mathbb{C}\Bigr).$$

Because $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$, we have

$$\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr) \simeq \operatorname{Hom}\bigl(\mathbf{L}f^* G, F\bigr).$$

If we now apply Serre duality on X, we get back to

$$\operatorname{Hom}\Bigl(\operatorname{Hom}\bigl(\mathbf{L}f^* G, F\bigr), \mathbb{C}\Bigr) \simeq \operatorname{Hom}\bigl(F, \mathbf{L}f^* G \otimes \omega_X[\dim X]\bigr).$$

Putting all three isomorphisms together, we obtain the desired formula for $f^! G$. $\square$

For a more concise statement, let $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$ denote the relative canonical bundle; then the formula in Theorem 17.2 becomes

$$f^! = \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*.$$

Note that $\dim X - \dim Y$ is simply the relative dimension of the morphism f. To summarize, we have a functorial isomorphism

$$\operatorname{Hom}\Bigl(\mathbf{R}f_* F, G\Bigr) \simeq \operatorname{Hom}\Bigl(F, \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^* G\Bigr)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. In this form, Grothendieck duality will appear frequently in the derived category calculations below.

Flat base change. Another technical result that we shall use below is the base change theorem. As in the case of Grothendieck duality, there is a very general statement (in the derived category); for our purposes, however, two special cases are enough, and so we shall restrict our attention to those.

The general problem addressed by the base change theorem is the following. Suppose we have a cartesian diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

We would like to compare the two functors $g^* f_*$ and $f'_* g'^*$; more generally, on the level of the derived category, the two functors $\mathbf{L}g^*\mathbf{R}f_*$ and $\mathbf{R}f'_*\mathbf{L}g'^*$. Using the adjointness of pullback and pushforward, we always have morphisms of functors

$$g^* f_* \to f'_* g'^* \quad \text{and} \quad \mathbf{L}g^*\mathbf{R}f_* \to \mathbf{R}f'_*\mathbf{L}g'^*,$$

but without some assumptions on f or g – or on the sheaves or complexes to which we apply the functors – they are not isomorphisms.

The simplest case where the two functors are isomorphic is when g (and hence also g') is flat. We begin by looking at the case of sheaves.

Lemma 17.3. *Suppose that g is flat, and that f is separated and quasi-compact. Then the base change morphism*

$$g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$$

is an isomorphism for every quasi-coherent sheaf $\mathscr{F}$ on X.

Proof. The statement is local on Y and Y', and so we may assume without loss of generality that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$ are affine, with A' flat over A. Let $\mathscr{F}' = g'^*\mathscr{F}$; then all sheaves involved are quasi-coherent on Y', and so it suffices to show that

$$\mathscr{F}(X) \otimes_A A' \to \mathscr{F}'(X')$$

is an isomorphism.

We first consider the case when $X = \operatorname{Spec} B$ is also affine; in that case, $X' = \operatorname{Spec} A' \otimes_A B$. We have $\mathscr{F} = \tilde{M}$ for some B-module M; then $g^* f_* \mathscr{F}$ is the quasi-coherent sheaf corresponding to the A'-module

$$A' \otimes_A M_A,$$

while $f'_* g'^* \mathscr{F}$ is the quasi-coherent sheaf corresponding to

$$(A' \otimes_A B) \otimes_B M.$$

The two are evidently isomorphic, which proves the assertion in case X is affine. In general, cover X by finitely many affine open subsets $U_1, \dots, U_n$. Because $\mathscr{F}$ is a sheaf, the complex of A-modules

$$0 \to \mathscr{F}(X) \to \bigoplus_{i=1}^n \mathscr{F}(U_i) \to \bigoplus_{i,j=1}^n \mathscr{F}(U_i \cap U_j)$$

is exact. Now A' is flat over A, and so

$$0 \to \mathscr{F}(X) \otimes_A A' \to \bigoplus_{i=1}^n \mathscr{F}(U_i) \otimes_A A' \to \bigoplus_{i,j=1}^n \mathscr{F}(U_i \cap U_j) \otimes_A A'$$

remains exact. We conclude from the affine case above that the kernel is isomorphic to $\mathscr{F}'(X')$, which is the result we were after. $\square$

In the derived category, we have the following version.

Proposition 17.4. *Suppose that g is flat, and the f is separated and quasi-compact. Then for any $F \in \mathrm{D}^+\big(\mathrm{QCoh}(X)\big)$, the base change morphism*

$$\mathbf{L}g^*\mathbf{R}f_* F \to \mathbf{R}f'_*\mathbf{L}g'^* F$$

is an isomorphism.

Proof. After replacing F by an injective resolution, we may assume without loss of generality that F is a complex of injective quasi-coherent sheaves. The result now follows by applying Lemma 17.3 termwise. $\square$

Mukai's Fourier transform. From now on, let's write $\mathrm{D}^b(X)$ for the bounded derived category of coherent sheaves on X. It may happen that two smooth projective varieties X and Y have isomorphic derived categories, without X and Y themselves being isomorphic.[1] The first interesting example of this was discovered by Mukai: if X is an abelian variety, and $\hat{X}$ the dual abelian variety, then

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(\hat{X}).$$

Here "isomorphic" means that there is an exact k-linear equivalence between the two categories. This equivalence comes from the Poincaré bundle P on the product $X \times \hat{X}$, using the projections to the two factors:

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{X} \\ X & \end{tikzcd}$$

Given a complex $K \in \mathrm{D}^b(X)$, we can define its "Fourier transform"

$$\mathbf{R}\Phi_P(K) = \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes P\big)$$

which is an object in $\mathrm{D}^b(\hat{X})$. Because p_1 is flat, the functor p_1^* is already exact; similarly, P is a line bundle, and so the tensor product with P is also exact. So the only genuinely derived functor is $\mathbf{R}(p_2)_*$, and so $\mathbf{R}\Phi_P$ really is the derived functor of the naive functor $\mathscr{F} \mapsto (p_2)_*(p_1^*\mathscr{F} \otimes P)$ on sheaves. Mukai called this the "Fourier transform" because of its formal similarities with the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$$

for L^1-functions on $\mathbb{R}$. (In this analogy, complexes of sheaves are functions; tensoring with P is multiplication by the exponential function; and the direct image along p_2 is integration along the fibers.)

With that in mind, Mukai's theorem is as follows.

Theorem 17.5 (Mukai)**.** *Let X be an abelian variety, and let $\hat{X}$ be the dual abelian variety. Then the Fourier transform*

$$\mathbf{R}\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$$

is an equivalence of categories.

Note that X and $\hat{X}$ are usually not isomorphic; but they are nevertheless related on the level of the derived category.

General integral transforms. The Fourier transform is an example of what people call an "integral transform" (or "Fourier-Mukai transform") between derived categories. Suppose that X and Y are two smooth projective varieties, and that $E \in \mathrm{D}^b(X \times Y)$ is an object on the product. We can then define an exact functor

$$\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

by the same formula as above:

$$\mathbf{R}\Phi_E(K) = \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* \overset{\mathbf{L}}{\otimes} E\big),$$

but now the tensor product is also derived (because E is no longer locally free). The object E is called the "kernel" of the transform; the name again comes from integral transforms on function spaces (where the kernel is some kind of function or distribution on the product).

[1] Bondal and Orlov proved that if the (anti-)canonical bundle of X is ample, then any isomorphism $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ comes from an isomorphism $X \cong Y$.

Example 17.6. A basic example is $E = \Delta_* \mathscr{O}_X$, the structure sheaf of the diagonal on the product $X \times X$. In this case, $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is the identity.

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\ & X & \end{tikzcd}$$

Indeed, for any object $K \in \mathrm{D}^b(X)$, we have

$$\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} \Delta_* \mathscr{O}_X \cong \mathbf{R}\Delta_* K$$

by the projection formula, and therefore

$$\mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_2)_* \mathbf{R}\Delta_* K \cong K.$$

Example 17.7. More generally, take a morphism $f \colon X \to Y$, look at its graph

$$\Gamma_f \colon X \to X \times Y, \quad \Gamma_f(x) = \big(x, f(x)\big),$$

and use the object $E = (\Gamma_f)_* \mathscr{O}_X$ on the product $X \times Y$ as the kernel of an integral transform. The following diagram shows the relevant morphisms:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

By exactly the same computation as above, we have $\mathbf{R}\Phi_E = \mathbf{R}f_* \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. If we swap the roles of X and Y, and denote by

$$\mathbf{R}\Psi_E \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$$

the integral transform with kernel E going the other way, then we have

$$\mathbf{L}p_2^* K \overset{\mathbf{L}}{\otimes} (\Gamma_f)_* \mathscr{O}_X \cong \mathbf{R}\Delta_* \mathbf{L}f^* K,$$

and therefore $\mathbf{R}\Psi_E(K) \cong \mathbf{R}(p_1)_* \mathbf{R}\Delta_* \mathbf{L}f^* K \cong \mathbf{L}f^* K$. So both $\mathbf{R}f_*$ and $\mathbf{L}f^*$ are special cases of integral transforms.

Let's check that the composition of two integral transforms is again an integral transform. Say $E \in \mathrm{D}^b(X \times Y)$ and $F \in \mathrm{D}^b(Y \times Z)$ are two kernels. Consider the composition

$$\mathbf{R}\Phi_F \circ \mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Z).$$

To work out what this does, we are going to use the following big diagram:

$$\begin{tikzcd} X \times Y \times Z \arrow[r, "p_{23}"] \arrow[rr, bend left, "p_3"] \arrow[d, "p_{12}"] \arrow[dd, bend right, "p_1"'] & Y \times Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Z \\ X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y & \\ X & & \end{tikzcd}$$

Let $K \in \mathrm{D}^b(X)$ be any object. Then

$$\mathbf{R}\Phi_E(K) = \mathbf{R}(p_2)_* \big(\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} E\big).$$

In order to compute $\mathbf{L}p_1^*$ of this complex, we can use flat base change (along the projection $p_1 \colon Y \times Z \to Y$). This gives

$$\mathbf{L}p_1^*\mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_{23})_*\big(\mathbf{L}p_{12}^*(\mathbf{L}p_1^*K \overset{\mathbf{L}}{\otimes} E)\big) \cong \mathbf{R}(p_{23})_*\big(\mathbf{L}p_1^*K \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{12}^*E\big).$$

Tensoring by F and pushing forward to Z then produces

$$\begin{aligned}\mathbf{R}\Phi_F\mathbf{R}\Phi_E(K) &\cong \mathbf{R}(p_2)_*\Big(\mathbf{R}(p_{23})_*\big(\mathbf{L}p_1^*K \otimes \mathbf{L}p_{12}^*E\big) \overset{\mathbf{L}}{\otimes} F\Big) \\ &\cong \mathbf{R}(p_2)_*\mathbf{R}(p_{23})_*\Big(\mathbf{L}p_1^*K \otimes \mathbf{L}p_{12}^*E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^*F\Big) \\ &\cong \mathbf{R}(p_3)_*\Big(\mathbf{L}p_1^*K \otimes \mathbf{L}p_{12}^*E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^*F\Big);\end{aligned}$$

to go from the first to the second line, we used the projection formula (for the morphism $p_{23} \colon X \times Y \times Z \to Y \times Z$. If we now use the factorization

$$\begin{tikzcd} X \times Y \times Z \arrow[r, "p_{13}"] \arrow[rr, bend left, "p_3"] \arrow[dr, bend right, "p_1"'] & X \times Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Z \\ & X & \end{tikzcd}$$

and apply the projection formula one more time, we can rewrite this as

$$\mathbf{R}\Phi_F\mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_2)_*\Big(\mathbf{L}p_1^* \overset{\mathbf{L}}{\otimes} \mathbf{R}(p_{13})_*\big(\mathbf{L}p_{12}^*E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^*F\big)\Big).$$

The composition is therefore again an integral transform, with kernel the object

$$E * F = \mathbf{R}(p_{13})_*\big(\mathbf{L}p_{12}^*E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^*F\big)\Big) \in \mathrm{D}^b(X \times Z).$$

This object is called the "convolution" of the two kernels $E \in \mathrm{D}^b(X \times Y)$ and $F \in \mathrm{D}^b(Y \times Z)$, again by analogy with the convolution of two functions (which is defined by integration over a common argument). With this notation, we have

$$\mathbf{R}\Phi_F \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*F}, \tag{17.8}$$

where "isomorphism" really means that the two functors are related by a natural isomorphism. This kind of computation – using flat base change and the projection formula – is very typical in the subject.

Example 17.9. In order to show that an integral transform $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence, it is enough to find an object $F \in \mathrm{D}^b(Y \times X)$ such that $E * F \cong \Delta_*\mathscr{O}_X$ is the structure sheaf of the diagonal on $X \times X$, and $F * E \cong \Delta_*\mathscr{O}_Y$ is the structure sheaf of the diagonal on $Y \times Y$. The reason is that the structure sheaf of the diagonal represents the identity.

Let me also mention, without proof, the following very nice theorem by Orlov.

Theorem 17.10. *Let X and Y be two smooth projective varieties. Then any (exact and k-linear) equivalence of categories $F \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is of the form $F \cong \mathbf{R}\Phi_E$ for an object $E \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism.*

Thinking of E as the family of objects

$$E_x = E|_{\{x\} \times Y} \in \mathrm{D}^b(Y),$$

parametrized by the closed point $x \in X(k)$, one necessarily has

$$E_x \cong F\big(k(x)\big),$$

the image of the skyscraper sheaf $k(x) \in \mathrm{D}^b(X)$ under the equivalence F. The difficult thing is to show that these objects actually fit together into a complex of coherent sheaves on $X \times Y$.

The symmetric Fourier transform. As a postdoc, when I was doing a lot of computations with Mukai's Fourier transform, I found that I could never remember all the formulas, and so each time I wanted to prove something, I had to go back to Mukai's paper and look up the correct formula. (There are shifts by $\pm \dim X$, signs, and inverses, and it is hard to remember which goes where.) This eventually led me to write a paper with the grand title "The Fourier-Mukai transform made easy", whose main point was that one can change the definition of the Fourier transform very slightly, and make all the formulas easy to remember. The idea is to use the (contravariant) Grothendieck duality functor

$$\mathbf{R}\Delta_X = \mathbf{R}\mathcal{H}om\big(-, \omega_X[\dim X]\big) \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)^{op},$$

where $\omega_X = \det \Omega^1_{X/k}$ is the canonical bundle of the smooth projective variety X. In the case $X = \operatorname{Spec} k$, we shall use the simplified notation $\mathbf{R}\Delta_k$.

Definition 17.11. Let X be an abelian variety, and let $P = P_X$ be the Poincaré bundle on $X \times \hat{X}$. The exact functor

$$\mathsf{FM}_X = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op}$$

is called the *symmetric Fourier-Mukai transform*.

Note that FM_X is a *contravariant* functor; this turns out to be quite useful in practice. The following theorem justifies the name "symmetric Fourier-Mukai transform"; it is of course equivalent to Mukai's theorem (because the duality functor is a contravariant equivalence of categories).

Theorem 17.12. *The composed functors* $\mathsf{FM}_{\hat{X}} \circ \mathsf{FM}_X$ *and* $\mathsf{FM}_X \circ \mathsf{FM}_{\hat{X}}$ *are naturally isomorphic to the identity. In other words,*

$$\mathsf{FM}_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op}$$

is an equivalence of categories, with quasi-inverse $\mathsf{FM}_{\hat{X}}$.

One advantage of the modified definition is that it respects the symmetry between the two abelian varieties X and $\hat{X}$. For example, one can show that

$$\mathsf{FM}_X\big(k(0)\big) = \mathscr{O}_{\hat{X}} \quad \text{and} \quad \mathsf{FM}_X(\mathscr{O}_X) = k(0). \tag{17.13}$$

Here $k(0) = e_* \mathscr{O}_{\operatorname{Spec} k}$ means the structure sheaf of the closed point $0 \in X(k)$; we use the same notation also on $\hat{X}$.

Let's verify the two identities in (17.13). The first one is very easy: Grothendieck duality, applied to the morphism $e \colon \operatorname{Spec} k \to X$, gives

$$\mathbf{R}\Delta_X\big(k(0)\big) = e_* \mathbf{R}\Delta_{\operatorname{Spec} k}\big(\mathscr{O}_{\operatorname{Spec} k}\big) = e_* \mathscr{O}_{\operatorname{Spec} k} = k(0),$$

and so the symmetric Fourier-Mukai transform of $k(0)$ is

$$\mathsf{FM}_X\big(k(0)\big) = \mathbf{R}\Phi_P\big(k(0)\big) = \mathscr{O}_X.$$

The second isomorphism comes from the fact that $(e \times \mathrm{id})^* P = P|_{\{0\} \times \hat{X}}$ is trivial. In exactly the same way, one can show that

$$\mathsf{FM}_X\big(k(x)\big) = P|_{\{x\} \times \hat{X}}.$$

The Fourier-Mukai transform therefore takes structure sheaves of points to line bundles in $\operatorname{Pic}^0(\hat{X})$.

For the second identity in (17.13), we need to compute

$$\mathsf{FM}_X(\mathscr{O}_X) = \mathbf{R}\Phi_P\Big(\omega_X[\dim X]\Big) = \mathbf{R}(p_2)_*\Big(P \otimes p_1^* \omega_X[\dim X]\Big).$$

Recall from (15.13) that we have

$$R^i(p_2)_* P \cong \begin{cases} 0 & \text{if } i \neq \dim X, \\ k(0) & \text{if } i = \dim X. \end{cases}$$

In terms of the derived category, this says that $\mathbf{R}(p_2)_* P \cong k(0)[-\dim X]$. If we put this together with the formula above, and remember that ω_X is trivial, we get

$$\mathsf{FM}_X(\mathscr{O}_X) \cong k(0),$$

as required. We will prove later that for any $L \in \operatorname{Pic}^0(X)$, one has

$$\mathsf{FM}_X(L) \cong k(\alpha),$$

where $\alpha \in \hat{X}(k)$ is the unique closed point corresponding to L under the isomorphism of groups $\hat{X}(k) \cong \operatorname{Pic}^0(X)$.

Lecture 18 (April 3)

Let X be an abelian variety, $\hat{X}$ the dual abelian variety, and P_X the Poincaré bundle on $X \times \hat{X}$. Last time, we introduced the symmetric Fourier-Mukai transform

$$\mathsf{FM}_X = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op},$$

which is defined as the composition of the (contravariant) duality functor

$$\mathbf{R}\Delta_X = \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(-, \omega_X[\dim X]\big) \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)^{op}$$

with Mukai's original Fourier transform

$$\mathbf{R}\Phi_P(K) = \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* K \otimes P\big).$$

The main theorem is that the two contravariant functors

$$\mathsf{FM}_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op} \quad \text{and} \quad \mathsf{FM}_{\hat{X}} \colon \mathrm{D}^b(\hat{X}) \to \mathrm{D}^b(X)^{op}$$

are mutually inverse equivalences of category.

Proof of Mukai's theorem. For clarity, let's denote the Poincaré bundle P_X on $X \times \hat{X}$ by the symbol P, and the Poincaré bundle $P_{\hat{X}}$ on $\hat{X} \times X$ by the symbol $\hat{P}$. The symmetric description of the dual abelian variety (in Lecture 14) shows that

$$\hat{P} \cong \sigma^* P$$

where $\sigma \colon X \times \hat{X} \to \hat{X} \times X$ swaps the two factors.

Now let's begin proving Mukai's theorem. Since we can interchange the role of X and $\hat{X}$, we only need to prove that the functor

$$\mathsf{FM}_{\hat{X}} \circ \mathsf{FM}_X = \mathbf{R}\Phi_{\hat{P}} \circ \mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \tag{18.1}$$

is naturally isomorphic to the identity. We are going to use the standard derived category tools (such as flat base change and Grothendieck duality) to show that the composition is an integral transform (with a kernel on $X \times X$); and then we'll use properties of the Poincaré bundle to prove that the kernel is the structure sheaf of the diagonal (and hence that the composition is the identity).

Let's first consider the last three terms; they give us a covariant functor

$$\mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X}).$$

A brief computation using Grothendieck duality shows that this functor is an integral transform, whose kernel is the complex

$$P^{-1} \otimes p_2^* \omega_{\hat{X}}[g] \tag{18.2}$$

on $X \times \hat{X}$; here $g = \dim X$. To see this, we take $K \in \mathrm{D}^b(X)$, and compute:

$$\begin{aligned}\mathbf{R}\Phi_P \mathbf{R}\Delta_X(K) &= \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^*\mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(K, \omega_X[g]) \otimes P\big)\\ &\cong \mathbf{R}(p_2)_*\big(\mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times\hat{X}}}(\mathbf{L}p_1^*K, p_1^*\omega_X[g]) \otimes P\big).\end{aligned}$$

The local version of Grothendieck duality gives $\mathbf{R}\Delta_{\hat{X}} \circ \mathbf{R}(p_2)_* \cong \mathbf{R}(p_2)_*\mathbf{R}\Delta_{X\times\hat{X}}$. If we apply this to the result of the preceding computation, we get

$$\begin{aligned}\mathbf{R}\Delta_{\hat{X}}\mathbf{R}\Phi_P\mathbf{R}\Delta_X(K) &\cong \mathbf{R}(p_2)_*\mathbf{R}\Delta_{X\times\hat{X}}\Big(\mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times\hat{X}}}(\mathbf{L}p_1^*K, p_1^*\omega_X[g]) \otimes P\Big)\\ &\cong \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^*K \otimes p_2^*\omega_{\hat{X}}[g] \otimes P^{-1}\big),\end{aligned}$$

because $\omega_{X\times\hat{X}} \cong p_1^*\omega_X \otimes p_2^*\omega_{\hat{X}}$, and because the two $\mathbf{R}\mathcal{H}om$'s cancel each other. This is an integral transform with kernel (18.2).

Now we need to compose this with $\mathbf{R}\Phi_{\hat{P}}$. By the computation from last time, the composition is again an integral transform; the kernel is the convolution of (18.2) on $X \times \hat{X}$ with the line bundle $\hat{P}$ on $\hat{X} \times X$. Thus (18.1) is also an integral transform, with kernel

$$\mathbf{R}(p_{13})_*\Big(p_{12}^*P^{-1} \otimes p_2^*\omega_{\hat{X}}[g] \otimes p_{23}^*\hat{P}\Big).$$

In order to avoid ridiculous notation in the computation below, we now swap the second and third factor in $X \times \hat{X} \times X$. Since $\hat{P} \cong \sigma^*P$, we can then rewrite the kernel for (18.1) as

$$\mathbf{R}(p_{12})_*\Big(p_3^*\omega_{\hat{X}}[g] \otimes p_{13}^*P^{-1} \otimes p_{23}^*P\Big) \tag{18.3}$$

on $X \times X$. Theorem 17.12 will be proved once we show that (18.3) is isomorphic to the structure sheaf of the diagonal in $X \times X$.

Let $s\colon X \times X \to X$ be defined as $s = m \circ (i \times \mathrm{id})$; the formula on closed points is $s(x, y) = y - x$. The theorem of the cube shows that

$$p_{13}^*P^{-1} \otimes p_{23}^*P \cong (s \times \mathrm{id})^*P.$$

We can now apply flat base change in the commutative diagram

$$\begin{array}{ccc} X \times X \times \hat{X} & \xrightarrow{p_{12}} & X \times X \\ \Big\downarrow{\scriptstyle s\times\mathrm{id}} & & \Big\downarrow{\scriptstyle s} \\ X \times \hat{X} & \xrightarrow{p_1} & X \end{array}$$

and conclude that the complex in (18.3) is isomorphic to

$$\mathbf{L}s^*\,\mathbf{R}(p_1)_*\Big(P \otimes p_2^*\omega_{\hat{X}}[g]\Big) = \mathbf{L}s^*\,\mathsf{FM}_{\hat{X}}(\mathscr{O}_{\hat{X}}) = \mathbf{L}s^*k(0) = \Delta_*\mathscr{O}_X,$$

where $\Delta\colon X \to X \times X$ is the diagonal embedding. Here we used the fact that the symmetric Fourier-Mukai transform of the structure sheaf $\mathscr{O}_{\hat{X}}$ is the structure sheaf $k(0)$ of the closed point $0 \in X(k)$, as in (17.13). Because the integral transform with kernel $\Delta_*\mathscr{O}_X$ is the identity, this concludes the proof of Theorem 17.12.

Properties of the Fourier-Mukai transform. If we wanted to summarize the above proof in one line, it would be that the Fourier-Mukai transform is an equivalence because of the identity $p_{13}^*P^{-1} \otimes p_{23}^*P \cong (s \times \mathrm{id})^*P$ for the Poincaré bundle. The other formulas involving the Poincaré bundle that we have proved also lead to interesting properties of FM_X.

The first topic is how the Fourier-Mukai transform interacts with pulling back or pushing forward by a homomorphism between abelian varieties. Mukai only looked at the case of isogenies; the general case is due to Chen and Jiang.

Proposition 18.4. *Let $f\colon X \to Y$ be a homomorphism of abelian varieties over k. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathsf{FM}_X \quad \textit{and} \quad \mathsf{FM}_X \circ \mathbf{L}f^* \cong \mathbf{R}\hat{f}_* \circ \mathsf{FM}_Y,$$

where $\hat{f}\colon \hat{Y} \to \hat{X}$ is the induced homomorphism between the dual abelian varieties.

Proof. It will be enough to show that

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathsf{FM}_X;$$

the second identity in the theorem follows from this with the help of Theorem 17.12. Using the definition of FM_Y and Grothendieck duality, we obtain

$$\mathsf{FM}_Y \circ \mathbf{R}f_* \cong \mathbf{R}\Phi_{P_X} \circ \mathbf{R}\Delta_Y \circ \mathbf{R}f_* \cong \mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* \circ \mathbf{R}\Delta_X.$$

This reduces the problem to proving that

$$\mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_X}. \tag{18.5}$$

We make use of the following commutative diagram:

$$\begin{tikzcd}
\hat{X} & \hat{Y} \arrow[l, "\hat{f}"'] & & \\
X \times \hat{X} \arrow[u, "p_2"] & X \times \hat{Y} \arrow[l, "\mathrm{id} \times \hat{f}"'] \arrow[u, "p_2"] \arrow[r, "f \times \mathrm{id}"] \arrow[d, "p_1"] & Y \times \hat{Y} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{Y} \\
 & X \arrow[r, "f"] & Y &
\end{tikzcd}$$

The identity in (14.2), which followed from the universal property of the dual abelian variety, gives us $(f \times \mathrm{id})^* P_Y \cong (\mathrm{id} \times \hat{f})^* P_X$. Using the projection formula and flat base change, we can write the following chain of isomorphisms:

$$\begin{aligned}
\mathbf{R}\Phi_{P_Y} \circ \mathbf{R}f_* &\cong \mathbf{R}(p_2)_*\bigl(P_Y \otimes p_1^* \mathbf{R}f_*\bigr) \cong \mathbf{R}(p_2)_*\bigl(P_Y \otimes \mathbf{R}(f \times \mathrm{id})_* p_1^*\bigr) \\
&\cong \mathbf{R}(p_2)_*\bigl((f \times \mathrm{id})^* P_Y \otimes p_1^*\bigr) \cong \mathbf{R}(p_2)_*\bigl((\mathrm{id} \times \hat{f})^* P_X \otimes p_1^*\bigr) \\
&\cong \mathbf{R}(p_2)_* \mathbf{L}(\mathrm{id} \times \hat{f})^* \bigl(P_X \otimes p_1^*\bigr) \cong \mathbf{L}\hat{f}^* \, \mathbf{R}(p_2)_*\bigl(P_X \otimes p_1^*\bigr) \\
&\cong \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_X}
\end{aligned}$$

This calculation establishes Proposition 18.4. ∎

The symmetric Fourier-Mukai transform also exchanges translations and tensoring by the corresponding line bundles. Any closed point $x \in X(k)$ determines a translation morphism $t_x\colon X \to X$; on closed points, it is given by the formula $t_x(y) = x + y$. Since $X(k) \cong \mathrm{Pic}^0(\hat{X})$, it also determines a line bundle $\hat{P}_x \in \mathrm{Pic}^0(\hat{X})$.

Proposition 18.6. *Let $x \in X(k)$ and $\alpha \in \hat{X}(k)$ be closed points. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_X \circ (t_X)_* = (\hat{P}_X \otimes -) \circ \mathsf{FM}_X \quad \textit{and} \quad \mathsf{FM}_X \circ (P_\alpha \otimes -) = (t_\alpha)_* \circ \mathsf{FM}_X,$$

where $\hat{P}_x \in \mathrm{Pic}^0(X)$, and $P_\alpha \in \mathrm{Pic}^0(\hat{X})$, are the corresponding line bundles.

Together with (17.13), this leads to the pleasant formulas

$$\mathsf{FM}_X\bigl(k(x)\bigr) = \hat{P}_x \quad \text{and} \quad \mathsf{FM}_X(P_\alpha) = k(\alpha),$$

for any pair of closed points $x \in X(k)$ and $\alpha \in \hat{X}(k)$. This symmetry is another reason for the name “symmetric” Fourier-Mukai transform.

Proof. Once again, it suffices to prove that

$$\mathsf{FM}_X \circ (t_x)_* = (\hat{P}_x \otimes -) \circ \mathsf{FM}_X$$

because the other identity follows from this with the help of Theorem 17.12. Using Grothendieck duality, we get a natural isomorphism of functors

$$\mathsf{FM}_X \circ (t_x)_* = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \circ (t_x)_* = \mathbf{R}\Phi_P \circ (t_x)_* \circ \mathbf{R}\Delta_X,$$

and so the problem is reduced to showing that

$$\mathbf{R}\Phi_P \circ (t_x)_* = (\hat{P}_x \otimes -) \circ \mathbf{R}\Phi_P.$$

We use the following commutative diagram:

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "t_x \times \mathrm{id}"] \arrow[d, "p_1"] & X \times \hat{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{X} \\ X \arrow[r, "t_x"] & X \end{tikzcd}$$

Since $(t_x \times \mathrm{id})^* P \cong p_2^* \hat{P}_x \otimes P$ (by the seesaw theorem), we have

$$\begin{aligned} \mathbf{R}\Phi_P \circ (t_x)_* &= \mathbf{R}(p_2)_* \Big(P \otimes p_1^* (t_x)_* \Big) = \mathbf{R}(p_2)_* \Big(P \otimes (t_x \times \mathrm{id})_* p_1^* \Big) \\ &= \mathbf{R}(p_2)_* \Big((t_x \times \mathrm{id})^* P \otimes p_1^* \Big) = \mathbf{R}(p_2)_* \Big(p_2^* \hat{P}_x \otimes P \otimes p_1^* \Big) \\ &= \hat{P}_x \otimes \mathbf{R}(p_2)_* \big(P \otimes p_1^* \big) = \hat{P}_x \otimes \mathbf{R}\Phi_P, \end{aligned}$$

which is exactly what we need. □

The third property is more of an extended example. Let L be an ample line bundle on the abelian variety X. Mukai's Fourier transform

$$\mathbf{R}\Phi_P(L) = \mathbf{R}(p_2)_* \big(p_1^* L \otimes P \big)$$

is a vector bundle of rank $\dim H^0(X, L)$ on the dual abelian variety $\hat{X}$. The reason is that on each fiber of $p_2 \colon X \times \hat{X} \to \hat{X}$, the line bundle $L \otimes P_\alpha$ is again ample, and so all of its higher cohomology groups vanish; we know this at least over $\mathbb{C}$, where it follows from the Kodaira vanishing theorem. By cohomology and base change, we therefore have $R^i(p_2)_* \big(p_1^* L \otimes P \big) = 0$ for $i \neq 0$; for $i = 0$, we get a locally free sheaf $\mathscr{E}_L$ of rank $\dim H^0(X, L)$.

To see what $\mathscr{E}_L$ actually looks like, let's pull it back by the isogeny

$$\phi_L \colon X \to \hat{X};$$

recall that this has the property that $t_x^* L \cong L \otimes P_{\phi_L(x)}$ for all closed points $x \in X(k)$. The key identity (which we used for the construction of the Poincaré bundle) is that

$$(\mathrm{id} \times \phi_L)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

Now let's do the computation, using the following commutative diagram:

$$\begin{tikzcd} X \times X \arrow[r, "p_2"] \arrow[d, "\mathrm{id} \times \phi_L"] & X \arrow[d, "\phi_L"] \\ X \times \hat{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{X} \\ X \end{tikzcd}$$

Applying flat base change and the projection formula, we get

$$\begin{aligned}\phi_L^* \mathscr{E}_L = \mathbf{L}\phi_L^* \mathbf{R}(p_2)_* \big(P \otimes p_1^* L\big) &\cong \mathbf{R}(p_2)_* \big((\mathrm{id} \times \phi_L)^* P \otimes p_1^* L\big) \\ &\cong \mathbf{R}(p_2)_* \big(m^* L \otimes p_2^* L^{-1}\big) \cong L^{-1} \otimes \mathbf{R}(p_2)_* m^* L.\end{aligned}$$

Now we need a small trick. We can write $m\colon X \times X \to X$ as a composition

$$X \times X \xrightarrow{f} X \times X \xrightarrow{p_1} X,$$

where $f\colon X \times X \to X \times X$ is the automorphism $f(x, y) = (x + y, y)$. Therefore

$$\mathbf{R}(p_2)_* m^* L \cong \mathbf{R}(p_2)_* \mathbf{R} f_* (f^* p_1^* L) \cong \mathbf{R}(p_2)_* p_1^* L \cong H^0(X, L) \otimes \mathscr{O}_X,$$

where the second step is the projection formula, and the third flat base change. So

$$\phi_L^* \mathscr{E}_L \cong H^0(X, L) \otimes L^{-1}. \tag{18.7}$$

Note that L was ample, but that Mukai's Fourier transform $\mathbf{R}\Phi_P$ takes it to the *dual* of an ample vector bundle.

Here is the result for the symmetric Fourier-Mukai transform; this is better, because positivity is preserved.

Proposition 18.8. *Let L be an ample line bundle on X. Then $\mathsf{FM}_X(L)$ is an ample vector bundle of rank $\dim H^0(X, L)$, and one has*

$$\phi_L^* \,\mathsf{FM}_X(L) \cong i^* L \otimes H^0(X, L)^*,$$

where $i\colon X \to X$ is the inversion morphism.

This follows directly from (18.7), together with the following formula (that gives an alternative description of the symmetric Fourier-Mukai transform):

$$\mathsf{FM}_X(K) \cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(i^* \mathbf{R}\Phi_P(K), \mathscr{O}_{\hat{X}}\big) \tag{18.9}$$

To prove it, we take an object $K \in \mathrm{D}^b(X)$ and start computing:

$$\mathsf{FM}_X(K) = \mathbf{R}(p_2)_* \Big(\mathbf{L}p_1^* \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(K, \omega_X[g]\big) \otimes P\Big)$$

We would like to interchange $\mathbf{R}(p_2)_*$ and $\mathbf{R}\mathcal{H}om$, and for that, we need to move all the terms on the right-hand side into the first argument of $\mathbf{R}\mathcal{H}om$. Here it helps that $\omega_{X \times \hat{X}} \cong p_1^* \omega_X \otimes p_2^* \omega_{\hat{X}}$ and that $P \cong (i \times \mathrm{id})^* P^{-1}$. Accordingly,

$$\begin{aligned}\mathbf{L}p_1^* \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(K, \omega_X[g]\big) \otimes P &\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X \times \hat{X}}}\Big(\mathbf{L}p_1^* K, p_1^* \omega_X[g]\Big) \otimes (i \times \mathrm{id})^* P^{-1}\Big) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X \times \hat{X}}}\Big(\mathbf{L}p_1^* K \otimes (i \times \mathrm{id})^* P \otimes p_2^* \omega_{\hat{X}}[g], \omega_{X \times \hat{X}}[2g]\Big) \\ &= \mathbf{R}\Delta_{X \times \hat{X}}\Big(\mathbf{L}p_1^* K \otimes (i \times \mathrm{id})^* P \otimes p_2^* \omega_{\hat{X}}[g]\Big)\end{aligned}$$

If we put this into the formula from above and use the relative version of Grothendieck duality, we obtain

$$\begin{aligned}\mathsf{FM}_X(K) &\cong \mathbf{R}\Delta_{\hat{X}} \mathbf{R}(p_2)_* \Big(\mathbf{L}p_1^* K \otimes (i \times \mathrm{id})^* P \otimes p_2^* \omega_{\hat{X}}[g]\Big) \\ &\cong \mathbf{R}\Delta_{\hat{X}}\Big(\omega_{\hat{X}}[g] \otimes \mathbf{R}(p_2)_* \big(\mathbf{L}p_1^* K \otimes (i \times \mathrm{id})^* P\big)\Big) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\hat{X}}}\Big(\mathbf{R}(p_2)_* \big(\mathbf{L}p_1^* K \otimes (i \times \mathrm{id})^* P\big), \mathscr{O}_{\hat{X}}\Big) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\hat{X}}}\Big(\mathbf{L}i^* \mathbf{R}\Phi_P(K), \mathscr{O}_{\hat{X}}\Big).\end{aligned}$$

The last step follows from the projection formula. So (18.9) is proved.

Remark. Observe that L has rank 1 and $h^0(X, L)$ many global sections, whereas $\mathsf{FM}_X(L)$ has rank $h^0(X, L)$ and one global section (by Proposition 18.4). So the Fourier-Mukai transform takes ample line bundles to ample vector bundles, but interchanges "rank" and "dimension of the space of global sections". More generally, FM_X tends to interchange "local" and "global" data. This can be very useful in geometric applications of the Fourier-Mukai transform (such as generic vanishing theory). The reason is that there are two sets of tools: local tools (such as commutative algebra in regular local rings) and global tools (such as vanishing theorems), and a local (or global) problem on X may become tractable once we convert it into a global (or local) problem on $\hat{X}$.

Lecture 19 (April 8)

Derived equivalences of abelian varieties. From Mukai's theorem, we know that an abelian variety X and its dual $\hat{X}$ have isomorphic derived categories. Let's say that two abelian varieties X and Y are *derived equivalent* if $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. We would like to know exactly when this happens. This question was completely answered by Orlov and Polishchuk. The general idea is that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ happens if and only if $X \times \hat{X} \cong Y \times \hat{Y}$ are isomorphic as abelian varieties (but only certain kinds of isomorphisms are allowed).

Let me first explain why the product $X \times \hat{X}$ shows up. This has to do with "automorphisms" of the derived category $\mathrm{D}^b(X)$, or more precisely auto-equivalences. A closed point $x \in X(k)$ defines an automorphism $t_x \colon X \to X$ by translation, and pullback along this automorphism is an auto-equivalence of the derived category:

$$t_x^* \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X).$$

Similarly, a closed point $\alpha \in \hat{X}(k)$ defines a line bundle $P_\alpha \in \mathrm{Pic}^0(X)$, and tensor product by P_α is also an auto-equivalence:

$$P_\alpha \otimes - \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X).$$

By composition, the closed points of $X \times \hat{X}$ therefore correspond to a family of auto-equivalences

$$T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T_{(x,\alpha)}(K) = P_\alpha \otimes t_x^* K \cong t_x^*(P_\alpha \otimes K).$$

Because X and $\hat{X}$ are varieties, this is a connected family; it contains $T_{(0,0)} = \mathrm{id}$. One can make sense of the group of auto-equivalences $\mathrm{Aut}\, \mathrm{D}^b(X)$ (using more fancy category theory); it has countably many connected components, and the neutral component (= the component containing the identity) is $X \times \hat{X}$. Now if $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$, then the automorphism groups of the two categories should be the same, and so $X \times \hat{X}$ should be isomorphic to $Y \times \hat{Y}$.

Orlov and Polishchuk make this heuristic argument precise, without actually defining the automorphism group $\mathrm{Aut}\, \mathrm{D}^b(X)$. It requires a careful study of the kernels of several different integral transforms. Each $T_{(x,\alpha)}$ is of course an integral transform: the kernel is the object

$$(t_x, \mathrm{id})_* P_\alpha \in \mathrm{D}^b(X \times X), \tag{19.1}$$

where the notation is as in the following diagram:

$$\begin{tikzcd} X \arrow[r, "{(t_x,\mathrm{id})}"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend right, "t_x"'] & X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\ & X & \end{tikzcd}$$

Indeed, with this choice, we get from the projection formula that

$$\mathbf{R}(p_2)_*\Big(p_1^*K \otimes (t_x, \mathrm{id})_* P_\alpha\Big) \cong \mathbf{R}(p_2)_*(t_x, \mathrm{id})_*\big(t_x^*K \otimes P_\alpha\big) \cong t_x^*K \otimes P_\alpha.$$

Now suppose that X and Y are two abelian varieties, whose derived categories $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ are equivalent. By Orlov's theorem, the equivalence is of the form

$$\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

for an object $E \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism. We are going to associate to E an isomorphism of abelian varieties

$$\varphi_E\colon X \times \hat{X} \to Y \times \hat{Y},$$

by the following device. For each pair of closed points $(x, \alpha) \in X(k) \times \hat{X}(k)$, consider the auto-equivalence

$$T_{(x,\alpha)}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$$

and its conjugate by $\mathbf{R}\Phi_E$, which is

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \circ \mathbf{R}\Phi_E^{-1}\colon \mathrm{D}^b(Y) \to \mathrm{D}^b(Y).$$

We'll argue below that this is again of the form $T_{\varphi_E(x,\alpha)}$ for a unique closed point $\varphi_E(x, \alpha) \in Y(k) \times \hat{Y}(k)$, starting from the fact that it is true for the closed point $(0, 0)$, because $T_{(0,0)} = \mathrm{id}$.

The following lemma will be useful in describing the quasi-inverse $\mathbf{R}\Phi_E^{-1}$ as an integral transform. For a complex $E \in \mathrm{D}^b(X \times Y)$, we define

$$E^\vee = \mathbf{R}\mathcal{H}om_{\mathscr{O}_{X\times Y}}\big(E, \mathscr{O}_{X\times Y}\big).$$

Compare the following lemma with the formula for the inverse of the Fourier-Mukai transform.

Lemma 19.2. *Let $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence of categories. Then the quasi-inverse is again an integral transform, with kernel*

$$E^\vee \otimes p_2^*\omega_Y[\dim Y].$$

Proof. The point is that the quasi-inverse $\mathbf{R}\Phi_E^{-1}\colon \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ is necessarily the left-adjoint of $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, because

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}\big(A, \mathbf{R}\Phi_E(B)\big) \cong \mathrm{Hom}_{\mathrm{D}^b(X)}\big(\mathbf{R}\Phi_E^{-1}(A), B\big).$$

We can easily derive a formula for the left-adjoint:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(Y)}\Big(A, \mathbf{R}\Phi_E(B)\Big) &\cong \mathrm{Hom}_{\mathrm{D}^b(Y)}\Big(A, \mathbf{R}(p_2)_*(E \overset{\mathbf{L}}{\otimes} p_1^*B)\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A, E \overset{\mathbf{L}}{\otimes} p_1^*B\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee, p_1^*B\Big).
\end{aligned}$$

The exceptional inverse image functor (from Grothendieck duality) is

$$p_1^!B = p_1^*B \otimes p_2^*\omega_Y[\dim Y],$$

and by using Grothendieck duality, we can continue the calculation from above:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee, p_1^*B\Big) &\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y], p_1^!B\Big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(X)}\Big(\mathbf{R}(p_1)_*\big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y]\big), B\Big)
\end{aligned}$$

This proves that

$$\mathbf{R}\Phi_E^{-1}(A) \cong \mathbf{R}(p_1)_*\big(p_2^*A \overset{\mathbf{L}}{\otimes} E^\vee \otimes p_2^*\omega_Y[Y]\big)$$

is equivalent to an integral transform. $\square$

Let's now return to our problem. Instead of trying to construct the isomorphism $\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}$ directly, we shall first define an equivalence

$$F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y}),$$

and then argue that F_E actually comes from an isomorphism between $X \times \hat{X}$ and $Y \times \hat{Y}$. This equivalence fits into the following commutative diagram:

$$\begin{tikzcd} \mathrm{D}^b(X \times \hat{X}) \arrow[r, "F_E"] \arrow[d, "\mathbf{R}\Phi_{A(X)}"] & \mathrm{D}^b(Y \times \hat{Y}) \\ \mathrm{D}^b(X \times X) \arrow[r, "\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}"] & \mathrm{D}^b(Y \times Y) \arrow[u, "\mathbf{R}\Phi_{A(Y)}^{-1}"'] \end{tikzcd} \tag{19.3}$$

The vertical arrow is an equivalence

$$\mathbf{R}\Phi_{A(X)} \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(X \times X)$$

that takes the skyscraper sheaf $k(x, \alpha)$ at a closed point $(x, \alpha) \in X(k) \times \hat{X}(\alpha)$ to the object in (19.1). Recall that this object is the kernel of the auto-equivalence $T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. Think of this as saying that $X \times \hat{X}$ is the parameter space for all of these auto-equivalences. The correct kernel is

$$A(X) = \mu_* \big(p_{32}^* P_X \big) \in \mathrm{D}^b(X \times \hat{X} \times X \times X),$$

where the notation is as follows:

$$\begin{tikzcd} X \times \hat{X} \times X \arrow[r, "\mu"] \arrow[d, "p_{32}"] & X \times \hat{X} \times X \times X \\ X \times \hat{X} & \end{tikzcd}$$

The two morphisms act on closed points as

$$\mu(x, \alpha, y) = (x, \alpha, x + y, y) \quad \text{and} \quad p_{32}(x, \alpha, y) = (y, \alpha).$$

With this choice, you can easily compute for yourself that

$$\mathbf{R}\Phi_{A(X)} \big(k(x, \alpha) \big) \cong (t_x, \mathrm{id})_* (P_\alpha).$$

The other vertical arrow in (19.3) is the quasi-inverse to $\mathbf{R}\Phi_{A(Y)}$; one can get an explicit formula for the kernel from Lemma 19.2.

Exercise 19.1. Verify that $\mathbf{R}\Phi_{A(X)}$ is indeed an equivalence. (*Hint:* Write it as the composition of an automorphism of $X \times X$ and Mukai's Fourier transform.)

The horizontal arrow in (19.3) is conjugation by $\mathbf{R}\Phi_E$. If we set $g = \dim Y$, then the kernel for $\mathbf{R}\Phi_E^{-1}$ is $E^\vee[g]$, and so the kernel representing conjugation is

$$p_{13}^* E^\vee[g] \overset{\mathbf{L}}{\otimes} p_{24}^* E \in \mathrm{D}^b(X \times X \times Y \times Y).$$

As the composition of three equivalences, $F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y})$ is an equivalence. It is also an integral transform for some $\tilde{E} \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y})$. One can in principle derive a formula for the kernel $\tilde{E}$ (using convolutions), but the actual formula doesn't matter for us. Here is Orlov's theorem.

Theorem 19.4. *There is an isomorphism of abelian varieties*

$$\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}$$

and a line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$*, such that*

$$F_E = \mathbf{R}(\varphi_E)_* \big(N_E \otimes - \big).$$

Equivalently, the kernel representing F_E is

$$(\mathrm{id}, \varphi_E)_* N_E \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y}).$$

We'll prove the theorem after looking at a few examples. A by-product of the construction is that for every pair of closed points $(x, \alpha) \in X(k) \times \hat{X}(k)$, the conjugated auto-equivalence

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{\varphi_E(x,\alpha)}$$

is again of the same form. We can rewrite this identity as

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \cong T_{\varphi_E(x,\alpha)} \circ \mathbf{R}\Phi_E. \tag{19.5}$$

As another exercise, you can compute the convolutions of the two kernels on each side. The result is that if $\varphi_E(x, \alpha) = (y, \beta)$, then one has

$$(t_x \times \mathrm{id})_* \bigl(p_1^* P_{X,\alpha} \otimes E\bigr) \cong (\mathrm{id} \times t_y)^* E \otimes p_2^* P_{Y,\beta} \tag{19.6}$$

in $\mathrm{D}^b(X \times Y)$. In other words, the automorphism φ_E records how the kernel E responds to translations and tensor products on both X and Y.

Example 19.7. Consider the Fourier transform $\mathbf{R}\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$. Here $Y = \hat{X}$ and $E = P$; by symmetry, $\hat{Y} \cong X$ and $\hat{P} = \sigma^* P$. What is the isomorphism

$$\varphi_P \colon X \times \hat{X} \to \hat{X} \times X$$

in this case? We can figure out the answer with very little pain if we make use of (19.6). Suppose that $\varphi_P(x, \alpha) = (\alpha', x')$. Then

$$(t_x \times \mathrm{id})_* \bigl(p_1^* P_\alpha \otimes P\bigr) \cong (\mathrm{id} \times t_{\alpha'})^* P \otimes p_2^* \hat{P}_{x'}$$

on $X \times \hat{X}$. By the seesaw theorem,

$$(\mathrm{id} \times t_{\alpha'})^* P \cong P \otimes p_1^* P_{\alpha'} \quad \text{and} \quad (t_x \times \mathrm{id})_* P \cong P \otimes p_2^* \hat{P}_{-x},$$

and so the identity from above becomes

$$p_1^* P_\alpha \otimes P \otimes p_2^* \hat{P}_{-x} \cong p_1^* P_{\alpha'} \otimes P \otimes p_2^* \hat{P}_{x'}.$$

Comparing the two sides, we find that $\alpha' = \alpha$ and $x' = -x$, and so

$$\varphi_P(x, \alpha) = (\alpha, -x).$$

This tells us how φ_P acts on closed points. Not surprisingly, one also has $N_P \cong P$ (but proving this takes a lot more work).

Here is another example where the line bundle N_E is nontrivial.

Example 19.8. Let $L \in \mathrm{Pic}(X)$, and consider $L \otimes - \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. In this case, $Y = X$ and $E = \Delta_* L$. Let's again determine

$$\varphi_E \colon X \times \hat{X} \to X \times \hat{X}$$

with the help of (19.6). Suppose that $\varphi_E(x, \alpha) = (y, \beta)$. Then

$$(t_x \times \mathrm{id})_* \bigl(p_1^* P_\alpha \otimes \Delta_* L\bigr) \cong (\mathrm{id} \times t_y)^* \Delta_* L \otimes p_2^* P_\beta$$

We can simplify the left-hand side using the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[rr, bend left, "(t_x, \mathrm{id})"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times X \arrow[r, "t_x \times \mathrm{id}"] \arrow[d, "p_1"] & X \times X \\ & X & \end{tikzcd}$$

From the projection formula, we get

$$(t_x \times \mathrm{id})_* \bigl(p_1^* P_\alpha \otimes \Delta_* L\bigr) \cong (t_x \times \mathrm{id})_* \Delta_* (L \otimes P_\alpha) \cong (t_x, \mathrm{id})_* (L \otimes P_\alpha).$$

We can also simplify the right-hand side using the Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{(t_y,\mathrm{id})} & X \times X \\ \big\downarrow t_y & & \big\downarrow \mathrm{id} \times t_y \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

Flat base change (for the automorphism $\mathrm{id} \times t_y$ gives

$$(\mathrm{id} \times t_y)^* \Delta_* L \otimes p_2^* P_\beta \cong (t_y, \mathrm{id})_* t_y^* L \otimes p_2^* P_\beta.$$

If we compare the two sides of our original identity, we get

$$(t_x, \mathrm{id})_* (L \otimes P_\alpha) \cong (t_y, \mathrm{id})_* t_y^* L \otimes p_2^* P_\beta,$$

and therefore $y = x$ and $L \otimes P_\alpha \cong t_x^* L \otimes P_\beta$. When we looked at line bundles on abelian varieties, we defined the homomorphism

$$\phi_L \colon X \to \hat{X}, \quad P_{\phi_L(x)} \cong t_x^* L \otimes L^{-1}.$$

Substituting this into the above formula, we get $\alpha = \beta + \phi_L(x)$, and so

$$\varphi_E \colon X \times \hat{X} \to X \times \hat{X}, \quad \varphi_E(x, \alpha) = \big(x, \alpha - \phi_L(x)\big).$$

When $L \in \mathrm{Pic}^0(X)$ is translation invariant, φ_E is the identity; but otherwise, it isn't. One can also check that $N_E = p_1^* L$, and so the line bundle in Theorem 19.4 is nontrivial in this example.

Example 19.9. For the shift functor $[n] \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$, we have $Y = X$ and $E = \Delta_* \mathscr{O}_X[n]$. In this case, $E^\vee$ has a shift by $-n$ in it, and so the two cancel out; the result is that $\varphi_E = \mathrm{id}$ and $N_E = \mathscr{O}_{X \times \hat{X}}$. From the point of view of Theorem 19.4, a shift is therefore indistinguishable from the identity.

Proof of Orlov's theorem. Let's now prove Theorem 19.4. The equivalence

$$F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y})$$

from (19.3) is an integral transform with a certain kernel $\tilde{E} \in \mathrm{D}^b(X \times \hat{X} \times Y \times \hat{Y})$. It has two additional properties that we can make use of. The first is that

$$F_E\big(k(0,0)\big) \cong k(0,0).$$

Indeed, $\mathbf{R}\Phi_{A(X)}\big(k(0,0)\big)$ is the kernel corresponding to $T_{(0,0)}$, which is the identity. Conjugating by $\mathbf{R}\Phi_E$ takes this to

$$\mathbf{R}\Phi_E \circ T_{(0,0)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{(0,0)},$$

because the identity of course commutes with $\mathbf{R}\Phi_E$. Under $\mathbf{R}\Phi_{A(Y)}^{-1}$, this goes back to the skyscraper sheaf $k(0,0)$ at the closed point $(0,0) \in Y(k) \otimes \hat{Y}(k)$.

The second property is that F_E is something like a homomorphism. Suppose that (x_1, α_2) and (x_2, α_2) are closed points such that

$$F_E\big(k(x_i, \alpha_i)\big) \cong k(y_i, \beta_i)$$

for closed points $(y_i, \beta_i) \in Y(k) \otimes \hat{Y}(k)$. This means that

$$\mathbf{R}\Phi_E \circ T_{(x_i,\alpha_i)} \circ \mathbf{R}\Phi_E^{-1} \cong T_{(y_i,\beta_i)}.$$

If we compose the two equivalences, we get

$$\begin{aligned} \mathbf{R}\Phi_E \circ T_{(x_1+x_2,\alpha_1+\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} &\cong \mathbf{R}\Phi_E \circ T_{(x_1,\alpha_1)} \circ T_{(x_2,\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} \\ &\cong \mathbf{R}\Phi_E \circ T_{(x_1,\alpha_1)} \circ \mathbf{R}\Phi_E^{-1} \circ \mathbf{R}\Phi_E \circ T_{(x_2,\alpha_2)} \circ \mathbf{R}\Phi_E^{-1} \\ &\cong T_{(y_1,\beta_1)} \circ T_{(y_2,\beta_2)} \cong T_{(y_1+y_2,\beta_1+\beta_2)}, \end{aligned}$$

because $t_{x_1} \circ t_{x_2} = t_{x_1+x_2}$ and $P_{\alpha_1} \otimes P_{\alpha_2} \cong P_{\alpha_1+\alpha_2}$. This is saying that the set

$$\left\{ (x, \alpha) \in X(k) \times \hat{X}(k) \;\middle|\; F_E\big(k(x, \alpha)\big) \cong k(y, \beta) \text{ for some } (y, \beta) \in Y(k) \times \hat{Y}(k) \right\}$$

is a *subgroup* of $X(k) \times \hat{X}(k)$. (In fact, we have shown that it contains the zero element and is closed under addition.)

Theorem 19.4 is therefore a consequence of the following abstract result about derived equivalences between abelian varieties. (The point is that the notation becomes much simpler if we consider arbitrary abelian varieties!)

Proposition 19.10. *Let X, Y be abelian varieties, and let $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence. If the set*

$$\left\{ x \in X(k) \;\middle|\; \mathbf{R}\Phi_E\big(k(x)\big) \cong k(y) \textit{ for some } y \in Y(k) \right\}$$

is a subgroup of $X(k)$, then $E \cong (\mathrm{id} \times \varphi)_* N$ for an isomorphism $\varphi \colon X \to Y$ and a line bundle $N \in \mathrm{Pic}(X)$.

Proof. For each closed point $x \in X(k)$, we set $E_x = E|_{\{x\} \times Y}$, so that

$$\mathbf{R}\Phi_P\big(k(x)\big) = E_x \in \mathrm{D}^b(Y).$$

As usual, we view these as a family of objects in the derived category $\mathrm{D}^b(Y)$, parametrized by the closed points of X. They form an algebraic family because $E \in \mathrm{D}^b(X \times Y)$ is a bounded complex of coherent sheaves on the product.

Let's first argue that E must be supported on the graph of a homomorphism $\varphi \colon X \to Y$. Let $S = \mathrm{Supp}\, E$ be the support of the complex E (= the union of the supports of all its cohomology sheaves). This is a closed subset of $X \times Y$. Consider the projection $p_1 \colon S \to X$. Because $E_0 \cong k(0)$, we know that $p_1^{-1}(0) = \{0\}$. By the theorem about fiber dimensions, the set of $x \in X(k)$ such that $\dim p_1^{-1}(x) = 0$ is the set of closed points of an open subscheme $U \subseteq X$; of course, $0 \in U(k)$. This means that E_x is supported on a finite set of points for $x \in U(k)$.

Because $\mathbf{R}\Phi_E$ is an equivalence, it is in particular fully faithful, and therefore

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}(E_x, E_x) \cong \mathrm{Hom}_{\mathrm{D}^b(X)}\big(k(x), k(x)\big) \cong k.$$

If $\mathrm{Supp}\, E_x$ was two or more points, then E_x would split as a direct sum of complexes supported at each point, and then the left-hand side would have dimension ≥ 2. Similarly, if E_x had more than one nontrivial cohomology sheaf, we could again decompose E_x and get too many endomorphisms. Since $E_0 \cong k(0)$, it follows that for $x \in U(k)$, the complex E_x is actually a sheaf supported at a single closed point in $Y(k)$. If we denote this closed point by $\varphi(x) \in Y(k)$, then $\varphi \colon U \to Y$ is a morphism (because its graph is $S \cap U \times Y$). Now in fact

$$E_x \cong k\big(\varphi(x)\big);$$

indeed, you can easily check that if M is a finitely-generated module over a local k-algebra $(A, \mathfrak{m})$ such that $\mathrm{Supp}\, M = \{\mathfrak{m}\}$ and $\mathrm{Hom}_A(M, M) \cong k$, then $M \cong k$.

This says of course that $U(k)$ is contained in the subgroup

$$\left\{ x \in X(k) \;\middle|\; \mathbf{R}\Phi_E\big(k(x)\big) \cong k(y) \text{ for some } y \in Y(k) \right\}.$$

Because X is an abelian variety, any open neighborhood of 0 generates X as a group; therefore $U = X$, the morphism φ is defined on all of X, and $E_x \cong k\big(\varphi(x)\big)$ for every $x \in X(k)$. Since we also know that $\varphi(0) = 0$, we see that $\varphi \colon X \to Y$ is a homomorphism. It then follows from Nakayama's lemma that

$$E \cong (\mathrm{id}, \varphi)_* N$$

for a line bundle $N \in \mathrm{Pic}(X)$. It is a line bundle because its stalk at every point is a one-dimensional k-vector space. Therefore

$$\mathbf{R}\Phi_E \cong \mathbf{R}\varphi_*(N \otimes -),$$

and this can only be an equivalence if $\varphi\colon X \to Y$ is an isomorphism. □

Lecture 20 (April 10)

More on derived equivalences between abelian varieties. Let X and Y be two abelian varieties (of the same dimension g). Last time, we associated to any equivalence $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ an isomorphism of abelian varieties

$$\varphi_E\colon X \times \hat{X} \to Y \times \hat{Y}.$$

The construction used the following commutative diagram:

$$\begin{tikzcd} \mathrm{D}^b(X \times \hat{X}) \arrow[r, "F_E"] \arrow[d, "\mathbf{R}\Phi_{A(X)}"] & \mathrm{D}^b(Y \times \hat{Y}) \\ \mathrm{D}^b(X \times X) \arrow[r, "\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}"] & \mathrm{D}^b(Y \times Y) \arrow[u, "\mathbf{R}\Phi_{A(Y)}^{-1}"'] \end{tikzcd} \tag{20.1}$$

Here $\mathbf{R}\Phi_{A(X)}$ is the equivalence that takes the structure sheaf of a closed point $(x, \alpha) \in X(k) \times \hat{X}(k)$ to the object $(t_x, \mathrm{id})_* P_\alpha$ on $X \times X$, which is the kernel of the auto-equivalence

$$T_{(x,\alpha)}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T_{(x,\alpha)}(K) = L \otimes t_x^* K.$$

We showed that the equivalence F_E, defined as in the diagram above, has the form

$$F_E(K) = \mathbf{R}(\varphi_E)_*(N_E \otimes K)$$

for a line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$. The isomorphism φ_E records how $\mathbf{R}\Phi_E$ interacts with translations and tensor product: one has $\varphi_E(x, \alpha) = (y, \beta)$ iff

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \cong T_{(y,\beta)} \circ \mathbf{R}\Phi_E.$$

Today, we are going to investigate this construction a bit further.

Property 1. The first observation is that the construction of F_E (and of φ_E) is compatible with composition, in the following sense. Suppose that

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_E"] \arrow[rr, bend left, "\mathbf{R}\Phi_{E*G}"] & \mathrm{D}^b(Y) \arrow[r, "\mathbf{R}\Phi_G"] & \mathrm{D}^b(Z) \end{tikzcd}$$

are two equivalences of derived categories, with composition $\mathbf{R}\Phi_G \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*G}$, where $E * G$ is the convolution of the two kernels. Then the induced equivalences

$$\begin{tikzcd} \mathrm{D}^b(X \times \hat{X}) \arrow[r, "F_E"] \arrow[rr, bend left, "F_{E*G}"] & \mathrm{D}^b(Y \times \hat{Y}) \arrow[r, "F_G"] & \mathrm{D}^b(Z \times \hat{Z}) \end{tikzcd}$$

are compatible (up to natural isomorphism). Because of the shape of (20.1), this comes down to the identity

$$\big(\mathbf{R}\Phi_G \times \mathbf{R}\Phi_G^{-1}\big) \circ \big(\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}\big) \cong \mathbf{R}\Phi_{E*G} \times \mathbf{R}\Phi_{E*G}^{-1}\big),$$

which holds because $\mathbf{R}\Phi_G \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*G}$. By looking at the kernels, we get

$$\varphi_G \circ \varphi_E = \varphi_{E*G},$$

and so the construction of F_E and φ_E respects composition.

Property 2. The next question is which homomorphisms $\varphi\colon X \times \hat{X} \to Y \times \hat{Y}$ can show up. We can write such a homomorphism as a matrix

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha\colon X \to Y$, $\beta\colon \hat{X} \to Y$, $\gamma\colon X \to \hat{Y}$, and $\delta\colon \hat{X} \to \hat{Y}$. Each of these four homomorphisms has a dual homomorphism: $\hat{\alpha}\colon \hat{Y} \to \hat{X}$, $\hat{\beta}\colon \hat{Y} \to X$, $\hat{\gamma}\colon Y \to \hat{X}$, and $\hat{\delta}\colon Y \to X$. We can put these together into a sort of "adjoint" matrix

$$\varphi^* = \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{pmatrix}$$

which then defines a homomorphism $\varphi^*\colon Y \times \hat{Y} \to X \times \hat{X}$.

Example 20.2. In the case of the Fourier transform $\mathbf{R}\Phi_P$, we had

$$\varphi_P\colon X \times \hat{X} \to \hat{X} \times X, \quad \varphi_P(x, \alpha) = (\alpha, -x).$$

Here we get $\varphi_P^* = -\varphi_P$ because

$$\varphi_P = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

Inside the group of all isomorphisms from $X \times \hat{X}$ to $Y \times \hat{Y}$, consider the subset

$$U(X \times \hat{X}, Y \times \hat{Y}) = \left\{ \varphi\colon X \times \hat{X} \to Y \times \hat{Y} \;\middle|\; \varphi^* \circ \varphi = \mathrm{id} \right\}.$$

It turns out that when $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence, then the associated isomorphism φ_E must lie in this set.

Lemma 20.3. *One has* $\varphi_E \in U(X \times \hat{X}, Y \times \hat{Y})$.

Proof. For any closed point $(x, \alpha) \in X(k) \times \hat{X}(k)$, we have the auto-equivalence $T_{(x,\alpha)}$ of $\mathrm{D}^b(X)$, and we let $F_{(x,\alpha)}$ be the associated auto-equivalence of $\mathrm{D}^b(X \times \hat{X})$. One can check that $\varphi_{(x,\alpha)} = \mathrm{id}$ and $N_{(x,\alpha)} = P_\alpha \boxtimes \hat{P}_{-x}$; in fact, we did half of this computation last time, when we looked at tensor products by line bundles.

The idea behind the proof is to use the fact that Orlov's construction respects compositions. Suppose that $\varphi_E(x, \alpha) = (y, \beta)$. Then

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \cong T_{(y,\beta)} \circ \mathbf{R}\Phi_E,$$

and therefore $F_E \circ F_{(x,\alpha)} \cong F_{(y,\beta)} \circ F_E$. Writing out both sides explicitly, we get

$$\mathbf{R}(\varphi_E)_*\bigl(N_E \otimes N_{(x,\alpha)} \otimes -\bigr) \cong N_{(y,\beta)} \otimes \mathbf{R}(\varphi_E)_*\bigl(N_E \otimes -\bigr)$$

and therefore $\varphi_E^* N_{(y,\beta)} \cong N_{(x,\alpha)}$ (by the projection formula). This gives

$$\varphi_E^*\bigl(P_\beta \boxtimes \hat{P}_{-y}\bigr) \cong P_\alpha \boxtimes \hat{P}_{-x},$$

or in terms of the dual homomorphism $\hat{\varphi}_E\colon \hat{Y} \times Y \to \hat{X} \times X$,

$$\hat{\varphi}_E(\beta, -y) = (\alpha, -x).$$

Because of how we defined φ_E^*, this becomes $\varphi_E^*(y, \beta) = (x, \alpha)$, and so $\varphi_E^* \circ \varphi_E$ is indeed the identity. $\square$

Property 3. In fact, Polishchuk and Orlov showed that *every* $\varphi \in U(X \times \hat{X}, Y \times \hat{Y})$ is equal to φ_E for some equivalence $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. To prove this, one has to construct sufficiently many kernels on $X \times Y$; in fact, they are all of the form, a vector bundle supported on an abelian subvariety of $X \times Y$.

Corollary 20.4. *One has* $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ *iff* $U(X \times \hat{X}, Y \times \hat{Y}) \neq \emptyset$.

Property 4. Let's prove that the kernel of any derived equivalence $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ between abelian varieties must be a vector bundle supported on an abelian subvariety. The first step is to compute the kernel of the induced equivalence F_E using convolutions, as in (20.1). We can then push this object forward along the projection $p_{13}\colon X \times \hat{X} \times Y \times \hat{Y} \to X \times Y$. At the same time, we know that the kernel is isomorphic to $(\mathrm{id}, \varphi_E)_* N_E$. If we use the diagram

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "{(\mathrm{id},\varphi_E)}"] \arrow[dr, bend right, "f_E"'] & X \times \hat{X} \times Y \times \hat{Y} \arrow[d, "p_{13}"] \\ & X \times Y \end{tikzcd}$$

to define a homomorphism $f_E = p_{13} \circ (\mathrm{id}, \varphi_E)$ from $X \times \hat{X}$ to $X \times Y$, then the result of this (big) computation is that

$$\mathbf{R}(f_E)_* N_E = \mathbf{R}(p_{13})_*\big((\mathrm{id}, \varphi_E)_* N_E\big) \cong E \otimes_k E^\vee|_{(0,0)}.$$

Here $E^\vee|_{(0,0)}$ means that we take the dual complex $E^\vee = \mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{O}_{X\times Y})$ and restrict it to the closed point $(0,0)$; this is just a complex of finite-dimensional k-vector spaces. So up to this small "error term", we can recover the kernel object E from the isomorphism φ_E and the line bundle N_E.

Example 20.5. The Fourier transform $\mathbf{R}\Phi_P$ had $\varphi_P(x,\alpha) = (\alpha, -x)$, and $N_P = P$. Here f_P is the identity, because $(\mathrm{id}, \varphi_P)(x,\alpha) = (x, \alpha, \alpha, -x)$.

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "{(\mathrm{id},\varphi_P)}"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times \hat{X} \times \hat{X} \times X \arrow[d, "p_{13}"] \\ & X \times \hat{X} \end{tikzcd}$$

So it is indeed the case that P is the pushforward of N_P.

We can now prove the following result, originally due to Orlov.

Proposition 20.6. *Suppose that $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence. Then up to a shift, E is a locally free sheaf supported on an abelian subvariety of $X \times Y$.*

We are going to abstract a bit, in order to simplify the notation. Consider a homomorphism $f\colon X \to Y$ between two abelian varieties of the same dimension g. Suppose that $\ker f$ has dimension n, so that $Z = \operatorname{im} f \subseteq Y$ is an abelian subvariety of codimension n. We are going to use the following morphisms:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[rr, bend left, "f"] & Z \arrow[r, "i"] & Y \end{tikzcd}$$

Suppose that $E \in \mathrm{D}^b(Y)$ is an object in the derived category, such that

$$E \otimes E^\vee|_0 \cong \mathbf{R}f_* L$$

for a line bundle $L \in \mathrm{Pic}(X)$. Then we claim that, up to a shift, E must be a locally free sheaf supported on Z.

Proof. After a shift, we may assume that $\mathcal{H}^i E = 0$ for $i > 0$, but that $\mathcal{H}^0 E \neq 0$. On a suitable affine open neighborhood of the point $0 \in Y(k)$, we can find a minimal locally free resolution for E, of the form

$$0 \to \mathscr{E}_p \to \mathscr{E}_{p-1} \to \cdots \to \mathscr{E}_0 \to 0.$$

The dual complex $E^\vee$ is then

$$0 \to (\mathscr{E}_0)^* \to (\mathscr{E}_1)^* \to \cdots \to (\mathscr{E}_p)^* \to 0,$$

and by minimality, $E^\vee|_0$ is the complex with terms $(\mathscr{E}_i)^*|_0$ and trivial differentials. So in the derived category of k-vector spaces, $E^\vee|_0$ decomposes as

$$E^\vee|_0 \cong \bigoplus_{i=0}^{p} \mathcal{H}^i\big(E^\vee|_0\big)[-i] \cong \bigoplus_{i=0}^{p} V_i[-i].$$

In particular, the 0-th cohomology V_0 of the complex $E^\vee|_0$ is nontrivial. Because tensor product over k is exact, it follows that

$$R^i f_* L \cong \mathcal{H}^i\big(E \otimes E^\vee|_0\big)$$

contains $\mathcal{H}^i E \otimes V_0$ as a direct summand. For obvious reasons, $R^i f_* L = 0$ for $i < 0$, and therefore $\mathcal{H}^i E = 0$ for $i < 0$; this means that E is isomorphic to a sheaf in degree 0. Since

$$\mathbf{R} f_* L \cong i_* \mathbf{R} p_* L,$$

this sheaf is isomorphic to a direct summand of $i_* \mathbf{R} p_* L$, and so it is annihilated by the ideal sheaf $\mathcal{I}_Z$. Consequently, $E \cong i_* \mathscr{F}$, where $\mathscr{F}$ is a coherent sheaf on Z. Moreover, $\mathscr{F}$ is isomorphic to a direct summand of $\mathbf{R} p_* L \in \mathrm{D}^b(Z)$.

Now we argue that Z is actually locally free. Since Z is smooth projective of dimension $g - n$, we can find a locally free resolution

$$0 \to \mathscr{E}_{g-n} \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Consider the dual complex $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$, which is isomorphic to

$$0 \to (\mathscr{E}_0)^* \to \cdots \to (\mathscr{E}_{g-n})^* \to 0.$$

We are going to argue that the dual complex is a sheaf. The key point is that

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathbf{R} p_* L, \mathscr{O}_Z) \cong \mathbf{R} p_* \mathcal{H}om_{\mathscr{O}_X}(L, p^! \mathscr{O}_Z) \cong \mathbf{R} p_* L^{-1}[n],$$

because $p\colon X \to Z$ is smooth of relative dimension n and the canonical bundles of X and Z are both trivial. For dimension reasons, the complex $\mathbf{R} p_* L^{-1}[n]$ is concentrated in degrees $-n, \dots, 0$. Because $\mathscr{F}$ is a direct summand of $\mathbf{R} p_* L$, it follows that $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$ is a direct summand of $\mathbf{R} p_* L^{-1}[n]$, and therefore also concentrated in deegree $-n, \dots, 0$. It follows that $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$ is a single locally free sheaf in degree 0; dualizing back, we find that $\mathscr{F}$ is itself locally free. $\square$

Property 5. One can use the results above to classify all auto-equivalences of the derived category $\mathrm{D}^b(X)$. Let's write $\operatorname{Aut} \mathrm{D}^b(X)$ for the group of all auto-equivalences. We showed above that the function

$$\operatorname{Aut} \mathrm{D}^b(X) \to U(X \times \hat{X}, X \times \hat{X}), \quad \mathbf{R}\Phi_E \mapsto \varphi_E,$$

is a group homomorphism; we also know (from Property 3) that it is surjective. One can show that the kernel consists exactly of the auto-equivalences $T_{(x,\alpha)}$, with $(x, \alpha) \in X(k) \times \hat{X}(k)$, and of the shift functors $[n]$ with $n \in \mathbb{Z}$. This makes precise the heuristic from last time that $X(k) \times \hat{X}(k)$ is the neutral component of the automorphism group of $\mathrm{D}^b(X)$.

Mukai's $\mathrm{SL}_2(\mathbb{Z})$-action. Suppose now that X is a principally polarized abelian variety; this means that we have an ample line bundle L such that $h^0(X, L) = 1$. Equivalently, the morphism

$$\phi_L\colon X \to \hat{X}, \quad t_x^* L \otimes L^{-1} \cong P_{\phi_L(x)},$$

is an isomorphism. (It is surjective by Theorem 11.7 and has degree $h^0(X, L)^2 = 1$.) In this case, we get several interesting auto-equivalences of the derived category, and Mukai noticed that they determine an action of the group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{D}^b(X)$.

The first auto-equivalence $S\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is the composition

$$\mathrm{D}^b(X) \xrightarrow{\mathbf{R}\Phi_P} \mathrm{D}^b(\hat{X}) \xrightarrow{\phi_L^*} \mathrm{D}^b(X).$$

Because of the formula $(\mathrm{id} \times \phi_L)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$, we can write this as

$$S(K) = \phi_L^* \mathbf{R}(p_2)_*(p_1^* K \otimes P) \cong \mathbf{R}(p_2)_*\bigl(p_1^* K \otimes m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}\bigr).$$

We also have a second auto-equivalence

$$T\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T(K) = L \otimes K.$$

Both S and T have associated automorphisms φ_S and φ_T in $\mathrm{Hom}(X \times \hat{X}, X \times \hat{X})$; using the isomorphism ϕ_L between X and $\hat{X}$, we may consider φ_S and φ_T as elements of $\mathrm{Hom}(X \times X, X \times X)$. We showed last time that $\varphi_P(x, \alpha) = (\alpha, -x)$, and after making the identifications, we get

$$\varphi_S = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

We also computed last time that $\varphi_T(x, \alpha) = (x, \alpha - \phi_L(x))$; after the appropriate identifications, this tells us that

$$\varphi_T = \begin{pmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{pmatrix}.$$

Now we observe that the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \bigl\{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \bigm| \det A = 1 \bigr\}$$

embeds into $\mathrm{Hom}(X \times X, X \times X)$. Indeed, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = ad - bc = 1$, then A defines an automorphism of the abelian variety $X \times X$, represented by the matrix

$$\begin{pmatrix} a_X & b_X \\ c_X & d_X \end{pmatrix};$$

on closed points, the formula is $A \cdot (x, y) = (ax + by, cx + dy)$. So φ_S and φ_T represent the action of the two matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is known that these two matrices together generate $\mathrm{SL}_2(\mathbb{Z})$; the relations are

$$S^4 = \mathrm{id} \quad \text{and} \quad (TS)^3 = \mathrm{id}.$$

Mukai's observation is that these identities already hold (up to a shift) for the two equivalences S and T. In this sense, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathrm{D}^b(X)$.

Proposition 20.7. *The two equivalences $S, T\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ satisfy*

$$S^4 \cong [-2g] \quad \textit{and} \quad (T \circ S)^3 \cong [-g].$$

Proof. Let $\mathbf{R}\Phi_P\colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$ and $\mathbf{R}\Phi_{\hat{P}}\colon \mathrm{D}^b(\hat{X}) \to \mathrm{D}^b(X)$ be the two Fourier transforms. From the proof of Mukai's theorem, we know that

$$\mathbf{R}\Phi_P \circ \mathbf{R}\Phi_{\hat{P}} \cong (-1)_{\hat{X}}^*[-g].$$

The isomorphism $\phi_L \colon X \to \hat{X}$ is self-dual, in the sense that $\hat{\phi}_L = \phi_L$. The identity in (18.5) therefore tells us that the diagram

$$\begin{tikzcd}
\mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_P"] \arrow[d, "(\phi_L)_*"] & \mathrm{D}^b(\hat{X}) \arrow[d, "\phi_L^*"] \\
\mathrm{D}^b(\hat{X}) \arrow[r, "\mathbf{R}\Phi_{\hat{P}}"] & \mathrm{D}^b(X)
\end{tikzcd}$$

is commutative. This gives

$$\begin{aligned}
S \circ S = \phi_L^* \circ \mathbf{R}\Phi_P \circ \phi_L^* \circ \mathbf{R}\Phi_P &\cong \phi_L^* \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Phi_{\hat{P}} \circ (\phi_L)_* \\
&\cong \phi_L^* \circ (-1)_{\hat{X}}^*[-g] \circ (\phi_L)_* \cong (-1)_X^*[-g].
\end{aligned}$$

So clearly $S^4 \cong [-2g]$, which is the first identity.

For the second identity, we note that

$$(T \circ S)(K) = L \otimes S(K) \cong \mathbf{R}(p_2)_*\big(p_1^* K \otimes m^* L \otimes p_1^* L^{-1}\big),$$

which means that $T \circ S$ is an integral transform with kernel $m^* L \otimes p_1^* L^{-1}$. The kernel of $(T \circ S)^3$ is therefore given by convolution: concretely, it is

$$\mathbf{R}(p_{14})_*\Big(m_{12}^* L \otimes m_{23}^* L \otimes m_{34}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes p_3^* L^{-1}\Big),$$

where $p_{14} \colon X \times X \times X \times X \to X \times X$ is the projection to the first and fourth factor, and m_{ij} is the morphism that adds the i-th and j-th coordinates. We can simplify the line bundle in parentheses using the seesaw theorem. For any two closed points $x, y \in X(k)$, its restriction to $\{x\} \times X \times X \times \{y\}$ is

$$p_1^* t_x^* L \otimes m^* L \otimes p_2^* t_y^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \cong m^* L \otimes p_1^* \phi_L(x) \otimes p_2^* \phi_L(y),$$

and under the natural isomorphism $X \times X \times X \times X \cong \hat{X} \times X \times X \times \hat{X}$, this is isomorphic to the restriction of $p_{12}^* P \otimes m_{23}^* L \otimes p_{34}^* P \otimes p_4^* L$. Both bundles also have the same restriction to $X \times \{0\} \times \{0\} \times X$, and so they are isomorphic by the seesaw theorem. Therefore the kernel of $(T \circ S)^3$ is

$$\begin{aligned}
&\mathbf{R}(p_{14})_*\Big(p_{12}^* \hat{P} \otimes p_{23}^* m^* L \otimes p_{34}^* P \otimes p_4^* L\Big) \\
&\qquad \cong p_2^* L \otimes \mathbf{R}(p_{14})_*\Big(p_{12}^* P \otimes p_{23}^* m^* L \otimes p_{34}^* P\Big) \cong p_2^* L \otimes \mathbf{R}\Phi_{P \boxtimes P}(m^* L).
\end{aligned}$$

The second factor is exactly the Fourier-Mukai transform of $m^* L \in \mathrm{D}^b(X \times X)$ using the Poincaré bundle $P \boxtimes P$ on $\hat{X} \times X \times X \times \hat{X}$.

The homomorphism $m \colon X \times X \to X$ is dual to the diagonal $\Delta \colon \hat{X} \to \hat{X} \times \hat{X}$, in the sense that $\hat{m} = \Delta$. Because we know from Proposition 18.4 how the Fourier-Mukai transform interacts with homomorphisms, we get

$$\mathbf{R}\Phi_{P \boxtimes P}(m^* L) \cong \Delta_* \Phi_P(L)[-g] \cong \Delta_* L^{-1}[-g].$$

Therefore the kernel of $(T \circ S)^3$ simplifies to

$$p_2^* L \otimes \Delta_* L^{-1}[-g] \cong \Delta_* \mathscr{O}_{\hat{X}}[-g],$$

which shows that $(T \circ S)^3 \cong [-g]$. $\square$

Exercise 20.1. As an exercise, you can try to figure out what the kernel for the equivalence corresponding to a general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

might look like. As a starting point, consider the diagram from Proposition 20.6, which now reads (after identifying X and $\hat{X}$)

$$\begin{tikzcd} X \times X \arrow[r, "(\mathrm{id},A)"] \arrow[dr, bend right, "f_A"'] & X \times X \times X \times X \arrow[d, "p_{13}"] \\ & X \times X. \end{tikzcd}$$

Here $f_A(x,y) = (x, ax+by)$, and if $b \neq 0$, then f_A is an isogeny of degree $\deg b_X = b^{2g}$, and so the kernel object must be a vector bundle on $X \times X$. What is its rank? Can you describe this vector bundle in some cases? What happens when $b = 0$?

Lecture 21 (April 15)

Derived equivalences and cohomology. Today, we go back to abelian varieties over the complex numbers. Let X and Y be two abelian varieties (of the same dimension g), and suppose that we have an integral transform

$$\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y).$$

We can use the kernel E on $X \times Y$ to construct an induced transformation

$$\Phi_E^H \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

in cohomology. Let $\mathrm{ch}(E) \in H^*(X \times Y, \mathbb{Q})$ be the Chern character of the complex $E \in \mathrm{D}^b(X \times Y)$, and let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the two projections. Then define

$$\Phi_E^H \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad \Phi_E^H(\alpha) = (p_2)_*\big(p_1^*(\alpha) \cup \mathrm{ch}(E)\big),$$

where $(p_2)_*$ is the Gysin map in cohomology. Note that Φ_E^H does not respects degrees in general, because $\mathrm{ch}(E)$ can have components in many different degrees.

Example 21.1. When L is a line bundle on X, one has

$$\mathrm{ch}(L) = \exp c_1(L) = 1 + c_1(L) + \frac{1}{2!} c_1(L)^2 + \cdots$$

For a vector bundle E, the Chern character is a certain polynomial (with rational coefficients) in the Chern classes: using the splitting principle, if $L_1, \dots, L_r$ are the Chern roots of E, then

$$\mathrm{ch}(E) = \sum_{j=1}^{r} \mathrm{ch}(L_j).$$

For an arbitrary complex E, we can define the Chern character by choosing a bounded complex $\mathscr{E}^\bullet$ of locally free sheaves that is quasi-isomorphic to E, and then setting

$$\mathrm{ch}(E) = \sum_{j \in \mathbb{Z}} (-1)^j \,\mathrm{ch}(\mathscr{E}^j).$$

It can be shown that the alternating sum on the right-hand side is the same for every locally free resolution.

The construction is compatible with composition, in the sense that if

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_E"] \arrow[rr, bend left, "\mathbf{R}\Phi_{E*F}"] & \mathrm{D}^b(Y) \arrow[r, "\mathbf{R}\Phi_F"] & \mathrm{D}^b(Z) \end{tikzcd}$$

are two integral transforms (so that their composition is an integral transform with kernel $E * F$), then the induced diagram

$$\begin{tikzcd} H^*(X,\mathbb{Q}) \arrow[r, "\Phi_E^H"] \arrow[rr, bend left, "\Phi_{E*F}^H"] & H^*(Y,\mathbb{Q}) \arrow[r, "\Phi_F^H"] & H^*(Z,\mathbb{Q}) \end{tikzcd}$$

is also commutative. This is a consequence of the Grothendieck-Riemann-Roch theorem for Chern characters. Recall that the convolution is defined as

$$E * F = \mathbf{R}(p_{13})_*\big(p_{12}^* E \otimes p_{23}^* F\big).$$

As easy computation reduces the problem to showing that

$$\operatorname{ch}(E * F) = (p_{13})_*\big(p_{12}^* \operatorname{ch}(E) \cup p_{23}^* \operatorname{ch}(F)\big).$$

The Chern character always commutes with pulling back; and on abelian varieties, it also commutes with pushing forward. Indeed, if $f\colon X \to Y$ is a morphism between abelian varieties, and $E \in \mathrm{D}^b(X)$ a bounded complex of coherent sheaves, then the Grothendieck-Riemann-Roch theorem gives

$$\operatorname{ch}\big(\mathbf{R}f_* E\big) = \operatorname{td}(\mathscr{T}_Y) \cup \operatorname{ch}\big(\mathbf{R}f_* E\big) = f_*\big(\operatorname{td}(\mathscr{T}_X) \cup \operatorname{ch}(E)\big) = f_* \operatorname{ch}(E).$$

This works because the Todd class of the tangent bundle $\operatorname{td}(\mathscr{T}_X)$ is trivial on an abelian variety, due to the tangent bundle itself being trivial.

In particular, if $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence, then the induced homomorphism $\Phi_E^H\colon H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ is an isomorphism. Let me point out again that it is usually not compatible with the grading.

Example 21.2. Let's consider the Fourier transform $\mathbf{R}\Phi_P\colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$. To compute the induced homomorphism on cohomology, we first need to know the first Chern class $c_1(P)$ of the Poincaré bundle. From the Künneth formula, we get

$$H^2(X \times \hat{X}, \mathbb{Z}) \cong H^2(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z}) \otimes H^1(\hat{X},\mathbb{Z}) \oplus H^2(\hat{X},\mathbb{Z}),$$

Now $c_1(P)$ belongs to the subspace $H^1(X,\mathbb{Z}) \otimes H^1(\hat{X},\mathbb{Z})$, because the restriction of P to each slice $X \times \{\alpha\}$ and $\{x\} \times \hat{X}$ has trivial first Chern class (in cohomology). We can rewrite this subspace if we remember that $\hat{X} \cong \operatorname{Pic}^0(X) \cong H^1(X,\mathscr{O}_X)/H^1(X,\mathbb{Z})$. This gives $H_1(\hat{X},\mathbb{Z}) \cong H^1(X,\mathbb{Z})$, and therefore

$$H^1(\hat{X},\mathbb{Z}) \cong H^1(X,\mathbb{Z})^*.$$

Under this isomorphism, the first Chern class

$$c_1(P) \in H^1(X,\mathbb{Z}) \otimes H^1(X,\mathbb{Z})^*$$

is the identity: if $e_1, \dots, e_{2g} \in H^1(X,\mathbb{Z})$ is a basis, and $e_1^*, \dots, e_{2g}^* \in H^1(X,\mathbb{Z})^*$ the dual basis, then one can show that

$$c_1(P) = \sum_{j=1}^{2g} e_j \otimes e_j^*.$$

From this, it is easy to see that

$$\frac{1}{n!} c_1(P)^n \in H^n(X,\mathbb{Z}) \otimes H^n(X,\mathbb{Z})^*$$

has integer coefficients, and hence that the Chern character $\operatorname{ch}(P) \in H^*(X \times \hat{X}, \mathbb{Z})$ is also integral. It follows that

$$\Phi_P^H\colon H^*(X,\mathbb{Z}) \to H^*(\hat{X},\mathbb{Z})$$

makes sense (and is an isomorphism) over the integers. Considering degrees, we get

$$\Phi_P^H\colon H^n(X,\mathbb{Z}) \to H^{2g-n}(\hat{X},\mathbb{Z}).$$

One can prove (using the formula for the first Chern class of P) that this isomorphism is basically Poincaré duality: the product $(-1)^{n(n+1)/2+g} \cdot \Phi_P^H$ is the Poincaré duality isomorphism

$$H^n(X, \mathbb{Z}) \cong H_n(X, \mathbb{Z})^* \cong H^{2g-n}(X, \mathbb{Z})^* \cong H^{2g-n}(\hat{X}, \mathbb{Z}).$$

In fact, something similar happens for an arbitrary derived equivalence between two abelian varieties, as our the next proposition shows.

Proposition 21.3. *Let $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence. Then*

$$\Phi_E^H \colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$$

is an isomorphism over the integers.

Proof. From $\mathbf{R}\Phi_E$, we constructed an induced equivalence

$$F_E \colon \mathrm{D}^b(X \times \hat{X}) \to \mathrm{D}^b(Y \times \hat{Y}),$$

with the help of the diagram in (19.3). We also showed that F_E is tensor product with a certain line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$, followed by pushforward along the isomorphism $\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}$. Just as for the Poincaré bundle, the Chern character $\mathrm{ch}(N_E)$ is a class in the cohomology of $X \times \hat{X}$ with integer coefficients. Therefore the homomorphism in cohomology associated to F_E is an isomorphism $H^*(X \times \hat{X}, \mathbb{Z}) \cong H^*(Y \times \hat{Y}, \mathbb{Z})$. The two vertical arrows in (19.3) also induce isomorphisms on integral cohomology (because they involve only isomorphisms and the Poincaré bundle); therefore the homomorphism associated to the equivalence $\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1} \colon \mathrm{D}^b(X \times X) \to \mathrm{D}^b(Y \times Y)$ is an isomorphism

$$H^*(X \times X, \mathbb{Z}) \cong H^*(Y \times Y, \mathbb{Z}).$$

A short computation shows that it acts as conjugation by Φ_E^H, and together with the Künneth decomposition, this is enough to conclude that

$$\Phi_E^H \colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$$

must be an isomorphism. $\square$

Exercise 21.1. Let L be a line bundle on abelian variety X. Show that

$$\frac{1}{n!} c_1(L)^n \in H^{2n}(X, \mathbb{Z}),$$

and conclude that the Chern character $\mathrm{ch}(L)$ is an element of $H^*(X, \mathbb{Z})$.

Finally, we can give a cohomological criterion for when two abelian varieties X and Y are derived equivalent. The main point is that a complex abelian variety X can be reconstructed from the Hodge structure on $H^1(X, \mathbb{Z})$, meaning from the Hodge decomposition on

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X).$$

Indeed, X is isomorphic to its own Albanese variety

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^* / H_1(X, \mathbb{Z}),$$

and we have $H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$ and $H^0(X, \Omega_X^1) \cong H^{1,0}(X)$. So if X and Y are abelian varieties, and if $H^1(X, \mathbb{Z})$ and $H^1(Y, \mathbb{Z})$ are isomorphic as Hodge structures, then $X \cong Y$. (This is an isomorphism of compact complex manifolds, but since X and Y are projective, the isomorphism is automatically algebraic as well, due to Chow's theorem.)

Now let's go back to the criterion in Corollary 20.4, which says that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ iff $U(X \times \hat{X}, Y \times \hat{Y}) \neq \emptyset$. This set of "unitary" isomorphisms was defined as follows. Write a given homomorphism $\varphi \colon X \times \hat{X} \to Y \times \hat{Y}$ in the form

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with $\alpha \colon X \to Y$, $\beta \colon \hat{X} \to Y$, $\gamma \colon X \to \hat{Y}$, and $\delta \colon \hat{X} \to \hat{Y}$; then take the dual homomorphisms $\hat{\alpha} \colon \hat{Y} \to \hat{X}$, $\hat{\beta} \colon \hat{Y} \to X$, $\hat{\gamma} \colon Y \to \hat{X}$, and $\hat{\delta} \colon Y \to X$, and assemble them into a second matrixp

$$\varphi^* = \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{pmatrix}$$

that represents a homomorphism $\varphi^* \colon Y \times \hat{Y} \to X \times \hat{X}$. Then if $\varphi^* \circ \varphi = \mathrm{id}$, we say that $\varphi \in U(X \times \hat{X}, Y \times \hat{Y})$.

According to the discussion above, an isomorphism $\varphi \colon X \times \hat{X} \to Y \times \hat{Y}$ is the same thing as an isomorphism $f \colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$ that respects the Hodge structures. The extra condition of being "unitary" can also be seen on cohomology. Writing

$$H^1(X \times \hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*,$$

we have a natural bilinear pairing q_X, defined by the rule

$$q_X\big((\alpha_1, \phi_1), (\alpha_2, \phi_2)\big) = \phi_1(\alpha_2) + \phi_2(\alpha_1).$$

We can then restate the criterion from Corollary 20.4 as follows.

Corollary 21.4. *Let X and Y be abelian varieties. We have $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ if and only if there is an isomorphism of Hodge structures*

$$f \colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$$

that is an isometry with respect to the bilinear pairings q_X and q_Y.

Proof. Choose a basis in $H^1(X, \mathbb{Z})$, and the dual basis in $H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$; then the pairing q_X is represented by the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

We can represent the isomorphism $f \colon H^1(X \times \hat{X}, \mathbb{Z}) \to H^1(Y \times \hat{Y}, \mathbb{Z})$ as a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a \colon H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$, $b \colon H^1(X, \mathbb{Z})^* \to H^1(Y, \mathbb{Z})$, and so on. The condition to be an isometry is then

$$\begin{pmatrix} c^* & a^* \\ d^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & b^* \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Here $a^* \colon H^1(Y, \mathbb{Z})^* \to H^1(X, \mathbb{Z})^*$ is the homomorphism dual to a, and so on. Now $a \colon H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$ is, by assumption, a morphism of Hodge structures, and so it corresponds to a morphism of abelian varieties $\alpha \colon X \to Y$. Under the isomorphisms $H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$ and $H^1(\hat{Y}, \mathbb{Z}) \cong H^1(Y, \mathbb{Z})^*$, the dual homomorphism a^* then corresponds exactly to the dual morphism $\hat{\alpha} \colon \hat{Y} \to \hat{X}$. Likewise, $b \colon H^1(X, \mathbb{Z})^* \to H^1(Y, \mathbb{Z})$ corresponds to a morphism $\beta \colon \hat{X} \to Y$, but the dual homomorphism $b^* \colon H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z})^{**} \cong H^1(X, \mathbb{Z})$ involves the isomorphism with the double dual, and for that reason, there is an extra sign: the corresponding morphism is $-\hat{\beta}\colon Y \to \hat{X}$. In this manner, the condition that f is an isometry turns into the identity

$$\begin{pmatrix} -\hat{\gamma} & \hat{\alpha} \\ \hat{\delta} & -\hat{\beta} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix},$$

which is exactly saying that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(X \times \hat{X}, Y \times \hat{Y}).$$

We now conclude by applying Corollary 20.4. $\square$

Deligne's theorem on absolute Hodge classes. The next topic, which is going to take up the rest of the semester, is a theorem by Deligne about Hodge classes on abelian varieties. We started with a brief overview. Let X be a smooth projective variety over the complex numbers. For $k \geq 0$, we have the Hodge decomposition

$$H^{2k}(X, \mathbb{C}) = \bigoplus_{p+q=2k} H^{p,q}(X).$$

A cohomology class $\alpha \in H^{2k}(X, \mathbb{Z})$ is called an (integral) *Hodge class* if its image in $H^{2k}(X, \mathbb{C})$ lands in the subspace $H^{k,k}(X)$ – in other words, if it has type (k,k) with respect to the Hodge decomposition. Any closed subvariety $Z \subseteq X$ of codimension k has a fundamental class

$$[Z] \in H^{2k}(X, \mathbb{Z}),$$

and this is always a Hodge class. (*Proof:* Let $\mu\colon \tilde{Z} \to Z$ be a resolution of singularities; then $[Z]$ is Poincaré dual to the image of μ, and so

$$\int_{\tilde{Z}} \mu^* \alpha = \int_X [Z] \wedge \alpha$$

for every closed form $\alpha \in A^{2n-2k}(X)$, where $n = \dim X$. As $\dim \tilde{Z} = n - k$, the integral vanishes except when $\alpha \in A^{n-k,n-k}(X)$.) Hodge asked whether every integral Hodge class is "algebraic", meaning a linear combination of fundamental classes of subvarieties. Over $\mathbb{Z}$, there are counterexamples: cases where $[Z]$ is torsion (and therefore a Hodge class for trivial reasons), and cases where some multiple of $[Z]$ is algebraic, but $[Z]$ cannot be a linear combination of fundamental classes for degree reasons. The Hodge conjecture is therefore properly stated over $\mathbb{Q}$.

Conjecture 21.5 (Hodge)**.** *Every Hodge class in $H^{2k}(X, \mathbb{Q})$ is a $\mathbb{Q}$-linear combination of fundamental classes of subvarieties of codimension k.*

For $k = 1$, this is true even over $\mathbb{Z}$, by the Lefschetz $(1,1)$-theorem: every Hodge class in $H^2(X, \mathbb{Z})$ is the first Chern class of a line bundle. This works even on compact Kähler manifolds. But for larger values of k, the Hodge conjecture is known to be false on compact Kähler manifolds. (In fact, one can find compact complex tori that have Hodge classes in $H^{2k}(X, \mathbb{Q})$, but that don't contain any closed analytic subsets of codimension k.) So one has to use the fact that X is projective, and one way of doing this is by looking at arithmetic aspects (that make sense for polynomials but not for holomorphic functions). Deligne's theory of "absolute Hodge classes" is one step in this direction.

The general idea is as follows. Let $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective variety. It is of course the common zero set of finitely many homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_N]$, and the finitely many coefficients of all these polynomials generate a subfield $k \subseteq \mathbb{C}$ that is finitely generated over $\mathbb{Q}$. By construction, X is defined over this much smaller field k. If we have an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we can apply it to the coefficients of the polynomials defining X, and obtain a new smooth projective variety X^σ. It is isomorphic to the original X as a variety over $\mathbb{Q}$ – if a point $[z_0, \dots, z_N]$ lies on X, then its image $[\sigma(z_0), \dots, \sigma(z_N)]$ lies on X^σ – but not over $\mathbb{C}$. In fact, not only are X and X^σ not isomorphic as complex manifolds, they are usually even not isomorphic as topological spaces.

The cohomology $H^*(X, \mathbb{C})$ can actually be computed algebraically (using the algebraic de Rham complex), and for that reason, $H^*(X, \mathbb{C}) \cong H^*(X^\sigma, \mathbb{C})$. The isomorphism is functorial, but it does not take the subspace $H^*(X, \mathbb{Q})$ to the subspace $H^*(X^\sigma, \mathbb{Q})$, because one needs the underlying topological space to define the cohomology with $\mathbb{Q}$-coefficients, and the underlying topological spaces of X and X^σ are not isomorphic. So if one has a Hodge class $\alpha \in H^{2k}(X, \mathbb{Q})$, there is no reason why its image $\alpha^\sigma \in H^{2k}(X^\sigma, \mathbb{C})$ should again be a Hodge class – it might not even be a rational cohomology class. On the other hand, the fundamental class $[Z]$ of a closed subvariety does remain a Hodge class, because the isomorphism $H^{2k}(X, \mathbb{C}) \cong H^{2k}(X^\sigma, \mathbb{C})$ takes $[Z]$ to $[Z^\sigma]$. This potentially different behavior between Hodge classes and algebraic classes motivates the following definition.

Definition 21.6. A Hodge class $\alpha \in H^{2k}(X, \mathbb{Q})$ is called *absolute* if for every $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, the image $\alpha^\sigma \in H^{2k}(X^\sigma, \mathbb{C})$ is again a Hodge class.

The Hodge conjecture then breaks up into two steps: (1) Show that every Hodge class is absolute. (2) Show that every absolute Hodge class is algebraic. Absolute Hodge classes don't make sense on compact Kähler manifolds, and so this limits the scope of the problem to smooth projective varieties.

On abelian varietes, the Hodge conjecture is still open (and while I am skeptical about the general case, I do think that the Hodge conjecture is true on abelian varieties). If you went to Markman's talk two weeks ago, you'll remember that he proved the Hodge conjecture for all 4-dimensional abelian varieties. The best general result that we have is the following cool theorem by Deligne.

Theorem 21.7 (Deligne). *Every Hodge class on an abelian variety is absolute.*

In the rest of the semester, we'll talk about the proof of Deligne's theorem. It involves moduli spaces of abelian varieties; complex multiplication (CM) on abelian varieties; certain special Hodge classes called "Weil classess"; and other things.

Lecture 22 (April 17)

Hodge structures. Let's start with a brief review of Hodge structures, because we are going to need the language. Let H be a finite-dimensional $\mathbb{Q}$-vector space. A *Hodge structure* of weight k on H is a decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

with the property that $\overline{H^{p,q}} = H^{q,p}$ for all $p + q = k$. We can describe a Hodge structure in terms of its *Hodge filtration* $F^\bullet H_{\mathbb{C}}$; this is the decreasing filtration with

$$F^p H_{\mathbb{C}} = H^{p,k-p} \oplus H^{p+1,k-p+1} \oplus H^{p+2,k-p+2} \oplus \cdots$$

One can recover the Hodge decomposition from the Hodge filtration because

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}.$$

All Hodge structures that are of interest in geometry are "polarized". By definition, a *polarization* of H is a $(-1)^k$-symmetric bilinear pairing

$$S \colon H \otimes_{\mathbb{Q}} H \to \mathbb{Q}$$

with the property that the hermitian form

$$h(v, w) = \sum_{p+q=k} i^{p-q} S\big(v^{p,q}, \overline{w^{p,q}}\big)$$

is positive definite and makes the Hodge decomposition into an orthogonal decomposition. Concretely, S is symmetric if k is even, and skew-symmetric if k is odd; $S(H^{p,q}, H^{p',q'}) = 0$ unless $p' = q$ and $q' = p$; and $i^{p-q} S(v, \bar{v}) > 0$ for nonzero $v \in H^{p,q}$. These conditions are coming from the Hodge-Riemann bilinear relations on the cohomology of smooth projective varieties.

Example 22.1. Suppose that $k = 2m$ is even. A class $h \in H$ is called a *Hodge class* if $h \in H^{m,m}$. This is equivalent to the condition that $h \in F^m H_{\mathbb{C}}$. Indeed, if we write $h = \sum_{p+q=2m} h^{p,q}$, then $h \in F^m H_{\mathbb{C}}$ means that $h^{p,q} = 0$ for $p < m$. Because $h = \bar{h}$, we also gett $h^{p,q} = \overline{h^{q,p}} = 0$ for $p > m$, and so $h \in H^{m,m}$.

For any integer $\ell \in \mathbb{Z}$, we have Tate's Hodge structure $\mathbb{Q}(\ell)$ on the $\mathbb{Q}$-vector space $\mathbb{Q}(\ell) = (2\pi i)^\ell \mathbb{Q} \subseteq \mathbb{C}$. It has weight -2ℓ: the complexification is $\mathbb{C}$, and the Hodge decomposition is $\mathbb{C} = \mathbb{C}^{-\ell,-\ell}$. For any Hodge structure H, we can then form the *Tate twist* $H(\ell) = H \otimes_{\mathbb{Q}} \mathbb{Q}(\ell)$. The Hodge decomposition of $H(\ell)_{\mathbb{C}} = H_{\mathbb{C}}$ stays the same, but we now view it as a Hodge structure of weight $k - 2\ell$ by setting

$$H(\ell)^{p,q} = H^{p+\ell,q+\ell}.$$

Lastly, a *morphism* between two Hodge structures H_1, H_2 of the same weight is a $\mathbb{Q}$-linear mapping $f\colon H_1 \to H_2$ such that $f(H_1^{p,q}) \subseteq H_2^{p,q}$ for all $p + q = k$. In geometry, one often encounters linear mappings that change the weight (such as the Gysin morphism); they are properly considered as morphisms $H_1 \to H_2(\ell)$.

Example 22.2. Let X be a smooth projective variety over $\mathbb{C}$. By Hodge's theorem, each cohomology group $H^k(X, \mathbb{Q})$ has a Hodge structure of weight k. The Hodge filtration

$$F^p H^k(X, \mathbb{C}) = H^{p,k-p}(X) \oplus H^{p+1,k-p-1}(X) \oplus \cdots$$

has an alternative description in terms of de Rham cohomology. Set $n = \dim X$, and consider the holomorphic de Rham complex

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0.$$

By the holomorphic Poincaré lemma, the complex $\Omega_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_X$, and so

$$H^k(X, \mathbb{C}) \cong H^k(X, \Omega_X^\bullet).$$

We can filter the holomorphic de Rham complex by the family of subcomplexes

$$0 \longrightarrow \Omega_X^p \xrightarrow{d} \Omega_X^{p+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0,$$

usually denoted $F^p \Omega_X^\bullet$. A basic result in Hodge theory is that the mapping

$$H^k(X, F^p \Omega_X^\bullet) \to H^k(X, \Omega_X^\bullet)$$

is injective, and that its image is exactly the Hodge filtration $F^p H^k(X, \mathbb{C})$. An equivalent formulation is that the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1, and that the filtration coming from the spectral sequence is the Hodge filtration.

Example 22.3. Let's also quickly review the formulas for the polarization. Let L be an ample line bundle on X, and set $\omega = c_1(L)$, which is a Hodge class in $H^2(X, \mathbb{Z}(1))$. The Hard Lefschetz theorem says that for $0 \leq k \leq n$, the mapping

$$\omega^{n-k}\colon H^k(X, \mathbb{Q}) \to H^{2n-k}\big(X, \mathbb{Q}(n-k)\big)$$

is an isomorphism of Hodge structures. (The Tate twist is needed to make the weight of the second Hodge structure equal to k.) The primitive cohomology is

$$H_0^k(X,\mathbb{Q}) = \ker\Big(\omega^{n-k+1}\colon H^k(X,\mathbb{Q}) \to H^{2n-k+2}\big(X,\mathbb{Q}(n-k+1)\big)\Big).$$

According to the Hodge-Riemann bilinear relations, the pairing

$$S_k(\alpha,\beta) = (-1)^{k(k-1)/2}\frac{1}{(2\pi i)^n}\int_X \alpha\wedge\beta\wedge\omega^{n-k}$$

is a polarization of the Hodge structure on $H_0^k(X,\mathbb{Q})$. (The sign and the factor i^{p-q} show up because the associated hermitian form is, up to a positive constant, exactly the hermitian inner product on $H_0^k(X,\mathbb{C})$ induced by the Kähler metric.)

One can get a polarization on all of $H^k(X,\mathbb{Q})$ by using the Lefschetz decomposition. Because of the Hard Lefschetz theorem, we only need to consider $0 \le k \le n$. In that case, the Lefschetz decomposition is

$$H^k(X,\mathbb{Q}) = H_0^k(X,\mathbb{Q}) \oplus \omega H_0^{k-2}(X,\mathbb{Q}(-1)) \oplus \omega^2 H_0^{k-4}(X,\mathbb{Q}(-2)) \oplus \cdots$$

and one can show that $(-1)^\ell S_k$ polarizes the summand $\omega^\ell H_0^{k-2\ell}(X,\mathbb{Q}(-\ell))$. If we define an involution σ of $H^k(X,\mathbb{Q})$ that acts on the ℓ-th summand as $(-1)^\ell$, this means that the bilinear form

$$(\alpha,\beta) \mapsto S_k\big(\alpha,\sigma(\beta)\big)$$

is a polarization for the Hodge structure on $H^k(X,\mathbb{Q})$.

Algebraic de Rham cohomology. The theory of absolute Hodge classes is based on the observation that one can compute the cohomology of a smooth projective variety algebraically. Let X be a smooth projective variety over $\operatorname{Spec} k$, where k is a field containing $\mathbb{Q}$. (For example, k could be a finitely-generated extension of $\mathbb{Q}$, or $k = \mathbb{C}$.) From the sheaf of Kähler differentials $\Omega^1_{X/k}$ and its wedge powers, one can form the *algebraic de Rham complex*

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/k} \longrightarrow 0.$$

Its hypercohomology groups

$$H^i_{dR}(X/k) = H^i(X,\Omega^\bullet_{X/k})$$

are called the *algebraic de Rham cohomology* of X. They are finite-dimensional k-vector spaces. As before, we can filter the algebraic de Rham complex by the family of subcomplexes

$$0 \longrightarrow \Omega^p_{X/k} \xrightarrow{d} \Omega^{p+1}_{X/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/k} \longrightarrow 0,$$

denoted $F^p\Omega^\bullet_{X/k}$, and define the *Hodge filtration* as

$$F^pH^i_{dR}(X/k) = \operatorname{im}\Big(H^i(X,F^p\Omega^\bullet_{X/k}) \to H^i(X,\Omega^\bullet_{X/k})\Big).$$

The point is of course that this agrees with the definitions we gave earlier. To see why, suppose that $k = \mathbb{C}$. For the sake of clarity, let's denote the compact complex manifold associated to the smooth projective variety X by the symbol X^{an}. The analytification of $\Omega^1_{X/\mathbb{C}}$ is the sheaf of holomorphic 1-forms $\Omega^1_{X^{an}}$, and the analytification of the complex $\Omega^\bullet_{X/\mathbb{C}}$ is the holomorphic de Rham complex $\Omega^\bullet_{X^{an}}$. By Serre's GAGA theorem, we get a natural isomorphism

$$H^i_{dR}(X/\mathbb{C}) = H^i(X,\Omega^\bullet_{X/\mathbb{C}}) \cong H^i(X^{an},\Omega^\bullet_{X^{an}}) \cong H^i(X^{an},\mathbb{C}).$$

This isomorphism takes the subspace $F^pH^i_{dR}(X/\mathbb{C})$ to the subspace $F^pH^i(X^{an},\mathbb{C})$, hence to the usual Hodge filtration.

Note. We can not get the rational cohomology $H^i(X^{an}, \mathbb{Q})$ in this way; for that, we need the underlying topological space of the complex manifold X^{an}.

In general, we can take any embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, and consider the base change

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec} k. \end{tikzcd}$$

We have $\Omega^1_{X_{\mathbb{C}}/\mathbb{C}} \cong f^*\Omega^1_{X/k}$ because Kähler differentials are compatible with base change; we therefore get a natural isomorphism

$$H^i_{dR}(X_{\mathbb{C}}/\mathbb{C}) \cong H^i_{dR}(X/k) \otimes_k \mathbb{C}.$$

So both the algebraic de Rham cohomology, and the Hodge filtration on it, are actually defined over the field k.

Conjugate varieties. We can now give a precise definition of conjugating a variety by an automorphism of $\mathbb{C}$. Let X be a smooth projective variety over $\operatorname{Spec}\mathbb{C}$, and let $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$ be an automorphism. We define the *conjugate variety* X^σ as the base change

$$\begin{tikzcd} X^{\sigma} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

While X and X^σ are isomorphic as abstracts schemes (or as schemes over $\operatorname{Spec}\mathbb{Q}$), they are *not* isomorphic as schemes over $\operatorname{Spec}\mathbb{C}$. Because algebraic de Rham cohomology is compatible with base change, we have

$$H^i_{dR}(X^\sigma/\mathbb{C}) \cong H^i_{dR}(X/\mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C}$$

with the tensor product being taken over the isomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$. We therefore get a natural isomorphism

$$H^i\big((X^\sigma)^{an}, \mathbb{C}\big) \cong H^i(X^{an}, \mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C},$$

and this allows us to associate to every cohomology class $\alpha \in H^i(X^{an}, \mathbb{C})$ a conjugate α^σ in the i-th cohomology of $(X^\sigma)^{an}$.

The small problem is that the only automorphisms of $\mathbb{C}$ that one can write down (without the axiom of choice) are the identity and complex conjugation. A simpler way to accomplish the same thing is to start from a smooth projective variety X defined over a field k; in practice, k is going to be finitely-generated over $\mathbb{Q}$. For any embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, we can define $X^\sigma_{\mathbb{C}}$ as the base change

$$\begin{tikzcd} X^{\sigma}_{\mathbb{C}} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec} k. \end{tikzcd}$$

As before, we get a natural isomorphism

$$H^i\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{C}\big) \cong H^i_{dR}(X^\sigma_{\mathbb{C}}/\mathbb{C}) \cong H^i_{dR}(X/k) \otimes_k \mathbb{C}.$$

This again allows us to take any class in the cohomology of one complex manifold $(X^\sigma_{\mathbb{C}})^{an}$, and transport it in a canonical way to the cohomology of any other conjugate. The advantage is that embeddings of a finitely-generated field k into the complex numbers are easy to describe.

Note. When $k = \mathbb{Q}$, we get two different $\mathbb{Q}$-structures on the $\mathbb{C}$-vector space

$$H^i(X^{an}, \mathbb{C}) \cong H^i(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{dR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

one coming from singular cohomology, the other from algebraic de Rham cohomology. The relation between the two is the subject of Grothendieck's period conjecture: roughly speaking, it says that unless there is a geometric reason, the entries of the matrix relating the two $\mathbb{Q}$-structures are as transcendental as possible.

Note. Somebody asked about an example of two conjugate varieties that are not homeomorphic. I recommend the paper "Conjugate varieties with distinct real cohomology algebras" by François Charles, which you can find here:

`https://www.math.ens.psl.eu/~charles/crll5855.pdf`

Absolute Hodge classes. We can now give a precise definition of absolute Hodge classes. One formulation starts from algebraic de Rham cohomology.

Definition 22.4. Let X be a smooth projective variety over a field k that is finitely-generated over $\mathbb{Q}$. A class $\alpha \in F^p H^{2p}_{dR}(X/k) \otimes_k \mathbb{C}$ is called an *absolute Hodge class* if, for every embedding $\sigma \colon k \hookrightarrow \mathbb{C}$, the image of α under the isomorphism

$$H^{2p}_{dR}(X/k) \otimes_k \mathbb{C} \cong H^{2p}\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{C}\big)$$

belongs to the subspace $H^{2p}\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{Q}\big)$, and is therefore a Hodge class on $(X^\sigma_{\mathbb{C}})^{an}$.

In practice, we are usually starting from a smooth projective variety over $\operatorname{Spec} \mathbb{C}$ and are interested in classes in the rational cohomology of X^{an}. So here is an equivalent definition that is closer to what we said last time.

Definition 22.5. Let X be a smooth projective variety over $\operatorname{Spec} \mathbb{C}$. A Hodge class $\alpha \in H^{2p}(X^{an}, \mathbb{Q}) \cap F^p H^{2p}(X^{an}, \mathbb{C})$ is called an *absolute Hodge class* if, for every automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, the image of α under the isomorphism

$$F^p H^{2p}(X^{an}, \mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C} \cong F^p H^{2p}\big((X^\sigma)^{an}, \mathbb{C}\big)$$

belongs to the subspace $H^{2p}\big((X^\sigma)^{an}, \mathbb{Q}\big)$, and is therefore a Hodge class on $(X^\sigma)^{an}$.

The main example are of course fundamental classes of algebraic subvarieties. Let's start with a simpler example.

Example 22.6. Let L be a line bundle on the smooth projective variety X. Let L^{an} denote the associated holomorphic line bundle on the complex manifold X^{an}. Then $c_1(L^{an})$ is an absolute Hodge class in $H^2(X^{an}, \mathbb{Q}(1))$. To see why, we need to look at the construction of the first Chern class, especially in algebraic de Rham cohomology. This will also explain where the $2\pi i$ comes from.

In the analytic topology, we can use the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{O}_{X^{an}} \xrightarrow{\exp} \mathscr{O}^\times_{X^{an}} \longrightarrow 0.$$

Here $\mathbb{Z}(1) = 2\pi i \mathbb{Z} \subseteq \mathbb{C}$ shows up as the kernel of the exponential function. The first Chern class is the connecting homomorphism

$$c_1 \colon \operatorname{Pic}(X^{an}) \cong H^1\big(X^{an}, \mathscr{O}^\times_{X^{an}}\big) \to H^2\big(X^{an}, \mathbb{Z}(1)\big).$$

To compute $c_1(L^{an})$, we take a good covering by contractible open subsets U_i on which L^{an} is trivial. The transition functions $g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}^\times_{X^{an}})$ form a 1-cocycle. We then write

$$g_{ij} = e^{f_{ij}}$$

for holomorphic functions $f_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_{X^{an}})$, and then the 2-cocycle

$$f_{jk} - f_{ik} + f_{ij} \in \mathbb{Z}(1)$$

represents $c_1(L^{an}) \in H^2(X^{an}, \mathbb{Z}(1))$. To convert this class to de Rham cohomology, we consider the holomorphic 1-forms

$$df_{ij} = \frac{dg_{ij}}{g_{ij}} \in \Gamma(U_i \cap U_j, \Omega^1_{X^{an}}).$$

They are closed, and form a 1-cocycle, and so they determine a class in $H^2(X^{an}, F^1\Omega^\bullet_{X^{an}})$ (using Čech cohomology). Since this computes $F^1H^2(X^{an}, \mathbb{C})$, the first Chern class of L^{an} is a Hodge class. Note that it naturally lives in $H^2(X^{an}, \mathbb{Q}(1))$, which is a Hodge structure of weight zero (because of the Tate twist).

We can imitate this construction in algebraic de Rham cohomology. Indeed, L is an algebraic line bundle, so there is a covering of X by affine open subsets U_i on which L is trivial. Denoting the transition functions again by $g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times)$, we get a 1-cocycle consisting of the closed algebraic 1-forms

$$\frac{dg_{ij}}{g_{ij}} \in \Gamma(U_i \cap U_j, \Omega^1_{X/\mathbb{C}}),$$

and using Čech cohomology, this again determines a class in

$$F^1H^2_{dR}(X/\mathbb{C}) = H^2(X, F^1\Omega^\bullet_{X/\mathbb{C}}).$$

Under the comparison isomorphism, the two classes $c_1(L^{an})$ and $c_1(L)$ then correspond to each other. For every automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we can pull L back along the morphism $X^\sigma \to X$ and obtain a line bundle L^σ. By the above, $c_1(L^{an})$ corresponds to the class $c_1(L)$ in de Rham cohomology; under σ, this goes to $c_1(L^\sigma)$, which in turn corresponds to $c_1\big((L^\sigma)^{an}\big)$. Therefore the conjugate of $c_1(L^{an})$ is again the first Chern class of a line bundle, and this means that $c_1(L^{an})$ is an absolute Hodge class. (With a slightly broader definition of absolute Hodge classes that allows Tate twists.)

More generally, the fundamental class of any closed subvariety is an absolute Hodge class. Let $Z \subseteq X$ be a closed subvariety of codimension p. For the sake of precision, let's denote by $Z^{an} \subseteq X^{an}$ the associated analytic subset of the compact complex manifold X^{an}. The fundamental class

$$[Z^{an}] \in H^{2p}\big(X^{an}, \mathbb{Q}(p)\big)$$

can be defined using Poincaré duality (which also explain the appearance of the Tate twist). Let $\mu\colon \tilde{Z} \to Z$ be a resolution of singularities, and denote by $f\colon \tilde{Z} \to X$ the composition. The linear functional

$$H^{2n-2p}(X^{an}, \mathbb{Q}) \to \mathbb{Q}, \quad \alpha \mapsto \int_{\tilde{Z}^{an}} f^*\alpha,$$

is represented by a unique cohomology class $\zeta \in H^{2p}(X^{an}, \mathbb{Q})$, which then satisfies

$$\int_{\tilde{Z}^{an}} f^*\alpha = \int_{X^{an}} \zeta \cup \alpha.$$

We saw in the discussion about polarization that it is better to divide the integral over X^{an} by $(2\pi i)^n$. We therefore define the fundamental class of Z as

$$[Z^{an}] = (2\pi i)^p \zeta \in H^{2p}\big(X^{an}, \mathbb{Q}(p)\big),$$

and this turns the identity from above into

$$\frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}^{an}} f^*\alpha = \frac{1}{(2\pi i)^n} \int_{X^{an}} [Z^{an}] \cup \alpha.$$

Now we need to define a corresponding class $[Z]$ in the algebraic de Rham cohomology of X. The easiest way to do this is to use Chern classes, which make sense both in usual cohomology and in algebraic de Rham cohomology. Starting from the case of line bundles (for which we have Chern classes in both theories), one first constructs Chern classes for vector bundles (using the splitting principle), and then Chern classes for arbitrary coherent sheaves (using locally free resolutions). Once this theory is in place, the Grothendieck-Riemann-Roch theorem implies that

$$[Z^{an}] = \frac{(-1)^{p-1}}{(p-1)!} c_p\bigl(\mathscr{O}_{Z^{an}}\bigr).$$

We can then simply define

$$[Z] = \frac{(-1)^{p-1}}{(p-1)!} c_p\bigl(\mathscr{O}_Z\bigr) \in F^p H^{2p}_{dR}(X/\mathbb{C}),$$

and then $[Z^{an}]$ and $[Z]$ correspond to each other under the comparison isomorphism. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we get a conjugate subvariety $Z^\sigma \subseteq X^\sigma$, and just as in the case of line bundles, this implies that $[Z^{an}]$ is an absolute Hodge class.

Lecture 23 (April 22)

Deligne's Principle B. The goal of today's lecture is to show that absolute Hodge classes behave well in families. Suppose that $f\colon X \to B$ is a smooth projective morphism (over $\mathbb{C}$); for each $b \in B$, we denote the fiber by $X_b = f^{-1}(b)$, which is a smooth projective variety. For simplicity, let's assume that the parameter space B is connected and quasi-projective; then X itself is also quasi-projective. The $2p$-th cohomology groups $H^{2p}(X_b, \mathbb{Q})$ of the fibers fit together into a local system $R^{2p} f_* \mathbb{Q}_X$ on B. If we have a global section $\alpha \in H^0(B, R^{2p} f_* \mathbb{Q}_X)$, we denote its value at a point $b \in B$ by $\alpha_b \in H^{2p}(X_b, \mathbb{Q})$. We think of α as being a family of cohomology classes on the fibers.

The following important result is known as "Deligne's Principle B". Informally, it says that if we have a family of cohomology classes (in the above sense), and if one of them is an absolute Hodge class, then all of them are absolute Hodge classes. (The analogue problem for algebraic classes is the so-called "variational Hodge conjecture"; this is wide open.)

Theorem 23.1 (Principle B)**.** *Let $f\colon X \to B$ be a smooth projective morphism, with B connected and quasi-projective, and let $\alpha \in H^0(B, R^{2p} f_* \mathbb{Q}_X)$. If there is a point $0 \in B$ such that $\alpha_0 \in H^{2p}(X_0, \mathbb{Q})$ is an absolute Hodge class, then $\alpha_b \in H^{2p}(X_b, \mathbb{Q})$ is an absolute Hodge class for every $b \in B$.*

In practice, this means that if α_0 is the class of an algebraic cycle (and therefore an absolute Hodge classe), then all the α_b are absolute Hodge classes. So Principle B allows us to bypass the Hodge conjecture in certain cases.

Properties of absolute Hodge classes. We are going to prove the theorem by studying the behavior of absolute Hodge classes under various operations.

Pullbacks. The most basic operation is pulling back along a morphism $f\colon X \to Y$ between two smooth projective varieties (over $\mathbb{C}$). Here the pullback morphism

$$f^*\colon H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

takes absolute Hodge classes on Y to absolute Hodge classes on X. This can be seen as follows. First, we have a pullback morphism in algebraic de Rham cohomology: the morphism of sheaves $f^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}}$ induces a morphism of complexes

$$f^*\Omega^\bullet_{Y/\mathbb{C}} \to \Omega^\bullet_{X/\mathbb{C}}$$

between the algebraic de Rham complexes of X and Y; passing to cohomology gives

$$H^k_{dR}(Y/\mathbb{C}) = H^k(Y, \Omega^\bullet_{Y/\mathbb{C}}) \to H^k(X, f^*\Omega^\bullet_{Y/\mathbb{C}}) \to H^k(X, \Omega^\bullet_{X/\mathbb{C}}) = H^k_{dR}(X/\mathbb{C}).$$

It is easy to see that this morphism is compatible with $f^*\colon H^k(Y,\mathbb{C}) \to H^k(X,\mathbb{C})$ under the comparison isomorphism with algebraic de Rham cohomology. Now if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, then we get a conjugate morphism $f^\sigma\colon X^\sigma \to Y^\sigma$, and the compatibility with algebraic de Rham cohomology implies that

$$f^*(\alpha^\sigma) = (f^*\alpha)^\sigma$$

So if $\alpha \in H^k(Y,\mathbb{Q})$ is an absolute Hodge class (and k is even), then $\alpha^\sigma \in H^k(Y^\sigma,\mathbb{Q})$, and so its pullback lies in $H^k(X^\sigma,\mathbb{Q})$, which shows that $f^*\alpha$ is again an absolute Hodge class.

Cup product. Similarly, the cup product morphism

$$H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \to H^{i+j}(X,\mathbb{Q}), \quad \alpha\otimes\beta \mapsto \alpha\cup\beta,$$

takes pairs of absolute Hodge classes to absolute Hodge classes. To see this, we rewrite the cup product as

$$H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \hookrightarrow H^{i+j}(X\times X,\mathbb{Q}) \xrightarrow{\Delta^*} H^{i+j}(X,\mathbb{Q}),$$

where the first morphism comes from the Künneth isomorphism, and the second is pullback along the diagonal $\Delta\colon X \to X\times X$. The Künneth isomorphism also holds in algebraic de Rham cohomology, in a way that is compatible with the comparison isomorphism; this is a consequence of the fact that

$$\Omega^1_{X\times X/\mathbb{C}} \cong p_1^*\Omega^1_{X/\mathbb{C}} \otimes p_2^*\Omega^1_{X/\mathbb{C}}.$$

For that reason, the inclusion $H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \hookrightarrow H^{i+j}(X\times X,\mathbb{Q})$ takes a pair of absolute Hodge classes to an absolute Hodge class; and because Δ^* preserves absolute Hodge classes, we get the result.

Poincaré duality. On a smooth projective variety X of dimension n, the pairing

$$H^k(X,\mathbb{Q}) \otimes H^{2n-k}(X,\mathbb{Q}) \to H^{2n}(X,\mathbb{Q}), \quad \alpha\otimes\beta \mapsto \alpha\cup\beta,$$

is nondegenerate, which means that

$$H^k(X,\mathbb{Q}) \to \mathrm{Hom}\big(H^{2n-k}(X,\mathbb{Q}), H^{2n}(X,\mathbb{Q})\big)$$

is an isomorphism. As Hodge structures of weight $2n$, we have $H^{2n}(X,\mathbb{Q}) \cong \mathbb{Q}(-n)$; an explicit isomorphism is given by

$$H^{2n}(X,\mathbb{Q}) \to \mathbb{Q}(-n), \quad \alpha \mapsto \frac{1}{(2\pi i)^n}\int_X \alpha.$$

Its inverse is represented by the fundamental class $[x] \in H^{2n}(X,\mathbb{Q}(n))$ of any point $x \in X$. Because cup product preserves absolute Hodge classes, and because the fundamental class of a point is of course an absolute Hodge class, it follows that the Poincaré duality isomorphism

$$H^k(X,\mathbb{Q}) \to \mathrm{Hom}\big(H^{2n-k}(X,\mathbb{Q}), \mathbb{Q}(-n)\big)$$

takes absolute Hodge classes to absolute Hodge classes. (The notion of absolute Hodge classes also makes sense for classes in the dual vector space.)

Example 23.2. Let $f\colon X \to Y$ be a morphism between smooth projective varieties, and set $r = \dim Y - \dim X$. Then the Gysin homomorphism $f_*\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ takes absolute Hodge classes to absolute Hodge classes. The reason is that f_* is the composition of Poincaré duality on X and Y and the homomorphism dual to $f^*\colon H^k(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})$.

Absolute homomorphisms. More generally, suppose that we have a homomorphism $\phi\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ between cohomology groups of two smooth projective varieties X and Y. The Tate twist changes the weight of the second cohomology group to $k+2r-2r=k$. Using Poincaré duality and the Künneth formula, we can associate to ϕ a cohomology class $\operatorname{cl}(\phi)$ in

$$\begin{aligned} H^k(X,\mathbb{Q})^\vee \otimes H^{k+2r}(Y,\mathbb{Q}(r)) &\cong H^{2n-k}(X,\mathbb{Q}(n)) \otimes H^{k+2r}(Y,\mathbb{Q}(r)) \\ &\subseteq H^{2n+2r}(X\times Y,\mathbb{Q}(n+r)), \end{aligned}$$

where $n=\dim X$. It is not hard to see that ϕ is a morphism of Hodge structures of weight k if and only if $\operatorname{cl}(\phi)$ is a Hodge class on $X\times Y$.

Definition 23.3. We will say that a morphism $\phi\colon H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q}(r))$ is *absolute* if $\operatorname{cl}(\phi) \in H^{2n+2r}(X\times Y,\mathbb{Q}(n+r))$ is an absolute Hodge class.

Example 23.4. The Gysin homomorphism is absolute.

One can recover the action of ϕ by a formula similar to an integral transform:

$$\phi(\alpha) = (p_2)_*\big(p_1^*(\alpha)\cup \operatorname{cl}(\phi)\big),$$

with $p_1\colon X\times Y\to X$ and $p_2\colon X\times Y\to Y$ the two projections. This shows that if ϕ is absolute, then it takes absolute Hodge classes in $H^k(X,\mathbb{Q})$ to absolute Hodge classes in $H^{k+2r}(Y,\mathbb{Q}(r))$ (when k is even). Indeed, all three operations on the right-hand side of the formula preserve absolute Hodge classes.

Composition and inverses. The composition of absolute morphisms is absolute. For simplicity, let's take the case where $\phi\colon H^k(X,\mathbb{Q})\to H^k(Y,\mathbb{Q})$ and $\psi\colon H^k(Y,\mathbb{Q})\to H^k(Z,\mathbb{Q})$ are homomorphisms between cohomology groups of the same degree. The associated cohomology classes are $\operatorname{cl}(\phi)\in H^{2n}(X\times Y,\mathbb{Q}(n))$ and $\operatorname{cl}(\psi)\in H^{2m}(Y\times Z,\mathbb{Q}(m))$, where $n=\dim X$ and $m=\dim Y$. Just as with integral transforms, the cohomology class of the composition $\psi\circ\phi$ is computed by a convolution:

$$\operatorname{cl}(\psi\circ\phi) = (p_{13})_*\big(p_{12}^*\operatorname{cl}(\phi)\cup p_{23}^*\operatorname{cl}(\psi)\big) \in H^{2n}(X\times Z,\mathbb{Q}(n)).$$

If $\operatorname{cl}(\phi)$ and $\operatorname{cl}(\psi)$ are absolute Hodge classes, then so is their convolution; therefore $\psi\circ\phi$ is again absolute. Similarly, one shows that if

$$\phi\colon H^k(X,\mathbb{Q})\to H^{k+2r}(Y,\mathbb{Q}(r))$$

is both absolute and an isomorphism, then the inverse homomorphism ϕ^{-1} is again absolute.

Images of absolute morphisms. We'll now use the facts from the previous section to prove the following result.

Proposition 23.5. *Let X and Y be smooth projective varieties. Suppose that $\phi\colon H^{2p}(X,\mathbb{Q})\to H^{2p}(Y,\mathbb{Q})$ is an absolute morphism of Hodge structures. If $\alpha\in H^{2p}(Y,\mathbb{Q})$ is an absolute Hodge class in the image of ϕ, then there is an absolute Hodge class $\beta\in H^{2p}(X,\mathbb{Q})$ such that $\phi(\beta)=\alpha$.*

In other words, any absolute Hodge class in the image of an absolute morphism is actually the image of an absolute Hodge class. The proof relies on the fact that the two Hodge structures can be polarized, in a way that is compatible with absolute Hodge classes.

Let's start with a few general remarks. Let H be a Hodge structure of weight k, with Hodge decomposition

$$H_{\mathbb{C}} = H\otimes_{\mathbb{Q}}\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}.$$

Define the Weil operator $C \in \operatorname{End}(H_{\mathbb{R}})$ by the formula $Cv = i^{p-q}v$ for $v \in H^{p,q}$. By Hodge symmetry, we have $C(\bar{v}) = \overline{Cv}$, and so C is a real operator with $C^2 = (-1)^k$. Now recall that a polarization is a $(-1)^k$-symmetric pairing

$$S \colon H \otimes_{\mathbb{Q}} H \to \mathbb{Q}(-k)$$

such that $\langle v, w \rangle = S(Cv, \bar{w})$ is a hermitian inner product on $H_{\mathbb{C}}$ that makes the Hodge decomposition into an orthogonal decomposition. This implies in particular that the polarization S is non-degenerate: if $S(v, w) = 0$ for all $w \in H_{\mathbb{C}}$, then $\|v\|^2 = S(Cv, \bar{v}) = 0$, and so $v = 0$. If we consider S as a homomorphism

$$S \colon H \to \operatorname{Hom}_{\mathbb{Q}}\bigl(H, \mathbb{Q}(-k)\bigr),$$

it is therefore an isomorphism of Hodge structures (of weight k).

If $V \subseteq H$ is a sub-Hodge structure, meaning a rational subspace such that $V^{p,q} = H^{p,q} \cap V_{\mathbb{C}}$, then the orthogonal complement

$$V^{\perp} = \bigl\{ h \in H \bigm| S(h, v) = 0 \text{ for all } v \in V \bigr\}$$

is again a sub-Hodge structure, and $H = V \oplus V^{\perp}$. This follows from the fact that

$$V^{\perp} \otimes_{\mathbb{Q}} \mathbb{C} = \bigl\{ h \in H_{\mathbb{C}} \bigm| \langle h, v \rangle = 0 \text{ for all } v \in V_{\mathbb{C}} \bigr\},$$

which holds because $\langle v, w \rangle = S(Cv, \bar{w})$ and because the Hodge decomposition is orthogonal with respect to the inner product.

In the geometric case, the polarization is itself absolute, in the sense we talked about earlier. Let's recall the construction; to keep down the notation, I am going to leave out the Tate twists in the formulas below. Let X be a smooth projective variety of dimension n, choose an ample line bundle $L \in \operatorname{Pic}(X)$, and let $\omega = c_1(L) \in H^2(X, \mathbb{Z}(1))$ be its first Chern class. The pairing

$$S_k(\alpha, \beta) = \frac{(-1)^{k(k-1)/2}}{(2\pi i)^n} \int_X \alpha \cup \beta \cup \omega^{n-k},$$

takes values in $\mathbb{Q}(-k)$, and is a polarization of the Hodge structure on the primitive cohomology

$$H_0^k(X, \mathbb{Q}) = \ker\bigl(\omega^{n-k+1} \colon H^k(X, \mathbb{Q}) \to H^{2n-k+2}(X, \mathbb{Q})\bigr)\bigr).$$

The formula for S_k only involves absolute operations: $\omega = c_1(L)$ is an absolute Hodge class, and the isomorphism

$$H^{2n}(X, \mathbb{Q}) \to \mathbb{Q}(-n), \quad \alpha \mapsto \frac{1}{(2\pi i)^n} \int_X \alpha,$$

is the inverse of the fundamental class of a point.

To get a polarization on all of $H^k(X, \mathbb{Q})$, we use the Lefschetz decomposition

$$H^k(X, \mathbb{Q}) = H_0^k(X, \mathbb{Q}) \oplus \omega H_0^{k-2}(X, \mathbb{Q}) \oplus \omega^2 H_0^{k-4}(X, \mathbb{Q}) \oplus \cdots$$

Define an involution $s \in \operatorname{End} H^k(X, \mathbb{Q})$ by acting as $(-1)^{\ell}$ on the subspace $\omega^{\ell} H_0^{k-2\ell}(X, \mathbb{Q})$ in the Lefschetz decomposition. Then

$$S(\alpha, \beta) = S_k\bigl(\alpha, s(\beta)\bigr)$$

polarizes the Hodge structure on $H^k(X, \mathbb{Q})$. As we said above, we can view S as an isomorphism

$$S \colon H^k(X, \mathbb{Q}) \to \operatorname{Hom}\bigl(H^k(X, \mathbb{Q}), \mathbb{Q}(-k)\bigr), \tag{23.6}$$

and this isomorphism is absolute; we'll abbreviate this by saying that the polarization is absolute.

Proposition 23.7. *The isomorphism in* (23.6) *is absolute.*

Proof. Because the formula for S_k only involves absolute operations, it suffices to prove that the involution s is absolute. Let $p_\ell \colon H^k(X,\mathbb{Q}) \to H^k(X,\mathbb{Q})$ be the projection to the subspace $\omega^\ell H_0^{k-2\ell}(X,\mathbb{Q})$ in the Lefschetz decomposition. Then

$$s = \sum_{\ell \in \mathbb{N}} (-1)^\ell p_\ell,$$

and so it is enough to prove that each p_ℓ is absolute. Take any $\alpha \in H^k(X,\mathbb{Q})$, and write its Lefschetz decomposition as

$$\alpha = \alpha_0 + \omega \cup \alpha_1 + \omega^2 \cup \alpha_2 + \cdots .$$

Here each $\alpha_\ell \in H_0^{k-2\ell}(X,\mathbb{Q})$ is primitive, which means that $\omega^{n-k+2\ell} \cup \alpha_\ell \neq 0$ and $\omega^{n-k+2\ell+1} \cup \alpha_\ell = 0$. For $r \geq 1$, we therefore have

$$\omega^{n-k+r} \cup \alpha = \sum_{\ell \geq r} \omega^{n-k+r+\ell} \cup \alpha_\ell \in H^{2n-k+2r}(X,\mathbb{Q}).$$

By the Hard Lefschetz theorem,

$$\omega^{n-k+2r} \colon H^{k-2r}(X,\mathbb{Q}) \to H^{2n-k+2r}(X,\mathbb{Q})$$

is an isomorphism, and we clearly have

$$(\omega^{n-k+2r})^{-1}\bigl(\omega^{n-k+r} \cup \alpha\bigr) = \sum_{\ell \geq r} \omega^{\ell-r} \alpha_\ell.$$

By comparing this with the original Lefschetz decomposition for α, we find that

$$(p_0 + \cdots + p_{r-1})(\alpha) = \alpha - \sum_{\ell \geq r} \omega^\ell \cup \alpha_\ell = \alpha - \omega^r \cup (\omega^{n-k+2r})^{-1}\bigl(\omega^{n-k+r} \cup \alpha\bigr).$$

This is clearly an absolute morphism, because it only involves cup product with the absolute Hodge class ω and the inverse of the absolute isomorphism ω^{n-k+2r}. By subtracting the formulas for r and $r+1$, we conclude that each projector p_r is absolute. $\square$

We can now show that if an absolute Hodge class lies in the image of an absolute morphism, then it must be the image of an absolute Hodge class.

Proof of Proposition 23.5. Let's denote by S_X and S_Y the polarizations on $H^{2p}(X,\mathbb{Q})$ and $H^{2p}(Y,\mathbb{Q})$. The absolute morphism $\phi \colon H^{2p}(X,\mathbb{Q}) \to H^{2p}(Y,\mathbb{Q})$ has an adjoint $\phi^\dagger \colon H^{2p}(Y,\mathbb{Q}) \to H^{2p}(X,\mathbb{Q})$ with respect to the polarizations, which satisfies

$$S_Y\bigl(\alpha, \phi(\beta)\bigr) = S_X\bigl(\phi^\dagger(\alpha), \beta\bigr).$$

The adjoint fits into a commutative diagram

$$\begin{tikzcd}
H^{2p}(Y,\mathbb{Q}) \arrow[r, "\phi^\dagger"] \arrow[d, "S_Y"] & H^{2p}(X,\mathbb{Q}) \arrow[d, "S_X"] \\
\operatorname{Hom}\bigl(H^{2p}(Y,\mathbb{Q}), \mathbb{Q}(-p)\bigr) \arrow[r, "\phi^*"] & \operatorname{Hom}\bigl(H^{2p}(X,\mathbb{Q}), \mathbb{Q}(-p)\bigr),
\end{tikzcd}$$

where ϕ^* is the morphism induced by ϕ. Because ϕ is absolute, the dual morphism ϕ^* is also absolute; and because S_X is absolute, its inverse S_X^{-1} is also absolute. Therefore $\phi^\dagger$ is absolute as well. Note that $\phi^\dagger$ is also the adjoint of ϕ with respect to the inner products on $H^{2p}(X,\mathbb{Q})$ and $H^{2p}(Y,\mathbb{Q})$.

Because ϕ is a morphism of Hodge structures, the polarization S_Y gives us an orthogonal decomposition

$$H^{2p}(Y,\mathbb{Q}) = \operatorname{im}\phi \oplus (\operatorname{im}\phi)^\perp.$$

Just as in linear algebra, the adjoint has the property that $(\operatorname{im}\phi)^\perp = \ker\phi^\dagger$. We can therefore rewrite the decomposition as

$$H^{2p}(Y,\mathbb{Q}) = \operatorname{im}\phi \oplus \ker\phi^\dagger.$$

Now consider the morphism $\phi\circ\phi^\dagger\colon H^{2p}(Y,\mathbb{Q}) \to H^{2p}(Y,\mathbb{Q})$. It is self-adjoint, and its kernel is exactly $\ker\phi^\dagger$, because of the identity

$$\big\langle \alpha, (\phi\circ\phi^\dagger)(\alpha)\big\rangle_Y = \big\langle \phi^\dagger(\alpha), \phi^\dagger(\alpha)\big\rangle_X.$$

Let $\pi\colon H^{2p}(Y,\mathbb{Q}) \to H^{2p}(Y,\mathbb{Q})$ denote the orthogonal projection to the subspace $\operatorname{im}\phi$. By the spectral theorem (for self-adjoint linear operators), π can be written as a polynomial in $\phi\circ\phi^\dagger$ without constant term, say

$$\pi = \sum_{n\geq 1} c_n(\phi\circ\phi^\dagger)^n.$$

Now if $\alpha \in H^{2p}(Y,\mathbb{Q})$ is an absolute Hodge class in the image of ϕ, then

$$\alpha = \pi(\alpha) = \sum_{n\geq 1} c_n(\phi\circ\phi^\dagger)^n\alpha,$$

which is equal to the image under ϕ of the absolute Hodge class

$$\beta = \sum_{n\geq 1} c_n(\phi^\dagger\circ\phi)^{n-1}\phi^\dagger(\alpha) \in H^{2p}(X,\mathbb{Q}).$$

This proves the proposition. □

Proof of Principle B. After all this work, it is now an easy matter to prove Deligne's Principle B. Let $f\colon X \to B$ be a smooth projective morphism, with B connected and quasi-projective. Let $\alpha \in H^0(B, R^{2p}f_*\mathbb{Q})$ be a section of the local system, and denote by $\alpha_b \in H^{2p}(X_b,\mathbb{Q})$ its value at a point $b \in B$. Suppose that $\alpha_0 \in H^{2p}(X_0,\mathbb{Q})$ is an absolute Hodge class for some $0 \in B$. The local system contains the same information as the monodromy action of $\pi_1(B,0)$ on the cohomology group $H^{2p}(X_0,\mathbb{Q})$, and a global section is the same as a cohomology class that is invariant under monodromy:

$$H^0(B, R^{2p}f_*\mathbb{Q}) \cong H^{2p}(X_0,\mathbb{Q})^{\pi_1(B,0)}$$

From the Leray spectral sequence (which degenerates at E_2 because f is smooth and projective), we get a surjection

$$H^{2p}(X,\mathbb{Q}) \to H^0(B, R^{2p}f_*\mathbb{Q}) \cong H^{2p}(X_0,\mathbb{Q})^{\pi_1(B,0)}.$$

Denoting by $i_b\colon X_b \hookrightarrow X$ the inclusion of the fiber, the composition is just i_0^*.

Now let $\bar{X}$ be a smooth projective variety containing X as a Zariski-open subset, and let $j\colon X \hookrightarrow \bar{X}$ be the open embedding. According to the global invariant cycle theorem, the composition

$$H^{2p}(\bar{X},\mathbb{Q}) \xrightarrow{j^*} H^{2p}(X,\mathbb{Q}) \xrightarrow{i_0^*} H^{2p}(X_0,\mathbb{Q})^{\pi_1(B,0)}$$

is surjective. (This theorem is also due to Deligne; it uses the fact that $H^{2p}(X,\mathbb{Q})$ has a mixed Hodge structure with weights $\geq 2p$, and that the part of weight $2p$ is exactly the image of j^*.) Our absolute Hodge class $\alpha_0 \in H^{2p}(X_0,\mathbb{Q})$ is invariant under monodromy (because it comes from a global section α), and so it belongs to the image; by Proposition 23.5, it is the image of an absolute Hodge class $\beta \in H^{2p}(\bar{X},\mathbb{Q})$. But then we have

$$\alpha_b = i_b^* j^*\beta$$

for every $b \in B$, due to the fact that B is connected; and this shows that each α_b is an absolute Hodge class.

Lecture 24 (April 24)

Outline of the proof of Deligne's theorem. After these preliminaries about absolute Hodge classes, we can now start talking about Deligne's theorem.

Theorem 24.1 (Deligne)**.** *All Hodge classes on abelian varieties are absolute.*

A key object in the proof are so-called "abelian varieties of CM-type", which are abelian varieties whose (rational) endomorphism algebra contains a CM-field. Let's first recall the necessary definitions.

Definition 24.2. A *CM field* is a number field E, such that for every embedding $s\colon E \hookrightarrow \mathbb{C}$, complex conjugation induces an automorphism of E that is independent of the embedding. In other words, E admits an involution $\iota \in \mathrm{Aut}(E/\mathbb{Q})$, such that for any embedding $s\colon E \hookrightarrow \mathbb{C}$, one has $\bar{s} = s \circ \iota$. Here $\bar{s}$ denotes the composition of the embedding s with complex conjugation on $\mathbb{C}$.

The fixed field of the involution is a totally real field F; concretely, this means that $F = \mathbb{Q}(\alpha)$, where α and all of its conjugates are real numbers. The field E is then of the form $F[x]/(x^2 - f)$, for some element $f \in F$ that is mapped to a negative number under all embeddings of F into $\mathbb{R}$. The simplest example of a CM-field is $\mathbb{Q}(\sqrt{-d})$ for a square-free positive integer d; the involution ι is just complex conjugation.

Definition 24.3. An abelian variety A is said to be *of CM-type* if a CM-field E is contained in $\mathrm{End}(A) \otimes \mathbb{Q}$, and if $H^1(A, \mathbb{Q})$ is one-dimensional as an E-vector space. In that case, we clearly have $2 \dim A = \dim_{\mathbb{Q}} H^1(A, \mathbb{Q}) = [E\colon \mathbb{Q}]$.

Example 24.4. It is easy to describe elliptic curves of CM-type. Write the elliptic curve as $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where τ is a complex number with $\operatorname{Im} \tau > 0$. Any rational endomorphism can be lifted to an endomorphism of $\mathbb{C}$, and is therefore of the form $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$. The lifting needs to preserve $\mathbb{Q} + \mathbb{Q}\tau$, and so we get $\lambda = a\tau + b$ and $\lambda\tau = c\tau + d$ for rational numbers $a, b, c, d \in \mathbb{Q}$. This gives

$$a\tau^2 + (b - c)\tau + d = 0,$$

and because τ has positive imaginary part, we get (for $a \neq 0$) that

$$\tau = \frac{c - b + \sqrt{(b-c)^2 + 4ad}}{2a}.$$

As long as $(b - c)^2 + 4ad < 0$, this is an imaginary quadratic extension of $\mathbb{Q}$, and therefore a CM-field. Observe that there are countably many possible values for τ, which are dense in the upper half-plane; so there are only countably many elliptic curves of CM-type, but they are dense in the space of all elliptic curves.

After this preliminary discussion of abelian varieties of CM-type, we return to Deligne's theorem. Let A be an abelian variety, and let $\alpha \in H^{2p}(A, \mathbb{Q})$ be a Hodge class. The proof consists of the following three steps.

1. The first step is to reduce the problem to abelian varieties of CM-type. This is done by constructing an algebraic family of abelian varieties that links a given A and a Hodge class in $H^{2p}(A, \mathbb{Q})$ to an abelian variety of CM-type and a Hodge class on it, and then applying Principle B.
2. The second step is to show that every Hodge class on an abelian variety of CM-type can be expressed as a sum of pullbacks of so-called split Weil classes. The latter are Hodge classes on certain special abelian varieties, constructed by linear algebra from the CM-field E and its embeddings into $\mathbb{C}$. This part of the proof is a simplification of Deligne's argument, due to Yves André.

3. The last step is to show that all split Weil classes are absolute. For a fixed CM-type, all abelian varieties of split Weil type are naturally parametrized by a certain hermitian symmetric domain; by Principle B, this allows to reduce the problem to split Weil classes on abelian varieties of a very specific form, for which the proof of the result is straightforward.

The original proof by Deligne uses Baily-Borel theory to show that certain families of abelian varieties are algebraic. In the presentation below, I am going to replace this by the following two results: the existence of a quasi-projective moduli space for polarized abelian varieties with level structure and the theorem of Cattani-Deligne-Kaplan concerning the algebraicity of Hodge loci.

Abelian varieties of CM-type. To motivate what follows, let's briefly look at a criterion for a simple abelian variety A to be of CM-type that involves the Mumford-Tate group $\mathrm{MT}(A)$. This is a certain algebraic group that serves as a sort of "symmetry group" of the Hodge structure on $H^1(A, \mathbb{Q})$.

Recall that a complex abelian variety A is uniquely determined by the Hodge structure on $H^1(A, \mathbb{Z})$, which is

$$H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A).$$

Indeed, writing $A = V/\Lambda$, we have natural isomorphisms

$$\begin{aligned} V &\cong H^0(A, \mathscr{T}_A) \cong H^0(A, \Omega_A^1)^* \cong H^{1,0}(A)^*, \\ \Lambda &\cong H_1(A, \mathbb{Z}) \cong H^1(A, \mathbb{Z})^*. \end{aligned}$$

Moreover, A is an abelian variety iff A is projective iff the Hodge structure on $H^1(A, \mathbb{Z})$ is polarized. From the rational Hodge structure $H^1(A, \mathbb{Q})$, we can only recover the lattice Λ up to finite index; therefore the Hodge structure on $H^1(A, \mathbb{Q})$ determines the abelian variety A only up to isogeny.

Now let's define the Mumford-Tate group. Suppose that V is a rational Hodge structure of weight n, with Hodge decomposition

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

We can encode the decomposition into a morphism of real Lie groups

$$h \colon U(1) \to \mathrm{GL}(V_{\mathbb{R}})$$

from the circle group, by letting a complex number z with $|z| = 1$ act on the subspace $V^{p,q}$ as multiplication by $z^{p-q} = z^p |z|^q$. Due to Hodge symmetry, each $h(z)$ is actually a real endomorphism, because

$$\overline{h(z)v} = \overline{z^{p-q}v} = \bar{z}^{p-q}\bar{v} = z^{q-p}\bar{v} = h(z)\bar{v}$$

for $v \in V^{p,q}$. The Hodge decomposition is then exactly the decomposition into common eigenspaces for the commuting endomorphisms $h(z)$ with $z \in U(1)$; these eigenspaces correspond to characters of $U(1)$, which are of the form $z \mapsto z^k$ for $k \in \mathbb{Z}$.

We define the *Mumford-Tate group* $\mathrm{MT}(V)$ as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ whose set of real points contains the image of h. In other words, we view $\mathrm{GL}(V)$ as an affine variety over $\operatorname{Spec} \mathbb{Q}$, defined by the determinant function on $\mathrm{End}(V)$, and $\mathrm{MT}(V)$ is the Zariski closure of $\operatorname{im} h$.

Lemma 24.5. *The Mumford-Tate group* $\mathrm{MT}(V)$ *is exactly the subgroup of* $\mathrm{GL}(V)$ *that fixes every Hodge class in every tensor product*

$$T^{k,\ell}(V) = V^{\otimes k} \otimes (V^*)^{\otimes \ell}.$$

Proof. One implication is easy. There are countably many Hodge classes in all the tensor products, and their joint stabilizer is defined by countably many algebraic equations with coefficients in $\mathbb{Q}$; therefore it is a $\mathbb{Q}$-algebraic subgroup M of $\mathrm{GL}(V)$. If we have a Hodge class of type (p,p) in some $T^{k,\ell}(V)$, then $h(z)$ acts on it as multiplication by $z^{p-p}=1$, and so the image of h is contained in M. Because $\mathrm{MT}(V)$ is the Zariski closure of the image, we get $\mathrm{MT}(V) \subseteq M$. The other implication needs a bit of theory of algebraic groups, so I won't present it here. □

The Mumford-Tate group of an abelian variety A is $\mathrm{MT}(A) = \mathrm{MT}\big(H^1(A,\mathbb{Q})\big)$. By the lemma, it is exactly the subgroup of $\mathrm{GL}\big(H^1(A,\mathbb{Q})\big)$ that fixes every Hodge class in every tensor product

$$T^{k,\ell}(A) = H^1(A,\mathbb{Q})^{\otimes k} \otimes H_1(A,\mathbb{Q})^{\otimes \ell}.$$

We have the following nice criterion for simple abelian varieties to be of CM-type.

Proposition 24.6. *A simple abelian variety is of CM-type if and only if its Mumford-Tate group* $\mathrm{MT}(A)$ *is an abelian group.*

Proof. Let's look at the proof of the interesting direction, namely that $\mathrm{MT}(A)$ abelian implies that A is of CM-type. Let $H = H^1(A,\mathbb{Q})$. The abelian variety A is simple, which implies that $E = \mathrm{End}(A) \otimes \mathbb{Q}$ is a division algebra. (This is just Schur's lemma: because A does not have nontrivial abelian subvarieties, any endomorphism must be surjective with finite kernel, hence an isogeny.) It is also the space of Hodge classes in $\mathrm{End}_{\mathbb{Q}}(H)$, and therefore consists exactly of those endomorphisms that commute with $\mathrm{MT}(A)$. Because the Mumford-Tate group is abelian, its action splits $H^1(A,\mathbb{C})$ into a direct sum of character spaces

$$H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\chi} H_\chi,$$

where $m \cdot h = \chi(m)h$ for $h \in H_\chi$ and $m \in \mathrm{MT}(A)$. Now any endomorphism of H_χ obviously commutes with $\mathrm{MT}(A)$, and is therefore contained in $E \otimes_{\mathbb{Q}} \mathbb{C}$. By counting dimensions, we find that

$$\dim_{\mathbb{Q}} E \geq \sum_{\chi} \big(\dim_{\mathbb{C}} H_\chi\big)^2 \geq \sum_{\chi} \dim_{\mathbb{C}} H_\chi = \dim_{\mathbb{Q}} H.$$

On the other hand, we have $\dim_{\mathbb{Q}} E \leq \dim_{\mathbb{Q}} H$; indeed, since E is a division algebra, the map $E \to H$, $e \mapsto e \cdot h$, is injective for every nonzero $h \in H$. Therefore $[E\colon \mathbb{Q}] = \dim_{\mathbb{Q}} H = 2\dim A$; moreover, each character space H_χ is one-dimensional, and this implies that E is commutative, hence a field. To construct the involution $\iota\colon E \to E$ that makes E into a CM-field, choose a polarization $\psi\colon H \times H \to \mathbb{Q}$, and define ι by the condition that, for every $h, h' \in H$,

$$\psi(e \cdot h, h') = \psi\big(h, \iota(e) \cdot h'\big).$$

The fact that $i\psi$ is positive definite on the subspace $H^{1,0}(A)$ can then be used to show that ι is nontrivial, and that $\bar{s} = s \circ \iota$ for any embedding of E into the complex numbers. (We'll prove this below.) □

Hodge structures of CM-type. When A is an abelian variety of CM-type, $H^1(A,\mathbb{Q})$ is an example of a Hodge structure of CM-type. We now undertake a more careful study of this class of Hodge structures. Let V be a rational Hodge structure of weight n, with Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

Because the weight n is fixed, there is a one-to-one correspondence between such decompositions and group homomorphisms $h\colon U(1) \to \mathrm{GL}(V_{\mathbb{R}})$, with $h(z)$ acting as multiplication by $z^{p-q} = z^{2p-n}$ on the subspace $V^{p,q}$.

Definition 24.7. We say that V is a *Hodge structure of CM-type* if the following two equivalent conditions are satisfied:

(a) The group of real points of $\mathrm{MT}(V)$ is a compact torus.
(b) $\mathrm{MT}(V)$ is abelian and V is polarizable.

We mostly use (b) in what follows; the equivalence between (a) and (b) needs a bit of structure theory for algebraic groups, and so we'll skip it.

It is not hard to see that any Hodge structure of CM-type is a direct sum of irreducible Hodge structures of CM-type. Indeed, since V is polarizable, it admits a finite decomposition $V = V_1 \oplus \cdots \oplus V_r$, with each V_i irreducible. As subgroups of $\mathrm{GL}(V) = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_r)$, we then have $\mathrm{MT}(V) \subseteq \mathrm{MT}(V_1) \times \cdots \times \mathrm{MT}(V_r)$, and since the projection to each factor is surjective, it follows that $\mathrm{MT}(V_i)$ is abelian. But this means that each V_i is again of CM-type. It is therefore sufficient to concentrate on irreducible Hodge structures of CM-type. For those, there is a nice structure theorem that we shall now explain.

Let V be an irreducible Hodge structure of weight n that is of CM-type, and as above, denote by $M = \mathrm{MT}(V)$ its Mumford-Tate group. Because V is irreducible, its algebra of endomorphisms

$$E = \mathrm{End}_{\mathbb{Q}\text{-HS}}(V)$$

must be a division algebra. In fact, since the endomorphisms of V as a Hodge structure are exactly the Hodge classes in $\mathrm{End}_{\mathbb{Q}}(V)$, we see that E consists of all rational endomorphisms of V that commute with $\mathrm{MT}(V)$. If $T_E = E^{\times}$ denotes the algebraic torus in $\mathrm{GL}(V)$ determined by E, then we get $\mathrm{MT}(V) \subseteq T_E$ because $\mathrm{MT}(V)$ is commutative by assumption.

Since $\mathrm{MT}(V)$ is commutative, it acts on $V \otimes_{\mathbb{Q}} \mathbb{C}$ by characters, and so we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\chi} V_{\chi},$$

where $m \in \mathrm{MT}(V)$ acts on $v \in V_{\chi}$ by the rule $m \cdot v = \chi(m)v$. Any endomorphism of V_{χ} therefore commutes with $\mathrm{MT}(V)$, and so $E \otimes_{\mathbb{Q}} \mathbb{C}$ contains the spaces $\mathrm{End}_{\mathbb{C}}(V_{\chi})$. As before, this leads to the inequality

$$\dim_{\mathbb{Q}} E \geq \sum_{\chi} \bigl(\dim_{\mathbb{C}} V_{\chi}\bigr)^2 \geq \sum_{\chi} \dim_{\mathbb{C}} V_{\chi} = \dim_{\mathbb{Q}} V.$$

On the other hand, we have $\dim_{\mathbb{Q}} V \leq \dim_{\mathbb{Q}} E$ because every nonzero element in E is invertible. It follows that each V_{χ} is one-dimensional, that E is commutative, and therefore that E is a field of degree $[E\colon \mathbb{Q}] = \dim_{\mathbb{Q}} V$. In particular, V is one-dimensional as an E-vector space.

The decomposition into character spaces can be made more canonical in the following way. Let $S = \mathrm{Hom}(E, \mathbb{C})$ denote the set of all complex embeddings of E; its cardinality is $[E\colon \mathbb{Q}]$. Then

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e)z,$$

is an isomorphism of E-vector spaces; E acts on each summand on the right through the corresponding embedding s. This decomposition induces an isomorphism

$$V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s,$$

where $V_s = V \otimes_{E,s} \mathbb{C}$ is a one-dimensional complex vector space on which E acts via s. The induced homomorphism $U(1) \to \mathrm{MT}(V) \to E^\times \to \mathrm{End}_{\mathbb{C}}(V_s)$ is a character of $U(1)$, hence of the form $z \mapsto z^k$ for some integer k. Solving $k = p - q$ and $n = p + q$, we find that $k = 2p - n$, which means that V_s is of type $(p, n-p)$ in the Hodge decomposition of V. Now define a function $\varphi\colon S \to \mathbb{Z}$ by setting $\varphi(s) = p$; then any choice of isomorphism $V \simeq E$ puts a Hodge structure of weight n on E, whose Hodge decomposition is given by

$$E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)}.$$

From the fact that $\overline{e \otimes z} = e \otimes \bar{z}$, we deduce that

$$\overline{\sum_{s \in S} z_s} = \sum_{s \in S} \overline{z_{\bar{s}}}.$$

Since complex conjugation has to interchange $\mathbb{C}^{p,q}$ and $\mathbb{C}^{q,p}$, this implies that $\varphi(\bar{s}) = n - \varphi(s)$, and hence that $\varphi(s) + \varphi(\bar{s}) = n$ for every $s \in S$.

Definition 24.8. Let E be a number field, and $S = \mathrm{Hom}(E, \mathbb{C})$ the set of its complex embeddings. Any function $\varphi\colon S \to \mathbb{Z}$ with the property that $\varphi(s) + \varphi(\bar{s}) = n$ defines a *Hodge structure E_φ of weight n* on the $\mathbb{Q}$-vector space E, whose Hodge decomposition is given by

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

By construction, the action of E on itself respects this decomposition.

In summary, we have $V \simeq E_\varphi$, which is an isomorphism both of E-modules and of Hodge structures of weight n. Next, we would like to prove that in all interesting cases, E must be a CM-field. Recall from Definition 24.2 that a field E is called a *CM-field* if there exists a nontrivial involution $\iota\colon E \to E$, such that complex conjugation induces ι under any embedding of E into the complex numbers. In other words, we must have $s(\iota e) = \bar{s}(e)$ for any $s \in S$ and any $e \in E$. We usually write $\bar{e}$ in place of ιe, and refer to it as complex conjugation on E. The fixed field of E is then a totally real subfield F, and E is a purely imaginary quadratic extension of F.

To prove that E is either a CM-field or $\mathbb{Q}$, we choose a polarization ψ on E_φ. We then define the so-called *Rosati involution* $\iota\colon E \to E$ by the condition that

$$\psi(e \cdot x, y) = \psi(x, \iota e \cdot y)$$

for every $x, y, e \in E$. Denoting the image of $1 \in E$ by $\sum_{s \in S} 1_s$, we have

$$\sum_{s \in S} \psi(1_s, 1_{\bar{s}}) s(e \cdot x) \bar{s}(y) = \sum_{s \in S} \psi(1_s, 1_{\bar{s}}) s(x) \bar{s}(\iota e \cdot y),$$

which implies that $s(e) = \bar{s}(\iota e)$. Now there are two cases: Either ι is nontrivial, in which case E is a CM-field and the Rosati involution is complex conjugation. Or ι is trivial, which means that $\bar{s} = s$ for every complex embedding. In the second case, we see that $\varphi(s) = n/2$ for every s, and so the Hodge structure must be $\mathbb{Q}(-n/2)$, being irreducible and of type $(n/2, n/2)$. This implies that $E = \mathbb{Q}$.

From now on, we exclude the trivial case $V = \mathbb{Q}(-n/2)$ and assume that E is a CM-field.

Definition 24.9. A *CM-type* of E is a mapping $\varphi\colon S \to \{0, 1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$ for every $s \in S$.

When φ is a CM-type, E_φ is a polarizable rational Hodge structure of weight 1. As such, it is the rational Hodge structure of an abelian variety with complex multiplication by E. This variety is unique up to isogeny. In general, we have the following structure theorem.

Proposition 24.10. *Any Hodge structure V of CM-type and of even weight $2k$ with $V^{p,q} = 0$ for $p < 0$ or $q < 0$ occurs as a direct factor of $H^{2k}(A, \mathbb{Q})$, where A is a finite product of simple abelian varieties of CM-type.*

Proof. In our classification of irreducible Hodge structures of CM-type above, there were two cases: $\mathbb{Q}(-n/2)$, and Hodge structures of the form E_φ, where E is a CM-field and $\varphi\colon S \to \mathbb{Z}$ is a function satisfying $\varphi(s)+\varphi(\bar{s}) = n$. Clearly φ can be written as a linear combination (with integer coefficients) of CM-types for E. Because of the relations

$$E_{\varphi+\psi} \simeq E_\varphi \otimes_E E_\psi \quad \text{and} \qquad E_{-\varphi} \simeq E_\varphi^\vee,$$

every irreducible Hodge structure of CM-type can thus be obtained from Hodge structures corresponding to CM-types by tensor products, duals, and Tate twists.

As we have seen, every Hodge structure of CM-type is a direct sum of irreducible Hodge structures of CM-type. The assertion follows from this by simple linear algebra. □

To conclude our discussion of Hodge structures of CM-type, we will consider the case when the CM-field E is a Galois extension of $\mathbb{Q}$. In that case, the Galois group $G = \operatorname{Gal}(E/\mathbb{Q})$ acts on the set of complex embeddings of E by the rule

$$(g \cdot s)(e) = s(g^{-1}e).$$

This action is simply transitive. Recall that we have an isomorphism

$$E \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \bigoplus_{g\in G} E, \quad x \otimes e \mapsto g(e)x.$$

For any E-vector space V, this isomorphism induces a decomposition

$$V \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \bigoplus_{g\in G} V, \quad v \otimes e \mapsto g(e)v.$$

When V is an irreducible Hodge structure of CM-type, a natural question is whether this decomposition is compatible with the Hodge decomposition. The following lemma shows that the answer to this question is yes.

Lemma 24.11. *Let E be a CM-field that is a Galois extension of $\mathbb{Q}$, with Galois group $G = \operatorname{Gal}(E/\mathbb{Q})$. Then for any $\varphi\colon S \to \mathbb{Z}$ with $\varphi(s) + \varphi(\bar{s}) = n$, we have*

$$E_\varphi \otimes_{\mathbb{Q}} E \simeq \bigoplus_{g\in G} E_{g\varphi}.$$

Proof. We chase the Hodge decompositions through the various isomorphisms that are involved in the statement. To begin with, we have

$$\bigl(E_\varphi \otimes_{\mathbb{Q}} E\bigr) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigl(E_\varphi \otimes_{\mathbb{Q}} \mathbb{C}\bigr) \otimes_{\mathbb{Q}} E \simeq \bigoplus_{s\in S} \mathbb{C}^{\varphi(s),n-\varphi(s)} \otimes_{\mathbb{Q}} E \simeq \bigoplus_{s,t\in S} \mathbb{C}^{\varphi(s),n-\varphi(s)},$$

and the isomorphism takes $(v \otimes e) \otimes z$ to the element

$$\sum_{s,t\in S} t(e) \cdot z \cdot s(v).$$

On the other hand,

$$\bigl(E_\varphi \otimes_{\mathbb{Q}} E\bigr) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{g\in G} E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{g\in G} \bigoplus_{s\in S} \mathbb{C}^{\varphi(s),n-\varphi(s)},$$

and under this isomorphism, $(v \otimes e) \otimes z$ is sent to the element

$$\sum_{g \in G} \sum_{s \in S} s(ge) \cdot s(v) \cdot z.$$

If we fix $g \in G$ and compare the two expressions, we see that $t = sg$, and hence

$$E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{t \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)} \simeq \bigoplus_{t \in S} \mathbb{C}^{\varphi(tg^{-1}), n-\varphi(tg^{-1})}.$$

But since $(g\varphi)(t) = \varphi(tg^{-1})$, this is exactly the Hodge decomposition of $E_{g\varphi}$. $\square$

Lecture 25 (April 29)

Last time, we talked about abelian varieties and Hodge structures of CM-type. Let E be a number field, and let $S = \mathrm{Hom}(E, \mathbb{C})$ be the set of its complex embeddings. The cardinality of S is equal to $[E \colon \mathbb{Q}]$. For every $s \in S$, we denote by $\bar{s}$ the conjugate embedding, meaning the composition of s with complex conjugation on $\mathbb{C}$. Recall that E is called a CM-field if there is an involution $\iota \in \mathrm{Aut}(E/\mathbb{Q})$ such that $\bar{s} = s \circ \iota$ for every $s \in S$; in other words, ι corresponds to complex conjugation under every embedding of E. From now on, we adopt the simplified notation

$$\bar{e} = \iota(e);$$

then we have $\overline{s(e)} = s(\bar{e})$ for every embedding $s \in S$. An abelian variety A is of CM-type if $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a CM-field E and if $V = H^1(A, \mathbb{Q})$ is 1-dimensional as an E-vector space; this is equivalent to the condition that the Mumford-Tate group $\mathrm{MT}(A)$ is abelian.

We also constructed all Hodge structures of CM-type explicitly, starting from the following class of examples. Recall that

$$E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) \cdot z,$$

is an isomorphism; the summand corresponding to an embedding $s \in S$ consists of all elements of $E \otimes_{\mathbb{Q}} \mathbb{C}$ on which every $e \in E$ acts as multiplication by the complex number $s(e)$. (In other words, this is exactly the decomposition into common eigenspaces for the action by E on itself.) By definition, a CM-type of E is a function $\varphi \colon S \to \{0, 1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$ for every $s \in S$. It determines a Hodge structure E_φ of weight 1 on the $\mathbb{Q}$-vector space E by setting

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

These are exactly the Hodge structures that appear as $H^1(A, \mathbb{Q})$, where A is an abelian variety of CM-type, with the CM-field being E. We proved last time that every Hodge structure of CM-type can be obtained (by duals, tensor products, and direct sums) from these basic Hodge structures of CM-type.

Moduli of abelian varieties. The proof of Deligne's theorem involves the construction of algebraic families of abelian varieties, in order to apply Principle B. For this, we shall use the existence of a fine moduli space for polarized abelian varieties with level structure.

Suppose that we are looking at moduli of some class of smooth projective varieties (such as abelian varieties). We would like to have a moduli space M whose points correspond to isomorphism classes of abelian varieties; and over M, there should be a universal family, such that every family of abelian varieties over some base S is the pullback of the universal family along a morphism $S \to M$. Because all abelian varieties have nontrivial isomorphisms, such a universal family cannot exist: from a nontrivial automorphism, one can construct a locally trivial family over $\mathbb{C}^*$ that is not globally trivial, and so this family cannot come from a morphism to M. The solution is to add some extra data that eliminates all nontrivial automorphisms.

Recall that if A is an abelian variety of dimension g, the subgroup $A[N]$ of its N-torsion points is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{\oplus 2g}$. A *level N-structure* is a choice of symplectic isomorphism $A[N] \simeq (\mathbb{Z}/N\mathbb{Z})^{\oplus 2g}$. If we write $A \cong V/\Lambda$, then we have

$$A[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda,$$

and so we can also think of $A[N] \cong H_1(A, \mathbb{Z}/N\mathbb{Z})$ as the first homology of A with coefficients in $\mathbb{Z}/N\mathbb{Z}$.

Suppose that A comes with a polarization, that is with the first Chern class of an ample line bundle L; recall that this determines as isogeny $\theta\colon A \to \hat{A}$ to the dual abelian variety. The degree of the polarization is the degree of this isogeny. The polarization determines a bilinear form $\psi\colon H^1(A,\mathbb{Z}) \otimes H^1(A,\mathbb{Z}) \to \mathbb{Z}$ that polarizes the Hodge structure on $H^1(A,\mathbb{Z})$ (by the Hodge-Riemann bilinear relations). As in Lemma 5.2, we can choose a symplectic basis in $H^1(A,\mathbb{Z})$, and then the polarization has a certain type $m = (m_1, \dots, m_g)$, with $m_1 \mid m_2 \mid \cdots \mid m_g$.

Now say we have an automorphism $f\colon A \to A$ that preserves the given polarization and the given level N-structure. It corresponds to an automorphism

$$f^*\colon H^1(A,\mathbb{Z}) \to H^1(A,\mathbb{Z}),$$

that is an isomorphism of Hodge structures and also compatible with ψ. From the polarization, we construct a hermitian inner product

$$\langle v, w\rangle = \psi\big(h(i)v, w\big),$$

where $h(i)$ acts as i on the subspace $H^{1,0}(A)$, and as $-i$ on the subspace $H^{0,1}(A)$. Because f^* preserves this inner product, it is unitary, and therefore diagonalizable with eigenvalues of absolute value 1. But all eigenvalues are also algebraic integers, and so they are roots of unity (by Kronecker's theorem). Because f preserves the level N-structure, we also get that

$$f^*\colon H^1(A,\mathbb{Z}/N\mathbb{Z}) \to H^1(A,\mathbb{Z}/N\mathbb{Z})$$

is the identity; in other words, f^* is congruent to the identity modulo N. We can now apply the following lemma and conclude that f^* (and hence f) must be the identity.

Lemma 25.1. *Let A be an $n \times n$-matrix with integer entries, all of whose eigenvalues are of absolute value 1. If A is congruent to the identity modulo an integer $N \geq 3$, then all eigenvalues of A are equal to 1.*

Adding a polarization and a level N-structure therefore eliminates all nontrivial automorphisms. One can then prove the following theorem.

Theorem 25.2. *Fix $g \geq 1$ and a type $m = (m_1, \dots, m_g)$. For any $N \geq 3$, there is a smooth quasi-projective variety $\mathcal{M}_{g,m,N}$ that is a fine moduli space for g-dimensional abelian varieties with polarization of type m and level N-structure. In particular, we have a universal family of abelian varieties over $\mathcal{M}_{g,m,N}$.*

The relationship of this result with Hodge theory is the following. Fix an abelian variety A of dimension g, with level N-structure and polarization of type m. The polarization corresponds to an antisymmetric bilinear form $\psi\colon H^1(A,\mathbb{Z}) \otimes H^1(A,\mathbb{Z}) \to \mathbb{Z}$ that polarizes the Hodge structure; we shall refer to ψ as a *Riemann form.* Define $V_{\mathbb{Z}} = H^1(A,\mathbb{Z})$, and let D be the period domain that parametrizes all possible Hodge structures of type $\{(1,0),(0,1)\}$ on $V_{\mathbb{Z}}$ that are polarized by the form

$\psi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}$. The period domain D is an open subset of a certain nonsingular algebraic subvariety of the Grassmannian $G(g, V_{\mathbb{C}})$. Indeed, a Hodge structure

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

on $V_{\mathbb{Z}}$ is completely determined by the g-dimensional subspace $V^{1,0}$; it has the property that $\psi(v, w) = 0$ for all $v, w \in V^{1,0}$, and that $i\psi(v, \bar{v}) > 0$ for every nonzero $v \in V^{1,0}$. The first condition defines a nonsingular closed subvariety; and the second condition an open subset of this subvariety (in the analytic topology). In particular, D is a complex manifold.

Example 25.3. When the polarization is principal (which is equivalent to ψ being unimodular), the period domain D is just the Siegel space $\mathcal{H}_g$

One can show that the period domain D is a simply connected Hermitian symmetric domain. In fact, D is isomorphic to the universal covering space of the quasi-projective variety $\mathcal{M}_{g,m,N}$. This is because a polarized abelian variety A' is completely determined by the polarized Hodge structure on $H^1(A', \mathbb{Z})$, but in order to associate to this Hodge structure a point in the period domain, we need to choose an isomorphism $H^1(A', \mathbb{Z}) \cong V_{\mathbb{Z}}$ that takes the given polarization to the pairing ψ. The additional data of such an isomorphism then turns D into an infinite-sheeted covering space of $\mathcal{M}_{g,m,N}$.

Reduction to abelian varieties of CM-type. We now come to the first step in the proof of Deligne's theorem, namely the reduction of the general problem to abelian varieties of CM-type. This is accomplished by the following theorem and Principle B (from Theorem 23.1).

Theorem 25.4. *Let A be an abelian variety, and let $\alpha \in H^{2p}(A, \mathbb{Q})$ be a Hodge class on A. Then there exists a family $\pi\colon \mathcal{A} \to B$ of abelian varieties, with B nonsingular, irreducible, and quasi-projective, such that the following is true:*

(a) $\mathcal{A}_0 \cong A$ for some point $0 \in B$.
(b) There is a Hodge class $\tilde{\alpha} \in H^{2p}(\mathcal{A}, \mathbb{Q})$ whose restriction to A equals α.
(c) For a dense set of $b \in B$, the abelian variety $\mathcal{A}_b = \pi^{-1}(b)$ is of CM-type.

Before giving the proof, let us briefly recall the following useful interpretation of period domains. Say D parametrizes all Hodge structures of weight n on a fixed rational vector space V that are polarized by a given bilinear form ψ. The set of real points of the $\mathbb{Q}$-algebraic group $G = \operatorname{Aut}(V, \psi)$ then acts transitively on D by the rule $(gV)^{p,q} = g \cdot V^{p,q}$ for $g \in G(\mathbb{R})$, and so $D \simeq G(\mathbb{R})/K$. Here K is the stabilizer of any given Hodge structure; this is contained in the unitary group for the inner product $\langle v, w\rangle = \psi(h(i)v, w)$, and therefore compact.

As we said last time, Hodge structures on V that are polarized by the bilinear form ψ are in one-to-one correspondence with homomorphisms of real algebraic groups $h\colon U(1) \to G(\mathbb{R})$; we denote the Hodge structure corresponding to h by the symbol V_h. Then $V_h^{p,q}$ is exactly the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts as multiplication by z^{p-q}, and from this, it is easy to verify that $gV_h = V_{ghg^{-1}}$. In other words, the points of the period domain D can be thought of as conjugacy classes of a fixed homomorphism $h\colon U(1) \to G(\mathbb{R})$ under the action by $G(\mathbb{R})$.

Proof of Theorem 25.4. After choosing an ample line bundle on A, we may assume that the Hodge structure on $V = H^1(A, \mathbb{Q})$ is polarized by a Riemann form ψ. Let $G = \operatorname{Aut}(V, \psi)$, and recall that $M = \operatorname{MT}(A)$ is the smallest $\mathbb{Q}$-algebraic subgroup of G whose set of real points $M(\mathbb{R})$ contains the image of the homomorphism $h\colon U(1) \to G(\mathbb{R})$. Let D be the period domain whose points parametrize all possible Hodge structures of type $\{(1,0),(0,1)\}$ on V that are polarized by the form ψ. With

$V_h = H^1(A, \mathbb{Q})$ as the base point, we then have $D \simeq G(\mathbb{R})/K$; the points of D are thus exactly the Hodge structures $V_{ghg^{-1}}$, for $g \in G(\mathbb{R})$ arbitrary.

The main idea of the proof is to consider the Mumford-Tate domain

$$D_h = M(\mathbb{R})/K \cap M(\mathbb{R}) \hookrightarrow D.$$

By definition, D_h consists of all Hodge structures of the form $V_{ghg^{-1}}$, for $g \in M(\mathbb{R})$. These are precisely the Hodge structures whose Mumford-Tate group is contained in M: indeed, for $g \in M(\mathbb{R})$, the image of the homomorphism $ghg^{-1} \colon U(1) \to G(\mathbb{R})$ is obviously contained in $M(\mathbb{R})$, and so the Mumford-Tate group of this Hodge structure must be contained in M (because the Mumford-Tate group is defined as the smallest $\mathbb{Q}$-algebraic subgroup containing the image, and M is one such subgroup). Note that D_h is a homogeneous space for the action of the real Lie group $M(\mathbb{R})$, and therefore a complex submanifold of the period domain D.

To find Hodge structures of CM-type in D_h, we appeal to a result by Borel. Since the image of h is abelian, it is contained in a maximal torus T of the real Lie group $M(\mathbb{R})$. One can show that, for a generic element ξ in the Lie algebra $\mathfrak{m}_{\mathbb{R}}$, this torus is the stabilizer of ξ under the adjoint action by $M(\mathbb{R})$. Now $\mathfrak{m}$ is defined over $\mathbb{Q}$, and so there exist elements $g \in M(\mathbb{R})$ arbitrarily close to the identity for which $\mathrm{Ad}(g)\xi = g\xi g^{-1}$ is rational. The stabilizer gTg^{-1} of such a rational point is then a maximal torus in M that is defined over $\mathbb{Q}$. The Hodge structure $V_{ghg^{-1}}$ is a point of the Mumford-Tate domain D_h, and by definition of the Mumford-Tate group, we have $\mathrm{MT}(V_{ghg^{-1}}) \subseteq gTg^{-1}$. In particular, $V_{ghg^{-1}}$ is of CM-type, because its Mumford-Tate group is abelian. This reasoning shows that D_h contains a dense set of points of CM-type.

To obtain an algebraic family of abelian varieties with the desired properties, we can now argue as follows. Let $\mathcal{M}$ be the moduli space of abelian varieties of dimension $\dim A$, with polarization of the same type as ψ, and level 3-structure. Then $\mathcal{M}$ is a smooth quasi-projective variety, and since it is a fine moduli space, it carries a universal family $\pi : \mathcal{A} \to \mathcal{M}$. The period domain D is the universal covering space of $\mathcal{M}$. Now we would like to replace $\mathcal{M}$ by the image of the Mumford-Tate domain D_h.

Recall from last time that the Mumford-Tate group $M = \mathrm{MT}(A)$ is the subgroup of $G = \mathrm{Aut}(V, \psi)$ that fixes all Hodge tensors, meaning all Hodge classes in all tensor powers

$$T^{p,q}(A) = H^1(A, \mathbb{Q})^{\otimes p} \otimes H_1(A, \mathbb{Q})^{\otimes q}.$$

Because M is an algebraic subgroup, we can finitely many such Hodge tensors $\tau_1, \dots, \tau_r$ such that M is exactly the stabilizer of $\tau_1, \dots, \tau_r$. A Hodge structure in D belongs to D_h iff its Mumford-Tate group is contained in M iff $\tau_1, \dots, \tau_r$ remain Hodge tensors. By the theorem of Cattani-Deligne-Kaplan, the subset of $\mathcal{M}$ where each τ_j stays a Hodge class is algebraic. So if we let $B \subseteq \mathcal{M}$ denote the connected component of this subvariety that contains the base point $0 \in \mathcal{M}$, then B is a quasi-projective algebraic variety; it is irreducible and nonsingular, being the image of the complex submanifold $D_h \subseteq D$. Let $\pi \colon \mathcal{A} \to B$ be the restriction of the universal family to B. Then (a) is clearly satisfied for this family.

Because B is the image of the Mumford-Tate domain D_h, the argument we gave above shows that B contains a dense set of points $b \in B$ such that the Mumford-Tate group of the abelian variety $\mathcal{A}_b = \pi^{-1}(b)$ is abelian; this means that $\mathcal{A}_b$ is an abelian variety of CM-type, and so we get (c). Since B is also contained in the Hodge locus of α, and since the monodromy action of $\pi_1(B, 0)$ on the space of Hodge classes has finite orbits, we can pass to a finite étale cover of B and arrange that α in invariant under monodromy. By the global invariant cycle theorem, it therefore comes from a Hodge class in $H^{2p}(\mathcal{A}, \mathbb{Q})$, and this gives (b). $\square$

Construction of split Weil classes. Let E be a CM-field; as usual, we let $S = \operatorname{Hom}(E, \mathbb{C})$ be the set of complex embeddings; it has $[E\colon \mathbb{Q}]$ elements.

Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$ whose endomorphism algebra contains E. We shall assume that $\dim_E V = d$ is an even number. Let $V_s = V \otimes_{E,s} \mathbb{C}$. Corresponding to the decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) z,$$

we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} V_s.$$

The isomorphism is E-linear, where $e \in E$ acts on the complex vector space V_s as multiplication by $s(e)$. Since $\dim_{\mathbb{Q}} V = [E\colon \mathbb{Q}] \cdot \dim_E V$, each V_s has dimension d over $\mathbb{C}$. By assumption, E respects the Hodge decomposition on V, and so we get an induced decomposition

$$V_s = V_s^{1,0} \oplus V_s^{0,1}.$$

Note that $\dim_{\mathbb{C}} V_s^{1,0} + \dim_{\mathbb{C}} V_s^{0,1} = d$.

Lemma 25.5. *The rational subspace* $\bigwedge_E^d V \subseteq \bigwedge_{\mathbb{Q}}^d V$ *is purely of type* $(d/2, d/2)$ *if and only if* $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$ *for every* $s \in S$.

Proof. We have

$$\Bigl(\bigwedge\nolimits_E^d V\Bigr) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigwedge\nolimits_{E \otimes_{\mathbb{Q}} \mathbb{C}}^d (V \otimes_{\mathbb{Q}} \mathbb{C}) \simeq \bigoplus_{s \in S} \bigwedge\nolimits_{\mathbb{C}}^d V_s \simeq \bigoplus_{s \in S} \Bigl(\bigwedge\nolimits_{\mathbb{C}}^{p_s} V_s^{1,0}\Bigr) \otimes \Bigl(\bigwedge\nolimits_{\mathbb{C}}^{q_s} V_s^{0,1}\Bigr),$$

where $p_s = \dim_{\mathbb{C}} V_s^{1,0}$ and $q_s = \dim_{\mathbb{C}} V_s^{0,1}$. The assertion follows because the Hodge type of each summand is evidently (p_s, q_s). ☐

Let A be an abelian variety such that $H^1(A, \mathbb{Q}) \cong V$. Assuming that the condition in the lemma is satisfied, we get a subspace in $H^d(A, \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}^d V$ of dimension $[E\colon \mathbb{Q}]$ that consists entirely of Hodge classes. These classes are called *Hodge classes of Weil type.* They are not known to be algebraic in general. We will study them in more detail next time.

Lecture 26 (May 1)

Let me quickly remind you of the construction of Weil classes from last time. Let E be a CM-field, and $S = \operatorname{Hom}(E, \mathbb{C})$ the set of its complex embeddings. We write $e \mapsto \bar{e}$ for the involution on E; for any $s \in S$, we then have $\overline{s(e)} = s(\bar{e})$.

Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$ whose endomorphism algebra contains E. We shall assume that $\dim_E V = d$ is an even number; then $\dim_{\mathbb{Q}} V = d \cdot [E\colon \mathbb{Q}]$. For every $s \in S$, we define $V_s = V \otimes_{E,s} \mathbb{C}$, which is a complex vector space of dimension d. The tensor product gives us the relation $ev \otimes z = v \otimes s(e) z$. Corresponding to the decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) z,$$

we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} V_s.$$

Complex conjugation, which acts on the left-hand side as $v \otimes z \mapsto v \otimes \bar{z}$, exchanges the summand V_s with the summand $V_{\bar{s}}$; this can be seen by conjugating the identity

$ev \otimes z = v \otimes s(e)z$ in V_s. By assumption, E respects the Hodge decomposition on V, and so we get an induced decomposition

$$V_s = V_s^{1,0} \oplus V_s^{0,1}.$$

Note that $\dim_{\mathbb{C}} V_s^{1,0} + \dim_{\mathbb{C}} V_s^{0,1} = d$.

Lemma 26.1. *The rational subspace $\bigwedge_E^d V \subseteq \bigwedge_{\mathbb{Q}}^d V$ is purely of type $(d/2, d/2)$ if and only if $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$ for every $s \in S$.*

When the condition in the lemma is satisfied, the subspace $\bigwedge_E^d V$ consists entirely of Hodge classes. These Hodge classes are called *Weil classes*. We are now going to give a linear algebra condition for this to be the case, using hermitian forms and polarizations.

Hermitian forms. This requires a little bit of background on hermitian forms. Throughout, E is a CM-field, with totally real subfield F and complex conjugation $e \mapsto \bar{e}$, and $S = \operatorname{Hom}(E, \mathbb{C})$ is the set of complex embeddings of E. An element $\zeta \in E^\times$ is called *totally imaginary* if $\bar{\zeta} = -\zeta$; concretely, this means that $\bar{s}(\zeta) = -s(\zeta)$ for every complex embedding s.

Definition 26.2. Let V be an E-vector space. A $\mathbb{Q}$-bilinear form $\phi\colon V \times V \to E$ is said to be *E-hermitian* if $\phi(e \cdot v, w) = e \cdot \phi(v, w)$ and $\phi(v, w) = \overline{\phi(w, v)}$ for every $v, w \in V$ and every $e \in E$. It follows that $\phi(v, e \cdot w) = \bar{e} \cdot \phi(v, w)$.

Now suppose that V is an E-vector space of dimension $d = \dim_E V$, and that ϕ is an E-hermitian form on V. We begin by describing the numerical invariants of the pair (V, ϕ). For any embedding $s\colon E \hookrightarrow \mathbb{C}$, we obtain a hermitian form ϕ_s in the usual sense on the complex vector space $V_s = V \otimes_{E,s} \mathbb{C}$. Concretely, we have

$$\phi_s\left(\sum_j v_j \otimes z_j, \sum_k v_k' \otimes z_k'\right) = \sum_{j,k} z_j \bar{z}_k' s\big(\phi(v_j, v_k')\big).$$

We let a_s and b_s be the dimensions of the maximal subspaces where ϕ_s is, respectively, positive and negative definite. Because $\dim_{\mathbb{C}} V_s = d$, the signature of the hermitian form ϕ_s is then $(a_s, b_s, d - a_s - b_s)$.

A second invariant of ϕ is its discriminant. To define it, note that ϕ induces an E-hermitian form on the one-dimensional E-vector space $\bigwedge_E^d V$, which up to a choice of basis vector, is of the form $(x, y) \mapsto f x \bar{y}$. The element f belongs to the totally real subfield F, and a different choice of basis vector only changes f by elements of the form $\operatorname{Nm}_{E/F}(e) = e \cdot \bar{e}$. Consequently, the class of f in $F^\times / \operatorname{Nm}_{E/F}(E^\times)$ is well-defined, and is called the *discriminant* of (V, ϕ). We denote it by the symbol $\operatorname{disc} \phi$. Equivalently, we can choose a basis for V and represent ϕ by a $d \times d$-matrix with entries in E; then $\operatorname{disc} \phi$ is the determinant of this matrix.

Now suppose that ϕ is nondegenerate. Let $v_1, \dots, v_d \in V$ be an orthogonal basis for ϕ, and set $c_i = \phi(v_i, v_i)$. Then we have $c_i \in F^\times$, and

$$a_s = \#\big\{\, i \;\big|\; s(c_i) > 0 \,\big\} \quad \text{and} \quad b_s = \#\big\{\, i \;\big|\; s(c_i) < 0 \,\big\}$$

satisfy $a_s + b_s = d$. Moreover, we have

$$f = \prod_{i=1}^d c_i \mod \operatorname{Nm}_{E/F}(E^\times);$$

this implies that $\operatorname{sgn}\big(s(f)\big) = (-1)^{b_s}$ for every $s \in S$. The following theorem by Landherr says that the discriminant and the integers a_s and b_s are a complete set of invariants for E-hermitian forms.

Theorem 26.3 (Landherr)**.** *Let $a_s, b_s \geq 0$ be a collection of integers, indexed by the set S, and let $f \in F^\times / \operatorname{Nm}_{E/F}(E^\times)$ be an arbitrary element. Suppose that they satisfy $a_s + b_s = d$ and $\operatorname{sgn}\bigl(s(f)\bigr) = (-1)^{b_s}$ for every $s \in S$. Then there exists a nondegenerate E-hermitian form ϕ on an E-vector space V of dimension d with these invariants; moreover, (V, ϕ) is unique up to isomorphism.*

This classical result has the following useful consequence.

Corollary 26.4. *If (V, ϕ) is nondegenerate, then the following two conditions are equivalent:*

(a) $a_s = b_s = d/2$ for every $s \in S$, and $\operatorname{disc} \phi = (-1)^{d/2}$.
(b) There is a totally isotropic subspace of V of dimension $d/2$.

Proof. If $W \subseteq V$ is a totally isotropic subspace of dimension $d/2$, then $v \mapsto \phi(-, v)$ induces an antilinear isomorphism $V/W \xrightarrow{\sim} W^\vee$. Thus we can extend a basis $v_1, \dots, v_{d/2}$ of W to a basis $v_1, \dots, v_d$ of V, with the property that

$$\begin{aligned} \phi(v_i, v_{i+d/2}) &= 1 \qquad \text{for } 1 \leq i \leq d/2, \\ \phi(v_i, v_j) &= 0 \qquad \text{for } \lvert i - j \rvert \neq d/2. \end{aligned}$$

We can use this basis to check that (a) is satisfied. For the converse, consider the hermitian space $(E^{\oplus d}, \phi)$, where

$$\phi(x, y) = \sum_{1 \leq i \leq d/2} \bigl(x_i \bar{y}_{i+d/2} + x_{i+d/2} \bar{y}_i \bigr)$$

for every $x, y \in E^{\oplus d}$. By Landherr's theorem, this space is (up to isomorphism) the unique hermitian space satisfying (a), and it is easy to see that it satisfies (b), too. $\square$

Definition 26.5. An E-hermitian form ϕ that satisfies the two equivalent conditions in Corollary 26.4 is said to be *split*.

We shall see below that E-hermitian forms are related to polarizations on Hodge structures of CM-type. We now describe one additional technical result that is going tobe useful in that context. Suppose that V is a Hodge structure of type $\{(1,0), (0,1)\}$ that is of CM-type and whose endomorphism ring contains E; let $h \colon U(1) \to E^\times$ be the corresponding homomorphism. Recall that a *Riemann form* for V is a $\mathbb{Q}$-bilinear antisymmetric form $\psi \colon V \otimes V \to \mathbb{Q}$, with the property that

$$(x, y) \mapsto \psi\bigl(h(i) \cdot x, \bar{y}\bigr)$$

is hermitian and positive definite on $V \otimes_{\mathbb{Q}} \mathbb{C}$. We only consider Riemann forms whose Rosati involution induces complex conjugation on E, meaning that

$$\psi(ev, w) = \psi(v, \bar{e}w).$$

The next result says that polarizations with that property are closely related to E-hermitian forms.

Lemma 26.6. *Let $\zeta \in E^\times$ be a totally imaginary element ($\bar{\zeta} = -\zeta$), and let ψ be a Riemann form for V as above. Then there exists a unique E-hermitian form ϕ with the property that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta \phi)$.*

Because the trace can be computed by summing over all complex embeddings, the formula $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta \phi)$ means concretely that

$$\psi(v, w) = \sum_{s \in S} s(\zeta) s\bigl(\phi(v, w)\bigr).$$

I did not give the proof in class, but I will include it here. We first prove a simpler statement about bilinear forms.

Lemma 26.7. *Let V and W be finite-dimensional vector spaces over E, and let $\psi\colon V \times W \to \mathbb{Q}$ be a $\mathbb{Q}$-bilinear form such that $\psi(ev, w) = \psi(v, ew)$ for every $e \in E$. Then there exists a unique E-bilinear form ϕ such that $\psi(v, w) = \mathrm{Tr}_{E/\mathbb{Q}} \phi(v, w)$.*

Proof. The trace pairing $E \times E \to \mathbb{Q}$, $(x, y) \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(xy)$, is nondegenerate. Consequently, composition with $\mathrm{Tr}_{E/\mathbb{Q}}$ induces an injective homomorphism

$$\mathrm{Hom}_E\bigl(V \otimes_E W, E\bigr) \to \mathrm{Hom}_{\mathbb{Q}}\bigl(V \otimes_E W, \mathbb{Q}\bigr),$$

which has to be an isomorphism because both vector spaces have the same dimension over $\mathbb{Q}$. By assumption, ψ defines a $\mathbb{Q}$-linear map $V \otimes_E W \to \mathbb{Q}$, and we let ϕ be the element of $\mathrm{Hom}_E\bigl(V \otimes_E W, E\bigr)$ corresponding to ψ under the above isomorphism. ☐

Proof of Lemma 26.6. We apply the preceding lemma with $W = V$, but with E acting on W through complex conjugation. This gives a sesquilinear form ϕ_1 such that $\psi(x, y) = \mathrm{Tr}_{E/\mathbb{Q}} \phi_1(x, y)$. Now define $\phi = \zeta^{-1}\phi_1$, so that we have $\psi(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(x, y)\bigr)$. The uniqueness of ϕ is obvious from the preceding lemma.

It remains to show that we have $\phi(y, x) = \overline{\phi(x, y)}$. Because ψ is antisymmetric, $\psi(y, x) = -\psi(x, y)$, which implies that

$$\mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(y, x)\bigr) = -\mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta\phi(x, y)\bigr) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\bar{\zeta}\phi(x, y)\bigr).$$

On replacing y by ey, for arbitrary $e \in E$, we obtain

$$\mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta e \cdot \phi(y, x)\bigr) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e} \cdot \phi(x, y)\bigr).$$

On the other hand, we have

$$\mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta e \cdot \phi(y, x)\bigr) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e \cdot \phi(y, x)}\bigr) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\overline{\zeta e} \cdot \overline{\phi(y, x)}\bigr).$$

Since $\overline{\zeta e}$ can be an arbitrary element of E, the nondegeneracy of the trace pairing implies that $\phi(x, y) = \overline{\phi(y, x)}$. ☐

Hodge classes of split Weil type. We will now describe a condition on V that guarantees that the space $\bigwedge_E^d V$ consists entirely of Hodge cycles.

Definition 26.8. Let V be a rational Hodge structure of type $\{(1, 0), (0, 1)\}$ with $E \hookrightarrow \mathrm{End}_{\mathbb{Q}\text{-HS}}(V)$ and $\dim_E V = d$ even. We say that V is *of split Weil type* relative to E if there exists a split E-hermitian form ϕ on V such that $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ defines a polarization on V for some totally imaginary element $\zeta \in E^\times$.

According to Corollary 26.4, the condition on the E-hermitian form ϕ is that there should exist a totally isotropic subspace $W \subseteq V$ with $\dim_E W = d/2$.

Proposition 26.9. *If V is of split Weil type relative to E, then the space*

$$\bigwedge\nolimits_E^d V \subseteq \bigwedge\nolimits_{\mathbb{Q}}^d V$$

consists of Hodge classes of type $(d/2, d/2)$.

Proof. For any $s \in S$, let ϕ_S be the induced hermitian form on $V_s = V \otimes_{E,s} \mathbb{C}$. The isomorphism

$$\alpha\colon V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s$$

respects the Hodge decomposition. According to Lemma 25.5, it suffices to prove that $\dim V_s^{1,0} = \dim V_s^{0,1} = d/2$. We are going to do this by showing that ϕ_s is positive/negative definite on these two subspaces. Since $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ defines a polarization, ϕ is nondegenerate; recall from above that the signature of ϕ_s is (a_s, b_s). Because ϕ is split, Corollary 26.4 shows that we have $a_s = b_s = d/2$ for every embedding $s \in S$. So the signature of ϕ_s is actually $(d/2, d/2)$.

Now let $x \in V_s^{1,0}$ be any nonzero element. Writing $x = \sum_j v_j \otimes z_j$, we have

$$\phi_s(x,x) = \sum_{j,k} z_j \bar{z}_k s\big(\phi(v_j, v_k)\big).$$

At the same time, the fact that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ is a polarization tells us that

$$\begin{aligned} \|x\|^2 = i\psi(x, \bar{x}) = i \sum_{j,k} z_j \bar{z}_k \psi(v_j, v_k) = i \sum_{j,k} z_j \bar{z}_k \operatorname{Tr}_{E/\mathbb{Q}}\big(\zeta\phi(v_j, v_k)\big) \\ = i \sum_{j,k} \sum_{s' \in S} z_j \bar{z}_k s'(\zeta) s'\big(\phi(v_j, v_k)\big) = i \sum_{s' \in S} s'(\zeta)\phi_{s'}(x,x) \end{aligned}$$

is positive. Because $x \in V_s$, the sum is equal to $is(\zeta)\phi_s(x,x)$, and so ϕ_s is either positive or negative definite on $V_s^{1,0}$, depending on the sign of $is(\zeta)$. Because we know the signature of ϕ_s, we get $\dim V_s^{1,0} \leq d/2$. For the same reason, we have $\dim V_s^{0,1} \leq d/2$; but because both dimensions must add up to d, we can then conclude that $\dim V_s^{1,0} = \dim V_s^{0,1} = d/2$. $\square$

These special Hodge classes are called *split Weil classes* or more precisely *Hodge classes of split Weil type*. They are the most important examples of Hodge classes on abelian varieties of CM-type; as I said before, the Hodge conjecture is *not* known for these classes except in dimension ≤ 4.

Are there any examples of Hodge structures of split Weil type? Fortunately, there is a simple numerical criterion that can be used to check this. Recall that a CM-type of E is a function $\varphi \colon S \to \{0,1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$. It determines a Hodge structure E_φ of CM-type on the $\mathbb{Q}$-vector space E, with Hodge decomposition

$$E_\phi \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

This is the Hodge structure on $H^1(A, \mathbb{Q})$, where A is a simple abelian variety of CM-type.

Now let $\varphi_1, \dots, \varphi_d$ be CM-types attached to E. Let $V_i = E_{\varphi_i}$ be the Hodge structure of CM-type corresponding to φ_i, and define

$$V = \bigoplus_{i=1}^d V_i.$$

Then V is a Hodge structure of CM-type with $\dim_E V = d$.

Proposition 26.10. *If $\sum \varphi_i$ is constant on S, then V is of split Weil type.*

Proof. To begin with, it is necessarily the case that $\sum \varphi_i = d/2$; indeed,

$$\sum_{i=1}^d \varphi_i(s) + \sum_{i=1}^d \varphi(\bar{s}) = \sum_{i=1}^d \big(\varphi_i(s) + \varphi_i(\bar{s})\big) = d,$$

and the two sums are equal by assumption. By construction, we have

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{i=1}^d \big(E_{\varphi_i} \otimes_{\mathbb{Q}} \mathbb{C}\big) \simeq \bigoplus_{i=1}^d \bigoplus_{s \in S} \mathbb{C}^{\varphi_i(s), \varphi_i(\bar{s})}.$$

This shows that

$$V_s = V \otimes_{E,s} \mathbb{C} \simeq \bigoplus_{i=1}^d \mathbb{C}^{\varphi_i(s), \varphi_i(\bar{s})}.$$

Therefore $\dim_{\mathbb{C}} V_s^{1,0} = \sum \varphi_i(s) = d/2$, and likewise $\dim_{\mathbb{C}} V_s^{0,1} = \sum \varphi_i(\bar{s}) = d/2$. Of course, this already implies that all classes in $\bigwedge_E^d V$ are Hodge classes.

Next, we construct the required E-hermitian form on V. For each i, choose a Riemann form ψ_i on V_i, whose Rosati involution acts as complex conjugation on E. Since $V_i = E_{\varphi_i}$, there exist totally imaginary elements $\zeta_i \in E^\times$, such that

$$\psi_i(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}\bigl(\zeta_i x \bar{y}\bigr)$$

for every $x, y \in E$. Set $\zeta = \zeta_d$, and define $\phi_i(x, y) = \zeta_i \zeta^{-1} x \bar{y}$, which is an E-hermitian form on V_i with the property that $\psi_i = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta \phi_i)$.

For any collection of totally positive elements $f_i \in F$,

$$\psi = \sum_{i=1}^{d} f_i \psi_i$$

is a Riemann form for V. As E-vector spaces, we have $V = E^{\bigoplus d}$, and so we can define a nondegenerate E-hermitian form on V by the rule

$$\phi(v, w) = \sum_{i=1}^{d} f_i \phi_i(v_i, w_i).$$

We then have $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$. By the same argument as before, $a_s = b_s = d/2$, since $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$. By construction, the form ϕ is diagonalized, and so its discriminant is easily found to be

$$\operatorname{disc}\phi = \zeta^{-d} \prod_{i=1}^{d} f_i \zeta_i \mod \mathrm{Nm}_{E/F}(E^\times).$$

On the other hand, we know from general principles that, for any $s \in S$,

$$\operatorname{sgn}\bigl(s(\operatorname{disc}\phi)\bigr) = (-1)^{b_s} = (-1)^{d/2}.$$

This means that $\operatorname{disc}\phi = (-1)^{d/2} f$ for some totally positive element $f \in F^\times$. Upon replacing f_d by $f_d f^{-1}$, we get $\operatorname{disc}\phi = (-1)^{d/2}$, which proves that (V, ϕ) is split. $\square$

26.1. André's theorem and reduction to split Weil classes. The second step in the proof of Deligne's theorem is to reduce the problem from arbitrary Hodge classes on abelian varieties of CM-type to Hodge classes of split Weil type. This is accomplished by the following pretty theorem due to Yves André.

Theorem 26.1 (André). *Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$, which is of CM-type. Then there exists a CM-field E, rational Hodge structures V_α of split Weil type relative to E, and morphisms of Hodge structure $V_\alpha \to V$, such that every Hodge class $\xi \in \bigwedge_{\mathbb{Q}}^{2p} V$ is a sum of images of Hodge classes $\xi_\alpha \in \bigwedge_{\mathbb{Q}}^{2p} V_\alpha$ of split Weil type.*

Proof. Let $V = V_1 \oplus \cdots \oplus V_r$, with V_i irreducible; then each $E_i = \mathrm{End}_{\mathbb{Q}\text{-HS}}(V_i)$ is a CM-field. Define E to be the Galois closure of the compositum of the fields $E_1, \ldots, E_r$. Since V is of CM-type, E is a CM-field which is Galois over $\mathbb{Q}$. Let G be its Galois group over $\mathbb{Q}$. After replacing V by $V \otimes_{\mathbb{Q}} E$ (of which V is a direct factor), we may assume without loss of generality that $E_i = E$ for all i.

As before, let $S = \mathrm{Hom}(E, \mathbb{C})$ be the set of complex embeddings of E; we then have a decomposition

$$V \simeq \bigoplus_{i \in I} E_{\varphi_i}$$

for some collection of CM-types φ_i. Applying Lemma 24.11, we get

$$V \otimes_{\mathbb{Q}} E \simeq \bigoplus_{i \in I} \bigoplus_{g \in G} E_{g\varphi_i}.$$

Since each $E_{g\varphi_i}$ is one-dimensional over E, we get

$$\Big(\bigwedge\nolimits_{\mathbb{Q}}^{2p} V\Big) \otimes_{\mathbb{Q}} E \simeq \bigwedge\nolimits_{E}^{2p} (V \otimes_{\mathbb{Q}} E) \simeq \bigwedge\nolimits_{E}^{2p} \bigoplus_{(i,g) \in I \times G} E_{g\varphi_i} \simeq \bigoplus_{\substack{\alpha \subseteq I \times G \\ |\alpha| = 2p}} \bigotimes_{(i,g) \in \alpha} E_{g\varphi_i}$$

where the tensor product is over E. If we now define Hodge structures of CM-type

$$V_\alpha = \bigoplus_{(i,g) \in \alpha} E_{g\varphi_i}$$

for any subset $\alpha \subseteq I \times G$ of size $2p$, then V_α has dimension $2p$ over E. The above calculation shows that

$$\Big(\bigwedge\nolimits_{\mathbb{Q}}^{2p} V\Big) \otimes_{\mathbb{Q}} E \simeq \bigoplus_{\alpha} \bigwedge\nolimits_{E}^{2p} V_\alpha,$$

which is an isomorphism both as Hodge structures and as E-vector spaces. Moreover, as V_α is a sub-Hodge structure of $V \otimes_{\mathbb{Q}} E$, we clearly have morphisms $V_\alpha \to V$, and any Hodge class $\xi \in \bigwedge_{\mathbb{Q}}^{2p} V$ is a sum of Hodge class $\xi_\alpha \in \bigwedge_{E}^{2p} V_\alpha$.

It remains to see that V_α is of split Weil type whenever ξ_α is nonzero. Fix a subset $\alpha \subseteq I \times G$ of size $2p$, with the property that $\xi_\alpha \neq 0$. Note that we have

$$\bigwedge\nolimits_{E}^{2p} V_\alpha \simeq \bigotimes_{(i,g) \in \alpha} E_{g\varphi_i} \simeq E_\varphi,$$

where $\varphi \colon S \to \mathbb{Z}$ is the function

$$\varphi = \sum_{(i,g) \in \alpha} g\varphi_i$$

The Hodge decomposition of E_φ is given by

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

The image of the Hodge cycle ξ_α in E_φ must be purely of type (p,p) with respect to this decomposition. But

$$\xi_\alpha \otimes 1 \mapsto \sum_{s \in S} s(\xi_\alpha),$$

and since each $s(\xi_\alpha)$ is nonzero (because $\xi_\alpha \neq 0$ and s is an embedding), we conclude that $\varphi(s) = p$ for every $s \in S$. This means that the sum of the $2p$ CM-types $g\varphi_i$, indexed by $(i,g) \in \alpha$, is constant on S. We conclude by the criterion in Proposition 26.10 that V_α is of split Weil type. □

In geometric terms, this is saying that if A is an abelian variety of CM-type, and if $\xi \in H^{2p}(A, \mathbb{Q})$ is a Hodge class, then there are abelian varieties A_α of split Weil type, and morphisms $q_\alpha \colon A \to A_\alpha$, such that $\xi = \sum_\alpha q_\alpha^*(\xi_\alpha)$, where $\xi_\alpha \in H^{2p}(A_\alpha, \mathbb{Q})$ are Hodge classes of split Weil type. So if we can show that all Hodge classes of split Weil type are absolute (or algebraic), then all Hodge classes on abelian varieties of CM-type will also be absolute (or algebraic).

Lecture 27 (May6)

Split Weil classes are absolute. The third step in the proof of Deligne's theorem is to show that split Weil classes are absolute. We begin by describing a special class of abelian varieties of split Weil type where this can be proved directly.

Let V_0 be a rational Hodge structure of even rank d and type $\{(1,0),(0,1)\}$. Let ψ_0 be a Riemann form that polarizes V_0, and W_0 an isotropic subspace of dimension $d/2$. (For example, $V_0^{1,0}$ is an isotropic subspace of dimension $d/2$ over

$\mathbb{C}$, and because ψ_0 is defined over $\mathbb{Q}$, it will also have isotropic subspaces of the same dimenension over $\mathbb{Q}$.) We also fix an element $\zeta \in E^\times$ with $\bar{\zeta} = -\zeta$.

Now set $V = V_0 \otimes_\mathbb{Q} E$, with Hodge structure induced by the isomorphism

$$V \otimes_\mathbb{Q} \mathbb{C} \simeq V_0 \otimes_\mathbb{Q} \big(E \otimes_\mathbb{Q} \mathbb{C}\big) \simeq \bigoplus_{s \in S} V_0 \otimes_\mathbb{Q} \mathbb{C}.$$

Define a $\mathbb{Q}$-bilinear form $\psi\colon V \times V \to \mathbb{Q}$ by the formula

$$\psi(v_0 \otimes e, v_0' \otimes e') = \operatorname{Tr}_{E/\mathbb{Q}}\big(e\overline{e'}\big) \cdot \psi_0(v_0, v_0').$$

This is a Riemann form on V, for which $W = W_0 \otimes_\mathbb{Q} E$ is an isotropic subspace of dimension $d/2$. By Lemma 26.6, there is a unique E-hermitian form $\phi\colon V \times V \to E$ such that $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$; clearly W is a totally isotropic subspace of dimension $d/2$ for ϕ. By Corollary 26.4, (V, ϕ) is split, and V is therefore of split Weil type. Let A_0 be an abelian variety with $H^1(A_0, \mathbb{Q}) = V_0$. The integral lattice of V_0 induces an integral lattice in $V = V_0 \otimes_\mathbb{Q} E$. We denote by $A_0 \otimes_\mathbb{Q} E$ the corresponding abelian variety. It is of split Weil type since V is.

The next result is the key to proving that split Weil classes are absolute.

Proposition 27.1. *Let A_0 be an abelian variety with $H^1(A_0, \mathbb{Q}) = V_0$ as above, and define $A = A_0 \otimes_\mathbb{Q} E$. Then the subspace $\bigwedge_E^d H^1(A, \mathbb{Q})$ of $H^d(A, \mathbb{Q})$ consists entirely of absolute Hodge classes.*

Proof. We have $H^d(A, \mathbb{Q}) \simeq \bigwedge_\mathbb{Q}^d H^1(A, \mathbb{Q})$, and the subspace

$$\bigwedge\nolimits_E^d H^1(A, \mathbb{Q}) \simeq \bigwedge\nolimits_E^d V_0 \otimes_\mathbb{Q} E \simeq \Big(\bigwedge\nolimits_\mathbb{Q}^d V_0\Big) \otimes_\mathbb{Q} E \simeq H^d(A_0, \mathbb{Q}) \otimes_\mathbb{Q} E$$

consists entirely of Hodge classes by Proposition 26.9. But since $\dim A_0 = d/2$, the space $H^d(A_0, \mathbb{Q})$ is generated by the fundamental class of a point, which is clearly absolute. This implies that every class in $\bigwedge_E^d H^1(A, \mathbb{Q})$ is absolute. □

The following theorem, together with Principle B (from Theorem 23.1), completes the proof of Deligne's theorem.

Theorem 27.2. *Let E be a CM-field, and let A be an abelian variety of split Weil type (relative to E). Then there exists a family $\pi\colon \mathcal{A} \to B$ of abelian varieties, with B irreducible and quasi-projective, such that the following three things are true:*

(a) $\mathcal{A}_0 = A$ for some point $0 \in B$.
(b) For every $t \in B$, the abelian variety $\mathcal{A}_t = \pi^{-1}(t)$ is of split Weil type (relative to E).
(c) The family contains an abelian variety of the form $A_0 \otimes_\mathbb{Q} E$.

In the remainder of the lecture, we are going to prove Theorem 27.2. Throughout, we let $V = H^1(A, \mathbb{Q})$, which is an E-vector space of some even dimension d. The polarization on A corresponds to a Riemann form $\psi\colon V \times V \to \mathbb{Q}$, with the property that the Rosati involution acts as complex conjugation on E. Fix a totally imaginary element $\zeta \in E^\times$; then $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ for a unique E-hermitian form ϕ by Lemma 26.6. Since A is of split Weil type, the pair (V, ϕ) is split.

As before, let D be the period domain, whose points parametrize Hodge structures of type $\{(1,0),(0,1)\}$ on V that are polarized by the form ψ. Let $D^{\mathrm{sp}} \subseteq D$ be the subset of those Hodge structures that are of split Weil type (relative to E, and with polarization given by ψ). Our first task is to show that D^{sp} is a complex manifold (and, in fact, a hermitian symmetric domain).

We begin by observing that there are essentially $2^{[E\colon\mathbb{Q}]}/2$ many different choices for the totally imaginary element ζ, up to multiplication by totally positive elements in $F^\times$. Indeed, if we fix a choice of $i = \sqrt{-1}$, and define $\varphi_\zeta \colon S \to \{0,1\}$ by the rule

$$\varphi_\zeta(s) = \begin{cases} 1 & \text{if } s(\zeta)/i > 0, \\ 0 & \text{if } s(\zeta)/i < 0, \end{cases} \tag{27.3}$$

then $\varphi_\zeta(s) + \varphi_\zeta(\bar{s}) = 1$ because $\bar{s}(\zeta) = -s(\zeta)$, and so φ_ζ is a CM-type for E. If we change ζ by a totally positive element $f \in F^\times$, then φ_ζ does not change (because $s(f) > 0$ for every $s \in S$). Conversely, one can show that any CM-type of the CM-field E is obtained in this way. Indeed, for a given CM-type $\varphi \colon S \to \{0,1\}$, we are looking for an element $f \in F^\times$ with the property that $s(f) > 0$ if $\varphi(s) = \varphi_\zeta(s)$, and $s(f) < 0$ if $\varphi(s) \neq \varphi_\zeta(s)$, because then $\varphi = \varphi_{f\zeta}$. The existence of such an element $f \in F^\times$ is an exercise in field theory.

Exercise 27.1. Let F be a totally real number field, and let $S = \operatorname{Hom}(F, \mathbb{R})$ be the set of all embeddings of F. Then for any function $\varphi \colon S \to \{-1,+1\}$, there is an element $f \in F^\times$ such that $\varphi(s) = \operatorname{sgn} s(f)$.

Lemma 27.4. *The subset D^{sp} of the period domain D is a hermitian symmetric domain; in fact, it is isomorphic to the product of $|S| = [E \colon \mathbb{Q}]$ many copies of Siegel upper halfspace.*

Proof. Recall that V is an E-vector space of even dimension d, and that the Riemann form is equal to $\psi = \operatorname{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ for a split E-hermitian form $\phi \colon V \times V \to E$ and a totally imaginary $\zeta \in E^\times$. The Rosati involution corresponding to ψ induces complex conjugation on E; this means that $\psi(ev, w) = \psi(v, \bar{e}w)$ for every $e \in E$.

By definition, D^{sp} parametrizes all Hodge structures of type $\{(1,0),(0,1)\}$ on V that admit ψ as a Riemann form and are of split Weil type relative to the given CM-field E. Such a Hodge structure amounts to a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

with $V^{0,1} = \overline{V^{1,0}}$, with the following two properties:

(a) The action by E preserves $V^{1,0}$ and $V^{0,1}$.
(b) The form $i\psi(x, \bar{y}) = \psi\big(h(i)x, \bar{y}\big)$ is positive definite on $V^{1,0}$.

Let $S = \operatorname{Hom}(E, \mathbb{C})$, and consider the isomorphism

$$V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s,$$

where $V_s = V \otimes_{E,s} \mathbb{C}$. Since V_s is exactly the subspace on which $e \in E$ acts as multiplication by $s(e) \in \mathbb{C}$, the condition in (a) is equivalent to demanding that each complex vector space V_s decomposes as $V_s = V_s^{1,0} \oplus V_s^{0,1}$.

On the other hand, ϕ induces a hermitian form ϕ_s on each V_s, and we have

$$\psi(v, w) = \operatorname{Tr}_{E/\mathbb{Q}}\big(\zeta\phi(v,w)\big) = \sum_{s \in S} s(\zeta)\phi_s(v \otimes 1, w \otimes 1).$$

Therefore ψ polarizes the Hodge structure $V^{1,0} \oplus V^{0,1}$ if and only if $i\psi(x, \bar{x}) > 0$ for every nonzero $x \in V_s^{1,0}$. Writing

$$x = \sum_j v_j \otimes z_j \in V \otimes_{\mathbb{Q}} \mathbb{C},$$

we computed last time that

$$i\psi(x, \bar{x}) = is(\zeta)\phi_s(x, x).$$

Remembering the definition of φ_ζ in (27.3), we see that this will be positive definite exactly when the hermitian form $(-1)^{\varphi_\zeta(s)}\phi_s$ is positive definite on $V_s^{1,0}$.

In summary, Hodge structures of split Weil type on V for which ψ is a Riemann form are parametrized by a choice of $d/2$-dimensional subspace $V_s^{1,0} \subseteq V_s$, one for each $s \in S$, with the property that the hermitian form $x \mapsto (-1)^{\varphi_\zeta(s)}\phi_s(x,x)$ is positive definite on $V_s^{1,0}$. This information determines the subspace $V_s^{0,1}$ as the orthogonal complement of $V_s^{1,0}$ with respect to ϕ_s. Since we have $a_s = b_s = d/2$ for every $s \in S$ (by Corollary 26.4), the hermitian form ϕ_s has signature $(d/2, d/2)$; this implies that the space

$$D_s = \left\{ W \in \mathrm{Grass}_{d/2}(V_s) \;\middle|\; (-1)^{\varphi_\zeta(s)}\phi_s(x,x) > 0 \text{ for } 0 \neq x \in W \right\}$$

is isomorphic to the usual Siegel upper halfspace. The parameter space D^{sp} for our Hodge structures is therefore the hermitian symmetric domain

$$D^{\mathrm{sp}} \simeq \prod_{s \in S} D_s.$$

In particular, it is a connected complex manifold. □

To be able to satisfy the final condition in Theorem 27.2, we need to know that D^{sp} contains Hodge structures of the form $V_0 \otimes_{\mathbb{Q}} E$. This is the content of the following lemma.

Lemma 27.5. *With notation as above, there is a rational Hodge structure V_0 of weight one, such that $V_0 \otimes_{\mathbb{Q}} E$ belongs to D^{sp}.*

Proof. Since the pair (V, ϕ) is split, there is a totally isotropic subspace $W \subseteq V$ of dimension $\dim_E W = d/2$. Arguing as in the proof of Corollary 26.4, we can therefore find a basis $v_1, \dots, v_d$ for the E-vector space V, with the property that

$$\begin{aligned} \phi(v_i, v_{i+d/2}) &= \zeta^{-1} \quad \text{for } 1 \leq i \leq d/2, \\ \phi(v_i, v_j) &= 0 \quad \text{for } |i-j| \neq d/2. \end{aligned}$$

Let V_0 be the $\mathbb{Q}$-linear span of $v_1, \dots, v_d$; then we have $V = V_0 \otimes_{\mathbb{Q}} E$. Now define $V_0^{1,0} \subseteq V_0 \otimes_{\mathbb{Q}} \mathbb{C}$ as the $\mathbb{C}$-linear span of the vectors $h_k = v_k + i v_{k+d/2}$ for $k = 1, \dots, d/2$. Evidently, this gives a Hodge structure of weight one on V_0, hence a Hodge structure on $V = V_0 \otimes_{\mathbb{Q}} E$. It remains to show that ψ polarizes this Hodge structure. But we compute that

$$\begin{aligned} i\psi\left(\sum_{j=1}^{d/2} a_j h_j, \sum_{k=1}^{d/2} \overline{a_k h_k}\right) &= \sum_{k=1}^{d/2} |a_k|^2 \psi\big(i v_k - v_{k+d/2}, v_k - i v_{k+d/2}\big) \\ &= 2\sum_{k=1}^{d/2} |a_k|^2 \psi(v_k, v_{k+d/2}) \\ &= 2\sum_{k=1}^{d/2} |a_k|^2 \operatorname{Tr}_{E/\mathbb{Q}}\big(\zeta\phi(v_k, v_{k+d/2})\big) = 2[E \colon \mathbb{Q}] \sum_{k=1}^{d/2} |a_k|^2, \end{aligned}$$

which proves that $x \mapsto i\psi(x, \bar{x})$ is positive definite on the subspace $V_0^{1,0}$. The Hodge structure $V_0 \otimes_{\mathbb{Q}} E$ therefore belongs to D^{sp} as desired. □

Finishing the proof of Deligne's theorem.

Proof of Theorem 27.2. As in Lecture 25, let $\mathcal{M}$ be the moduli space of abelian varieties of dimension $d/2 \cdot [E \colon \mathbb{Q}]$, with polarization of the same type as ψ, and level 3-structure. Then $\mathcal{M}$ is a quasi-projective complex manifold, and the period domain D is its universal covering space (with the Hodge structure on $H^1(A, \mathbb{Q})$ mapping to the point A). Let $B \subseteq \mathcal{M}$ be the locus of those abelian varieties whose endomorphism algebra contains E. Note that the original abelian variety A is contained in B. Since every element $e \in E$ is a Hodge class in $\mathrm{End}(A) \otimes \mathbb{Q}$, it is clear that B is a Hodge locus; in particular, B is a quasi-projective variety by the theorem of Cattani-Deligne-Kaplan. As before, we let $\pi\colon \mathcal{A} \to B$ be the restriction of the universal family of abelian varieties to B.

Now we claim that the preimage of B in D is precisely the set D^{sp} of Hodge structures of split Weil type. Indeed, the endomorphism ring of any Hodge structure in the preimage of B contains E by construction; since it is also polarized by the form ψ, all the conditions in Definition 26.8 are satisfied, and so the Hodge structure in question belongs to D^{sp}. Because D is the universal covering space of $\mathcal{M}$, this implies in particular that B is connected and smooth, hence a quasi-projective complex manifold.

The first two assertions are obvious from the construction, whereas the third follows from Lemma 27.5. This concludes the proof. $\square$

To complete the proof of Deligne's theorem, we have to show that every split Weil class is an absolute Hodge class. For this, we argue as follows. Consider the family of abelian varieties $\pi\colon \mathcal{A} \to B$ from Theorem 27.2. By Proposition 26.9, the space of split Weil classes $\bigwedge_E^d H^1(\mathcal{A}_t, \mathbb{Q})$ consists of Hodge classes for every $t \in B$. The family also contains an abelian variety of the form $A_0 \otimes_{\mathbb{Q}} E$, and according to Proposition 27.1, all split Weil classes on this particular abelian variety are absolute. But now B is irreducible, and so Principle B applies and shows that for every $t \in B$, all split Weil classes on $\mathcal{A}_t$ are absolute. This finishes the third step of the proof, and finally establishes Deligne's theorem.

Lecture 28 (May 8)

The Hodge conjecture for abelian varieties. In the final lecture, I surveyed what is known about the Hodge conjecture for abelian varieties. An important role is played by abelian varieties of "Weil type", but the definition is slightly broader than the one we used during the previous lectures. Let's briefly look at this, in case you want to read some of the papers later on. Let A be an abelian variety of even dimension $2n$. Then A is said to be of *Weil type* if there is an embedding

$$\eta\colon \mathbb{Q}(\sqrt{-d}) \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of an imaginary quadratic field (with $d \geq 2$ a square-free integer) into the rational endomorphism ring of A, such that both eigenspaces for the action of $\eta(\sqrt{-d})$ on $H^{1,0}(A)$ have dimension n. Note that $\eta(\sqrt{-d})^2 = \eta(-d)$ acts on $H^1(A, \mathbb{Q})$ as multiplication by $-d$, and so the two possible eigenvalues of $\eta(\sqrt{-d})$ are exactly $\pm\sqrt{-d}$. If we set $V = H^1(A, \mathbb{Q})$, this is exactly the condition that $\dim V_s^{1,0} = \dim V_s^{0,1} = n$ for each of the two complex embeddings of $\mathbb{Q}(\sqrt{-d})$. Note that $\mathbb{Q}(\sqrt{-d})$ is a CM-field of degree 2.

A *polarization* on an abelian variety of Weil type is by definition an ample divisor class $h \in H^2(A, \mathbb{Z})$ such that $\eta(\sqrt{-d})^* h = d \cdot h$. This may look different, but it is actually the same as our condition that the Rosati involution needs to act as complex conjugation on the CM-field. Let's do the computation. The ample class h defines a polarization on $V = H^1(A, \mathbb{Q})$ by the formula

$$\psi(v, w) = [A] \cap \bigl(v \cup w \cup h^{2n-1}\bigr).$$

Here $[A]$ is the fundamental class of A; over the real or complex numbers, this is basically the integral over A. We would like to show that

$$\psi\Bigl(\eta(\sqrt{-d})^* v, w\Bigr) = \psi\Bigl(v, \eta(-\sqrt{-d})^* w\Bigr).$$

We first observe that $\eta(\sqrt{-d})$ acts on $H^{4n}(A, \mathbb{Q})$ as multiplication by d^{2n}. Indeed, $\eta(\sqrt{-d})$ must be multiplication by some positive integer N, and because $\eta(\sqrt{-d})^2 =$

$\eta(-d)$ acts as multiplication by d^{4n}, we get $N = d^{2n}$. This gives

$$[A] \cap \Big(\eta(\sqrt{-d})^* v \cup \eta(\sqrt{-d})^* w \cup \eta(\sqrt{-d})^* h^{2n-1}\Big) = d^{2n} \cdot [A] \cap \big(v \cup w \cup h^{2n-1}\big).$$

If we now replace w by $\eta(-\sqrt{-d})^* w$, and remember that $\eta(\sqrt{-d})^* h = dh$ and $\eta(d)^* w = dw$, we obtain

$$d^{2n} \cdot [A] \cap \Big(\eta(\sqrt{-d})^* v \cup w \cup h^{2n-1}\Big) = d^{2n} \cdot [A] \cap \Big(v \cup \eta(-\sqrt{-d})^* w \cup h^{2n-1}\Big).$$

This shows that the Rosati involution for ψ is complex conjugation on $\mathbb{Q}(\sqrt{-d})$.

As in the previous lectures, the polarization ψ can be written as

$$\psi = \operatorname{Tr}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\big(\sqrt{-d}\phi\big)$$

for a unique hermitian form $\phi\colon V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(\sqrt{-d})$. The discrete invariants of the polarized abelian variety (A, h) of Weil type are therefore the integer d, as well as the discriminant $\operatorname{disc}\phi$, which is an element in $\mathbb{Q}^\times$ modulo rational numbers of the norm $a^2 + db^2$ with $a, b \in \mathbb{Q}$. (In the "split" case, which is the one we were considering earlier, the discriminant is always $(-1)^n$.)

One can show (by a dimension count) that the space of polarized abelian varieties of Weil type has dimension n^2. The 3-dimensional subspace

$$\Big\langle h^n, \bigwedge\nolimits^{2n}_{\mathbb{Q}(\sqrt{-d})} H^1(A, \mathbb{Q}) \Big\rangle \subseteq H^{2n}(A, \mathbb{Q})$$

consists of Hodge classes; these are again called Hodge classes of Weil type. For a general (A, h), one can show moreover (by computing the Mumford-Tate group) that these are all the Hodge classes in $H^{2n}(A, \mathbb{Q})$.

Remark. All Hodge classes in $H^2(A, \mathbb{Q})$ are algebraic (by the Lefschetz $(1,1)$-theorem). Since the intersection of algebraic classes is algebraic, every Hodge class in the image of $\operatorname{Sym}^2 H^2(A, \mathbb{Q}) \to H^4(A, \mathbb{Q})$ is also algebraic. Mumford constructed the first example of an abelian fourfold that has extra Hodge classes in $H^4(A, \mathbb{Q})$. Weil realized the importance of CM-fields in Mumford's construction, which is why these classes are now called Hodge classes of Weil type.

Here are some known results about the Hodge conjecture for abelian varieties. Let's write $H^{k,k}(A, \mathbb{Q}) = H^{2k}(A, \mathbb{Q}) \cap H^{k,k}(A)$ for the space of Hodge classes in $H^{2k}(A, \mathbb{Q})$. In order to know all the Hodge classes on A, it is enough to know the Mumford-Tate group $\operatorname{MT}(A) = \operatorname{MT}\big(H^1(A, \mathbb{Q})\big)$. The reason is that

$$H^{2k}(A, \mathbb{Q}) = \bigwedge^{2k} H^1(A, \mathbb{Q}),$$

and so the Hodge classes are exactly the classes in $H^{2k}(A, \mathbb{Q})$ that are invariant under the action by $\operatorname{MT}(A)$. Unfortunately, a lot of proofs in this subject work by first classifying all possible Mumford-Tate groups (and their possible representations), and then doing a case-by-case analysis.

(1) Tate proved that the Hodge conjecture is true if A is isogeneous to a product of elliptic curves.
(2) Mari Rámon proved that the Hodge conjecture is true if A is isogeneous to a product of abelian surfaces.
(3) Tankeev proved that the Hodge conjecture holds on simple abelian varieties such that $\dim A$ is a prime number.
(4) Moonen and Zarhin showed that if A is a simple abelian 4-fold such that $\operatorname{Sym}^2 H^{1,1}(A, \mathbb{Q}) \to H^{2,2}(A, \mathbb{Q})$ is not surjective, then A is of Weil type, and $H^{2,2}(A, \mathbb{Q})$ is spanned by the image of $\operatorname{Sym}^2 H^{1,1}(A, \mathbb{Q})$ together with the Hodge classes of Weil type. (Note that A can be of Weil type for several

different values of d, and we are supposed to take the Hodge classes of Weil type for all such values.)

(5) Moonen and Zarhin also showed that this holds when A is isogeneous to the product of an elliptic curve with a simple abelian threefold.

Altogether, these results reduce the Hodge conjecture on abelian fourfolds to the case of abelian fourfolds of Weil type, and to proving that all Hodge classes of Weil type are algebraic. This result was recently announced by Markman, after many earlier results (especially by Schoen). Markman proves this for all imaginary quadratic fields and all values of the discriminant, by reducing the problem to abelian sixfolds of Weil type with discriminant -1. A lot of the earlier work was for specific fields and/or specific values of the discriminant. The simplest example is the following result by van der Geemen.

Example 28.1. Van der Geemen gave a nice geometric proof for the following result: On a general principally polarized abelian fourfold of Weil type, with $E = \mathbb{Q}(i)$, all Hodge classes of Weil type are algebraic. In outline, the argument goes like this. The principal polarization can be represented by a symmetric theta divisor Θ, with $h^0\big(A, \mathscr{O}_A(\Theta)\big) = 1$. The line bundle $L = \mathscr{O}_A(2\Theta)$ is then base-point free and has $h^0(A, L) = 2^4 = 16$. The endomorphism $\eta(i)$ acts on $H^0(A, L)$ as an involution, and in the eigenspace decomposition

$$H^0(A, L) = H^0(A, L)^+ \oplus H^0(A, L)^-,$$

the first summand has dimension 10, the second dimension 6. Consider now the rational mapping

$$A \to \mathbb{P}^{15} \to \mathbb{P}^5$$

given by the linear system $|2\Theta|$ followed by projection to the second summand. One can show that the closure of the image is a smooth 4-dimensional quadric. The pullback of one of the two rulings then gives a subvariety of codimension 2 in A, whose class is not a multiple of $h^2 = \Theta^2$. For general A, the space $H^{2,2}(A, \mathbb{Q})$ is generated by h^2 and Weil classes, so at least one Weil class is algebraic; one can then use the monodromy action to conclude that all Weil classes must be algebraic for general A.

Remark. Somebody asked whether the Hodge conjecture is known for Jacobians of curves. I said yes, but that was wrong: the Hodge conjecture for Jacobians is equivalent to the Hodge conjecture for symmetric products of curves, but that's only known in certain special cases.