Lecture 9: September 25

We continue studying a polarized variation of Hodge structure of weight n on the punctured disk Δ^*. We keep the notation $\mathscr{V}$ for the holomorphic vector bundle, ∇ for the connection, $F^p\mathscr{V}$ for the Hodge bundles, and $h_{\mathscr{V}}$ for the polarization. Recall that we defined V as the space of flat sections of $\exp^* \mathscr{V}$ on the halfspace $\tilde{\mathbb{H}} = \{ z \in \mathbb{C} \mid \operatorname{Re} z < 0 \}$. The polarization induces a nondegenerate hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$, and the action by $\mathbb{Z}(1)$ induces a monodromy transformation $T \in \operatorname{End}(V)$, according to the rule

$$v(z - 2\pi i) = (Tv)(z).$$

From this data, we constructed the period mapping

$$\Phi\colon \tilde{\mathbb{H}} \to D,$$

which has the property that $\Phi(z + 2\pi i) = T\Phi(z)$. (Remember that $T \in G_{\mathbb{R}}$, and that the group $G_{\mathbb{R}}$ acts on D.)

For a choice of half-open interval $I \subseteq \mathbb{R}$ of length 1, we also constructed the canonical extension

$$\tilde{\mathscr{V}} = \mathscr{O}_\Delta \otimes_{\mathbb{C}} V,$$

and we saw that the connection on $\mathscr{V}$ extends to a logarithmic connection with

$$\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv.$$

It is also possible to run the entire construction backwards, starting from the canonical extension. This has the advantage that the canonical extension is unique. Starting from $(\mathscr{V}, \nabla)$, let $\tilde{\mathscr{V}}$ be the canonical extension (for some choice of interval I), and define $V = \tilde{\mathscr{V}}\big|_0$ as the fiber over the origin. The residue of the connection gives us an endomorphism $R = \operatorname{Res}_0 \nabla \in \operatorname{End}(V)$, whose eigenvalues are contained in I. (This only exists if the eigenvalues of the monodromy transformation have absolute value 1, of course.) As I mentioned last time, $\tilde{\mathscr{V}}$ has a distiguished trivialization

$$\mathscr{O}_\Delta \otimes_{\mathbb{C}} V \cong \tilde{\mathscr{V}},$$

such that the logarithmic connection takes the simple form

$$\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv.$$

(To construct this trivialization in general, one has to solve a system of ordinary differential equations.) For every vector $v \in V$, we can now construct a flat section of $\exp^* \mathscr{V}$ on $\tilde{\mathbb{H}}$, by defining

$$\tilde{s}_v(z) = e^{-zR}(1 \otimes v) = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!} \otimes R^j v.$$

This is flat because $(\exp^* \nabla)(1 \otimes v) = dz \otimes Rv$. The resulting mapping

$$V \to H^0\big(\tilde{\mathbb{H}}, \exp^* \mathscr{V}\big)^{\exp^* \nabla}, \quad v \mapsto \tilde{s}_v,$$

is an isomorphism. Since

$$\tilde{s}_v(z - 2\pi i) = e^{2\pi i R} e^{-zR}(1 \otimes v) = e^{2\pi i R} \tilde{s}_v(z),$$

the monodromy transformation is therefore again $T = e^{2\pi i R}$.

Hodge bundles and canonical extension. Either way, the canonical extension leads to a preferred trivialization

$$\mathscr{O}_{\Delta^*} \otimes_{\mathbb{C}} V \cong \mathscr{V}$$

of our vector bundle, which gives us another way to view the Hodge bundles $F^p\mathscr{V}$ as subbundles of a trivial bundle with fiber V. Remembering that the compact dual $\check{D}$ parametrizes all filtrations of the given type, we get a holomorphic mapping

$$\Psi \colon \Delta^* \to \check{D}.$$

It is not hard to describe Ψ in terms of the period mapping Φ: the two trivializations (of $\exp^* \mathscr{V}$ by flat sections, and of $\mathscr{V}$ from the canonical extension) are related by the factor e^{-zR}, and therefore

$$\Psi(e^z) = e^{-zR}\Phi(z).$$

Here we are considering $\Phi(z) \in D$ as a point in $\check{D}$, and act by $e^{-zR} \in \mathrm{GL}(V)$; note that e^{-zR} only belongs to $G_{\mathbb{R}}$ when z is purely imaginary. We can also see directly that the mapping $z \mapsto e^{-zR}\Phi(z)$ descends to Δ^*, because

$$e^{-(z+2\pi i)R}\Phi(z+2\pi i) = e^{-zR}e^{-2\pi i R}T\Phi(z) = e^{-zR}\Phi(z).$$

The first result that Schmid proves is that the canonical extension is a good place to compare the different filtrations: the limit $\lim_{t\to 0}\Psi(t)$ exists.

Theorem 9.1. *The mapping $\Psi \colon \Delta^* \to \check{D}$ extends holomorphically to Δ.*

The main ingredient in the proof is the distance decreasing property of period mappings. The limit $\Psi(0) \in \check{D}$ is a well-defined filtration on V; unfortunately, it is basically never the Hodge filtration of a polarized Hodge structure.

Polarization and canonical extension. In order to understand why $\Psi(0)$ does not give us a polarized Hodge structure, we need to look at the pairing. Suppose that, in addition to $(\mathscr{V}, \nabla)$, we also have a nondegenerate flat hermitian pairing

$$h_{\mathscr{V}} \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^{\infty}_{\Delta^*}.$$

If we let V be the fiber of the canonical extension over the origin, then just as before, we get an induced hermitian pairing

$$h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C},$$

as follows. For any $v \in V$, we have a flat section $e^{-zR}(1\otimes v)$ of $\exp^* \mathscr{V}$, and since the pairing $h_{\mathscr{V}}$ is flat, the function

$$z \mapsto (\exp^* h_{\mathscr{V}})\big(e^{-zR}(1\otimes v'), e^{-zR}(1\otimes v'')\big)$$

is constant on $\tilde{\mathbb{H}}$. Define $h(v',v'') \in \mathbb{C}$ to be that constant value. Since $h_{\mathscr{V}}$ is nondegenerate, the same thing is true for h.

Lemma 9.2. *For every $v', v'' \in V$, one has $h(Rv', v'') = h(v', Rv'')$.*

Proof. The proof is similar to that of Lemma 8.1. The function that we used to define $h(v',v'')$ is constant, and therefore

$$h(v',v'') = h(e^{2\pi i m R}v', e^{2\pi i m R}v'')$$

for every $m \in \mathbb{Z}$. If we expand the right-hand side into a power series in m, we get

$$h(v',v'') = h(v',v'') + m\Big(2\pi i h(Rv',v'') - 2\pi i h(v',Rv'')\Big) + \cdots$$

Since this has to hold for every $m \in \mathbb{Z}$, we get the result. ∎

To analyze the behavior of e^{zR}, it is helpful to work with the (additive) Jordan decomposition

$$R = R_S + R_N$$

of the endomorphism $R \in \operatorname{End}(V)$ into its semisimple (= diagonalizable) and nilpotent parts. You may remember from linear algebra that R_S and R_N can both be written as polynomials in R, and therefore commute with each other. We also get

$$h(R_S v', v'') = h(v', R_S v'') \quad \text{and} \quad h(R_N v', v'') = h(v', R_N v'').$$

The eigenvalues of R_S are real and contained in the interval I; since R_S is diagonalizable, it gives us a decomposition

$$V = \bigoplus_{\alpha \in I} E_\alpha(R_S).$$

This decomposition is orthogonal with respect to h.

Note. The point of the Jordan decomposition is that

$$e^{zR} v = e^{zR_S} e^{zR_N} v = e^{\alpha z} e^{zR_N} v$$

when $v \in E_\alpha(R_S)$. The first factor is exponential in z, whereas e^{zR_N} is polynomial in z (because R_N is nilpotent); the Jordan decomposition allows us to see the difference clearly.

That said, let us now see how the flat pairing $h_{\mathscr{V}}$ looks like in the trivialization of $\mathscr{V}$ given by the canonical extension. For the sake of clarity, I am going to replace z by the equivalent expression $\log t$. Here we go:

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = h\big(e^{R \log t} v', e^{R \log t} v''\big) = h\big(v', e^{R \log |t|^2} v''\big) = h\big(v', e^{-L(t) R} v''\big).$$

I used Lemma 9.2, and the fact that h is conjugate-linear in the second argument, to move $e^{R \log t}$ over to the second argument as $e^{R \log \bar{t}}$. I also introduced the function

$$L(t) = -\log |t|^2,$$

which has the advantage of being positive on Δ^*. Remember that the different eigenspaces $E_\alpha(R_S)$ are orthogonal under h. Let

$$v' = \sum_{\alpha \in I} v'_\alpha \quad \text{and} \quad v'' = \sum_{\alpha \in I} v''_\alpha$$

be the eigenspace decompositions of v' and v''; then

$$e^{-L(t) R_S} v''_\alpha = e^{\alpha \log |t|^2} v''_\alpha = |t|^{2\alpha} \cdot v''_\alpha.$$

Continuing from above, we have

$$\begin{aligned} h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') &= \sum_{\alpha \in I} h\big(v'_\alpha, e^{-L(t) R_N} e^{-L(t) R_S} v''_\alpha\big) = \sum_{\alpha \in I} |t|^{2\alpha} h\big(v'_\alpha, e^{-L(t) R_N} v''_\alpha\big) \\ &= \sum_{\alpha \in I} |t|^{2\alpha} \sum_{j=0}^{\infty} \frac{L(t)^j}{j!} (-1)^j h\big(v'_\alpha, R_N^j v''_\alpha\big). \end{aligned}$$

When $t \to 0$, this expression is all over the place: some of the terms $|t|^{2\alpha} L(t)^j$ are going to zero, some to infinity, and they all do so at different rates. This explains why one cannot expect to get any sort of Hodge structure in the limit: we have the limit filtration $\Psi(0) \in \check{D}$, but our Hodge structures are supposed to be polarized, and the pairing is very far from converging to anything.

For an analyst, it is of course completely clear what should be done: one should "renormalize" the pairing, meaning switch to a different reference frame in which the pairing also behaves nicely. We will see in a moment how to do this. For now, let me first make a few other observations. The formula for the pairing shows that the eigenspace decomposition

$$V = \bigoplus_{\alpha \in I} E_\alpha(R_S)$$

of the semisimple part of R is important, because it controls the leading term $|t|^{2\alpha}$ (which of course dominates any power of $L(t)$ as $t \to 0$). We can easily renormalize this part of the expression by dividing by $|t|^\alpha$ on the α-eigenspace. For the terms involving $L(t)$, the situation is less clear.

To see what can happen, choose $\ell \geq 0$ in such a way that $R_N^{\ell+1} = 0$ but $R_N^\ell \neq 0$; recall that R_N is nilpotent. Also, let me suppose that $v', v'' \in E_\alpha(R_S)$, and only concentrate on the expression

$$\sum_{j=0}^{\infty} \frac{L(t)^j}{j!} (-1)^j h(v', R_N^j v''). \tag{9.3}$$

For general $v', v'' \in V$, the coefficient $h(v', R_N^\ell v'')$ will be nonzero, and so we do get a term with $L(t)^\ell$. But if either v' or v'' happens to lie in the subspace $\ker R_N^\ell$, then the coefficient is zero, and the highest power of $L(t)$ that can show up is $L(t)^{\ell-1}$. In other words, if we are interested in the rate of growth of (9.3), we naturally end up with a filtration

$$V \supseteq \ker R_N^\ell \supseteq \cdots$$

where the typical rate of growth is $L(t)^\ell$, which drops to $L(t)^{\ell-1}$ on the subspace $\ker R_N^\ell$, etc. The filtration that we end up with here is called the *monodromy weight filtration*. Let me give the definition first, so that we know what we are talking about.

Proposition 9.4. *Let $N \in \operatorname{End}(V)$ be a nilpotent endomorphism of a finite-dimensional complex vector space V. Then there is a unique increasing filtration $W_\bullet = W_\bullet(N)$ of V with the following two properties:*

(a) For every $j \in \mathbb{Z}$, one has $N(W_j) \subseteq W_{j-2}$.
(b) For every $j \geq 1$, one has an isomorphism

$$N^j \colon \operatorname{gr}_j^W \xrightarrow{\cong} \operatorname{gr}_{-j}^W,$$

where $\operatorname{gr}_j^W = W_j / W_{j-1}$.

Proof. Define $\ell \in \mathbb{N}$ by the condition that $N^{\ell+1} = 0$ but $N^\ell \neq 0$. The proof is by induction on ℓ. If $\ell = 0$, then $N = 0$, and so we can take $W_0 = V$ and $W_{-1} = \{0\}$. In the general case, we always have an isomorphism

$$N^\ell \colon V / \ker N^\ell \xrightarrow{\cong} \operatorname{im} N^\ell,$$

and so we should define $W_\ell = V$, $W_{\ell-1} = \ker N^\ell$, $W_{-\ell} = \operatorname{im} N^\ell$, and $W_{-\ell-1} = \{0\}$, to get (b) for $j = \ell$. Now consider the quotient $\tilde{V} = \ker N^\ell / \operatorname{im} N^\ell$. The induced endomorphism $\tilde{N} \in \operatorname{End}(\tilde{V})$ satisfies $\tilde{N}^\ell = 0$, and so $\tilde{W}_\bullet = W_\bullet(\tilde{N})$ exists by induction. For $-\ell \leq j \leq \ell - 1$, we now define $W_j \subseteq \ker N^\ell$ as the unique subspace with $W_j / \operatorname{im} N^\ell = \tilde{W}_j$. Since $\operatorname{gr}_j^W \cong \operatorname{gr}_j^{\tilde{W}}$, this has all the desired properties. □

The subspaces in the monodromy weight filtration are certain (somewhat complicated) expressions in $\ker N^j$ and $\operatorname{im} N^j$, as the following examples show.

Example 9.5. If $N \neq 0$ but $N^2 = 0$, the monodromy weight filtration is

$$\begin{aligned} W_1 &= V \\ W_0 &= \ker N \\ W_{-1} &= \operatorname{im} N \\ W_{-2} &= \{0\}. \end{aligned}$$

If $N^2 \neq 0$ but $N^3 = 0$, the weight filtration is

$$\begin{aligned} W_2 &= V \\ W_1 &= \ker N^2 \\ W_0 &= \ker N^2 \cap N^{-1}(\operatorname{im} N^2) + \operatorname{im} N^2 \\ W_{-1} &= N(\ker N^2) + \operatorname{im} N^2 \\ W_{-2} &= \operatorname{im} N^2 \\ W_{-3} &= \{0\}. \end{aligned}$$

Exercise 9.1. Prove the following properties of the monodromy weight filtration:

(a) For every $j \geq 1$, one has $\ker N^j \subseteq W_{j-1}$
(b) For every $j \geq 1$, one has $W_{-j} \subseteq \operatorname{im} N^j$.
(c) One has $W_{-1} \cap \ker N = N(\ker N^2)$.

With the language now in place, let us return to the expression

$$\sum_{j=0}^{\ell} \frac{L(t)^j}{j!} (-1)^j h(v', N^j v''),$$

where $N = R_N$. Let $W_\bullet$ be the monodromy weight filtration of $N \in \operatorname{End}(V)$; since $N^\ell \neq 0$ and $N^{\ell+1} = 0$, we have $W_\ell = V$, $W_{\ell-1} = \ker N^\ell$, etc. The maximal power of $L(t)$ that can appear is $L(t)^\ell$, with coefficient $(-1)^\ell h(v', N^\ell v'')$. In fact, it is not hard to see that

$$(-1)^\ell h(v', N^\ell v'')$$

defines a nondegenerate hermitian pairing on $V / \ker N^\ell = \operatorname{gr}_\ell^W$. Now suppose that $v', v'' \in \ker N^\ell$. Since the subspaces $\ker N^\ell$ and $\operatorname{im} N^\ell$ are orthogonal with respect to h, we get an induced hermitian pairing $\tilde{h}$ on the quotient $\tilde{V} = \ker N^\ell / \operatorname{im} N^\ell$, still nondegenerate, and the expression from above simplifies to

$$\sum_{j=0}^{\ell-1} \frac{L(t)^j}{j!} (-1)^j \tilde{h}\big(\tilde{v}', \tilde{N}^j \tilde{v}''\big),$$

where $\tilde{v}'$ is the image of v' in $\tilde{V}$. Now the highest power of $L(t)$ that can appear is $L(t)^{\ell-1}$, and the expression

$$(-1)^{\ell-1} h(v', N^{\ell-1} v'')$$

defines a nondegenerate hermitian pairing on $\operatorname{gr}_{\ell-1}^W$. Continuing in this manner, we see that the filtration by the maximal power of $L(t)$ is precisely the positive part of the monodromy weight filtration; the negative part shows up as the subspaces on which the induced pairing is trivial in each case.

Renormalizing the pairing. Let us come back to the question of how to "renormalize" the pairing, in order to remove the different terms $|t|^{2\alpha} L(t)^j$. I already said that we should multiply by $|t|^{-\alpha}$ on the α-eigenspace $E_\alpha(R_S)$, to cancel out the $|t|^{2\alpha}$. We also found that the generic behavior on $W_j \setminus W_{j-1}$ is to have a term with $L(t)^j$, but to eliminate this, we need to have a decomposition

$$W_j = E_j \oplus W_{j-1},$$

so that we have a preferred subspace on which we can multiply by $L(t)^{-j/2}$. In order to simplify the formulas, it would also be good if the decomposition was compatible with R_S and with the pairing h. The following result, a special case of the *Jacobson-Morozov theorem*, helps us out.

Proposition 9.6. *It is possible to choose a semisimple operator $H \in \operatorname{End}(V)$, with integer eigenvalues, and the following three properties:*

(a) *For every $j \in \mathbb{Z}$, one has $W_j = E_j(H) \oplus W_{j-1}$.*
(b) *One has $[H, R_N] = -2R_N$ and $[H, R_S] = 0$.*
(c) *One has $h(Hv', v'') = -h(v', Hv'')$ for every $v', v'' \in V$.*

The condition $[H, R_N] = -2R_N$ means that R_N maps $E_j(H)$ into $E_{j-2}(H)$, in a way that is compatible with the monodromy weight filtration. The two operators H and R_N together determine a representation of $\mathfrak{sl}_2(\mathbb{C})$ in which

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

acts as R_N. Moreover, R_S is an endomorphism of this representation (because it commutes with both R_N and H).

The following sequence of exercises outlines a proof of Proposition 9.6.

Exercise 9.2. Let $N \in \operatorname{End}(V)$ be a nilpotent endomorphism of a finite-dimensional vector space. Denote by $\Sigma(N) \subseteq \operatorname{End}(V)$ the set of all semisimple endomorphisms H with integer eigenvalues, such that $[H, N] = -2N$ and $W_j = E_j(H) \oplus W_{j-1}$ for every $j \in \mathbb{Z}$. Show that $\Sigma(N) \neq \emptyset$. (Hint: Choose a basis that puts N into Jordan canonical form.)

Exercise 9.3. The vector space $\operatorname{End}(V)$ also has a nilpotent endomorphism $\operatorname{ad} N$, defined as $(\operatorname{ad} N)(A) = [N, A]$. Show that

$$W_\ell(\operatorname{ad} N) = \left\{ A \in \operatorname{End}(V) \;\middle|\; A(W_j) \subseteq W_{j+\ell} \text{ for all } j \in \mathbb{Z} \right\}.$$

Deduce that if $A \in \operatorname{End}(V)$ commutes with N and satisfies $A(W_j) \subseteq W_{j-1}$ for all $j \in \mathbb{Z}$, then $A = (\operatorname{ad} N)(B)$ for some $B \in \ker(\operatorname{ad} N)^2$.

Exercise 9.4. Let $H, H' \in \Sigma(N)$. Show that there is some $B \in \ker(\operatorname{ad} N)^2$ such that $H' - H = (\operatorname{ad} N)(B)$. Conversely, show that if $H \in \Sigma(N)$ and $B \in \ker(\operatorname{ad} N)^2$, then also $H + (\operatorname{ad} N)(B) \in \Sigma(N)$.

Exercise 9.5. Suppose that V comes with a nondegenerate hermitian pairing

$$h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$$

such that $h(Nv', v'') = h(v', Nv'')$ for all $v', v'' \in V$. For $A \in \operatorname{End}(A)$, denote by $A^* \in \operatorname{End}(V)$ the adjoint with respect to h; thus $N^* = N$. Choose any $H_0 \in \Sigma(N)$.

(a) Show that $-H_0^* \in \Sigma(N)$, and deduce that $H_0 + H_0^* = (\operatorname{ad} N)(B)$ for some endomorphism $B \in \ker(\operatorname{ad} N)^2$.
(b) Show that $H = H_0 + \frac{1}{2}(\operatorname{ad} N)(B^*)$ lies in $\Sigma(N)$ and satisfies $H = -H^*$.
(c) Conclude that there exists a semisimple endomorphism $H \in \operatorname{End}(V)$ with integer eigenvalues, such that $[H, N] = -2N$ and $W_j = E_j(H) \oplus W_{j-1}$ for every $j \in \mathbb{Z}$, and moreover $h(Hv', v'') = -h(v', Hv'')$ for all $v', v'' \in V$.

Exercise 9.6. Finally, suppose that $S \in \operatorname{End}(V)$ is semisimple, commutes with N, and satisfies $S^* = S$. Show that one can arrange, in the conclusion of the preceding exercise, that moreover $[H, S] = 0$. (Hint: Look at the eigenspaces of S.)

Now H and R_S are commuting semisimple endomorphisms of V, and so they have a simultaneous eigenspace decomposition

$$V = \bigoplus_{\substack{\alpha \in I \\ j \in \mathbb{Z}}} V_{\alpha,j}.$$

We will see next time that the divergent terms in the formula for the pairing all go away if we multiply by $|t|^{-\alpha} L(t)^{-j/2}$ on the subspace $V_{\alpha,j}$.