Lecture 6: September 16

The Hodge structure on the endomorphism algebra. Our goal today is to construct a $G_{\mathbb{R}}$-invariant metric on the period domain D. Let us first finish up the linear algebra construction from last time. Recall that

$$V = \bigoplus_{p+q=n} V^{p,q}$$

is a Hodge structure of weight n, polarized by a hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$. Let $C \in \operatorname{End}(V)$ be the "Weil operator", defined by the rule

$$C(v) = (-1)^p v \quad \text{for } v \in V^{p,q}.$$

Since h is a polarization, the expression

$$\langle v', v'' \rangle = h(Cv', v'')$$

defines a *positive definite* inner product on the vector space V; the Hodge decomposition is orthogonal with respect to this inner product.

Last time, we showed that $E = \operatorname{End}(V)$ has a Hodge structure of weight 0 with

$$E^{\ell,-\ell} = \left\{ A \in \operatorname{End}(V) \;\middle|\; A(V^{p,q}) \subseteq V^{p+\ell,q-\ell} \text{ for all } p,q \in \mathbb{Z} \right\}$$

and with Hodge filtration

$$F^k E = \left\{ A \in \operatorname{End}(V) \;\middle|\; A(F^p V) \subseteq F^{p+k} V \text{ for all } p \in \mathbb{Z} \right\}.$$

In particular, $F^0 E$ is the subspace of those endomorphisms that preserve the Hodge filtration on V. For $A \in E$, we defined $A^* \in E$ as the adjoint with respect to the non-degenerate hermitian pairing h, and we noted that

$$E_{\mathbb{R}} = \left\{ A \in \operatorname{End}(V) \;\middle|\; A = A^* \right\}$$

gives E a natural real structure. We had also started to prove the following lemma.

Lemma 6.1. *The pairing*

$$E \otimes_{\mathbb{C}} \overline{E} \to \mathbb{C}, \quad (A,B) \mapsto \operatorname{tr}(AB^*),$$

polarizes the Hodge structure on E.

Proof. It is easy to see that $\operatorname{tr} A^* = \overline{\operatorname{tr} A}$ for every $A \in \operatorname{End}(V)$. Therefore

$$\operatorname{tr}(BA^*) = \operatorname{tr}\bigl((AB^*)^*\bigr) = \overline{\operatorname{tr}(AB^*)},$$

and so the pairing is hermitian symmetric. Last time, we already proved that the Hodge decomposition is orthogonal with respect to the trace pairing. The point was that if $A \in E^{\ell,-\ell}$ and $B \in E^{k,-k}$, then $AB^* \in E^{\ell-k,k-\ell}$ is a nilpotent operator when $k \neq \ell$, and so $\operatorname{tr}(AB^*) = 0$.

Finally, we need to explain why $(-1)^\ell \operatorname{tr}(AA^*) > 0$ if $A \in E^{\ell,-\ell}$ is any nonzero endomorphism. The point is that $(-1)^\ell A^*$ is exactly the adjoint of A with respect to the inner product on V. Indeed, for $x \in V^{p,q}$ and $y \in V^{p+\ell,q-\ell}$, we have

$$\langle Ax, y \rangle = (-1)^{p+\ell} h(Ax, y) = (-1)^{p+\ell} h(x, A^*y) = (-1)^\ell \langle x, A^*y \rangle.$$

Consequently, the endomorphism $(-1)^\ell A^*A$ is self-adjoint with respect to the inner product, and also positive definite, because

$$\bigl\langle (-1)^\ell A^*Ax, x \bigr\rangle = \langle Ax, Ax \rangle = \|Ax\|^2.$$

This clearly implies that $(-1)^\ell \operatorname{tr}(AA^*) = (-1)^\ell \operatorname{tr}(A^*A) > 0$. I invite you to check, with the help of an orthonormal basis, that this expression is exactly the operator norm of A with respect to the inner product on V. $\square$

The conclusion is that if V is a polarized Hodge structure, then $E = \operatorname{End}(V)$ inherits a positive definite inner product, which is given by the simple formula

$$\langle A, B \rangle = \sum_{\ell \in \mathbb{Z}} (-1)^\ell \operatorname{tr}(A_\ell B_\ell^*);$$

here A_ℓ and B_ℓ are the components of A and B in the subspace $E^{\ell,-\ell}$. Of course, the Hodge decomposition of E is orthogonal with respect to this inner product.

Brief review of homogeneous spaces. Let X be a complex manifold, acted on by a real Lie group G, in such a way that for every $g \in G$, the diffeomorphism $g \colon X \to X$ is a biholomorphism. The action of $G_{\mathbb{R}}$ on the period domain D is an example of this type. Suppose that the action is transitive; then for every point $x \in X$, we get an isomorphism of smooth manifolds

$$G/H_x \cong X,$$

where $H_x \subseteq G$ is the stabilizer of the point x. If $\mathfrak{g}$ denotes the Lie algebra of G, and $\mathfrak{h}_x \subseteq \mathfrak{g}$ the Lie algebra of the subgroup $H_x \subseteq G$, then

$$\mathfrak{g}/\mathfrak{h}_x \cong T_x X$$

as $\mathbb{R}$-vector spaces; in particular, the quotient on the left has the structure of a complex vector space. Denote by $\operatorname{Ad}(g) \in \operatorname{End}(\mathfrak{g})$ the adjoint action of an element $g \in G$ on the Lie algebra; in the case of $\operatorname{GL}(V)$, we have $\operatorname{Ad}(g)A = gAg^{-1}$. Since $H_{gx} = gH_x g^{-1}$, the Lie algebras of the stabilizers are related by $\mathfrak{h}_{gx} = \operatorname{Ad}(g)\mathfrak{h}_x$.

Example 6.2. Suppose that G is a matrix group, and $g \colon [0,1] \to G$ is a smooth curve. Fix a point $x \in X$, and consider the function

$$f \colon [0,1] \to X, \quad f(t) = g(t) \cdot x,$$

I claim that under the isomorphism

$$\mathfrak{g}/\mathfrak{h}_{g(t_0)x} \cong T_{g(t_0)x} X,$$

the tangent vector $f_* \frac{d}{dt}\big|_{t=t_0} \in T_{g(t_0)}X$ is the image of

$$\frac{dg}{dt} g^{-1}\big|_{t=t_0} \in \mathfrak{g}.$$

To understand why this should be the case, write

$$f(t) = g(t)g(t_0)^{-1} \cdot g(t_0)x.$$

Now the curve $g(t)g(t_0)^{-1}$ goes through the identity element of G for $t = t_0$, and so its derivative at that point is an element of $\mathfrak{g}$.

For every $x \in X$ and every $g \in G$, we have an induced isomorphism

$$g_* \colon T_x X \to T_{gx} X$$

on holomorphic tangent spaces. It is not hard to see that the diagram

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h}_x & \xrightarrow{\operatorname{Ad}(g)} & \mathfrak{g}/\operatorname{Ad}(g)\mathfrak{h}_x \\ \downarrow \cong & & \downarrow \cong \\ T_x X & \xrightarrow{g_*} & T_{gx} X \end{array}$$

is commutative. This observation implies that

$$G \times (\mathfrak{g}/\mathfrak{h}_x) \to TX, \quad \big(g, A + \mathfrak{h}_x\big) \mapsto \big(gx, \operatorname{Ad}(g)A + \mathfrak{h}_{gx}\big),$$

is surjective, and hence that the tangent bundle of X satisfies

$$TX \cong \big(G \times (\mathfrak{g}/\mathfrak{h}_x)\big)/H_x,$$

where H_x acts on the product by the rule

$$\big(g, A + \mathfrak{h}_x\big) \cdot h = \big(gh, \operatorname{Ad}(h^{-1})A + \mathfrak{h}_x\big).$$

We say that a hermitian metric h on the complex manifold X is *G-invariant* if, for every pair of tangent vectors $\xi, \eta \in T_x X$, one has

$$h_x(\xi, \eta) = h_{gx}\big(g_* \xi, g_* \eta\big)$$

for all group elements $g \in G$. Using the above description of the tangent bundle, we see that a G-invariant hermitian metric on X is the same thing as a hermitian inner product on the (complex) vector space $\mathfrak{g}/\mathfrak{h}_x$ that is invariant under the adjoint action of the stabilizer H_x.

Period domains and their tangent spaces. We return to the study of period domains. Fix a complex vector space V and a hermitian pairing $h\colon V \otimes_{\mathbb{C}} V \to \mathbb{C}$. Also fix a reference Hodge structure

$$V = \bigoplus_{p+q=n} V_o^{p,q}$$

of weight n that is polarized by h. To keep the notation consistent, let us denote the resulting Hodge filtration by F_o, and the corresponding point in the period domain by $o \in D$; this will be our reference point. We also use the notation

$$\langle x, y \rangle_o = \sum_{p+q=n} (-1)^p h(x^{p,q}, y^{p,q})$$

for the resulting positive definite inner product on V. For clarity, we may denote the Hodge filtration corresponding to a point $z \in D$ by the symbol F_z, the Hodge decomposition by $V_z^{p,q}$, etc.

Recall from last time that the homogeneous space

$$\check{D} \cong \operatorname{GL}(V)/B$$

parametrizes all filtrations F with the property that $\dim F^p = \dim F_o^p$ for every $p \in \mathbb{Z}$. By construction, $\check{D}$ is a closed submanifold of a product of Grassmannians, and therefore a compact complex manifold. Here

$$B = B_o = \big\{\, g \in \operatorname{GL}(V) \,\big|\, gF_o = F_o \,\big\}$$

is the stabilizer of the reference Hodge filtration F_o. The period domain $D \subseteq \check{D}$ is the open subset of those filtrations that correspond to Hodge structures of weight n polarized by h. We also observed that

$$D = G_{\mathbb{R}}/H$$

is a homogeneous space for the real Lie group

$$G_{\mathbb{R}} = \big\{\, g \in \operatorname{GL}(V) \,\big|\, h(gx, gy) = h(x, y) \text{ for all } x, y \in V \,\big\},$$

which is one of the many different real forms of $\operatorname{GL}(V)$. Since $G_{\mathbb{R}}$ is a subgroup of $\operatorname{GL}(V)$, it acts on D via biholomorphisms. We showed last time that the subgroup $H = H_o = G_{\mathbb{R}} \cap B_o$ is compact. In fact, $\check{D}$ is also a homogeneous space for a real Lie group, namely the unitary group

$$U = U\big(V, \langle\,\rangle_o\big) = \big\{\, g \in \operatorname{GL}(V) \,\big|\, \langle gx, gy \rangle_o = \langle x, y \rangle_o \text{ for all } x, y \in V \,\big\},$$

which is the unique compact real form of $\operatorname{GL}(V)$.

Lemma 6.3. *We have $\check{D} \cong U/H$.*

Proof. It is easy to see that the action of U on $\check{D}$ is transitive: choose an orthonormal basis of V adapted to the filtration F_0, and another orthonormal basis adapted to the filtration F, and consider the linear transformation that takes one to the other. I will leave it as an exercise to check that

$$U \cap B = G_{\mathbb{R}} \cap B,$$

which means that the stabilizer of F_0 is the same subgroup H. □

The Lie algebra of $\mathrm{GL}(V)$ is of course $\mathrm{End}(V)$, and as we discussed above, the holomorphic tangent space at an arbitrary point $z \in D$ is therefore

$$T_z D = T_z \check{D} \cong \mathrm{End}(V)/F_z^0 \, \mathrm{End}(V) \cong \bigoplus_{\ell \leq -1} \mathrm{End}(V)_z^{\ell,-\ell},$$

where the subscript z means that we give $\mathrm{End}(V)$ the Hodge structure of weight 0 that comes from the Hodge structure on V corresponding to the point $z \in D$. This Hodge structure is polarized by the trace pairing, and so it has a positive definite inner product $\langle\,\rangle_z$. Its restriction to the subspace

$$\bigoplus_{\ell \leq -1} \mathrm{End}(V)_z^{\ell,-\ell}$$

therefore induces, via the above isomorphism, a positive definite inner product on the holomorphic tangent space $T_z D$. It is not hard to see that the Hodge decomposition, and therefore also the inner product, depend real-analytically on the point $z \in D$.

Proposition 6.4. *The resulting hermitian metric h_D on D is $G_{\mathbb{R}}$-invariant.*

Proof. For any $g \in G_{\mathbb{R}}$ and any $z \in D$, we have

$$\mathrm{End}(V)_{gz}^{\ell,-\ell} = g \cdot \mathrm{End}(V)_z^{\ell,-\ell} \cdot g^{-1}.$$

For $A, B \in \mathrm{End}(V)_z^{\ell,-\ell}$, we thus have

$$\begin{aligned} \left\langle gAg^{-1}, gBg^{-1} \right\rangle_{gz} &= (-1)^\ell \operatorname{tr}\Big(gAg^{-1}(gBg^{-1})^*\Big) = (-1)^\ell \operatorname{tr}\Big(gAg^{-1}gB^*g^{-1}\Big) \\ &= (-1)^\ell \operatorname{tr}(AB^*) = \langle A, B \rangle_z, \end{aligned}$$

because $g^* = g^{-1}$ for elements $g \in G_{\mathbb{R}}$. This is enough to conclude that the metric h_D is $G_{\mathbb{R}}$-invariant. □

Similarly, we can construct a U-invariant hermitian metric on $\check{D}$.

Proposition 6.5. *The compact dual $\check{D}$ has a U-invariant hermitian metric $h_{\check{D}}$.*

Proof. As explained above, the holomorphic tangent bundle of $\check{D} \cong U/H$ is isomorphic to the quotient

$$(U \times T_{F_0}\check{D})/H,$$

where H acts on $T_{F_0}\check{D} \cong \mathrm{End}(V)/F^0 \, \mathrm{End}(V)$ by conjugation. The inner product on $T_{F_0}\check{D}$ that we constructed above is invariant under the group U; this follows from the fact that the inner product on $\mathrm{End}(V)$ is induced by the inner product $\langle\,\rangle_0$ on V, which is by definition invariant under U. Since H is a subgroup of U, we obtain a U-invariant hermitian metric on the complex manifold $\check{D}$. □

Exercise 6.1. Check that the inner product on $\mathrm{End}(V)$, which is given by

$$\langle A, B \rangle = (-1)^\ell \operatorname{tr}(AB^*)$$

for $A, B \in \mathrm{End}(V)^{\ell,-\ell}$, is invariant under conjugation by the unitary group U.

Exercise 6.2. In Example 5.4, we found that the unit disk is an example of a period domain, with compact dual $\mathbb{P}^1$. What are the hermitian metrics on D and $\check{D}$ in this case?

Example 6.6. Consider Hodge structures of the form $H^{1,0} \oplus H^{0,1}$ on $\mathbb{C}^{n+1}$ that are polarized by the pairing

$$x_1' \overline{x_1''} + \cdots + x_n' \overline{x_n''} - x_{n+1}' \overline{x_{n+1}''}$$

and satisfy $\dim H^{1,0} = 1$. Clearly, the compact dual is $\mathbb{P}^n$ in this case. Describe the period domain D and the hermitian metric h_D explicitly.

Poincaré metrics. Suppose that M is a one-dimensional complex manifold with a hermitian metric h_M. Pick a local coordinate z, and consider the smooth function $h = h_M\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)$. The expression

$$K = -\frac{1}{h} \cdot \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log h$$

is called the *sectional curvature* of the hermitian metric.

Example 6.7. On the unit disk $\Delta = \left\{ t \in \mathbb{C} \;\middle|\; |t| < 1 \right\}$, the *Poincaré metric*

$$h_\Delta\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{2}{(1-|t|^2)^2}$$

has constant sectional curvature -1. Indeed,

$$\frac{\partial}{\partial \bar{t}} \log h = -2 \frac{\partial}{\partial \bar{t}} \log(1-|t|^2) = \frac{2t}{1-|t|^2},$$

and therefore

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \bar{t}} \log h = \frac{2(1-|t|^2) + 2|t|^2}{(1-|t|^2)^2} = \frac{2}{(1-|t|^2)^2} = h.$$

This shows that $K \equiv -1$.

Example 6.8. The half space $\tilde{\mathbb{H}} = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z < 0 \right\}$ is of course isomorphic to the unit disk, for example via the function

$$\tilde{\mathbb{H}} \to \Delta, \quad z \mapsto \frac{z+1}{z-1}.$$

The induced hermitian metric on $\tilde{\mathbb{H}}$ is also called the Poincaré metric; you can check that it is given by the formula

$$h_{\tilde{\mathbb{H}}}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{2(\operatorname{Re} z)^2}.$$

Of course, the sectional curvature is still -1 everywhere.

Example 6.9. The exponential function $\exp\colon \tilde{\mathbb{H}} \to \Delta^*$ realizes the half space as the universal covering space of the punctured disk $\Delta^* = \left\{ t \in \mathbb{C} \;\middle|\; 0 < |t| < 1 \right\}$. The group of deck transformations is $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z} \subseteq \mathbb{C}$, which acts on $\tilde{\mathbb{H}}$ by translations. The Poincaré metric on $\tilde{\mathbb{H}}$ is invariant under these translations, and therefore descends to a hermitian metric on Δ^*. Again, you should verify that

$$h_{\Delta^*}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{1}{2|t|^2(\log|t|)^2}.$$

This metric is also called the Poincaré metric.