Lecture 22: November 18

In the definition of polarized Hodge modules, we needed the induced pairings

$$h_\alpha\colon \operatorname{gr}_V^\alpha \mathcal{M} \otimes \overline{\operatorname{gr}_V^\alpha \mathcal{M}} \to \mathbb{C}$$

for $-1 \leq \alpha \leq 0$. The purpose of today's lecture is to explain how these pairings are constructed. We will focus on the case $-1 < \alpha \leq 0$, and exclude the (more involved) case $\alpha = -1$.

To get a feeling for the problem, let us go back to the example of a polarized variation of Hodge structure on Δ^*. Here $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, and for $\alpha > -1$, the V-filtration is given by $V^\alpha \mathcal{M} = \tilde{\mathscr{V}}^\alpha$. In particular,

$$\operatorname{gr}_V^\alpha \mathcal{M} \cong E_\alpha(R),$$

where $R \in \operatorname{End}(V)$ is the residue of the logarithmic connection on $\tilde{\mathscr{V}}^\alpha$. The pairing that appears in Schmid's results, specifically in Theorem 10.3, is the restriction of the hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ to the eigenspace $E_\alpha(R)$. The question is how we can recover this pairing from the pairing on the $\mathscr{D}$-module $\mathcal{M}$.

Recall from Lecture 9 that, with respect to the trivialization $\tilde{\mathscr{V}}^\alpha \cong \mathscr{O}_\Delta \otimes_{\mathbb{C}} V$, the polarization of the variation of Hodge structure takes the form

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = \sum_{\alpha \leq \beta < \alpha+1} |t|^{2\beta} \sum_{j=0}^{\infty} \frac{L(t)^j}{j!} (-1)^j h\big(v'_\beta, R_N^j v''_\beta\big),$$

where v'_β and v''_β are the components in $E_\beta(R)$. Since $\alpha > -1$, all the functions in this expression are locally integrable, and so the right-hand side is a distribution on Δ; the sesquilinear pairing on $\mathcal{M}$ was defined in such a way that $h_{\mathcal{M}} = h_{\mathscr{V}}$ on $V^\alpha \mathcal{M} = \tilde{\mathscr{V}}^\alpha$ for $\alpha > -1$. The pairing on $E_\alpha(R)$, which is $h(v'_\alpha, v''_\alpha)$, appears in this expression as the coefficient of the term $|t|^{2\alpha}$. So we need a way to extract this particular coefficient from the distribution. If someone gives us a distribution with an "asymptotic expansion" as above, in terms of the functions $|t|^{2\beta} L(t)^j$, then this is easy: simply take the coefficient at $|t|^{2\alpha}$. But for arbitrary Hodge modules, we need a construction that works even if we don't have the asymptotic expansion.

The key idea, due in this context to Daniel Barlet and Claude Sabbah, is to consider the function $|t|^{2s-2}$, where $s \in \mathbb{C}$ is a complex parameter. This function is locally integrable on the halfspace $\operatorname{Re} s > 0$, but is *not* locally integrable for $s = 0$. This sharp distinction gives us a way to pick out specific values of the exponent. Indeed, if we consider the expression

$$|t|^{2s-2} \cdot \sum_{\alpha \leq \beta < \alpha+1} |t|^{2\beta} \sum_{j=0}^{\infty} \frac{L(t)^j}{j!} (-1)^j h\big(v'_\beta, R_N^j v''_\beta\big),$$

then all the terms with $\beta > \alpha$ are locally integrable on a halfspace of the form $\operatorname{Re} s > -\beta$, whereas the terms with $|t|^{2\alpha}$ stop being locally integrable at $s = -\alpha$.

We also need a way to distinguish the term $|t|^{2\alpha}$ from the other terms $|t|^{2\alpha} L(t)^j$ with $j \geq 1$. Here the general idea is to consider the expression above as a *function* of the complex parameter s, and so see what happens near $s = -\alpha$. The following example shows how this works in practice.

Example 22.1. Let $\varphi \in C_c^\infty(\Delta)$ be a compactly supported function on Δ, and consider the expression

$$F(s) = -\frac{1}{2\pi i} \int_\Delta |t|^{2s-2} \varphi \, dt \wedge d\bar{t}.$$

as a function of the complex parameter s. For $\operatorname{Re} s > 0$, the function is well-defined, and by differentiating under the integral sign, one sees that $F(s)$ is a holomorphic function on the halfspace $\operatorname{Re} s > 0$. Similarly, we obtain

$$F^{(j)}(s) = (-1)^{j-1} \frac{1}{2\pi i} \int_\Delta L(t)^j |t|^{2s-2} \varphi dt \wedge d\bar{t}$$

by differentiating under the integral sign j times.

We want to understand how $F(s)$ behaves near the critical line $\operatorname{Re} s = 0$. We can use integration by parts to prove the following nice identity:

$$s^2 F(s) = -\frac{1}{2\pi i} \int_\Delta |t|^{2s} \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} dt \wedge d\bar{t}. \tag{22.2}$$

Here is how this goes. Observe that

$$d\Big(t|t|^{2s-2} \varphi d\bar{t}\Big) = s|t|^{2s-2} \varphi dt \wedge d\bar{t} + t|t|^{2s-2} \frac{\partial \varphi}{\partial t} dt \wedge d\bar{t}.$$

Recall that φ has compact support. From Stokes' theorem, applied on the annulus with inner radius $\varepsilon > 0$ and outer radius close to 1, we get

$$\frac{1}{2\pi i} \int_{|t|=\varepsilon} t|t|^{2s-2} \varphi d\bar{t} = sF(s) - \frac{1}{2\pi i} \int_\Delta t|t|^{2s-2} \frac{\partial \varphi}{\partial t} dt \wedge d\bar{t}.$$

The left-hand side goes to zero with ε as long as $\operatorname{Re} s > 0$, hence

$$sF(s) = \frac{1}{2\pi i} \int_\Delta t|t|^{2s-2} \frac{\partial \varphi}{\partial t} dt \wedge d\bar{t}.$$

If we repeat this argument, swapping the role of t and $\bar{t}$, we arrive at (22.2).

The point is that the function on the right-hand side of (22.2) is holomorphic on the larger halfspace $\operatorname{Re} s > -1$. If we divide both sides by s^2, we therefore get a *meromorphic* extension of $F(s)$ to the halfspace $\operatorname{Re} s > -1$, with a single pole at $s = -1$. To see what $F(s)$ looks like near $s = -1$, we use the exponential series

$$|t|^{2s} = e^{-sL(t)} = \sum_{j=0}^\infty (-1)^j s^j \frac{L(t)^j}{j!}.$$

Substituting this into (22.2) gives

$$F(s) = \sum_{j=0}^\infty s^{j-2} (-1)^{j-1} \frac{1}{2\pi i} \int_\Delta \frac{L(t)^j}{j!} \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} dt \wedge d\bar{t}.$$

The term with $j = 0$ vanishes (because the integral of an exact form with compact support is zero); for the term $j = 1$, another application of Stokes' theorem yields

$$\frac{1}{2\pi i} \int_\Delta L(t) \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} dt \wedge d\bar{t} = \varphi(0).$$

Nnear $s = -1$, the function $F(s)$ therefore looks like

$$F(s) = \frac{\varphi(0)}{s} + \sum_{j=2}^\infty s^{j-2} (-1)^{j-1} \frac{1}{2\pi i} \int_\Delta \frac{L(t)^j}{j!} \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} dt \wedge d\bar{t}.$$

It has a simple pole at $s = 0$, with residue $\operatorname{Res}_0 F(s) = \varphi(0)$. By contrast,

$$-\frac{1}{2\pi i} \int_\Delta L(t)^j |t|^{2s-2} \varphi dt \wedge d\bar{t} = (-1)^j F^{(j)}(s)$$

has a pole of order $j + 1$ at $s = 0$, but the residue is zero. So the conclusion is that the term with $j = 0$ is distinguished by the presence of a residue.

Going back to variations of Hodge structure, we can now pick out the coefficient at $|t|^{2\alpha}$ as follows. Choose a compactly supported function $\varphi \in C_c^\infty(\Delta)$ that is identically 1 in a neighborhood of the origin; such a function is often called a *cutoff* function. The calculation from above gives

$$h(v'_\alpha, v''_\alpha) = \operatorname{Res}_{s=-\alpha}\left(-\frac{1}{2\pi i}\int_\Delta h_{\mathscr{V}}(1\otimes v', 1\otimes v'')|t|^{2s-2}\varphi dt\wedge d\bar{t}\right),$$

because the terms with $|t|^{2\beta}L(t)^j$ and $\beta > \alpha$ contribute functions that are holomorphic at $s = -\alpha$, and the terms with $|t|^{2\alpha}L(t)^j$ and $j \geq 1$ contribute functions with poles of order $j+1$ and vanishing residue.

Definition of the induced pairing. We are now ready for the general definition. It does not matter that $\mathcal{M}$ is a polarized Hodge module – all we need is the existence of a V-filtration. So let us suppose that $\mathcal{M}$ is a coherent $\mathscr{D}$-module on the disk Δ, and that $\mathcal{M}$ admits a V-filtration $V^\bullet\mathcal{M}$ with the same properties as in Lecture 21. Consider a distribution-valued sesquilinear pairing

$$h_{\mathcal{M}} \colon \mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}} \to \mathrm{Db}_\Delta\,.$$

For each real number $\alpha > -1$, we are going to construct an induced pairing

$$h_\alpha \colon \operatorname{gr}_V^\alpha \otimes_{\mathbb{C}} \overline{\operatorname{gr}_V^\alpha} \to \mathbb{C}$$

on the vector space $\operatorname{gr}_V^\alpha = V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$, with the property that

$$h_\alpha(Nx, y) = h_\alpha(x, Ny);$$

here $N = t\partial_t - \alpha$ is the nilpotent operator on $\operatorname{gr}_V^\alpha$. The general idea should be clear at this point. Choose a cutoff function $\varphi \in C_c^\infty(\Delta)$. For two local sections $m', m'' \in V^\alpha\mathcal{M}$, we are going to show that the function

$$F_{m',m''}(s) = \left\langle h_{\mathcal{M}}(m', m''), |t|^{2s-2}\varphi dt\wedge d\bar{t}\right\rangle \tag{22.3}$$

extends to a meromorphic function on the complex plane; that it is holomorphic on the halfspace $\operatorname{Re} s > -\alpha$; and that it has a pole at the point $s = -\alpha$. Moreover, we will see that the residue of $F_{m',m''}(s)$ at the point $s = -\alpha$ only depends on the image of m', m'' in the quotient $\operatorname{gr}_V^\alpha$. We can then define

$$h_\alpha(m', m'') = \operatorname{Res}_{s=-\alpha}\left(-\frac{1}{2\pi i}F_{m',m''}(s)\right) \tag{22.4}$$

to obtain the desired pairing on $\operatorname{gr}_V^\alpha$. For the sake of clarity, I will divide the construction into several steps.

Step 1. Fix two real numbers $\alpha, \beta > -1$, as well as two sections $m' \in V^\alpha\mathcal{M}$ and $m'' \in V^\beta\mathcal{M}$. Initially, the problem with (22.3) is that distributions act on compactly supported *smooth* forms, but the function $|t|^{2s-2}$ is not smooth at $t = 0$. Here we need to use a basic fact about distributions: every distribution is locally of finite order. In other words, if $D \in \mathrm{Db}(\Delta)$ is a distribution, then for every compact subset $K \subseteq \Delta$, there is a constant $C > 0$ and an integer $p \in \mathbb{N}$ such that for every test function η with $\operatorname{Supp}\eta \subseteq K$, one has

$$\left|\langle D, \eta dt\wedge d\bar{t}\rangle\right| \leq C\|\eta\|_{C^p(K)}, \tag{22.5}$$

where the norm on the right-hand side is the supremum over all derivatives of η of order $\leq p$. In particular, D extends uniquely to bounded linear functional on the space $C^p(K)$. The smallest such integer p is called the *order* of D on K.

In our setting, we can use the support of the cutoff function φ as the compact set K. Since $V^\alpha\mathcal{M}$ and $V^\beta\mathcal{M}$ are coherent $\mathscr{O}_X$-modules, hence generated by finitely many sections, the orders of all the distributions $h_{\mathcal{M}}(m', m'')$ can be bounded by a fixed integer $p \in \mathbb{N}$, and an estimate as in (22.5) will hold with a constant $C > 0$

that is independent of m', m''. Because the function $|t|^{2s-2}$ is certainly in $C^p(\Delta)$ as long as $\operatorname{Re}(2s-2) \geq p$, the expression

$$F_{m',m''}(s) = \left\langle h_{\mathcal{M}}(m', m''), |t|^{2s-2} \varphi dt \wedge d\bar{t} \right\rangle$$

therefore makes sense for $\operatorname{Re} s > 1 + \frac{p}{2}$. The estimate in (22.5) then allows us to "differentiate under the integral sign" and conclude that $F_{m',m''}(s)$ is holomorphic on the halfspace $\operatorname{Re} s > 1 + \frac{p}{2}$.

Step 2. We use the $\mathscr{D}$-module structure on $\mathcal{M}$ to find relations between different functions of this kind, similar to what we did in the example above. The sesquilinearity of the pairing gives $h_{\mathcal{M}}(t\partial_t m', m'') = t\partial_t h_{\mathcal{M}}(m', m'')$, and therefore

$$\begin{aligned}
F_{t\partial_t m',m''}(s) &= \left\langle t\partial_t h_{\mathcal{M}}(m', m''), |t|^{2s-2} \varphi dt \wedge d\bar{t} \right\rangle \\
&= -\left\langle h_{\mathcal{M}}(m', m''), \partial_t \bigl(t|t|^{2s-2} \varphi\bigr) \, dt \wedge d\bar{t} \right\rangle \\
&= -\left\langle h_{\mathcal{M}}(m', m''), \left(s|t|^{2s-2} \varphi + t|t|^{2s-2} \frac{\partial \varphi}{\partial t}\right) dt \wedge d\bar{t} \right\rangle \\
&= -sF_{m',m''}(s) - \left\langle h_{\mathcal{M}}(m', m''), t|t|^{2s-2} \frac{\partial \varphi}{\partial t} dt \wedge d\bar{t} \right\rangle.
\end{aligned}$$

Now the derivative $\frac{\partial \varphi}{\partial t}$ is identically zero in a neighborhood of the origin, and so the product with $t|t|^{2s-2}$ is a smooth function on the disk. In the above formula, the last term is therefore holomorphic for every $s \in \mathbb{C}$. So if we use $\equiv$ to mean "modulo entire functions", we can write the result in the form

$$F_{t\partial_t m',m''}(s) \equiv -sF_{m',m''}(s) \equiv F_{m',t\partial_t m''}(s).$$

The second half is proved in the same way, of course.

Step 3. The next task is to construct a meromorphic extension of $F_{m',m''}(s)$ to the complex plane. Here the key point is that $t\partial_t - \alpha$ acts nilpotently on $\operatorname{gr}_V^\alpha$. We can therefore find a number $e_\alpha \in \mathbb{N}$ such that

$$(t\partial_t - \alpha)^{e_\alpha} m' \in V^{>\alpha}\mathcal{M},$$

If we keep applying this observation, we can produce a polynomial $b(s) \in \mathbb{R}[s]$, with roots in the interval $[\alpha, \infty)$, such that

$$b(t\partial_t) m' \in V^{\alpha+k}\mathcal{M},$$

where $k \in \mathbb{N}$ can be made as large as we please. Remembering that the multiplication map $t^k \colon V^\alpha\mathcal{M} \to V^{\alpha+k}\mathcal{M}$ is an isomorphism for $\alpha > -1$, we get

$$b(t\partial_t) m' = t^k m'_k,$$

for some $m'_k \in V^\alpha\mathcal{M}$. (If you took my course on $\mathscr{D}$-modules last semester, you may recognize this as a special case of the Bernstein polynomial.) The identity in Step 2 now gives us, modulo entire functions,

$$b(-s) F_{m',m''}(s) \equiv F_{b(t\partial_t)m',m''}(s) = \left\langle h_{\mathcal{M}}(m'_k, m''), t^k |t|^{2s-2} \varphi dt \wedge d\bar{t} \right\rangle$$

Because the function on the right-hand side is holomorphic on the halfspace $\operatorname{Re} s > 1 + \frac{p}{2} - k$, this shows that $F_{m',m''}(s)$ extends meromorphically to this larger halfspace; by increasing k, we therefore get a meromorphic extension to the complex plane. Moreover, all roots of the polynomial $b(-s)$ belong to the interval $(-\infty, -\alpha]$, and so $F_{m',m''}(s)$ is actually holomorphic on the halfspace $\operatorname{Re} s > -\alpha$, possibly with a pole at the point $s = -\alpha$. By applying the same construction to m'', the function $F_{m',m''}(s)$ is also holomorphic on the halfspace $\operatorname{Re} s > -\beta$.

Step 4. We can now complete the construction of the induced pairing. Suppose that $m', m'' \in V^\alpha \mathcal{M}$. The function $F_{m',m''}(s)$ is holomorphic on the halfspace $\operatorname{Re} s > -\alpha$, and possibly has a pole at the point $s = -\alpha$, and so the residue

$$\operatorname{Res}_{s=-\alpha}\left(-\frac{1}{2\pi i} F_{m',m''}(s)\right) \in \mathbb{C}$$

has a meaning. If either m' or m'' belong to $V^{>\alpha}\mathcal{M}$, then the function $F_{m',m''}(s)$ is holomorphic on a slightly larger halfspace, hence the residue is zero. The residue therefore depends only on the image of m' and m'' in the quotient $\operatorname{gr}_V^\alpha = V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$, and so (22.4) does define the desired pairing

$$h_\alpha \colon \operatorname{gr}_V^\alpha \otimes_{\mathbb{C}} \overline{\operatorname{gr}_V^\alpha} \to \mathbb{C}.$$

Let us check that $N = t\partial_t - \alpha$ is its own adjoint. By the result from Step 2,

$$F_{(t\partial_t - \alpha)m', m''}(s) \equiv F_{m', (t\partial_t - \alpha)m''}(s),$$

which only works of course because $\alpha \in \mathbb{R}$. Both sides therefore have the same residue, and so $h_\alpha(Nm', m'') = h_\alpha(m', Nm'')$.

Example 22.6. In the special case of a polarized variation of Hodge structure on Δ^*, we showed at the beginning of the lecture that the general construction recovers the pairing on $\operatorname{gr}_V^\alpha \cong E_\alpha(R_S)$ that is used in Schmid's results.

Example 22.7. Suppose that $\mathcal{M}$ is a vector bundle with connection on Δ. In this case, $\mathcal{M} = V^0\mathcal{M}$, and we can take m', m'' to be flat sections. Then $h_\mathcal{M}(m', m'')$ is a constant function, and the example from earlier shows that the residue of

$$-\frac{1}{2\pi i}\left\langle h_\mathcal{M}(m', m''), |t|^{2s-2}\varphi dt \wedge d\bar{t}\right\rangle = -\frac{1}{2\pi i}\int_\Delta h_\mathcal{M}(m', m'')|t|^{2s-2}\varphi dt \wedge d\bar{t}\cdot$$

at the point $s = 0$ is equal to the constant $h_\mathcal{M}(m', m'')$. So in this case, the effect of the construction is to simply restrict the flat pairing on the vector bundle to the fiber over the origin.