Lecture 21: November 13

Hodge modules with support a point. Last time, we introduced polarized Hodge modules. The definition contains the – at first glance somewhat mysterious – conditions that $F_\bullet\mathcal{M}$ needs to respect the local V-filtrations. Recall that this means that

$$t\colon F_k V^\alpha \to F_k V^{\alpha+1}$$

should be an isomorphism for $\alpha > -1$, and that

$$\partial_t\colon F_k \operatorname{gr}_V^\alpha \to F_{k+1} \operatorname{gr}_V^{\alpha-1}$$

should be an isomorphism for $\alpha < 0$. I mentioned that these conditions are need to make the filtration $F_\bullet\mathcal{M}$ see the properties of the $\mathscr{D}$-module $\mathcal{M}$. We also saw one example of this: if $\mathcal{M}$ is a bundle with connection, then the first condition forces each $F_k\mathcal{M}$ to be a subbundle. Here is another example.

Example 21.1. Suppose that $\mathcal{M} = H \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is a $\mathscr{D}_\Delta$-module supported at the origin, and that $F_\bullet\mathcal{M}$ is a "good" filtration by coherent $\mathscr{O}_\Delta$-modules. I claim that if $F_\bullet\mathcal{M}$ respects the local V-filtrations, then it must come from a filtration $F_\bullet H$ on the vector space H, as in the construction from last week. Recall that $V^{-1}\mathcal{M} = H \otimes 1$, and more generally

$$V^{-(\ell+1)}\mathcal{M} = \sum_{j=0}^{\ell} H \otimes \partial_t^j.$$

We first construct a filtration on H. Let $p \in \mathbb{Z}$ be such that $F_{p-1}\mathcal{M} = 0$ but $F_p\mathcal{M} \neq 0$. The inclusion $i\colon H \hookrightarrow \mathcal{M}$, given by $i(h) = h \otimes 1$, allows us to define

$$F_k H = i^{-1}(F_k\mathcal{M}).$$

By construction, $F_{p-1}H = 0$, and $F_k\mathcal{M} \supseteq F_k H \otimes 1$; since $F_\bullet\mathcal{M}$ is compatible with the action by differential operators, this gives

$$F_k\mathcal{M} \supseteq \sum_{j=0}^{\infty} F_{k-j} H \otimes \partial_t^j.$$

We are going to prove that the two sides are equal, by induction on $k \geq p$.

The first case is $k = p$. Let us show that $F_p\mathcal{M} \subseteq V^{-1}\mathcal{M}$. Since the V-filtration exhausts $\mathcal{M}$, we certainly have $F_p\mathcal{M} \subseteq V^\alpha\mathcal{M}$ for some $\alpha \ll 0$. By assumption,

$$\partial_t\colon F_{p-1} \operatorname{gr}_V^{\alpha+1} \mathcal{M} \to F_p \operatorname{gr}_V^\alpha \mathcal{M}$$

is an isomorphism as long as $\alpha < -1$; because the left-hand side is zero, this means that $F_p\mathcal{M} \subseteq V^{>\alpha}\mathcal{M}$. We can repeat this argument as long as $\alpha < -1$; eventually, we reach the conclusion that $F_p\mathcal{M} \subseteq V^{-1}\mathcal{M}$. But $V^{-1}\mathcal{M} = H \otimes 1$, and so

$$F_p\mathcal{M} = F_p H \otimes 1.$$

Now let us deal with the general case. From the fact that

$$\partial_t\colon F_k \operatorname{gr}_V^{\alpha+1} \mathcal{M} \to F_{k+1} \operatorname{gr}_V^\alpha \mathcal{M}$$

is an isomorphism for $\alpha > -1$, we deduce that

$$F_{k+1}\mathcal{M} \cap V^\alpha\mathcal{M} = F_{k+1}\mathcal{M} \cap V^{>\alpha}\mathcal{M} + \partial_t\big(F_k\mathcal{M} \cap V^{\alpha+1}\mathcal{M}\big),$$

and therefore (by gradually increasing α as before) that

$$F_{k+1}\mathcal{M} = F_{k+1}\mathcal{M} \cap V^{-1}\mathcal{M} + \partial_t\big(F_k\mathcal{M}\big).$$

Since $F_{k+1}\mathcal{M} \cap V^{-1}\mathcal{M} = F_{k+1}H \otimes 1$, we get

$$F_{k+1}\mathcal{M} = F_{k+1}H \otimes 1 + \partial_t\big(F_k\mathcal{M}\big),$$

which gives the desired result by induction.

Now let us suppose that $\mathcal{M} = H \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is a polarized Hodge module of weight w. It is not hard to see that the pairing $h_{\mathcal{M}}$ is induced from a pairing on the vector space H. Indeed, for any $x, y \in H$, the two sections $x \otimes 1$ and $y \otimes 1$ are annihilated by t, and therefore

$$t \cdot h_{\mathcal{M}}(x \otimes 1, y \otimes 1) = \bar{t} \cdot h_{\mathcal{M}}(x \otimes 1, y \otimes 1) = 0$$

by sesquilinearity. Therefore $h_{\mathcal{M}}(x \otimes 1, y \otimes 1)$ must be a multiple of the δ-function, and we obtain a well-defined pairing $h \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ with the property that

$$\big\langle h_{\mathcal{M}}(x \otimes 1, y \otimes 1), \varphi dt \wedge d\bar{t} \big\rangle = h(x, y) \cdot \varphi(0).$$

By sesquilinearity, the entire pairing $h_{\mathcal{M}}$ is then determined by h, as in the construction from last week.

The definition of a polarized Hodge module now implies that H is actually a polarized Hodge structure of weight w. Indeed, we have $\operatorname{gr}_V^{-1} \mathcal{M} \cong H$, and since the operator $N = t\partial_t - (-1) = \partial_t t$ acts trivially, we get $\operatorname{gr}_V^{-1} \mathcal{M} = \operatorname{gr}_0^W \operatorname{gr}_V^{-1} \mathcal{M}$. One can check that the induced pairing on H is just the pairing h from above. Since $\mathcal{M}$ is a polarized Hodge module of weight w, it follows that H has a Hodge structure of weight w, polarized by the pairing h (which must therefore be hermitian). The Hodge filtration is induced by $F_{\bullet+1}\mathcal{M}$, hence equal to $F_{\bullet+1}H$ in the notation from above. Since

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} F_{k-j}H \otimes \partial_t^j,$$

we find that the Hodge filtration on the Hodge structure H and the Hodge filtration on $\mathcal{M}$ are off by -1; this is consistent with the construction from last week.

The limiting mixed Hodge structure. Let $\mathcal{M}$ be a polarized Hodge module of weight w on Δ. Our goal is to analyze what the definition tells us about the two vector spaces $H = \operatorname{gr}_V^0 \mathcal{M}$ and $H' = \operatorname{gr}_V^{-1} \mathcal{M}$, and about the linear mappings

$$t \colon \operatorname{gr}_V^{-1} \to \operatorname{gr}_V^0 \quad \text{and} \quad \partial_t \colon \operatorname{gr}_V^0 \to \operatorname{gr}_V^{-1}.$$

On H, we have the nilpotent operator $N = t\partial_t$, its monodromy weight filtration $W_\bullet H$, and the filtration $F_\bullet H$ induced by $F_\bullet \mathcal{M}$; by definition,

$$\bigoplus_{\ell \in \mathbb{Z}} H_\ell = \bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^W H$$

is a Hodge-Lefschetz structure of central weight $w - 1$. In particular, each H_ℓ is a Hodge structure of weight $w - 1 + \ell$, whose Hodge filtration $F_\bullet H_\ell$ is induced by $F_\bullet H$. On H', we have the nilpotent operator $N' = t\partial_t + 1 = \partial_t t$, its monodromy weight filtration $W_\bullet H'$, and the filtration $F_\bullet H'$ induced by $F_\bullet \mathcal{M}$; by definition

$$\bigoplus_{\ell \in \mathbb{Z}} H'_\ell = \bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^W H'$$

is a Hodge-Lefschetz structure of central weight w. In particular, each H'_ℓ is a Hodge structure of weight $w + \ell$, whose Hodge filtration $F_{\bullet+1} H'_\ell$ is induced by $F_{\bullet+1} H'$.

It is customary to denote the linear mapping $\partial_t \colon H \to H'$ by the symbol $c \colon H \to H'$, as an abbreviation for "canonical"; likewise, the mapping $t \colon H' \to H$ is denote by $v \colon H' \to H$, as an abbrevation for "variation". The commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\;c\;} & H' \\ \big\downarrow N & \swarrow v & \big\downarrow N' \\ H & \xrightarrow{\;c\;} & H' \end{array}$$

expresses the fact that $N = vc$ and $N' = cv$. In this setting, the weight filtrations of the two nilpotent operators N and N' are related as follows.

Lemma 21.2. *One has* $c(W_\ell H) \subseteq W_{\ell-1}H'$ *and* $v(W_\ell H') \subseteq W_{\ell-1}H$.

We therefore get an induced mapping

$$c\colon H_\ell \to H'_{\ell-1};$$

both sides are polarized Hodge structures of weight $w-1+\ell$. Moreover, c maps $F_k H_\ell$ into $F_{k+1}H'_{\ell-1}$, due to the fact that $\partial_t \cdot F_k\mathcal{M} \subseteq F_{k+1}\mathcal{M}$; from the compatibility of c with the polarizations, one deduces that c is actually a morphism of Hodge structures. Similarly, we get an induced mapping

$$v\colon H'_\ell \to H_{\ell-1},$$

where the left-hand side is polarized Hodge structure of weight $w+\ell$ with Hodge filtration $F_{\bullet+1}H'_\ell$, and the right-hand side a polarized Hodge structure of weight $w+\ell-2$ with Hodge filtration $F_\bullet H_{\ell-1}$. Since v maps $F_k H'_\ell$ into $F_k H_\ell$, we can add a Tate twist to get a morphism of Hodge structures

$$c\colon H'_\ell \to H_{\ell-1}(-1).$$

One can then show that

$$c\colon \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_\ell^W H \to \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_{\ell-1}^W H'$$

is a morphism of Hodge-Lefschetz structures of central weight $w-1$, and that

$$v\colon \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_\ell^W H' \to \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_{\ell-1}^W H(-1)$$

is a morphism of Hodge-Lefschetz structures of central weight w.

Let us note the following important consequence of the fact that c is a morphism of Hodge structures.

Lemma 21.3. *We have* $\partial_t(\mathrm{gr}_V^0) \cap F_k \mathrm{gr}_V^{-1} = \partial_t(F_{k-1}\mathrm{gr}_V^0)$.

Proof. The statement is that $c\colon H \to H'$ is strictly compatible with the filtrations $F_\bullet H$ and $F_{\bullet+1}H'$. Since $c(W_\ell H) \subseteq W_{\ell-1}H')$, it suffices to show that this is true for $c\colon H_\ell \to H'_{\ell-1}$. But this is a morphism of Hodge structures, and morphisms of Hodge structures are always strictly compatible with the Hodge filtrations. ☐

Polarized Hodge modules with strict support. It is possible to characterize those polarized Hodge modules on Δ that come from a variation of Hodge structure on Δ^* purely in terms of the V-filtration. Let me explain next how this works. Suppose that $\mathcal{M}$ is a polarized Hodge module on Δ. The general properties of the V-filtration imply that $\mathcal{M} = \mathscr{D}_\Delta \cdot V^{-1}\mathcal{M}$, which means concretely that

$$\mathcal{M} = \sum_{j=0}^\infty \partial_t^j \cdot V^{-1}\mathcal{M}.$$

Let us briefly recall the argument. As long as $\alpha < -1$, the mapping

$$\partial_t\colon \mathrm{gr}_V^{\alpha+1}\mathcal{M} \to \mathrm{gr}_V^\alpha \mathcal{M}$$

is an isomorphism; this gives $V^\alpha\mathcal{M} = V^{>\alpha}\mathcal{M} + \partial_t \cdot V^{\alpha+1}\mathcal{M}$. We can iterate this by gradually increasing the value of α, until we get to

$$V^\alpha\mathcal{M} = V^{-1}\mathcal{M} + \partial_t \cdot V^{\alpha+1}\mathcal{M}.$$

From this, it is easy to deduce that

$$V^\alpha\mathcal{M} \subseteq \sum_{j=0}^\infty \partial_t^j \cdot V^{-1}\mathcal{M}$$

for any $\alpha < -1$. Since the V-filtration is exhaustive, this gives the desired result.

In fact, the same thing is true for the filtration $F_\bullet\mathcal{M}$, because of the condition that $F_\bullet\mathcal{M}$ respects the local V-filtrations. As before, we set $F_k V^\alpha\mathcal{M} = F_k\mathcal{M} \cap V^\alpha\mathcal{M}$. In the above argument,

$$\partial_t \colon F_{k-1}\operatorname{gr}_V^{\alpha+1}\mathcal{M} \to F_k \operatorname{gr}_V^\alpha \mathcal{M}$$

is an isomorphism for $\alpha < -1$, and as before, this leads to

$$F_k V^\alpha\mathcal{M} = F_k V^{-1}\mathcal{M} + \partial_t \cdot F_{k-1}V^{\alpha+1}\mathcal{M}.$$

Since the V-filtration is exhaustive, one deduces that

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{k-j}V^{-1}\mathcal{M},$$

which describes the entire filtration $F_\bullet\mathcal{M}$ in terms of the filtration $F_\bullet V^{-1}\mathcal{M}$ on the coherent $\mathscr{O}_\Delta$-module $V^{-1}\mathcal{M}$. (By the noetherian property of coherent sheaves, we have $F_k V^{-1}\mathcal{M} = V^{-1}\mathcal{M}$ for $k \gg 0$; this shows again that the first so many steps in the filtration $F_\bullet\mathcal{M}$ determine the whole thing.)

In the example from last week where $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1} \subseteq \tilde{\mathscr{V}}$, the $\mathscr{D}$-module was generated by $V^{>-1}\mathcal{M} = \tilde{\mathscr{V}}^{>-1}$ (by definition), and the filtration $F_\bullet\mathcal{M}$ was given by the better formula

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{k-j}V^{>-1}\mathcal{M},$$

This gives a necessary condition for a polarized Hodge module on Δ to come from a variation of Hodge structure on Δ^*. This condition can be formulated more nicely as follows.

Definition 21.4. Let X be a Riemann surface, and let $\mathcal{M}$ be a polarized Hodge module on X. We say that $\mathcal{M}$ has *strict support* X if $\mathcal{M}$ does not have any nontrivial subobject or quotient object whose support is a point.

Let us see how to express this condition in terms of the local V-filtration. After restricting to a neighborhood of a given point, we can assume that $\mathcal{M}$ is a polarized Hodge module on Δ, with V-filtration $V^\bullet\mathcal{M}$. If $\mathcal{M}$ has a nontrivial submodule supported on the origin, then we can find a local section $m \in \mathcal{M}$ such that $tm = 0$ but $m \neq 0$. Since $t\colon \operatorname{gr}_V^\alpha \mathcal{M} \to \operatorname{gr}_V^{\alpha+1}\mathcal{M}$ is an isomorphism except when $\alpha = -1$, we get $m \in V^{-1}\mathcal{M}$; and since $t\colon V^\alpha\mathcal{M} \to V^{\alpha+1}\mathcal{M}$ is an isomorphism for $\alpha > -1$, we must have $m \notin V^{>-1}\mathcal{M}$. This means that the image of m in $\operatorname{gr}_V^{-1}\mathcal{M}$ is a nonzero element in the kernel of

$$t\colon \operatorname{gr}_V^{-1}\mathcal{M} \to \operatorname{gr}_V^0 \mathcal{M}.$$

Therefore injectivity of this mapping implies that $\mathcal{M}$ does not have nontrivial subobjects supported on the origin; in fact, the two conditions are equivalent. By a similar argument, a nontrivial quotient object supported on the origin gives a nontrivial element in the cokernel of

$$\partial_t\colon \operatorname{gr}_V^0\mathcal{M} \to \operatorname{gr}_V^{-1}\mathcal{M},$$

and so surjectivity of this mapping implies (and is actually equivalent to) that there are no such quotients. We can summarize this as follows.

Lemma 21.5. *A polarized Hodge module $\mathcal{M}$ on a Riemann surface X has strict support X iff at any point $x \in X$, the mapping $t\colon \operatorname{gr}_V^{-1} \to \operatorname{gr}_V^0$ is injective and the mapping $\partial_t\colon \operatorname{gr}_V^0 \to \operatorname{gr}_V^{-1}$ is surjective.*

Since $\partial_t \colon \operatorname{gr}_V^0 \mathcal{M} \to \operatorname{gr}_V^{-1} \mathcal{M}$ is surjective, our earlier argument proves that

$$\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot V^{>-1}\mathcal{M},$$

and so $\mathcal{M}$ is generated as a $\mathscr{D}_\Delta$-module by $V^{>-1}\mathcal{M}$. We already know that outside the origin, $\mathcal{M}$ is a vector bundle with connection. Let us denote this vector bundle by $\mathscr{V}$, and let $\tilde{\mathscr{V}}^\alpha$ be the canonical extension.

Lemma 21.6. *For $\alpha > -1$, we have $V^\alpha \mathcal{M} = \tilde{\mathscr{V}}^\alpha$.*

Proof. The injectivity of $t \colon \operatorname{gr}_V^{-1} \mathcal{M} \to \operatorname{gr}_V^0 \mathcal{M}$ implies that $t \colon \mathcal{M} \to \mathcal{M}$ is injective; therefore each $V^\alpha\mathcal{M}$ is a torsion-free $\mathscr{O}_\Delta$-module, hence locally free. The action by $t\partial_t$ defines a logarithmic connection

$$\nabla \colon V^\alpha \mathcal{M} \to \Omega^1_\Delta(\log 0) \otimes_{\mathscr{O}_\Delta} V^\alpha \mathcal{M}$$

on this bundle, and for $\alpha > -1$, we have

$$V^\alpha \mathcal{M}/tV^\alpha\mathcal{M} = V^\alpha\mathcal{M}/V^{\alpha+1}\mathcal{M}.$$

Therefore the residue $\operatorname{Res}_0 \nabla$, which acts as multiplication by $t\partial_t$, has eigenvalues in the interval $[\alpha, \alpha+1)$, and since the conditions uniquely characterize the canonical extension, we get $V^\alpha\mathcal{M} = \tilde{\mathscr{V}}^\alpha$. □

What about the filtration? If we knew that

$$\partial_t \colon F_k \operatorname{gr}_V^0 \mathcal{M} \to F_{k+1} \operatorname{gr}_V^{-1} \mathcal{M}$$

was surjective for every $k \in \mathbb{Z}$, the same reasoning as before would show that

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{k-j} V^{>-1}\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{k-j} \tilde{\mathscr{V}}^{>-1},$$

as in the construction from last week. The problem is that this surjectivity is not part of the definition of a polarized Hodge module. Fortunately, the result is still true, by virtue of Lemma 21.3 (in the special case where ∂_t is surjective).

So far, we know that $\mathcal{M}$ restricts to a polarized variation of Hodge structure $\mathscr{V}$ on the punctured disk, and that both $\mathcal{M}$ and $F_\bullet\mathcal{M}$ are obtained from $\mathscr{V}$ by the construction from last week. One can show moreover that the pairing $h_\mathcal{M}$ is determined by the polarization on $\mathscr{V}$ in the same way, and so our polarized Hodge module with strict support Δ is actually the polarized Hodge module associated to $\mathscr{V}$ by the construction from last week. This is the essential step in the proof of the following theorem.

Theorem 21.7 (Saito)**.** *Let X be a Riemann surface.*

(a) *If $Z \subseteq X$ is a discrete subset, then a polarized variation of Hodge structure of weight n on $X \setminus Z$ extends uniquely to a polarized Hodge module of weight $n+1$ with strict support X.*
(b) *Every polarized Hodge module of weight $n+1$ with strict support X arises in this way.*