Lecture 20: November 11

Polarized Hodge modules. Today, we are going to look at the definition of polarized Hodge modules (on curves). Let X be a Riemann surface, for example a smooth algebraic curve. Last week, we constructed two kinds of examples: Hodge modules on X, coming from a polarized variation of Hodge structure on the complement of a discrete set of points; and Hodge modules on X with support at a point $x \in X$, coming from a polarized Hodge structure. In general, we want to allow only direct sums of these two kinds of examples: an arbitrary polarized Hodge module on X should be the direct sum of an object that is generically a polarized variation of Hodge structure, and objects that are polarized Hodge structures supported at points. To have a good theory, we need a way to define this class of objects by local conditions, so that we can check whether a given object is a polarized Hodge module or not.

By analogy with polarized variations of Hodge structure, our objects will be coherent $\mathscr{D}$-modules with a filtration and a pairing. The Hodge theory conditions will be implemented with the help of an auxilliary filtration called the V-filtration.

Let X be a Riemann surface, and denote by $\mathscr{D}_X$ the sheaf of linear partial differential operators on with holomorphic coefficients; if t is a local coordinate on X, then this is just $\mathscr{O}_X\langle\partial_t\rangle$, where $\partial_t = \frac{\partial}{\partial t}$. Also denote by Db_X the sheaf of distributions on X; both $\mathscr{D}_X$ and its conjugate $\mathscr{D}_{\bar{X}}$ act on Db_X from the left, and the two actions commute.

A polarized Hodge module is going to consist of a left $\mathscr{D}_X$-module $\mathcal{M}$, an increasing filtration $F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_X$-modules, and a distribution-valued pairing $h_{\mathcal{M}} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathrm{Db}_X$. Let me first spell out what this means. First, we want $\mathcal{M}$ to be a coherent left $\mathscr{D}_X$-module; this is the same thing as an $\mathscr{O}_X$-module with a left action by the tangent sheaf $\mathscr{T}_X$, subject to the commutator relation $[\xi, f] = \xi(f)$ for any two local sections $\xi \in \mathscr{T}_X$ and $f \in \mathscr{O}_X$. Next, we want each subsheaf $F_k\mathcal{M}$ to be a coherent $\mathscr{O}_X$-module, and we want the filtration to be compatible with the action by $\mathscr{D}_X$, in the sense that

$$\mathscr{T}_X \cdot F_k\mathcal{M} \subseteq F_{k+1}\mathcal{M},$$

with equality for $k \gg 0$ locally on X. This means that the filtration is locally determined by a finite amount of data. We also require that

$$\mathcal{M} = \bigcup_{k\in\mathbb{Z}} F_k\mathcal{M}$$

and that $F_k\mathcal{M} = 0$ for $k \ll 0$ locally on X. (When X is compact, this is the same thing as saying that $F_k\mathcal{M} = 0$ for $k \ll 0$.) Lastly, the pairing

$$h_{\mathcal{M}} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathrm{Db}_X$$

should be sesquilinear over $\mathscr{D}_X$, meaning that

$$h_{\mathcal{M}}(\xi m', m'') = \xi \cdot h_{\mathcal{M}}(m', m'') \quad \text{and} \quad h_{\mathcal{M}}(m', \xi m'') = \overline{\xi} \cdot h_{\mathcal{M}}(m', m'')$$

for local sections $m', m'' \in \mathcal{M}$ and $\xi \in \mathscr{T}_X$. On the right-hand side, ξ and $\overline{\xi}$ are considered as differential operators, acting on the distribution $h_{\mathcal{M}}(m', m'')$.

Now we need to single out the polarized Hodge modules among all the triples $(\mathcal{M}, F_\bullet\mathcal{M}, h_{\mathcal{M}})$. As in the examples from last week, this can be done with the help of an auxilliary local filtration, called the *V-filtration.* The first condition is the definition is the existence of such a filtration. We require that every point $x \in X$ has an open neighborhood $U \cong \Delta$, with coordinate t, such that $\mathcal{M}_U = \mathcal{M}\big|_U$ has a decreasing filtration $V^\bullet\mathcal{M}_U$, indexed by $\mathbb{R}$, with the following properties:

(1) Each $V^\alpha = V^\alpha \mathcal{M}_U$ is a coherent $\mathscr{O}_U$-module, and
$$\bigcap_{\alpha \in \mathbb{R}} V^\alpha = 0 \quad \text{and} \quad \bigcup_{\alpha \in \mathbb{R}} V^\alpha = \mathcal{M}_U.$$
(2) The filtration is decreasing and discrete, meaning that $V^\alpha \subseteq V^\beta$ for $\alpha \geq \beta$, and that there is some $\varepsilon > 0$ with the property that
$$V^{>\alpha} \underset{\text{def}}{=} \bigcup_{\beta > \alpha} V^\beta = V^{\alpha+\varepsilon}.$$
The jumps in the filtration therefore happen along a discrete subset of $\mathbb{R}$.
(3) One has $t \cdot V^\alpha \subseteq V^{\alpha+1}$ and $\partial_t \cdot V^\alpha \subseteq V^{\alpha-1}$, and $t \cdot V^\alpha = V^{\alpha+1}$ for $\alpha \gg 0$. In particular, each subsheaf V^α is preserved by $t\, \partial_t$.
(4) The operator $t\partial_t - \alpha$ acts nilpotently on $\operatorname{gr}_V^\alpha = V^\alpha / V^{>\alpha}$, which is a finite-dimensional $\mathbb{C}$-vector space.

It is not hard to see that there can be at most one filtration with these properties. If such a filtration $V^\bullet \mathcal{M}_U$ does exist on a neighborhood U of a given point $x \in X$, we say that $\mathcal{M}$ *admits a local V-filtration* at the point $x \in X$.

Exercise 20.1. Show that $t \colon V^\alpha \to V^{\alpha+1}$ is an isomorphism for $\alpha > -1$, and that $\partial_t \colon \operatorname{gr}_V^\alpha \to \operatorname{gr}_V^{\alpha-1}$ is an isomorphism for $\alpha \neq 0$. Prove that $\mathcal{M}_U = \mathscr{D}_U \cdot V^{-1}$.

For those of you who know something about $\mathscr{D}$-modules, I should mention that a V-filtration in the above sense exists if and only if $\mathcal{M}$ is regular holonomic and if the eigenvalues of the local monodromy transformation around any singular point are complex numbers of absolute value 1. In particular, $\mathcal{M}$ is generically a vector bundle with connection. Let us briefly discuss this important point.

Lemma 20.1. *Let $\mathcal{M}$ be a coherent $\mathscr{D}_X$-module. If $\mathcal{M}$ admits a V-filtration with the above properties at every point of X, then $\mathcal{M}$ is generically a vector bundle with connection.*

Proof. The question is local, and so we can work on an open neighborhood U of a given point $x \in X$. After replacing $\mathcal{M}$ by its restriction to U, we can assume that $\mathcal{M}$ is a $\mathscr{D}$-module on the disk Δ, and that a filtration $V^\bullet \mathcal{M}$ with the above properties exists. Now I claim that
$$\mathcal{M}\big|_{\Delta^*} = V^\alpha \mathcal{M}\big|_{\Delta^*}$$
for every $\alpha \in \mathbb{R}$. To see this, let $m \in \mathcal{M}$ be any local section defined on an open subset of Δ^*. Then $m \in V^\beta \mathcal{M}$ for some $\beta \ll 0$, and as long as $k \geq \alpha - \beta$, we have $t^k m \in V^\alpha \mathcal{M}$. But now t^{-1} is holomorphic on Δ^*, and therefore preserves the subsheaf $V^\alpha \mathcal{M}\big|_{\Delta^*}$, and so we get $m = t^{-k}(t^k m) \in V^\alpha \mathcal{M}$.

Since $V^\alpha \mathcal{M}$ is a coherent $\mathscr{O}_\Delta$-module, it follows that $\mathscr{F} = \mathcal{M}\big|_{\Delta^*}$ is coherent over $\mathscr{O}_{\Delta^*}$. The action by ∂_t can be viewed as a connection $\nabla \colon \mathscr{F} \to \Omega^1_{\Delta^*} \otimes_{\mathscr{O}_{\Delta^*}} \mathscr{F}$ that satisfies the Leibniz rule. Now Lemma 4.3 implies that $\mathscr{F}$ is locally free. □

It particular, the set of points where $\mathcal{M}$ is singular, meaning not a vector bundle with connection, is discrete. This also implies that $\mathcal{M}$ is holonomic: its characteristic variety in the cotangent bundle T^*X is 1-dimensional, because it is contained in the union of the zero section and the contangent spaces at the singular points of $\mathcal{M}$. Each singular point of $\mathcal{M}$ is a regular singularity, because $\mathcal{M}_U$ is generated by the coherent $\mathscr{O}_U$-module $V^{-1}\mathcal{M}$ which is preserved by the operator $t\partial_t$. The converse is a theorem by Masaki Kashiwara.

Example 20.2. Suppose that $\mathcal{M}_U$ is a vector bundle with connection. In that case, the V-filtration is just the filtration by powers of t, with $V^k \mathcal{M}_U = t^k \mathcal{M}_U$ for all

$k \in \mathbb{N}$. The formula for arbitrary $\alpha \in \mathbb{R}$ would be

$$V^\alpha \mathcal{M}_U = t^{\max(\lceil \alpha \rceil, 0)} \mathcal{M}_U.$$

You should check that this satisfies all the conditions in the definition. Note that

$$\operatorname{gr}_V^0 \mathcal{M}_U = V^0 \mathcal{M}_U / V^{>0} \mathcal{M}_U = \mathcal{M}_U / t\mathcal{M}_U$$

is just the fiber of the $\mathcal{M}_U$ at the origin.

The next set of conditions has to do with how the filtration $F_\bullet \mathcal{M}_U = F_\bullet \mathcal{M}\big|_U$ on the $\mathscr{D}$-module interacts with the V-filtration. This is needed because we want $F_\bullet \mathcal{M}$ to reflect some of the properties of the $\mathscr{D}$-module. For every $\alpha \in \mathbb{R}$, we get an induced filtration

$$F_\bullet V^\alpha = F_\bullet \mathcal{M}_U \cap V^\alpha \mathcal{M}_U$$

on the coherent $\mathscr{O}_U$-module $V^\alpha = V^\alpha \mathcal{M}_U$, as well as an induced filtration

$$F_\bullet \operatorname{gr}_V^\alpha = \frac{F_\bullet \mathcal{M}_U \cap V^\alpha \mathcal{M}_U + V^{>\alpha} \mathcal{M}_U}{V^{>\alpha} \mathcal{M}_U}$$

on the finite-dimensional vector space $\operatorname{gr}_V^\alpha = V^\alpha / V^{>\alpha}$. We will say that the filtration $F_\bullet \mathcal{M}$ *respects the local V-filtrations* if

$$t \colon F_k V^\alpha \to F_k V^{\alpha+1}$$

is an isomorphism for $\alpha > -1$ and $k \in \mathbb{Z}$, and if

$$\partial_t \colon F_k \operatorname{gr}_V^\alpha \to F_{k+1} \operatorname{gr}_V^{\alpha-1}$$

is an isomorphism for $\alpha < 0$ and $k \in \mathbb{Z}$. Both mappings are isomorphisms without the filtration; the condition is that they are actually *filtered* isomorphisms.

Example 20.3. Here is one example how these conditions force the filtration $F_\bullet \mathcal{M}$ to respect the properties $\mathcal{M}$. Suppose that $\mathcal{M}$ is a vector bundle with connection. I claim that each $F_k \mathcal{M}$ is then necessarily a subbundle of $\mathcal{M}$. Clearly, $F_k \mathcal{M} \subseteq \mathcal{M}$ is torsion-free, hence locally free (because $\dim X = 1$); the point is to show that the subquotients $\operatorname{gr}_k^F \mathcal{M} = F_k \mathcal{M} / F_{k-1} \mathcal{M}$ are also locally free. For that, it suffices to show that the stalk at each point $x \in X$ is torsion-free.

Fix a point $x \in X$. After replacing $\mathcal{M}$ by its restriction to some open neighborhood U, we can assume that $\mathcal{M}$ is a vector bundle with connection on Δ. Recall that $V^j \mathcal{M} = t^j \mathcal{M}$ for every $j \in \mathbb{N}$. The condition that $F_\bullet \mathcal{M}$ respects the local V-filtrations therefore amounts to the identity

$$F_k \mathcal{M} \cap t\mathcal{M} = t(F_k \mathcal{M}),$$

as subsheaves of $\mathcal{M}$. We can use this to show that $\operatorname{gr}_k^F \mathcal{M}$ is torsion-free. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{k-1}\mathcal{M} & \longrightarrow & F_k\mathcal{M} & \longrightarrow & \operatorname{gr}_k^F \mathcal{M} & \longrightarrow & 0 \\
 & & \big\downarrow t & & \big\downarrow t & & \big\downarrow t & & \\
0 & \longrightarrow & F_{k-1}\mathcal{M} & \longrightarrow & F_k\mathcal{M} & \longrightarrow & \operatorname{gr}_k^F \mathcal{M} & \longrightarrow & 0.
\end{array}$$

Since $F_{k-1}\mathcal{M} \cap t(F_k \mathcal{M}) = t(F_{k-1}\mathcal{M})$, a simple diagram chase shows that the vertical arrow $t \colon \operatorname{gr}_k^F \mathcal{M} \to \operatorname{gr}_k^F \mathcal{M}$ is injective. But this says exactly that the stalk of $\operatorname{gr}_k^F \mathcal{M}$ at the origin is torsion-free, proving the claim.

Exercise 20.2. Suppose that $\mathcal{M}$ is the $\mathscr{D}_\Delta$-module $H \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$, where H is a finite-dimensional vector space. Show that if $F_\bullet \mathcal{M}$ respects the local V-filtrations, then

$$F_k \mathcal{M} = \sum_{j=0}^{\infty} F_{k-1-j} H \otimes \partial_t^j$$

for a unique filtration $F_\bullet H$.

Now let us discuss what additional structure the vector space $\operatorname{gr}_V^\alpha$ has. From the fact that $t\partial_t - \alpha$ is nilpotent on $\operatorname{gr}_V^\alpha$, it follows that

$$t\colon \operatorname{gr}_V^{\alpha-1} \to \operatorname{gr}_V^\alpha \quad \text{and} \quad \partial_t\colon \operatorname{gr}_V^\alpha \to \operatorname{gr}_V^{\alpha-1}$$

are isomorphisms as long as $\alpha \neq 0$. Therefore we only need to consider $\operatorname{gr}_V^\alpha$ in the range $-1 \leq \alpha \leq 0$. The nilpotent operator $N = t\partial_t - \alpha$ has a monodromy weight filtration, which we denote by the symbol $W_\bullet \operatorname{gr}_V^\alpha$. From the distribution-valued sesquilinear pairing $h_\mathcal{M}$, one can also construct an induced hermitian pairing

$$h_\alpha\colon \operatorname{gr}_V^\alpha \otimes_\mathbb{C} \overline{\operatorname{gr}_V^\alpha} \to \mathbb{C};$$

this takes some work, and so we are going to leave this point to Wednesday. Right now, we are only going to consider one special case.

Example 20.4. Suppose that $\mathcal{M}$ is a vector bundle with connection. I claim that any sesquilinear pairing $h_\mathcal{M}\colon \mathcal{M} \otimes_\mathbb{C} \overline{\mathcal{M}} \to \mathrm{Db}_X$ actually takes values in the sheaf of smooth functions $\mathscr{C}_X^\infty$. This is a local problem, and so we may assume that $\mathcal{M}$ is a vector bundle with connection on Δ. If r denotes the rank of the bundle, we can then find a trivialization by flat sections $m_1, \dots, m_r$; in terms of the $\mathscr{D}$-module structure, this means that $\partial_t m_j = 0$. The sesquilinearity of the pairing gives

$$\partial_t h_\mathcal{M}(m_j, m_k) = h_\mathcal{M}(\partial_t m_j, m_k) = 0,$$
$$\overline{\partial}_t h_\mathcal{M}(m_j, m_k) = h_\mathcal{M}(m_j, \partial_t m_k) = 0.$$

It follows that $h_\mathcal{M}(m_j, m_k)$ is (the distribution defined by) a constant function, and in particular smooth. From this, it is easy to deduce that $h_\mathcal{M}$ takes values in the sheaf of smooth functions.

In all the examples from last week,

$$\bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^W \operatorname{gr}_V^\alpha$$

was always a polarized Hodge-Lefschetz structure of some weight, and so we make this behavior part of the definition of a polarized Hodge module.

Definition 20.5. Let X be a Riemann surface, and let $(\mathcal{M}, F_\bullet\mathcal{M}, h_\mathcal{M})$ be a filtered $\mathscr{D}_X$-module with a sesquilinear pairing. This is called a *polarized Hodge module of weight w* if the following four conditions are satisfied:

(a) The $\mathscr{D}$-module $\mathcal{M}$ admits a local V-filtration at every point $x \in X$.
(b) The filtration $F_\bullet\mathcal{M}$ respects the local V-filtrations.
(c) For each $\alpha \in (-1, 0]$, the graded vector space

$$\bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^W \operatorname{gr}_V^\alpha \mathcal{M}_U,$$

with the filtration induced by $F_\bullet \operatorname{gr}_V^\alpha$, the pairing induced by h_α, and the $\mathfrak{sl}_2(\mathbb{C})$-action induced by $N = t\partial_t - \alpha$, is a polarized Hodge-Lefschetz structure of central weight $w - 1$.
(d) For $\alpha = -1$, the graded vector space

$$\bigoplus_{\ell \in \mathbb{Z}} \operatorname{gr}_\ell^W \operatorname{gr}_V^\alpha \mathcal{M}_U,$$

with the filtration induced by $F_{\bullet+1} \operatorname{gr}_V^\alpha$, the pairing induced by h_α, and the $\mathfrak{sl}_2(\mathbb{C})$-action induced by $N = t\partial_t - \alpha$, is a polarized Hodge-Lefschetz structure of central weight w.

The reason for separating $\alpha = -1$ is to make the same definition work for all the different examples from last week. Let us go through the examples one by one, to see how this works in practice.

Example 20.6. Suppose that $\mathcal{M}$ is a vector bundle with connection. We have already shown that each $F_k\mathcal{M}$ is a subbundle, and that the pairing $h_{\mathcal{M}}$ is actually a flat pairing with values in the sheaf $\mathscr{C}_X^\infty$. Moreover, the local V-filtration at a point $x \in X$ is the filtration by powers of t, and so $\operatorname{gr}_V^\alpha$ is nonzero only for $\alpha \in \mathbb{N}$, and gr_V^0 is the fiber of the vector bundle at the point x, with $N = t\partial_t$ acting trivially. It is easy to see that each subspace $F_k \operatorname{gr}_V^0$ is just the fiber of the subbundle $F_k\mathcal{M}$ at x, and so the condition in (c) is saying that we get a polarized Hodge structure of weight $w - 1$. So in this case, our polarized Hodge module of weight w is just a polarized variation of Hodge structure of weight $w - 1$.

Example 20.7. Suppose that H is a polarized Hodge structure of weight n, with Hodge filtration $F_\bullet H = F^{-\bullet}H$. Last time, we defined $\mathcal{M} = H \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$, which is a $\mathscr{D}$-module supported on the origin in Δ, with filtration

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} F^{k-1-j}H \otimes \partial_t^j.$$

Let us see that this is a polarized Hodge module of weight n. We have $\operatorname{gr}_V^{-1}\mathcal{M} \cong H$, with $N = t\partial_t + 1$ acting trivially; moreover, $\operatorname{gr}_V^\alpha \mathcal{M} = 0$ for $-1 < \alpha \leq 0$. Since the two shifts cancel each other out, the filtration $F_{\bullet+1} \operatorname{gr}_V^{-1}$ is exactly $F_\bullet H$, and so the condition in (d) is saying that H should be a polarized Hodge structure of weight n, which is true.

Example 20.8. Let $\mathscr{V}$ be a polarized variation of Hodge structure of weight n on the punctured disk Δ^*. Last week, we defined $\mathcal{M} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{>-1}$, with filtration

$$F_k\mathcal{M} = \sum_{j=0}^{\infty} \partial_t^j \cdot F_{k-j}\tilde{\mathscr{V}}^{>-1}.$$

This is a polarized Hodge module on Δ of weight $n + 1$. Indeed, we showed last week that the condition in (c) holds as a consequence of Schmid's theorems, and that the condition in (d) follows from the vanishing cycle lemma.