Lecture 15: October 23

Recall the following definition from last time. A polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight n is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on a finite-dimensional vector space V, together with a compatible hermitian pairing $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ and a filtration F, subject to three conditions:

1. For every $p \in \mathbb{Z}$, one has $H(F^p) \subseteq F^p$.
2. For every $p \in \mathbb{Z}$, one has $Y(F^p) \subseteq F^{p-1}$.
3. The filtration $e^{-\frac{1}{2}Y} F$ is the Hodge filtration of a Hodge structure of weight n, polarized by h.

We showed last time that the subspace $V^{\mathfrak{sl}_2(\mathbb{C})}$ of $\mathfrak{sl}_2(\mathbb{C})$-invariants has a polarized Hodge structure of weight n, whose Hodge filtration is induced by F (or, equivalently, $e^{-\frac{1}{2}Y} F$). This has many useful consequences. For example, we can show that the $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure on the irreducible representation S_ℓ that we constructed last time is essentially unique.

Corollary 15.1. *Suppose we have a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure on S_ℓ of weight ℓ. Then the filtration F agrees with the filtration constructed in Example 14.7 up to a shift, and the pairing h agrees with the pairing constructed there up to rescaling by a nonzero real number.*

Proof. On the vector space $\operatorname{Hom}_{\mathbb{C}}(S_\ell, S_\ell)$, we get a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight $\ell - \ell = 0$ by using the given $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure constructed on the first argument, and the one constructed in Example 14.7 on the second argument. If we denote by F_0 the filtration constructed there, we have as usual

$$F^k \operatorname{Hom}_{\mathbb{C}}(S_\ell, S_\ell) = \left\{ A\colon S_\ell \to S_\ell \;\middle|\; AF^p \subseteq F_0^{p+k} \text{ for every } p \in \mathbb{Z} \right\}.$$

As S_ℓ is irreducible, Schur's lemma gives

$$\operatorname{Hom}_{\mathbb{C}}(S_\ell, S_\ell)^{\mathfrak{sl}_2(\mathbb{C})} = \mathbb{C} \cdot \mathrm{id},$$

and according to Proposition 14.8, this one-dimensional subspace has a Hodge structure of weight 0. For dimension reasons, id must therefore be of Hodge type $(k, -k)$ for some integer k; but this says exactly that $F^p = F_0^{p+k}$ for every $p \in \mathbb{Z}$. So the two filtrations are the same up to a shift by k steps.

Now let us consider the pairing. Since the given pairing h is compatible with the action by $\mathfrak{sl}_2(\mathbb{C})$, all its values are determined by $h(e_0, e_\ell)$, which is necessarily real. Here $e_0, e_1, \dots, e_\ell \in S_\ell$ is the basis constructed in Example 14.7. So the two pairings are the same up to multiplication by the real number $h(e_0, e_\ell)$. In fact, we can be more precise about the sign. Namely, we have $e_0 \in F_0^\ell = F^{\ell-k}$, which means that $e^{-\frac{1}{2}Y} e_0$ has Hodge type $(\ell - k, k)$ for the given $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure. Since h is a polarization, this gives

$$(-1)^{\ell-k} h(e^{-\frac{1}{2}Y} e_0, e^{-\frac{1}{2}Y} e_0) > 0.$$

After simplifying the expression, we arrive at $(-1)^k h(e_0, e_\ell) > 0$. ∎

Proof of Theorem 14.1. We can now prove Theorem 14.1. Let V be any polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight n. Our starting point is the decomposition

$$V \cong \bigoplus_{\ell \in \mathbb{N}} S_\ell \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, V).$$

Fix some $\ell \geq 0$. From the polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures on S_ℓ and V, the space $\operatorname{Hom}_{\mathbb{C}}(S_\ell, V)$ inherits a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight $n - \ell$. As usual, the filtration is given by the formula

$$F^k \operatorname{Hom}_{\mathbb{C}}(S_\ell, V) = \left\{ f\colon S_\ell \to V \;\middle|\; f(F^p S_\ell) \subseteq F^{p+k} V \text{ for all } p \in \mathbb{Z} \right\},$$

and the hermitian pairing

$$\operatorname{Hom}_{\mathbb{C}}(S_\ell, V) \otimes_{\mathbb{C}} \overline{\operatorname{Hom}_{\mathbb{C}}(S_\ell, V)} \to \mathbb{C}, \quad (f,g) \mapsto \frac{1}{\ell+1} \operatorname{tr}(g^* \circ f)$$

is a polarization. Here $g^* \colon V \to S_\ell$ is the adjoint of $g \colon S_\ell \to V$ with respect to the pairings on S_ℓ and V; the reason for the factor $\frac{1}{\ell+1}$ will become clear in a moment.

Proposition 14.8 tells us that the subspace

$$W_\ell = \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(S_\ell, V) = \operatorname{Hom}_{\mathbb{C}}(S_\ell, V)^{\mathfrak{sl}_2(\mathbb{C})}$$

has a Hodge structure of weight $n-\ell$, with Hodge filtration

$$F^k W_\ell = \left\{ f \in W_\ell \;\middle|\; f(F^p S_\ell) \subseteq F^{p+k} V \text{ for every } p \in \mathbb{Z} \right\},$$

and polarized by the restriction of the pairing $\frac{1}{\ell+1} \operatorname{tr}(g^* \circ f)$. But for $f, g \in W_\ell$, the composition $g^* \circ f$ is an endomorphism of S_ℓ as an $\mathfrak{sl}_2(\mathbb{C})$-representation, hence (by Schur's lemma) a multiple of the identity. Thus $g^* \circ f = c(f,g) \operatorname{id}$ for some constant $c(f,g) \in \mathbb{C}$, and because of the factor $\frac{1}{\ell+1}$, the trace of this operator equals $c(f,g)$.

Lemma 15.2. *With the above Hodge structures on W_ℓ, the evaluation mapping*

$$\bigoplus_{\ell \in \mathbb{N}} S_\ell \otimes_{\mathbb{C}} W_\ell \to V$$

is an isomorphism of polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures of weight n.

Proof. We know that the mapping is an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-representations. Let us first show that this isomorphism is compatible with the hermitian pairings on both sides. Given $x, y \in S_\ell$ and $f, g \in W_\ell$, the pairing between $x \otimes f$ and $y \otimes g$ is

$$\begin{aligned} h_{S_\ell}(x,y) \cdot \frac{1}{\ell+1} \operatorname{tr}(g^* \circ f) &= h_{S_\ell}(x,y) \cdot c(f,g) = h_{S_\ell}\bigl(c(f,g)x, y\bigr) \\ &= h_{S_\ell}\bigl(g^* f(x), y\bigr) = h_V\bigl(f(x), g(y)\bigr). \end{aligned}$$

Here we used the fact that $g^* \circ f = c(f,g) \operatorname{id}$. Since the different isotypical components are orthogonal with respect to h_V, this is enough to conclude that the isomorphism respects the pairings.

Now we only have to prove that the mapping is an isomorphism of Hodge structures of weight n. Let $p \in \mathbb{Z}$ be an integer. As with any tensor product, the $(p, n-p)$-subspace in the Hodge decomposition of the left-hand side is

$$\bigoplus_{\ell \in \mathbb{N}} \bigoplus_{k \in \mathbb{Z}} S_\ell^{k, \ell-k} \otimes_{\mathbb{C}} W_\ell^{p-k, n-\ell-p+k},$$

because the Hodge structure on S_ℓ has weight ℓ, and the Hodge structure on W_ℓ has weight $n - \ell$. But

$$W_\ell^{p-k, n-\ell-p+k} = \left\{ f \colon S_\ell \to V \;\middle|\; f(S_\ell^{j, \ell-j}) \subseteq V^{j+p-k, n-j-p+k} \text{ for all } j \in \mathbb{Z} \right\},$$

and so the evaluation mapping takes $S_\ell^{k,\ell-k} \otimes_{\mathbb{C}} W_\ell^{p-k, n-\ell-p+k}$ into $V^{p,n-p}$, and is therefore a morphism of Hodge structures of weight n. □

We already know from Lecture 14 that each S_ℓ is actually a polarized Hodge-Lefschetz structure of weight ℓ. Since $\mathfrak{sl}_2(\mathbb{C})$ acts trivially on the Hodge structures W_ℓ, it follows that

$$\bigoplus_{\ell \in \mathbb{N}} S_\ell \otimes_{\mathbb{C}} W_\ell$$

is a polarized Hodge-Lefschetz structure of weight $\ell + (n - \ell) = n$. Because of the lemma, the same thing is then true for V. The last assertion in Theorem 14.1 is left as an exercise.

Exercise 15.1. Let $S \in \operatorname{End}(V)$ be an endomorphism of the $\mathfrak{sl}_2(\mathbb{C})$-representation that is compatible with the pairing h and satisfies $S(F^pV) \subseteq F^pV$ for all $p \in \mathbb{Z}$. Prove that S is automatically an endomorphism of the Hodge-Lefschetz structure on V.

General facts about $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures. In this section, we prove two small results about $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures that were needed above. Suppose that V and W are polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures of the same weight n, with polarizations h_V and h_W; for the sake of clarity, we denote the two filtrations by $F^\bullet V$ and $F^\bullet W$.

Definition 15.3. A linear mapping $f\colon V \to W$ is a *morphism* of $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures of weight n if f is a morphism of $\mathfrak{sl}_2(\mathbb{C})$-representations and also a morphism of Hodge structures of weight n.

It follows from the definition that morphisms of $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures are strictly compatible with the filtrations F_V and F_W.

Lemma 15.4. *If $f\colon V \to W$ is a morphism of $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures, then f is a filtered morphism, meaning that $f(V) \cap F^pW = f(F^pV)$ for all $p \in \mathbb{Z}$.*

Proof. Morphisms of Hodge structures are filtered, and so $f(V) \cap e^{-\frac{1}{2}Y}F^pW = f(e^{-\frac{1}{2}Y}F^pV)$. The claim follows by applying the operator $e^{\frac{1}{2}Y}$ to both sides. □

Now suppose that V and W are polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures of weight n respectively m. Let us describe the induced $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure on $\operatorname{Hom}_{\mathbb{C}}(V,W)$ in more detail. As usual, the filtration is given by

$$F^k \operatorname{Hom}_{\mathbb{C}}(V,W) = \left\{ f\colon V \to W \;\middle|\; f(F^pV) \subseteq F^{p+k}W \text{ for all } p \in \mathbb{Z} \right\}.$$

The induced representation of $\mathfrak{sl}_2(\mathbb{C})$ is easy to describe: for $f\colon V \to W$, one has

$$(Hf)(v) = Hf(v) - f(Hv), \quad (Xf)(v) = Xf(v) - f(Xv), \quad (Yf)(v) = Yf(v) - f(Yv).$$

Observe that $\mathfrak{sl}_2(\mathbb{C})$ acts trivially on a linear mapping $f\colon V \to W$ exactly when f is a morphism of $\mathfrak{sl}_2(\mathbb{C})$-representations; therefore $\operatorname{Hom}_{\mathbb{C}}(V,W)^{\mathfrak{sl}_2(\mathbb{C})} = \operatorname{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(V,W)$.

As in Lecture 6, the induced pairing on $\operatorname{Hom}_{\mathbb{C}}(V,W)$ can again be expressed in terms of the trace. Given a linear mapping $f\colon V \to W$, we denote by $f^*\colon W \to V$ the adjoint with respect to the (nondegenerate) pairings h_V and h_W; to be precise,

$$h_W\bigl(f(v), w\bigr) = h_V\bigl(v, f^*(w)\bigr) \quad \text{for all } v \in V \text{ and } w \in W.$$

On $\operatorname{Hom}_{\mathbb{C}}(V,W)$, we have the hermitian pairing

$$\operatorname{Hom}_{\mathbb{C}}(V,W) \otimes_{\mathbb{C}} \overline{\operatorname{Hom}_{\mathbb{C}}(V,W)} \to \mathbb{C}, \quad (f,g) \mapsto \operatorname{tr}(g^* \circ f). \tag{15.5}$$

Lemma 15.6. *Suppose that V and W are polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structures of weight n respectively m. Then $\operatorname{Hom}_{\mathbb{C}}(V,W)$ is a polarized $\mathfrak{sl}_2(\mathbb{C})$-Hodge structure of weight $m-n$, polarized by hermitian pairing in (15.5).*

Proof. We need to check that the filtration on $\operatorname{Hom}_{\mathbb{C}}(V,W)$ satisfies the three conditions in the definition (from Lecture 14). Suppose that $f \in F^k \operatorname{Hom}_{\mathbb{C}}(V,W)$. For any $v \in F^pV$, we have $f(v) \in F^{p+k}W$, and therefore

$$(Hf)(v) = Hf(v) - f(Hv) \subseteq H(F^{p+k}W) + f(F^pV) \subseteq F^{p+k}W,$$

which proves that $Hf \in F^k \operatorname{Hom}_{\mathbb{C}}(V,W)$. Similarly, $Yf \in F^{k-1} \operatorname{Hom}_{\mathbb{C}}(V,W)$.

It remains to show that the filtration $e^{-\frac{1}{2}Y}F^\bullet \operatorname{Hom}_{\mathbb{C}}(V,W)$ defines a Hodge structure of weight $m-n$ on $\operatorname{Hom}_{\mathbb{C}}(V,W)$, polarized by the pairing in (15.5). Since

$$(e^{-\frac{1}{2}Y}f)(v) = e^{-\frac{1}{2}Y}f(v) - f(e^{\frac{1}{2}Y}v),$$

it is not hard to see that

$$e^{-\frac{1}{2}Y} F^k \operatorname{Hom}_{\mathbb{C}}(V, W)$$
$$= \big\{ f \colon V \to W \;\big|\; f\big(e^{-\frac{1}{2}Y} F^p V\big) \subseteq e^{-\frac{1}{2}Y} F^{p+k} W \text{ for all } p \in \mathbb{Z} \big\};$$

but the right-hand side is obviously the Hodge filtration of the induced Hodge structure on $\operatorname{Hom}_{\mathbb{C}}(V, W)$, which has weight $m - n$. The proof that the pairing in (15.5) polarizes this Hodge structure is similar to the proof of Lemma 6.1. □

The limiting mixed Hodge structure. I already mentioned that Schmid states his results in the language of mixed Hodge structures. You probably know that a mixed Hodge structure over $\mathbb{Q}$ or $\mathbb{R}$ is described by two filtrations: an increasing weight filtration $W_\bullet$, and a decreasing Hodge filtration $F^\bullet$, such that each

$$\operatorname{gr}_\ell^W = W_\ell / W_{\ell-1}$$

has a Hodge structure of weight ℓ, whose Hodge filtration is

$$F^\bullet \operatorname{gr}_\ell^W = (F^\bullet \cap W_\ell + W_{\ell-1}) / W_{\ell-1}.$$

Since the Hodge filtration alone does not determine the Hodge decomposition for arbitrary (complex) Hodge structures, we need three filtrations to describe (complex) mixed Hodge structures.

Definition 15.7. A *mixed Hodge structure* on a finite-dimensional vector space H consists of an increasing filtration $W_\bullet$ with $W_\ell = 0$ for $\ell \ll 0$ and $W_\ell = H$ for $\ell \gg 0$, and two decreasing filtrations $F^\bullet$ and $G^\bullet$, such that each subquotient

$$\operatorname{gr}_\ell^W = W_\ell / W_{\ell-1}$$

has a Hodge structure of weight ℓ, given by the two induced filtrations

$$F^\bullet \operatorname{gr}_\ell^W = (F^\bullet \cap W_\ell + W_{\ell-1}) / W_{\ell-1}$$
$$G^\bullet \operatorname{gr}_\ell^W = (G^\bullet \cap W_\ell + W_{\ell-1}) / W_{\ell-1}.$$

The filtration $W_\bullet$ is called the *weight filtration.*

Note. The (p, q)-subspace in the Hodge decomposition of gr_ℓ^W is

$$F^p \operatorname{gr}_\ell W \cap G^q \operatorname{gr}_\ell^W = \frac{(F^p \cap W_\ell + W_{\ell-1}) \cap (G^q \cap W_\ell + W_{\ell-1})}{W_{\ell-1}}.$$

In order to have a mixed Hodge structure on V, the direct sum of these subspaces (over $p + q = \ell$) must equal gr_ℓ^W, which means concretely that

$$W_\ell = \sum_{p+q=\ell} (F^p \cap W_\ell + W_{\ell-1}) \cap (G^q \cap W_\ell + W_{\ell-1})$$

and that, whenever $p + q = \ell + 1$, one has

$$(F^p \cap W_\ell + W_{\ell-1}) \cap (G^q \cap W_\ell + W_{\ell-1}) = W_{\ell-1}.$$

If you think about it, this is actually a fairly complicated set of conditions.

Example 15.8. An $\mathbb{R}$-mixed Hodge structure is a finite-dimensional $\mathbb{R}$-vector space $H_{\mathbb{R}}$, together with a mixed Hodge structure on $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, such that the weight filtration $W_\bullet$ is defined over $\mathbb{R}$, and $G^\bullet = \overline{F^\bullet}$. In that case, the Hodge structure on each gr_ℓ^W is an $\mathbb{R}$-Hodge structure of weight ℓ.

Example 15.9. A mixed Hodge structure is called *split* if it is a direct sum of Hodge structures of different weights, with the obvious weight filtration. This is equivalent to having a decomposition

$$H = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}$$

with the property that

$$W_\ell = \bigoplus_{i+j\leq \ell} H^{i,j}, \quad F^p = \bigoplus_{i\geq p,j} H^{i,j}, \quad G^q = \bigoplus_{j\geq q,i} H^{i,j}.$$

Let us return to the case of a polarized variation of Hodge structure of weight n on the punctured disk. Recall that we decomposed $R \in \operatorname{End}(V)$ as $R = R_S + R_N$, with R_S semisimple and R_N nilpotent, and that we chose another semisimple element $H \in \operatorname{End}(V)$ that commutes with R_S and satisfies $[H, R_N] = -2R_N$. The monodromy weight filtration $W_\bullet = W_\bullet(R_N)$ is split by the eigenspaces of H, in the sense that

$$W_\ell = E_\ell(H) + W_{\ell-1}$$

for every $\ell \in \mathbb{Z}$. We also constructed the limiting Hodge filtration F_{lim}, by making the filtration $F_{\Psi(0)}$ (from Theorem 9.1) compatible with the operator R_S. We then showed that each eigenspace $E_\ell(H)$ has a polarized Hodge structure of weight $n+\ell$, whose Hodge filtration is induced by F_{lim}. Since

$$E_\ell(H) \cong W_\ell/W_{\ell-1},$$

it follows that gr^W_ℓ has a Hodge structure of weight $n+\ell$, whose Hodge filtration is induced by F_{lim}. It can be shown that the second filtration G_{lim} is given by

$$G^q_{lim} = (F^{n+1-q}_{lim})^\perp,$$

where the orthogonal complement is with respect to the pairing h. The conclusion is that we get a mixed Hodge structure on V with weight filtration $W_{\bullet-n}$ and Hodge filtrations F_{lim} and G_{lim}. Schmid calls this the *limiting mixed Hodge structure*. Moreover, $R_S \in \operatorname{End}(V)$ is an endomorphism of this mixed Hodge structure, in the sense that it preserves all three filtrations. Each eigenspace $E_\alpha(R_S)$ is therefore itself a mixed Hodge structure, with V being the direct sum.

The pairing $h\colon V \otimes_\mathbb{C} \overline{V} \to \mathbb{C}$ induces hermitian pairings

$$h_\ell\colon \operatorname{gr}^W_\ell \otimes_\mathbb{C} \overline{\operatorname{gr}^W_{-\ell}} \to \mathbb{C},$$

and the results in Theorem 10.3 can be summarized by saying that

$$R^\ell_N\colon \operatorname{gr}^W_\ell \to \operatorname{gr}^W_{-\ell}(-\ell)$$

is an isomorphism of Hodge structures, and that for each $\ell \geq 0$, the pairing $(v', v'') \mapsto (-1)^\ell h_\ell(v', R^\ell_N v'')$ polarizes the Hodge structure on the primitive part

$$\ker\Big(R^{\ell+1}_N\colon \operatorname{gr}^W_\ell \to \operatorname{gr}^W_{-\ell-2}\Big).$$

Schmid says that the limiting mixed Hodge structure is "polarized by the pairing h and the nilpotent operator R_N". The advantage of this formulation is that it does not mention the semisimple operator H (which represented an additional choice).