Lecture 11: October 2

Two applications. As a motivation for going through the (rather involved) proof of Theorem 9.1 and Theorem 10.2, let me first give two applications. The first is the following "removable singularities" theorem.

Corollary 11.1. *If a polarized variation of Hodge structure on Δ^* has trivial monodromy, then it extends to a polarized variation of Hodge structure on the entire disk Δ.*

Proof. We use the notation from last time. Since $T = \mathrm{id}$, we can take our interval to be $I = [0, 1)$, and then $R = H = 0$. Now the residue of the logarithmic connection on $\tilde{\mathscr{V}}$ is zero, and so we get an actual connection

$$\nabla \colon \tilde{\mathscr{V}} \to \Omega^1_\Delta \otimes_{\mathscr{O}_\Delta} \tilde{\mathscr{V}}.$$

Since the exponential factors in Theorem 10.2 are all trivial, the mapping Ψ takes values in D, and extends to a holomorphic mapping $\Psi \colon \Delta \to D$. This gives us the desired extension to a polarized variation of Hodge structure on Δ. □

The next application are the famous "Hodge norm estimates". They play an important role in many applications of Hodge theory, for example in the theorem of Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan about the algebraicity of the locus of Hodge classes, or in Steven Zucker's results about L^2-cohomology with coefficients in a polarized variation of Hodge structure. (Time permitting, we are going to discuss both of these topics later in the semester.)

Let me first recall the definition of "Hodge norms". Suppose that

$$V = \bigoplus_{p+q=n} V^{p,q}$$

is a Hodge structure of weight n, polarized by a hermitian pairing $h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$. The Weil operator $C \in \mathrm{End}(V)$ acts on the subspace $V^{p,q}$ as multiplication by $(-1)^p$, and the fact that h is a polarization means that

$$(v', v'') \mapsto \langle v', v'' \rangle = h(Cv', v'')$$

is a positive definite inner product on V. The quantity

$$\|v\| = \sqrt{h(Cv, v)}$$

is called the *Hodge norm* of the vector $v \in V$. When the vector space and the pairing are fixed, but the Hodge filtration F is variable, we typically use the notation

$$C_F, \quad \langle v', v'' \rangle_F, \quad \|v\|_F.$$

Now suppose that we have a polarized variation of Hodge structure on the punctured disk Δ^*. As in Lecture 10, we trivialize the vector bundle $\mathscr{V}$ using the canonical extension; then the Hodge bundles $F^p\mathscr{V}$ are encoded by the "untwisted period mapping" $\Psi \colon \Delta^* \to \check{D}$. At each point $t \in \Delta^*$, the filtration $F_{\Psi(t)}$ puts a Hodge structure of weight n on the vector space V, which is polarized by the pairing

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = h\big(g(t)v', g(t)v''\big), \tag{11.2}$$

where I have denoted the exponential factor in Theorem 10.2 by the symbol

$$g(t) = e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2} \log L(t) H} \in \mathrm{GL}(V).$$

So for each $t \in \Delta^*$, we obtain a positive definite inner product $\langle v', v'' \rangle_t$ on the vector space V, and we denote by $\|v\|_t$ the Hodge norm of a given vector $v \in V$. For simplicity, let me write $F = e^{-\frac{1}{2} R_N} \hat{F}_H \in D$ for the Hodge filtration that appears as the limit in Theorem 10.2.

Corollary 11.3. *There is a constant $C > 0$ such that*

$$\frac{1}{C}\sum_{\alpha,j}|t|^{\alpha}L(t)^{j/2}\|v_{\alpha,j}\|_F \leq \|v\|_t \leq C\sum_{\alpha,j}|t|^{\alpha}L(t)^{j/2}\|v_{\alpha,j}\|_F$$

for every $v \in V$ and every $t \in \Delta$ sufficiently close to the origin.

The amazing thing is that, despite the sign changes in passing from the polarization to the inner product, the Hodge norm has *exactly* the same asymptotic behavior as the pairing (which we had studied in Lecture 9). After the fact, this explains again why the eigenspace decomposition of R_S (which is responsible for the term $|t|^{\alpha}$) and the monodromy weight filtration of R_N (which reflects the largest power of $L(t)$ that appears) show up in the study of polarized variations of Hodge structure on Δ^*.

Note. Two things are worth pointing out. First, the Hodge norm estimates are for *holomorphic* sections of the canonical extension (which induce the trivialization of $\mathscr{V}$ that we are working with), unlike in Schmid's paper, where the estimates are stated for multivalued flat sections. Second, the Hodge norm of a vector $v \in E_{\alpha}(R_S)$ grows like $|t|^{\alpha}L(t)^{\ell/2}$ if and only if $v \in W_{\ell}$, the ℓ-th part of the monodromy weight filtration. This means that the monodromy weight filtration controls, in a very precise way, the asymptotic behavior of the Hodge norm.

Example 11.4. An important special case is that if we are working with the interval $I = [0, 1)$, and if a vector $v \in V$ is monodromy invariant, in the sense that $Tv = v$, then the Hodge norm $\|v\|_t$ remains bounded near the origin. The reason is that $R_S v = R_N v = 0$, and hence $v \in W_0$ (because $\ker R_N \subseteq W_0$).

Proof of Corollary 11.3. We begin with a general observation. Suppose that

$$V = \bigoplus_{p+q=n} V^{p,q}$$

is a Hodge structure of weight n, polarized by h, with Hodge filtration F, Weil operator C_F, and Hodge norm $\|v\|_F$. Then for any $g \in \mathrm{GL}(V)$, the decomposition

$$V = \bigoplus_{p+q=n} gV^{p,q}$$

is another Hodge structure of weight n, which is now polarized by the pairing $(v', v'') \mapsto h(g^{-1}v', g^{-1}v'')$. Moreover, the Hodge filtration is gF, the Weil operator $C_{gF} = gC_F g^{-1}$, and the new inner product satisfies

$$\langle gv', gv''\rangle_{gF} = h\big(g^{-1}C_{gF}gv', g^{-1}gv''\big) = h(C_F v', v'') = \langle v', v''\rangle_F.$$

In particular, we have the identity $\|gv\|_{gF} = \|v\|_F$ relating the two Hodge norms.

Now we can prove the Hodge norm estimates. Because of the identity in (11.2), we have

$$\|v\|_t = \|g(t)v\|_{g(t)\Psi(t)}$$

for every $t \in \Delta^*$ and every $v \in V$. According to Theorem 10.2, the points $g(t)\Psi(t)$ converge, in the period domain D, to $F = e^{-\frac{1}{2}R_N}\hat{F}_H$, and so for t sufficiently close to the origin, the norms $\| \ \|_{g(t)\Psi(t)}$ are comparable to $\| \ \|_F$. In other words, there is a constant $C > 0$ such that

$$\frac{1}{C}\|v\|_F \leq \|v\|_{g(t)\Psi(t)} \leq C\|v\|_F$$

for all $v \in V$. Substituting the result from above, we get

$$\frac{1}{C}\|g(t)v\|_F \leq \|v\|_t \leq C\|g(t)v\|_F.$$

Now expand $g(t)v = e^{-\frac{1}{2}R_N} e^{-\frac{1}{2}L(t)R_S} e^{\frac{1}{2}\log L(t)H} v$ using the simultaneous eigenspace decomposition with respect to R_S and H. We then get the desired result (with a larger value for C) by noting that $e^{-\frac{1}{2}R_N}$ is a constant invertible operator, and that the different subspaces $V_{\alpha,j}$ are orthogonal with respect to the inner product $\langle\,\rangle_F$; this latter fact will become clear when we prove Theorem 10.3. □

Towards the proof of Theorem 9.1. Now we can start working on the problem of extending $\Psi\colon \Delta^* \to \check{D}$ to a holomorphic mapping from the entire disk Δ into the compact dual $\check{D}$. Since Ψ is holomorphic, it will be enough to prove that Ψ extend continuously; after that, we can use the Riemann extension theorem to conclude that the extension is holomorphic. To prove continuity, we need to show that $\Psi(t_1)$ and $\Psi(t_2)$ are close to each other when t_1 and t_2 are close to the origin; for that purpose, we need a distance function on $\check{D}$. Pretty much anything would do, but here is a construction that makes things especially convenient.

Recall that $\check{D} \cong \mathrm{GL}(V)/B$ is a homogeneous space, which means that at a point $z \in \check{D}$, we have the identification

$$T_z\check{D} \cong \mathrm{End}(V) / \left\{ A \in \mathrm{End}(V) \;\middle|\; AF_z^p \subseteq F_z^p \text{ for all } p \in \mathbb{Z} \right\}.$$

Our base point $o \in D$ determines a reference Hodge structure on V (polarized by h), and as we discussed in Lecture 6, this induces a polarized Hodge structure of weight 0 on $\mathrm{End}(V)$; the polarization is given by the trace pairing $\mathrm{tr}(AB^*)$. As usual, I will denote by $\langle v', v''\rangle_o$ respectively $\langle A', A''\rangle_o$ the positive definite inner products on V respectively $\mathrm{End}(V)$ induced by these polarized Hodge structures. By orthogonal projection, our fixed inner product on $\mathrm{End}(V)$ induced an inner product on each $T_z\check{D}$, and hence a hermitian metric on $\check{D}$. Let us write $d_{\check{D}}$ for the resulting distance function; since $\check{D}$ is compact and connected, the distance between any two points is finite. (Note that, unlike in the case of D, the distance function is *not* invariant under the group.)

Recall that the hermitian metric on D is $G_{\mathbb{R}}$-invariant, which means that translation by an element $g \in G_{\mathbb{R}}$ leaves distances unchanged. The first problem we need to study is how translation by elements $g \in \mathrm{GL}(V)$ affects the distance $d_{\check{D}}$. Since points in $\check{D}$ correspond to filtrations on V, the relevant quantity is not by how much g expands distances in V, but rather by how much the *adjoint action*

$$\mathrm{Ad}\, g\colon \mathrm{End}(V) \to \mathrm{End}(V), \quad (\mathrm{Ad}\, g)(A) = gAg^{-1},$$

expands distances in $\mathrm{End}(V)$. Let

$$\|\mathrm{Ad}\, g\| = \sup \left\{ \frac{\|gAg^{-1}\|_o}{\|A\|_o} \;\middle|\; A \in \mathrm{End}(V) \text{ with } A \neq 0 \right\}$$

be the operator norm of $\mathrm{Ad}\, g$, taken with respect to our fixed norm on $\mathrm{End}(V)$.

Lemma 11.5. *If $g \in \mathrm{GL}(V)$, then we have*

$$d_{\check{D}}(gz_1, gz_2) \leq \|\mathrm{Ad}\, g\| \cdot d_{\check{D}}(z_1, z_2)$$

for any pair of points $z_1, z_2 \in \check{D}$.

Proof. Let us first prove the infinitesimal version of this. As discussed in Lecture 6, the differential of the mapping $g\colon \check{D} \to \check{D}$, $z \mapsto gz$, fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{End}(V) & \xrightarrow{\mathrm{Ad}\, g} & \mathrm{End}(V) \\ \big\downarrow & & \big\downarrow \\ T_z\check{D} & \xrightarrow{g_*} & T_{gz}\check{D} \end{array}$$

in which the two vertical arrows are surjective. By definition of the operator norm, $\mathrm{Ad}\, g$ expands the length of any element $A \in \mathrm{End}(V)$ by at most $\|\mathrm{Ad}\, g\|$; since we defined the inner product on $T_z \check{D}$ by orthogonal projection, it follows that g_* can also expand the length of any tangent vector by at most $\|\operatorname{Ad} g\|$. The global result follows from the infinitesimal version by integrating along curves. $\square$

There is another issue that will come up during the proof of Theorem 9.1, having to do with the operator norms of the two operators

$$\operatorname{Ad} e^{-\frac{1}{2}L(t)R_S} \quad \text{and} \quad \operatorname{Ad} e^{\frac{1}{2}\log L(t)H}.$$

The following lemma is easy.

Lemma 11.6. *Let $S \in \operatorname{End}(V)$ be a semisimple with real eigenvalues. Then*

$$\|\operatorname{Ad} e^{xS}\| \leq C e^{(\lambda_{max}-\lambda_{min})\cdot|x|},$$

where λ_{min} and λ_{max} are the smallest and largest eigenvalue of S.

Proof. Let $A \in \operatorname{End}(V)$ be arbitrary. Write any vector $v \in V$ as $v = \sum v_\lambda$, with $S v_\lambda = \lambda v_\lambda$. Then

$$e^{xS} A e^{-xS} v = \sum_{\lambda,\mu} e^{(\mu-\lambda)x} (A v_\lambda)_\mu,$$

and therefore $\|e^{xS} A e^{-xS} v\| \leq C e^{(\lambda_{max}-\lambda_{min})\cdot|x|} \cdot \|A\| \|v\|$, where the constant C depends on the eigenspace decomposition of V. This gives

$$\|e^{xS} A e^{-xS}\| \leq C e^{(\lambda_{max}-\lambda_{min})\cdot|x|} \cdot \|A\|,$$

and hence the desired bound on the operator norm of $\operatorname{Ad} e^{xS}$. $\square$

The eigenvalues of H are integers, and therefore

$$\|\operatorname{Ad} e^{\frac{1}{2}\log L(t)H}\| \leq C \cdot L(t)^m$$

for some $m \in \mathbb{N}$. The right-hand side has only logarithmic growth, and we will see that such terms do not cause any problems. On the other hand, the eigenvalues of R_S lie in the interval I, and therefore

$$\|\operatorname{Ad} e^{-\frac{1}{2}L(t)R_S}\| \leq C \cdot \frac{1}{|t|^{\alpha_{max}-\alpha_{min}}}.$$

The exponent $\alpha_{max} - \alpha_{min}$ can be very close to 1, and since we are going to have some estimates involving $|t|^\varepsilon$ in the proof, this is unacceptably large. The idea for getting around this is to replace T by T^m; this replaces the eigenvalues of T by their m-th powers, and for well-chosen m, all the eigenvalues will be close to 1 (and hence $\alpha_{max} - \alpha_{min}$ will be very small). Geometrically, this amounts to making a base change by the mapping

$$\pi_m \colon \Delta^* \to \Delta^*, \psi_m(t) = t^m.$$

Let us study this problem right now. The diagram

$$\begin{array}{ccc} \tilde{\mathbb{H}} & \xrightarrow{m\cdot \mathrm{id}} & \tilde{\mathbb{H}} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ \Delta^* & \xrightarrow{\pi_m} & \Delta^* \end{array}$$

is obviously commutative, and so if we replace a variation of Hodge structure $\mathscr{V}$ by its pullback $\mathscr{V}_m = \pi_m^* \mathscr{V}$, the period mapping is essentially unchanged, except all distances need to be rescaled by a factor of m. In other words, if $\Phi \colon \tilde{\mathbb{H}} \to D$ and $\Phi_m \colon \tilde{\mathbb{H}} \to D$ are the two period mappings, then

$$\Phi_m(z) = \Phi(mz).$$

The translation $z \mapsto z + 2\pi i$ has the same effect on the new period mapping Φ_m as the translation $z \mapsto z + 2\pi i m$ has on Φ, and in particular, the new monodromy operator is indeed T^m. In this connection, the following lemma is very useful.

Lemma 11.7. *Let $R_m \in \mathrm{End}(V)$ have eigenvalues in a half-open interval $J \subseteq \mathbb{R}$ of length 1, and satisfy $T^m = e^{2\pi i R_m}$. Write*

$$\Psi_m(e^z) = e^{-zR_m}\Phi_m(z),$$

where $\Psi_m \colon \Delta^* \to \check{D}$ is holomorphic. If Ψ_m extends holomorphically over the origin, then so does our original mapping $\Psi \colon \Delta^* \to \check{D}$.

Proof. We have $e^{2\pi i m R} = T^m = e^{2\pi i R_m}$, which means that $S = mR - R_m$ acts on each eigenspace of R as multiplication by some integer. Consequently, the operator $t^{-S} = e^{-zS}$ acts on each eigenspace of R as multiplication by some (positive or negative) power of $t = e^z$. Since

$$e^{-zS}\Psi_m(e^z) = e^{-mzR}\Phi(mz) = \Psi(e^{mz})$$

we have $\Psi(t^m) = t^{-S}\Psi_m(t)$. By assumption, $\Psi_m \colon \Delta \to \check{D}$ is holomorphic. Because $\check{D}$ is a submanifold of projective space, and because the entries of the matrix t^{-S} are (restrictions of) algebraic functions, it follows that $\Psi(t^m)$ extends holomorphically over the origin. But then Ψ is continous there, and so it extends holomorphically by Riemann's extension theorem. $\square$