Lecture 1: August 26

Overview. The title of the course is "Introduction to Hodge Modules". Hodge modules, invented by Morihiko Saito in the late 1980s, provide a convenient formalism for doing Hodge theory for arbitrary projective morphisms. But the theory is somewhat complicated, and one needs some background in $\mathscr{D}$-module theory in order to understand it. For that reason, we are going to focus mostly on the things that go into Saito's theory, namely the theory of variations of Hodge structure (VHS). Hodge modules themselves will only appear towards the end of the semester. By analogy, think of an introduction to schemes that is mostly a course in commutative algebra. Since we are only going to work with VHS and Hodge modules over curves, you will not need to know anything about $\mathscr{D}$-modules in order to follow this course.

Here is the rough plan for the semester. After a brief review of Hodge structures and polarizations, we will start looking at variations of Hodge structure over curves. One source of examples is families of smooth projective varieties (or compact Kähler manifolds) over curves, but it turns out to be more convenient to study abstract VHS. We are going to discuss both the local theory (near a puncture, in case the base curve is not compact) and the global theory. The local results are due to Wilfried Schmid (in the 1970s), who gave a very precise asymptotic description of how a VHS behaves near a puncture. We will see several applications of the local theory, for example the local invariant cycle theorem, the so-called Clemens-Schmid exact sequence (which relates Schmid's results to the cohomology of singular fibers in a family), and the famous theorem of Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan about the locus of Hodge classes. The global results are semisimplicity of the monodromy representation (due to Deligne) and Steven Zucker's theorem that the cohomology groups of a VHS are themselves Hodge structures. Towards the end of the semester, we will use these results to define Hodge modules over curves, and I will explain how to translate the results of Schmid and Zucker into theorems about Hodge modules.

The website for the course,

`http://www.math.stonybrook.edu/~cschnell/mat690`,

contains a list of references, to be updated over the course of the semester.

Hodge structures and polarizations. Let me start by recalling the definition of a (complex) Hodge structure. Suppose that H is a finite-dimensional complex vector space.

Definition 1.1. A *Hodge structure of weight* k on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

into subspaces $H^{p,q} \subseteq H$.

For example, if X is a compact Kähler manifold, then the k-th singular cohomology group $H^k(X, \mathbb{C})$ has a Hodge structure of weight k. Note that the weight of a Hodge structure is not determined by the decomposition itself: for example, one can turn a Hodge structure of weight k into one of weight 0 by defining $H^{p,-p} = H^{p,k-p}$.

Definition 1.2. Let H be a Hodge structure of weight k. A *polarization* of H is a hermitian pairing $h \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ with the following two properties:

(a) The Hodge decomposition is orthogonal with respect to h.
(b) The hermitian pairing $(-1)^p h$ is positive definite on $H^{p,q}$.

The word “hermitian” means that $h(\lambda x, y) = \lambda h(x, y)$ and $h(y, x) = \overline{h(x, y)}$ for $x, y \in H$ and $\lambda \in \mathbb{C}$. We are going to look at the example of compact Kähler manifolds in a moment, to see where the factor $(-1)^p$ comes from. But let me first answer an obvious question.

Example 1.3. The usual definition of a Hodge structure, which you have surely seen before, contains some extra assumptions. Let $H_{\mathbb{Q}}$ be a finite-dimensional $\mathbb{Q}$-vector space. Then a *$\mathbb{Q}$-Hodge structure of weight k* on $H_{\mathbb{Q}}$ is a decomposition

$$H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

with the property that $\overline{H^{p,q}} = H^{q,p}$. Here complex conjugation is defined as $\overline{x \otimes \lambda} = x \otimes \bar{\lambda}$ for $x \in H_{\mathbb{Q}}$ and $\lambda \in \mathbb{C}$. This definition is motivated by the example of cohomology, where $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. But I prefer to work with arbitrary (complex) Hodge structures, and I hope to convince you over the course of the semester that this clarifies many things.

Example 1.4. For $\mathbb{Q}$-Hodge structures, the usual definition of a polarization also looks different. Suppose that $H_{\mathbb{Q}}$ has a $\mathbb{Q}$-Hodge structure of weight k. Then a polarization is usually defined to be a bilinear pairing

$$S \colon H_{\mathbb{Q}} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \to \mathbb{Q}$$

with the following three properties:

(a) S is $(-1)^k$-symmetric, meaning that $S(y, x) = (-1)^k S(x, y)$.
(b) The Hodge decomposition is orthogonal with respect to S.
(c) For $x \in H^{p,q}$, one has $S(i^{p-q} x, \bar{x}) \geq 0$, with equality only for $x = 0$.

You should convince yourself that if we set $H = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, then

$$h \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}, \quad h(x, y) = (2\pi i)^{-k} S(x, \bar{y}),$$

is a polarization of H in the sense of Definition 1.2.

Morphisms of Hodge structures are defined in the obvious way. If H_1 and H_2 are two Hodge structures of weight k, then a *morphism of Hodge structures* is a linear mapping $f \colon H_1 \to H_2$ with the property that $f(H_1^{p,q}) \subseteq H_2^{p,q}$ whenever $p + q = k$. One can show that Hodge structures of a given weight form an abelian category. There are many other exercises to be done here: the tensor product of two Hodge structures of weight k and ℓ is a Hodge structure of weight $k + \ell$; the dual of Hodge structure of weight k is a Hodge structure of weight $-k$; and so on.

Example 1.5. Let H be a Hodge structure of weight k, and $n \in \mathbb{Z}$ an integer. We can get a new Hodge structure of weight $k - 2n$ on H, denoted $H(n)$, by setting

$$H(n)^{p,q} = H^{p+n,q+n}.$$

This operation is called the *n-th Tate twist*.

Compact Kähler manifolds. Now let us discuss the example of compact Kähler manifolds in more detail. Suppose that X is a compact Kähler manifold of dimension n, with Kähler form ω. Recall that ω is a closed $(1,1)$-form that contains the same information as the Kähler metric h: in local coordinates $z_1, \dots, z_n$, the metric is represented by an $n \times n$-matrix with entries the smooth functions

$$h_{j,k} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right),$$

and the Kähler form is given by the formula

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k.$$

Classical Hodge theory gives us the following four results:

(1) The k-th cohomology group $H^k(X, \mathbb{Q})$ has a $\mathbb{Q}$-Hodge structure of weight k, given by

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Here $H^{p,q}(X)$ contains all those cohomology classes that are representable by a closed form of type (p, q). The proof of this fact is based on the theory of harmonic forms. The hermitian metric h gives rise to an inner product on the space $A^k(X, \mathbb{C})$ of all smooth k-forms, and a closed form is called *harmonic* if its norm is minimal among all forms in the same cohomology class. One then proves, using analysis, that every cohomology class contains a unique harmonic representative. The Kähler condition ensures that if we decompose a harmonic form $\alpha \in A^k(X, \mathbb{C})$ according to type as

$$\alpha = \sum_{p+q=k} \alpha^{p,q},$$

with $\alpha^{p,q} \in A^{p,q}(X)$, then each $\alpha^{p,q}$ is again harmonic. This implies the desired decomposition of $H^k(X, \mathbb{C})$. Since the conjugate of a (p, q)-form is a (q, p)-form, we get the Hodge symmetry $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

(2) The Hodge decomposition has another interpretation, observed by Deligne. Recall that, by the holomorphic Poincaré lemma, one has a resolution

$$0 \to \mathbb{C} \to \mathscr{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^n \to 0$$

of the constant sheaf, and therefore a convergent spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C}).$$

On compact Kähler manifolds, this so-called *Hodge-de Rham spectral sequence* degenerates at E_1. The reason is that $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$, and because of the the Hodge decomposition,

$$\sum_{p+q=k} \dim E_1^{p,q} = \dim H^k(X, \mathbb{C}) = \sum_{p+q=k} \dim E_\infty^{p,q}.$$

Therefore, $E_1^{p,q} = E_\infty^{p,q}$, and so the spectral sequence degenerates at E_1.

(3) There are some additional symmetries among the different cohomology groups. The *Lefschetz operator*

$$L\colon A^k(X, \mathbb{C}) \to A^{k+2}(X, \mathbb{C}), \quad L(\alpha) = \omega \wedge \alpha,$$

takes closed forms to closed forms (because $d\omega = 0$), and therefore induces a morphism on cohomology. The *Hard Lefschetz theorem* says that

$$L^k\colon H^{n-k}(X, \mathbb{C}) \to H^{n+k}(X, \mathbb{C})$$

is an isomorphism for every $k \geq 1$. In fact, one can be a bit more precise. Since ω is a $(1, 1)$-form, the Lefschetz operator maps $H^{p,q}(X)$ into $H^{p+1,q+1}(X)$; it is therefore not a morphism of Hodge structures, but after adding a Tate twist,

$$L\colon H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})(1)$$

is a morphism of Hodge structures of weight k. With this observation, the Hard Lefschetz theorem actually says that

$$L^k\colon H^{n-k}(X, \mathbb{C}) \to H^{n+k}(X, \mathbb{C})(k)$$

is an isomorphism of Hodge structures of weight $n - k$, for every $k \geq 1$.

(4) Finally, we come to the topic of polarizations. For every $k \geq 0$, one has the so-called *primitive subspace*

$$H_0^{n-k}(X, \mathbb{C}) = \ker\big(L^{k+1} \colon H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})(k+1)\big);$$

it is again a Hodge structure of weight $n-k$ (as the kernel of a morphism of Hodge structures). The so-called *Hodge-Riemann bilinear relations* say that the bilinear pairing

$$(\alpha, \beta) \mapsto (-1)^{(n-k)(n-k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^k$$

is a polarization of the $\mathbb{Q}$-Hodge structure on $H_0^{n-k}(X, \mathbb{Q})$. (Although, technically, it takes values in $\mathbb{R}$, unless the cohomology class of ω belongs to $H^2(X, \mathbb{Q})$, which only happens when X is projective.)

Example 1.6. For example, when X is a surface, this says that the intersection pairing is negative-definite on the part of the Néron-Severi group that is perpendicular to the class of ω.

Let me go through the proof of the fourth property, to show you that the polarization comes directly from the hermitian metric on X. The metric induces an inner product on $A^k(X, \mathbb{C})$, which can be written in the form

$$h(\alpha, \beta) = \int_X \alpha \wedge *\bar{\beta},$$

where $* \colon A^k(X, \mathbb{C}) \to A^{2n-k}(X, \mathbb{C})$ is the so-called *Hodge $*$-operator*. This operator is defined pointwise, and has a simple expression in terms of an orthonormal basis in the cotangent space at any point. If $\alpha \in A^{p,q}(X)$, then $*\alpha \in A^{n-q,n-p}(X)$. Define the adjoint

$$\Lambda \colon A^k(X, \mathbb{C}) \to A^{k-2}(X, \mathbb{C})$$

of the Lefschetz operator by the rule that $h(\Lambda\alpha, \beta) = h(\alpha, L\beta)$ for every $\alpha \in A^k(X, \mathbb{C})$ and every $\beta \in A^{k-2}(X, \mathbb{C})$.

Definition 1.7. A k-form $\alpha \in A^k(X, \mathbb{C})$ is called *primitive* if $\Lambda\alpha = 0$.

One can show that nonzero primitive forms only exist for $k \leq n$. Now a crucial fact is that one can describe the effect of the $*$-operator on primitive forms: if $\alpha \in A^{p,q}(X)$ satisfies $\Lambda\alpha = 0$, then one has

$$* \, \alpha = i^{q-p} \varepsilon(k) \frac{L^{n-k}}{(n-k)!} \alpha, \tag{1.8}$$

where $k = p + q$ and $\varepsilon(k) = (-1)^{k(k-1)/2}$. This fact is known as *Weil's identity*. The proof is tricky, but not deep; unlike other results in Hodge theory, no analysis is required. In fact, Weil's identity is a pointwise statement, which holds for in the wedge algebra of any complex vector space with a hermitian inner product.

Now let us use Weil's identity to derive the formula for the polarization. Let $\alpha \in A^{p,q}(X)$ be the harmonic representative of a primitive cohomology class in $H_0^{n-k}(X, \mathbb{C})$. One can show that $\Lambda\alpha = 0$, and so α is also primitive in the above sense. Since $p + q = n - k$, Weil's identity gives

$$*\bar{\alpha} = i^{p-q} \varepsilon(n-k) \frac{L^k}{k!} \bar{\alpha}.$$

If we put this into the formula for the inner product, we get

$$k! \cdot h(\alpha, \alpha) = i^{p-q} \varepsilon(n-k) \int_X \alpha \wedge \bar{\alpha} \wedge \omega^k,$$

and so the expression on the right-hand side is positive definite, as required.

In summary, the existence of the Hodge decomposition is a deep result, based on hard analysis, whereas the polarization comes basically for free, from the Kähler metric. For that reason, people often talk about Hodge structures without mentioning the polarization; but as we will see throughout the semester, a Hodge structure by itself is of very little use, and all the interesting results are actually coming from the polarization.

Representation theory. So far, we only have a polarization on the primitive part of the cohomology. There is a nice way to construct a polarization on the entire cohomology of X, with the help of some basic representation theory. The starting point is the following result, which is again fairly elementary: the commutator

$$[L, \Lambda] = L\Lambda - \Lambda L \colon A^k(X, \mathbb{C}) \to A^k(X, \mathbb{C})$$

is simply multiplication by the integer $k - n$. So if we define a new operator

$$H \colon A^k(X, \mathbb{C}) \to A^k(X, \mathbb{C}), \quad H(\alpha) = (k - n)\alpha,$$

we get the three relations

$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda.$$

Jean-Pierre Serre observed that these are exactly the relations among the three standard generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and so the cohomology ring $H^*(X, \mathbb{C})$ becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$.

More precisely, recall that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of complex 2×2-matrices with trace zero. It is 3-dimensional, with basis the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y,$$

which are exactly the same as the relations among H, L, and Λ.

Let us quickly review the finite-dimensional representation theory of $\mathfrak{sl}_2(\mathbb{C})$, since it turns out to be extremely useful for Hodge theory. Recall that a representation of $\mathfrak{sl}_2(\mathbb{C})$ on a finite-dimensional complex vector space V is a linear mapping

$$\rho \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}_{\mathbb{C}}(V)$$

that is compatible with taking commutators; the representation is of course determined by knowing $\rho(H)$, $\rho(X)$, and $\rho(Y)$. The first fact is that every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of irreducible representations. The reason is that $\mathfrak{sl}_2(\mathbb{C})$ is the complexification of the real Lie algebra $\mathfrak{su}_2$, and that every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ lifts to a representation of the Lie group SU_2. Since SU_2 is compact, one can then average over the group to obtain an invariant inner product, which can be used to decompose V into irreducible representations.