## Lecture 9: March 4

**Local coordinates on algebraic varieties.** Let $X$ be an algebraic variety over a field $k$, with structure sheaf $\mathscr{O}_X$. More precisely, $X$ is a scheme of finite type over $k$, meaning that for every affine open subset $U \subseteq X$, the ring of functions $\Gamma(U, \mathscr{O}_X)$ is a finitely generated $k$-algebra, or in other words, a quotient of a polynomial ring. We say that $X$ is *nonsingular* of dimension $n$ if, at each closed point $x \in X$, the stalk

$$\mathscr{O}_{X,x} = \lim_{U \ni x} \Gamma(U, \mathscr{O}_X)$$

is a regular local ring of dimension $n$; in other words, if $\mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$ denotes the maximal ideal, then

$$\dim_{\mathscr{O}_{X,x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = n = \dim \mathscr{O}_{X,x}.$$

When the field $k$ is perfect (which is always the case in characteristic zero), an equivalent condition is that the sheaf of Kähler differentials $\Omega^1_{X/k}$ is locally free of rank $n$.

Since we are going to need this in a moment, let me briefly review derivations and Kähler differentials. Let $A$ be a finitely generated $k$-algebra. A *derivation* from $A$ into an $A$-module $M$ is a $k$-linear mapping $D\colon A \to M$ such that $\delta(fg) = f\delta(g) + g\delta(f)$ for every $f, g \in A$. We denote by $\operatorname{Der}_k(A, M)$ the set of all such derivations; this is an $A$-module in the obvious way. In the special case $M = A$, we use the notation $\operatorname{Der}_k(A)$ for the derivations from $A$ to itself. In view of the formula $\delta(fg) = f\delta(g) + g\delta(f)$, such a derivation is the algebraic analogue of a vector field, acting on the set of functions in $A$. We have $\operatorname{Der}_k(A) \subseteq \operatorname{End}_k(A)$, and one can check that if $\delta_1, \delta_2 \in \operatorname{Der}_k(A)$, then their commutator

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \operatorname{End}_k(A)$$

is again a derivation. It is the analogue of the Lie bracket on complex manifolds.

The module of *Kähler differentials* $\Omega^1_{A/k}$ represents the functor $M \mapsto \operatorname{Der}_k(A, M)$, in the sense that one has a functorial isomorphism

$$\operatorname{Der}_k(A, M) \cong \operatorname{Hom}_A\bigl(\Omega^1_{A/k}, M\bigr).$$

In other words, $\Omega^1_{A/k}$ is an $A$-module, together with a derivation $d\colon A \to \Omega^1_{A/k}$, such that every derivation $\delta \in \operatorname{Der}_k(A, M)$ factors uniquely as $\delta = \tilde{\delta} \circ d$ for a unique $A$-linear map $\tilde{\delta}\colon \Omega^1_{A/k} \to M$. Concretely, $\Omega^1_{A/k}$ can be constructed by taking the free $A$-module on the set of generators $df$, for $f \in A$, and imposing the relations $d(fg) = fdg + gdf$ and $d(f+g) = df + dg$ for every $f, g \in A$, and $df = 0$ for every $f \in k$. By construction, one has

$$\operatorname{Der}_k(A) \cong \operatorname{Hom}_A(\Omega^1_{A/k}, A),$$

which makes the module of Kähler differentials dual to the module of derivations.

Globally, $\Omega^1_{X/k}$ is a coherent sheaf of $\mathscr{O}_X$-modules, such that for every affine open subset $U \subseteq X$, one has $\Gamma(U, \Omega^1_{X/k}) = \Omega^1_{A/k}$, where $A = \Gamma(U, \mathscr{O}_X)$. There is again a universal derivation $d\colon \mathscr{O}_X \to \Omega^1_{X/k}$. Think of $\Omega^1_{X/k}$ as an algebraic analogue of the sheaf of holomorphic one-forms on a complex manifold. The *tangent sheaf*

$$\mathscr{T}_X = \mathcal{H}om_{\mathscr{O}_X}(\Omega^1_{X/k}, \mathscr{O}_X)$$

is defined as the dual of the sheaf of Kähler differentials; on affines, one has $\Gamma(U, \mathscr{T}_X) = \operatorname{Der}_k(A)$, using the notation from above. This is an algebraic analogue of the sheaf of holomorphic tangent vector fields on a complex manifold.

Now suppose that $X$ is nonsingular of dimension $n$, or equivalently, that $\Omega^1_{X/k}$ is locally free of rank $n$. At every closed point $x \in X$, one can choose *local coordinates* in the following way: there is an affine open neighborhood $U$ of $x$, together with $n$ regular functions $x_1, \dots, x_n \in \Gamma(U, \mathscr{O}_X)$, such that

$$\Omega^1_{X/k}\big|_U \cong \bigoplus_{i=1}^n \mathscr{O}_X\big|_U \cdot dx_i.$$

Dually, we have derivations $\partial_1, \dots, \partial_n \in \operatorname{Der}_k\bigl(\Gamma(U, \mathscr{O}_X)\bigr)$, such that

$$\mathscr{T}_X\big|_U \cong \bigoplus_{i=1}^n \mathscr{O}_X\big|_U \cdot \partial_i.$$

This says that $df = \partial_1(f) \cdot dx_1 + \dots + \partial_n(f) \cdot dx_n$ for every $f \in \Gamma(U, \mathscr{O}_X)$, and so the derivation $\partial_i$ plays the role of the partial derivative operator $\partial/\partial x_i$. One can choose the functions $x_1, \dots, x_n \in \Gamma(U, \mathscr{O}_X)$ in such a way that they generate the maximal ideal $\mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$. Keep in mind that the morphism $U \to \mathbb{A}^n_k$ defined by the local coordinates is étale, but not usually an embedding (because open sets in the Zariski topology are too big).

**The sheaf of differential operators.** Let $X$ be a nonsingular algebraic variety. Our goal is to define the sheaf of differential operators $\mathscr{D}_X$, which is a global analogue of the Weyl algebra $A_n(k)$. This will be a quasi-coherent sheaf of $\mathscr{O}_X$-modules $\mathscr{D}_X$, together with an increasing filtration $F_\bullet \mathscr{D}_X$ by coherent $\mathscr{O}_X$-modules, where $F_j \mathscr{D}_X$ consists of differential operators of order $\leq j$.

We start by considering the affine case. So let $U \subseteq X$ be an affine open subset, and set $A = \Gamma(U, \mathscr{O}_X)$, which is a finitely generated $k$-algebra. We are going to define an $A$-module $D(A) \subseteq \operatorname{End}_k(A)$, whose elements are the algebraic differential operators of finite order on $A$. It will satisfy

$$D(A) = \bigcup_{j=0}^\infty F_j D(A),$$

where $F_j D(A)$ is the submodule of operators of order $\leq j$. The idea is that operators of order $0$ should be multiplication by elements in $A$, and that if $P \in F_i D(A)$ and $Q \in F_j D(A)$, then their commutator $[P, Q] = P \circ Q - Q \circ P \in \operatorname{End}_k(A)$ should belong to $F_{i+j-1} D(A)$. This is consistent with what happens for the Weyl algebra.

For an element $f \in A$, we also use the symbol $f \in \operatorname{End}_k(A)$ to denote the operator of multiplication by $f$. Observe that $P \in \operatorname{End}_k(A)$ is multiplication by the element $P(1) \in A$ if and only if $P$ is $A$-linear if and only if $[P, f] = 0$ for every $f \in A$. We can therefore define

$$F_0 D(A) = \bigl\{\, P \in \operatorname{End}_k(A) \bigm| [P, f] = 0 \text{ for every } f \in A \,\bigr\} \cong A.$$

We then define $F_j D(A)$ recursively by saying that

$$F_j D(A) = \bigl\{\, P \in \operatorname{End}_k(A) \bigm| [P, f] \in F_{j-1} D(A) \text{ for every } f \in A \,\bigr\}.$$

This construction of differential operators is due to Grothendieck.

*Example* 9.1. Let us work out the relationship between $F_1 D(A)$ and $\operatorname{Der}_k(A)$. Every derivation $\delta \in \operatorname{Der}_k(A)$ is also a differential operator of order 1, because

$$[\delta, f](g) = \delta(fg) - f\delta(g) = \delta(f) \cdot g$$

for every $f, g \in A$, which shows that $[\delta, f] = \delta(f) \in F_0 D(A)$. Conversely, suppose that we have some $P \in F_1 D(A)$. By definition, for every $f \in A$, there exists some $p_f \in A$ such that $[P, f] = p_f$. Concretely, this means that

$$P(fg) - fP(g) = p_f \cdot g$$

for every $f, g \in A$. Taking $g = 1$, we get $p_f = P(f) - fP(1)$, and so

$$P(fg) - fP(g) - gP(f) + fgP(1) = 0.$$

It is then easy to check that $P - P(1)$ is a derivation. The conclusion is that

$$F_1 D(A) \cong A \oplus \operatorname{Der}_k(A)$$

with $P \in F_1 D(A)$ corresponding to the pair $\bigl(P(1), P - P(1)\bigr)$.

It is easy to see that each $F_j D(A)$ is a finitely generated $A$-module, and that composition in $\operatorname{End}_k(A)$ has the following effect: if $P \in F_i D(A)$ and $Q \in F_j D(A)$, then $P \circ Q \in F_{i+j} D(A)$ and $[P, Q] \in F_{i+j-1} D(A)$. With some more work, one can prove the following result.

**Proposition 9.2.** *Let $A$ be a finitely generated $k$-algebra. If $A$ is nonsingular of dimension $n$, then the following is true:*

(a) *As an $A$-algebra, $D(A) \subseteq \operatorname{End}_k(A)$ is generated by $\operatorname{Der}_k(A)$, subject to the relations $[\delta, f] = \delta(f)$ for every $\delta \in \operatorname{Der}_k(A)$ and every $f \in A$.*
(b) *One has $F_j D(A)/F_{j-1} D(A) \cong \operatorname{Sym}^j \operatorname{Der}_k(A)$ for $j \geq 0$.*
(c) *One has an isomorphism of graded $A$-algebras*

$$\operatorname{gr}^F D(A) = \bigoplus_{j=0}^{\infty} F_j D(A)/F_{j-1} D(A) \cong \operatorname{Sym} \operatorname{Der}_k(A)$$

*between the associated graded algebra of $D(A)$ and the symmetric algebra on $\operatorname{Der}_k(A)$.*

Here, for any $A$-module $M$, the $j$-th symmetric power $\operatorname{Sym}^j M$ is the $A$-module obtained by quotienting $M \otimes_A \cdots \otimes_A M$ by the submodule generated by elements of the form $m_1 \otimes \cdots m_j - m_{\sigma(1)} \otimes \cdots m_{\sigma(j)}$, for all permutations $\sigma \in S_j$. The *symmetric algebra* on $M$ is the graded $A$-algebra

$$\operatorname{Sym} M = \bigoplus_{j=0}^{\infty} \operatorname{Sym}^j M.$$

It has the following universal property: if $B$ is any $A$-algebra, then every morphism of $A$-modules $M \to B$ extends uniquely to a morphism of $A$-algebras $\operatorname{Sym} M \to B$. For example, one has $\operatorname{Sym} A^{\oplus r} \cong A[x_1, \dots, x_r]$.

Let us give a concrete description of differential operators in local coordinates. Let $U \subseteq X$ be an affine open, with local coordinates $x_1, \dots, x_n$, and set $A = \Gamma(U, \mathscr{O}_X)$. The $A$-module $\operatorname{Der}_k(A)$ is free of rank $n$, generated by the derivations $\partial_1, \dots, \partial_n$, and so $D(A)$ is freely generated over $A$ by products of these. In other words, every $P \in F_j D(A)$ can be written uniquely in the form

$$P = \sum_{|\alpha| \leq j} f_\alpha \partial^\alpha,$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and where $f_\alpha \in A$. The only difference with the case of the Weyl algebra is that the coefficients now belong to the ring $A$, instead of to the polynomial ring.

*Example* 9.3. In the case $A = k[x_1, \dots, x_n]$, we have $D(A) = A_n(k)$, and the filtration $F_\bullet D(A)$ agrees with the order filtration.

Now we would like to say that $\mathscr{D}_X$ is the unique sheaf of $\mathscr{O}_X$-modules with the property that $\Gamma(U, \mathscr{D}_X) = D\bigl(\Gamma(U, \mathscr{O}_X)\bigr)$ for every affine open $U \subseteq X$. For this to work, one needs the following compatibility result.

**Proposition 9.4.** *Let $A$ be a finitely generated $k$-algebra that is nonsingular of dimension $n$. For nonzero $f \in A$, set $A_f = A[f^{-1}]$. Then one has isomorphisms*

$$D(A_f) \cong A_f \otimes_A D(A) \quad \text{and} \quad F_j D(A_f) \cong A_f \otimes_A F_j D(A).$$

The content of this is that every differential operator on $A_f$ extends, after multiplication by a sufficiently large power of $f$, to a differential operator on $A$. (The analogous result for Kähler differentials is that $\Omega^1_{A_f/k} \cong A_f \otimes_A \Omega^1_{A/k}$; you can find this in Hartshorne, who quotes Matsumura for the proof.)

*Note.* Unless $X$ is affine, $\Gamma(X, \mathscr{D}_X)$ does not embed into the $k$-linear endomorphisms of $\Gamma(X, \mathscr{O}_X)$. For example, we shall see below that there are many algebraic differential operators on $\mathbb{P}^n_k$, but since $\mathbb{P}^n_k$ is proper, every regular function on $\mathbb{P}^n_k$ is constant. This is why differential operators are defined locally.

The proposition implies that $\mathscr{D}_X$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules, and that each $F_j\mathscr{D}_X$ is coherent. Indeed, recall that a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is called *quasi-coherent* if, for every affine open subset $U \subseteq X$, the restriction of $\mathscr{F}$ to $U$ is the sheaf of $\mathscr{O}_X$-modules associated with the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \mathscr{F})$. On an affine scheme $\operatorname{Spec} A$, a necessary and sufficient condition for $\mathscr{F}$ to be quasi-coherent is that

$$\Gamma\bigl(D(f), \mathscr{F}\bigr) \cong A_f \otimes_A \Gamma(\operatorname{Spec} A, \mathscr{F})$$

for every $f \in A$, where $D(f) \subseteq \operatorname{Spec} A$ denotes the principal affine open defined by $f$. When $X$ is noetherian, which is the case for schemes of finite type over a field, $\mathscr{F}$ is *coherent* if each $\Gamma(U, \mathscr{F})$ is finitely generated over $\Gamma(U, \mathscr{O}_X)$. So the proposition says exactly that $\mathscr{D}_X$ is quasi-coherent and that each $F_j\mathscr{D}_X$ is coherent.

The isomorphisms in Proposition 9.2 globalize as follows. One has $F_0\mathscr{D}_X = \mathscr{O}_X$, and for every $j \geq 0$, one has

$$\operatorname{gr}^F_j \mathscr{D}_X = F_j\mathscr{D}_X / F_{j-1}\mathscr{D}_X \cong \operatorname{Sym}^j \mathscr{T}_X,$$

where $\mathscr{T}_X$ is the tangent sheaf. One also has an isomorphism of graded $\mathscr{O}_X$-algebras

$$\operatorname{gr}^F \mathscr{D}_X \cong \operatorname{Sym} \mathscr{T}_X,$$

and so the associated graded algebra of $\mathscr{D}_X$ is again commutative, as in the case of the Weyl algebra. Since $X$ is nonsingular, $\mathscr{T}_X$ is locally free of rank $n$, and the symmetric algebra on $\mathscr{T}_X$ can be interpreted as the sheaf of algebraic functions on the cotangent bundle. Let us denote by $p\colon T^*X \to X$ the cotangent bundle of $X$, with its natural projection to $X$. This is again a nonsingular algebraic variety, now of dimension $2n$, locally isomorphic to the product of $X$ and affine space $\mathbb{A}^n_k$. By the correspondence between vector bundles and locally free sheaves (from Hartshorne's book), one has an isomorphism

$$T^*X \cong \mathbb{V}(\mathscr{T}_X) = \mathbf{Spec}_X \operatorname{Sym} \mathscr{T}_X,$$

and therefore $p_*\mathscr{O}_{T^*X} \cong \operatorname{Sym} \mathscr{T}_X$ as $\mathscr{O}_X$-algebras. This is why people sometimes refer to $\mathscr{D}_X$ as a "noncommutative deformation" of the cotangent bundle.

*Example* 9.5. Let us consider the example $X = \mathbb{P}^n_k$. The $k$-vector space $\Gamma(X, \mathscr{D}_X)$ of global differential operators on projective space is infinite-dimensional. There are several ways to see this. One way is by diagram chasing. We have $F_0\mathscr{D}_X = \mathscr{O}_X$, and therefore $\Gamma(X, F_0\mathscr{D}_X) = k$. For each $j \geq 1$, we have a short exact sequence

$$0 \to F_{j-1}\mathscr{D}_X \to F_j\mathscr{D}_X \to \operatorname{Sym}^j \mathscr{T}_X \to 0.$$

One can show by induction that $H^1(X, F_j\mathscr{D}_X) = 0$ for $j \geq 0$, and so

$$H^0(X, F_j\mathscr{D}_X)/H^0(X, F_{j-1}\mathscr{D}_X) \cong H^0(X, \operatorname{Sym}^j \mathscr{T}_X).$$

These vector spaces can then be computed using the Euler sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(1)^{\oplus(n+1)} \to \mathscr{T}_X \to 0.$$

For example, $\dim H^0(X, \mathscr{T}_X) = (n+1)^2 - 1$, and so $\dim H^0(X, F_1\mathscr{D}_X) = (n+1)^2$.

Another way is to use the standard open covering $X = U_0 \cup U_1 \cup \cdots \cup U_n$. Since each $U_i$ is isomorphic to $\mathbb{A}_k^n$, one has $\Gamma(U_i, \mathscr{D}_X) \cong A_n(k)$, and so an element of $\Gamma(X, \mathscr{D}_X)$ can be described by $(n+1)$ elements of the Weyl algebra that are related to each other by the coordinate transformations among the $U_i$. (See the exercises.)

The third way is to use the presentation of $X$ as a quotient of $\mathbb{A}_k^{n+1}$ minus the origin, by identifying points of $\mathbb{P}_k^n$ with lines in $\mathbb{A}_k^{n+1}$. Recall how this works in the case of the Euler sequence. Once $n \geq 1$, a vector field on $\mathbb{A}_k^{n+1}$ minus the origin is the same thing as a vector field on $\mathbb{A}_k^{n+1}$, hence of the form

$$f_0 \partial_0 + f_1 \partial_1 + \cdots + f_n \partial_n,$$

for polynomials $f_0, \dots, f_n \in k[x_0, \dots, x_n]$. Such a vector field descends to $X$ if and only if it is homogeneous of degree 0, where $\deg x_j = 1$ and $\deg \partial_j = -1$. At the same time, the Euler vector field

$$x_0 \partial_0 + x_1 \partial_1 + \cdots + x_n \partial_n$$

is tangent to the lines through the origin, and therefore descends to the zero vector field. This shows that $\Gamma(X, \mathscr{T}_X)$ is generated by the $(n+1)^2$ vector fields $x_i \partial_j$, subject to the single relation $x_0 \partial_0 + \cdots + x_n \partial_n = 0$. In the same way, one can show that $\Gamma(X, \mathscr{D}_X)$ is isomorphic to the space of differential operators on $\mathbb{A}_k^{n+1}$ that are homogeneous of degree 0, modulo the ideal generated by the Euler vector field. Concretely, an element $P \in \Gamma(X, F_j \mathscr{D}_X)$ can be written in the form

$$P = \sum_{|\alpha| = |\beta| \leq j} c_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n} \partial_0^{\beta_0} \cdots \partial_n^{\beta_n}$$

and this expression is unique modulo multiples of $x_0 \partial_0 + \cdots + x_n \partial_n$. The restriction of $P$ to the standard affine open $U_0$ is obtained by setting $x_0 = 1$ and using the relation $\partial_0 = -(x_1 \partial_1 + \cdots + x_n \partial_n)$.

**Algebraic $\mathscr{D}_X$-modules.** Let me end with the following definition. An *algebraic $\mathscr{D}$-module* on a nonsingular algebraic variety $X$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules $\mathcal{M}$, together with a (left or right) action by the sheaf of differential operators $\mathscr{D}_X$. In other words, for every affine open subset $U \subseteq X$, with $A = \Gamma(U, \mathscr{O}_X)$, we get an $A$-module $M$, together with a (left or right) action by the module of differential operators $D(A)$.

**Exercises.**

*Exercise* 9.1. Show that one has $\operatorname{Der}_k(A_f) \cong A_f \otimes_A \operatorname{Der}_k(A)$ for every $f \in A$.

*Exercise* 9.2. For $X = \mathbb{P}_k^n$, compute $\dim_k \Gamma(X, F_j \mathscr{D}_X)$ as a function of $j \geq 0$.

*Exercise* 9.3. Consider the example $X = \mathbb{P}_k^1$. If we use the symbol $x_0$ for the coordinate on $U_0 = \mathbb{A}_k^1$, and $x_1$ for the coordinate on $U_1 = \mathbb{A}_k^1$, then $\Gamma(U_0, \mathscr{D}_X)$ is the Weyl algebra on $x_0$ and $\partial_0$, and $\Gamma(U_1, \mathscr{D}_X)$ is the Weyl algebra on $x_1$ and $\partial_1$. Using the coordinate change $x_1 = x_0^{-1}$, decide when two differential operators

$$P = \sum_{i,j} a_{i,j} x_0^i \partial_0^j \quad \text{and} \quad Q = \sum_{i,j} b_{i,j} x_1^i \partial_1^j$$

have the same restriction to $U_0 \cap U_1$. Use this to describe the space $\Gamma(X, \mathscr{D}_X)$ of global differential operators on $\mathbb{P}_k^1$.