Lecture 8 (February 20)

Abelian varieties. We are now going to look at abelian varieties from the point of view of algebraic geometry. Let k be an algebraically closed field; the theory can be developed in that generality, but some of the results are going to be a bit different when $\operatorname{char} k \neq 0$.

Definition 8.1. An *abelian variety* is a complete variety X (over the field k) that has the structure of a group, such that the group operations

$$m\colon X \times X \to X, \quad m(x,y) = xy, \qquad i\colon X \to X, \quad i(x) = x^{-1},$$

are morphisms (= regular maps).

Nonsingular cubic curves in $\mathbb{P}^2$ (in characteristic different from 2 and 3) are an example: the group law on the points of a nonsingular cubic can be described by morphisms. We can get higher-dimensional abelian varieties by taking products; other examples are less easy to come by.

We are going to show later that every abelian variety is projective; but in the definition, we only assume that X is complete (or, in scheme language, proper over $\operatorname{Spec} k$). We are mostly going to work with varieties, and not with schemes, so all the points of X are closed points. Generally speaking, we want to prove the same kind of results that we proved in the complex-analytic setting: the structure of X as a group; line bundles and their global sections; maps to projective space; etc.

Let's start with a few basic observations. First, X is always nonsingular. By definition, X is a variety, so it is reduced and irreducible. The set of nonsingular points is therefore Zariski-open and dense in X. Now X, being a group, is homogeneous, and so the existence of one nonsingular point implies that all points are nonsingular. More precisely, for any $x \in X$, consider the translation morphism

$$t_x\colon X \to X, \quad t_x(y) = m(x,y).$$

This is an automorphism (with inverse $t_{i(x)}$). Choose a nonsingular point $x_0 \in X$, and let $x \in X$ be an arbitrary point. Then translation by $m(x, i(x_0))$ takes the point x_0 to the point x, and since x_0 is nonsingular, x must also be nonsingular.

Second, let's prove that X is an abelian group. We will give two proofs for this; you should remember the technique, because it is very useful for studying group actions on algebraic varieties.

Lemma 8.2. *The group operation on an abelian variety is commutative.*

Proof. As in the complex case, we look at the conjugation morphism

$$C_x\colon X \to X, \quad C_x(y) = xyx^{-1}.$$

This is an automorphism, with inverse $C_{x^{-1}}$. It takes the identity element $e \in X$ to itself, and so it acts (by pullback of regular functions) on the local ring $\mathscr{O}_{X,e}$. The idea is to show that this action is trivial, by proving that it is trivial modulo larger and larger powers of the maximal ideal $\mathfrak{m}_e$. Because $e \in X$ is a nonsingular point, the quotient $\mathfrak{m}_e/\mathfrak{m}_e^2$ is a k-vector space of dimension $n = \dim X$; by Nakayama's lemma, we have $\mathfrak{m}_e = (f_1, \dots, f_n)$ for a system of parameters $f_1, \dots, f_n \in \mathscr{O}_{X,e}$. Now the automorphism

$$C_x^*\colon \mathscr{O}_{X,e} \to \mathscr{O}_{X,e}$$

preserves the maximal ideal $\mathfrak{m}_e$, and so for each $\ell \in \mathbb{N}$, it induces an automorphism

$$C_x^*\colon \mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1} \to \mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}.$$

Thinking of the elements in the quotient as polynomials of degree $\leq \ell$ in n-variables, we see that the quotient on the right-hand side is a finite-dimensional k-vector space of dimension $\binom{n+\ell}{\ell}$. So we get a function

$$f\colon X \to \operatorname{End}_k\bigl(\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}\bigr)$$

that sends a point $x \in X$ to the endomorphism C_x^* modulo $\mathfrak{m}_e^{\ell+1}$, viewed as an element of the k-vector space on the right-hand side. It is not hard to see that c is a morphism of algebraic varieties. Indeed, the mapping

$$C\colon X \times X \to X, \quad C(x,y) = C_x(y) = xyx^{-1},$$

is a morphism (by the definition of abelian varieties). Choose affine open neighborhoods V, W of the point $e \in X$, and U of the point $x \in X$, such that $C(U \times V) \subseteq W$. Then pullback of regular functions gives a morphism of k-algebras

$$C^*\colon k[W] \to k[U \times V] \cong k[U] \otimes_k k[V],$$

where $k[U] = \Gamma(U, \mathscr{O}_X)$ is the k-algebra of regular functions on U. Since $C(x,e) = e$, this induces a morphism of k-algebras

$$\mathscr{O}_{X,e} \to k[U] \otimes_k \mathscr{O}_{X,e},$$

and from this, it is easy to see that if we view $f|_U\colon U \to \operatorname{End}_k\bigl(\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}\bigr)$ as a matrix of size $\binom{n+\ell}{\ell}$, then the entries are regular functions on U. This means that f is a morphism of algebraic varieties.

The rest of the proof is easy. By assumption, X is complete, and so the morphism f must be constant; because $f(e) = \mathrm{id}$, it follows that C_x^* acts as the identity on $\mathscr{O}_{X,e}/\mathfrak{m}_e^{\ell+1}$. By Krull's intersection theorem, we have

$$\bigcap_{\ell \in \mathbb{N}} \mathfrak{m}_e^{\ell+1} = (0),$$

and so it follows that C_x^* is the identity on $\mathscr{O}_{X,e}$. Therefore C_x acts as the identity on a Zariski-open neighorhood of $e \in X$, and because X is a variety, C_x is the identity everywhere. But then $C_x(y) = y$, and this means that X is commutative. □

From now on, we are going to write the group operation on an abelian variety additively: so $m(x,y) = x + y$ and $i(x) = -x$, and the identity element is $0 \in X$.

As in the complex case, we can describe the tangent and cotangent bundles of an abelian variety. Let $T = T_{X,0}$ be the Zariski tangent space at $0 \in X$; if we set $\Omega_0 = \mathfrak{m}_0/\mathfrak{m}_0^2$, then $T = \operatorname{Hom}_k(\Omega_0, k)$, and both are k-vector spaces of dimension $\dim X$. For every $x \in X$, translation induces an isomorphism

$$t_{-x}^*\colon \mathfrak{m}_0/\mathfrak{m}_0^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2,$$

and so a cotangent vector $\theta \in \Omega_0$ defines an algebraic 1-form ω_θ by the rule $(\omega_\theta)_x = t_{-x}^*(\theta)$. As before, one can check on affines that ω_θ is a global section of the sheaf of Kähler differentials $\Omega^1_{X/k}$, and that this procedure defines a morphism of sheaves

$$\Omega_0 \otimes_k \mathscr{O}_X \to \Omega^1_{X/k}.$$

By construction, it is an isomorphism on fibers, meaning after tensoring by $\mathscr{O}_{X,x}/\mathfrak{m}_x$; by Nakayama's lemma, it is therefore an isomorphism of sheaves. After dualizing, we find that

$$T_X \cong T \otimes_k \mathscr{O}_X,$$

and so the tangent bundle of X is trivial. Similarly, we can take wedge powers to get

$$\Omega^p_{X/k} \cong \bigwedge^p \Omega_0 \otimes_k \mathscr{O}_X.$$

On global sections, this gives

$$H^0\bigl(X, \Omega^p_{X/k}\bigr) \cong \bigwedge^p \Omega_0,$$

because $H^0(X, \mathscr{O}_X) = k$ by completeness of X. All global algebraic p-forms on X are therefore translation invariant, exactly as on compact complex tori.

A fourth result, with a similar infinitesimal proof, is that the group of points of an abelian variety is divisible, provided we avoid the characteristic of the field k.

Lemma 8.3. *As long as n is not divisible by* $\mathrm{char}(k)$*, the homomorphism*

$$n_X \colon X \to X, \quad x \mapsto n \cdot x,$$

is surjective.

Proof. The morphism $m \colon X \times X \to X$ induces a k-linear mapping

$$dm \colon T_{X \times X,(0,0)} \to T_{X,0}$$

on tangent spaces. Set $T = T_{X,0}$. The tangent space to $X \times X$ at the point $(0,0)$ is isomorphic to $T \oplus T$, with the two copies given by the images of $T_{X,0}$ under the two inclusions $i_1 \colon X \to X \times X$, $i_1(x) = (x, 0)$, and $i_2 \colon X \to X \times X$, $i_2(x) = (0, x)$. Because $m \circ i_1 = m \circ i_2 = \mathrm{id}$, it follows that

$$dm \colon T \oplus T \to T$$

is just the sum map $(t_1, t_2) \mapsto t_1 + t_2$. From this, it is easy to see that

$$dn_X \colon T \to T$$

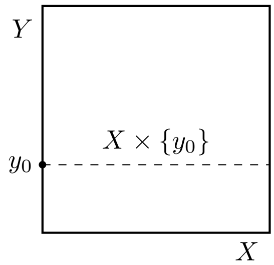
is multiplication by the integer n. Therefore dn_X is an isomorphism if n is not divisible by $\mathrm{char}(k)$. For dimension reasons, this means that n_X must be surjective: otherwise, the dimension of the image would be strictly less that $\dim X$, and so all fibers of n_X would have dimension ≥ 1. But if the fiber through the point $0 \in X$ has positive dimension, we can find a tangent vector $t \in T$ such that $dn_X(t) = 0$, and this contradicts the fact that dn_X is an isomorphism. $\square$

The proof shows more: because n_X is a homomorphism, the differential dn_X is actually an isomorphism at every point of X, and so $n_X \colon X \to X$ is finite étale. (In the case of compact complex tori, multiplication by n was a finite covering space.) We will later compute the degree of n_X, but this is more involved than on compact complex tori.

The rigidity theorem and its consequences. In order to go further, we need the following somewhat technical result, called the *rigidity theorem*. It is one of the important properties of complete varieties.

Theorem 8.4. *Let X be a complete variety over k, let Y, Z be varieties, and let $f \colon X \times Y \to Z$ be a morphism. Suppose that there is a point $y_0 \in Y$ such that $f(X \times \{y_0\})$ is a single point $z_0 \in Z$. Then $f = g \circ p_2$ for a morphism $g \colon Y \to Z$.*

This is saying that if one of the slices $X \times \{y_0\}$ is contracted to a point, then all slides $X \times \{y\}$ are contracted to a point (and $g(y)$ is that point).



Proof. Choose a point $x_0 \in X$ and define $g\colon Y \to Z$ by the formula $g(y) = f(x_0, y)$. Let $p_2\colon X \times Y \to Y$ be the second projection. In order to prove that $f = g \circ p_2$, it is enough to show that this holds on a Zariski-open set containing $X \times \{y_0\}$; the reason is that $X \times Y$ is irreducible. Choose an affine open set $U \subseteq Z$ such that $z_0 \in U$. The idea is to show that all nearby slices $X \times \{y\}$ also map into U.

The complement $Z \setminus U$ is a closed subset of Z. Because X is complete, the morphism $p_2\colon X \times Y \to Y$ is proper, which means that the image of any closed subset is closed. For that reason,

$$W = p_2\bigl(f^{-1}(Z \setminus U)\bigr) \subseteq Y$$

is a closed subset of Y. It does not contain the point y_0, because f maps $X \times \{y_0\}$ to the point $z_0 \in U$, and so $V = Y \setminus W$ is a Zariski-open set containing y_0. By construction, we have $f(X \times \{y\}) \subseteq U$ for every $y \in V$. Because U is affine and X is complete, f is therefore constant on $X \times \{y\}$. This shows that we have $f(x, y) = f(x_0, y) = g(y)$ for every $y \in V$. The identity $f = g \circ p_2$ therefore holds on the open set $X \times V$, as required. □

This has several useful consequences for abelian varieties.

Corollary 8.5. *Every morphism between two algebraic varieties is a group homomorphism composed with a translation.*

Proof. Let $f\colon X \to Y$ be a morphism from an abelian variety to an abelian variety. After composing f with the translation $t_{-f(e)}\colon Y \to Y$, we may assume that $f(e) = e$. We then claim that f must be a group homomorphism. To see that this is true, consider the morphism

$$F\colon X \times X \to Y, \quad F(x, y) = f(xy) - f(x) - f(y).$$

We have $F(x, e) = F(e, x) = e$, and so F contracts both $X \times \{e\}$ and $\{e\} \times X$. By the rigidity theorem, we must have $F(x, y) = e$ for all $x, y \in X$, and so f is a group homomorphism. □

We can also give another proof for the fact that X is commutative.

Corollary 8.6. *The group structure on an abelian variety is commutative.*

Proof. For the sake of clarity, let's briefly revert to multiplicative notation. Consider the morphism $i\colon X \to X$, $i(x) = x^{-1}$. It satisfies $i(0) = 0$, and so it must be a group homomorphism (by the previous corollary). This gives

$$y^{-1}x^{-1} = i(xy) = i(x)i(y) = x^{-1}y^{-1},$$

which obviously implies that the group operation is commutative. □

The last result for today is another special property of abelian varieties. If S and T are varieties, we can describe morphisms into the product $S \times T$ (which, in scheme-theoretic language, would be the fiber product over $\operatorname{Spec} k$). Indeed, the universal property says that a morphism $X \to S \times T$ is the same thing as a pair of morphisms $X \to S$ and $X \to T$ (all over k, of course); in other words, we have an isomorphism of sets

$$\operatorname{Hom}(X, S \times T) \cong \operatorname{Hom}(X, S) \times \operatorname{Hom}(X, T).$$

For abelian varieties, there is a similar result for maps *from* a product. Suppose that S and T are *complete* varieties, and that each comes with a choice of base point $s_0 \in S$ and $t_0 \in T$. We'll write (S, s_0) for the variety together with the point. Now suppose we have two morphisms $f\colon S \to X$ and $g\colon T \to X$ such that $f(s_0) = g(t_0) = 0$. The composition

$$S \times T \xrightarrow{f \times g} X \times X \xrightarrow{m} X$$

gives us a morphism $h\colon S \times T \to X$ with $h(s_0, t_0) = 0$. More concretely, we have

$$h\colon S \times T \to X, \quad h(s,t) = f(s) + g(t).$$

From h, we can of course recover f and g because $f(s) = h(s, t_0)$ and $g(t) = h(s_0, t)$. This shows that the function

$$\begin{aligned} \operatorname{Hom}\big((S, s_0), (X, 0)\big) \times \operatorname{Hom}\big((T, t_0), (X, 0)\big) &\to \operatorname{Hom}\big((S \times T, s_0 \times t_0), (X, 0)\big) \\ (f, g) &\mapsto m \circ (f \times g), \end{aligned}$$

is injective. It is also surjective: Given $h\colon S \times T \to X$ with $h(s_0, t_0) = 0$, we define $f(s) = h(s, t_0)$ and $g(t) = h(s_0, t)$, and then $h(s,t) = f(s) + g(t)$ by the rigidity theorem. (The difference $h(s,t) - f(s) - g(t)$ again contracts both $S \times \{t_0\}$ and $\{s_0\} \times T$, and so it must be constant.)

So, in somewhat more fancy language, the functor

$$(S, s_0) \mapsto \operatorname{Hom}\big((S, s_0), (X, 0)\big),$$

from the category of complete varieties with base point to the category of sets takes products to products.