Lecture 7 (February 18)

Principally polarized abelian varieties. Let $X = V/\Gamma$ be a compact complex torus. Recall from Lecture 5 that a polarization is a positive definite hermitian form $H \colon V \times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. (In Lecture 5, I said incorrectly that a polarization was an ample line bundle; instead, the polarization is just the first Chern class of an ample line bundle.) According to Lemma 5.2, we can always find a basis for Γ such that

$$E = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $m_1 \mid m_2 \mid \cdots \mid m_n$, and therefore $\det E = (m_1 \cdots m_n)^2$. The polarization is principal if $m_1 = \cdots = m_n = 1$. In that case, the homomorphism

$$E \colon \Gamma \to \Gamma^*, \quad \gamma \mapsto E(\gamma, -),$$

is an isomorphism. For any choice of $\alpha \colon \Gamma \to U(1)$ with the property that

$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)e^{i\pi E(\gamma,\delta)},$$

the line bundle $L = L(H, \alpha)$ is then ample, and $\dim H^0(X, L) = 1$. The divisor of the (essentially unique) nontrivial section of L is called a *theta divisor*.

Note that the principal polarization does not uniquely determine an ample line bundle. Since $e^{i\pi E(\gamma,\delta)} = \pm 1$, we can cut down on the number of choices by requiring that $\alpha(\Gamma) \subseteq \{\pm 1\}$. But even then, there are still 2^{2n} possible choices for α, and there is no way to pick a canonical one. On the other hand, the line bundle $L^2 = L(2H, 1)$ is uniquely determined by H, because $e^{2\pi i E(\gamma,\delta)} = 1$, and so $\alpha \equiv 1$ works for $2H$.

When H is a principal polarization, the holomorphic group homomorphism

$$\varphi_L \colon X \to \operatorname{Pic}^0(X), \quad \varphi_L(x) = t_x^* L \otimes L^{-1},$$

is also an isomorphism. This means that every line bundle with trivial first Chern class can be written in the form $t_x^* L \otimes L^{-1}$ for a unique point $x \in X$.

Let's now determine all possible principally polarized abelian varieties in a given dimension. It is customary to call this dimension g, as in the case of Jacobians (where g is the genus of the compact Riemann surface in question).

Example 7.1. As a warm-up, let's do the case $g = 1$. Any elliptic curve can be written in the form $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$ is a point in the upper halfplane (so $\operatorname{Im} \tau > 0$). The principal polarization is $E(\tau, 1) = 1$, and then $H(1, 1) = 1/\operatorname{Im} \tau$. We can choose a different basis for the lattice, say of the form

$$a\tau + b \quad \text{and} \quad c\tau + d,$$

for integers $a, b, c, d \in \mathbb{Z}$, subject to the condition that

$$1 = E(a\tau + b, c\tau + d) = ad - bc.$$

In terms of matrices, this is saying that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

If we again use the second basis vector in the lattice as the basis for the vector space, then the new lattice is $\mathbb{Z} + \mathbb{Z}\tau'$, where

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

This is still a point on the upper halfplane $\mathbb{H}$, because

$$\operatorname{Im}\tau' = \frac{\operatorname{Im}\tau}{|c\tau+d|^2} > 0.$$

The isomorphism between the two elliptic curves is then

$$\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau' \to \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau, \quad z \mapsto (c\tau+d)z.$$

To summarize, every elliptic curve can be written as $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ for $\tau \in \mathbb{H}$, and two such curves are isomorphic if and only if τ and τ' belong to the same orbit of the group $\mathrm{SL}_2(\mathbb{Z})$. In that sense, the moduli space of elliptic curves is the quotient $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This statement needs to be taken with some care, though, because $\mathrm{SL}_2(\mathbb{Z})$ does not act freely: for example, the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

acts trivially on $\mathbb{H}$, but on the level of the elliptic curves, it acts nontrivially (as the automorphism $z \mapsto -z$). There are also special points, for example $\tau = i$ or $\tau = \frac{1}{2}(-1+\sqrt{-3})$, whose stabilizer is even larger (and where the elliptic curve has additional automorphisms).

Let's now describe all principally polarized abelian varieties of dimension $g \geq 1$. We start from $X = V/\Gamma$, where $\dim V = g$ and $\operatorname{rk}\Gamma = 2g$. The principal polarization is $H\colon V \times V \to \mathbb{C}$, and $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. Using Lemma 5.2, but changing the order of the basis elements, we can find a basis $e_1, \dots, e_{2g} \in \Gamma$ such that E becomes the block matrix (of size $2g \times 2g$)

$$E = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right]$$

where I_g is the identity matrix of size $g \times g$. We can now use the lattice elements $e_{g+1}, \dots, e_{2g}$ as a basis for the $\mathbb{C}$-vector space $\mathbb{C}$, and write the other g lattice elements $e_1, \dots, e_g$ in terms of that basis as

$$e_k = \sum_{j=1}^{g} \Omega_{j,k} e_{g+j}$$

for certain complex numbers $\Omega_{j,k} \in \mathbb{C}$. With this convention, we have $V = \mathbb{C}^n$, and the lattice takes the form $\Gamma = \mathbb{Z}^n + \Omega\mathbb{Z}^n$; in other words, the lattice is spanned by the columns of the following block matrix (of size $g \times 2g$):

$$\left[\begin{array}{c|c} \Omega & I_g \end{array}\right]$$

The positive definite hermitian form H is then represented by the $g \times g$-matrix with entries $H_{j,k} = H(e_{g+j}, e_{g+k}) \in \mathbb{R}$ (because $E = \operatorname{Im} H$ vanishes on these vectors). This matrix is symmetric and positive definite. From

$$1 = E(e_k, e_{g+k}) = \operatorname{Im} H(e_k, e_{g+k}) = \sum_{j=1}^{g} \operatorname{Im}\Omega_{j,k} \cdot H_{j,k}, = \sum_{j=1}^{g} \operatorname{Im}\Omega_{j,k} \cdot H_{k,j},$$

we see that H is the inverse matrix to $\operatorname{Im}\Omega$. Therefore $\operatorname{Im}\Omega$ must be positive definite, and the polarization is represented, in the basis $e_{g+1}, \dots, e_{2g} \in V$, by the matrix $(\operatorname{Im}\Omega)^{-1}$.

For a similar reason, the matrix Ω is also symmetric. Indeed,

$$H(e_j, e_k) = \sum_{p,q=1}^{g} \Omega_{p,j}\overline{\Omega}_{q,k} H(e_{g+p}, e_{g+q}) = \sum_{p,q=1}^{g} \Omega_{p,j}\overline{\Omega}_{q,k} H_{p,q}$$

is also real (for $1 \leq j, k \leq g$), and therefore

$$0 = \sum_{p,q=1}^{g} \left(\operatorname{Im}\Omega_{p,j} \operatorname{Re}\Omega_{q,k} - \operatorname{Re}\Omega_{p,j} \operatorname{Im}\Omega_{q,k}\right) H_{p,q} = \operatorname{Re}\Omega_{j,k} - \operatorname{Re}\Omega_{k,j},$$

remembering that $\operatorname{Im}\Omega$ and H are inverse matrices. Therefore $\Omega^t = \Omega$.

The analogue of the upper halfplane is the so-called *Siegel space*

$$\mathcal{H}_g = \left\{ \Omega \in \operatorname{Mat}_{g\times g}(\mathbb{C}) \;\middle|\; \Omega^t = \Omega, \text{ and } \operatorname{Im}\Omega \text{ is positive definite} \right\}.$$

Every principally polarized abelian variety can be written in the form

$$\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g),$$

where $\Omega \in \mathcal{H}_g$. In the standard basis on $\mathbb{C}^g$, the polarization is represented by the positive definite matrix $(\operatorname{Im}\Omega)^{-1}$.

What happens when we choose a different basis for the lattice? Suppose that $e_1', \dots, e_{2g}' \in \Gamma$ is another basis, still with the property that $E(e_j', e_{g+j}') = 1$. We can represent the change of basis by the block matrix (of size $2g \times 2g$)

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with $A, B, C, D \in \operatorname{Mat}_{g\times g}(\mathbb{Z})$, and the condition that $E\colon \Gamma \times \Gamma \to \mathbb{Z}$ has the same shape as before translates into the matrix equation

$$\left[\begin{array}{c|c} A^t & C^t \\ \hline B^t & D^t \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right].$$

In other words, $M \in \operatorname{Sp}_g(\mathbb{Z})$ is an element of the symplectic group. A brief computation shows that the matrix Ω gets transformed into the new matrix

$$\Omega' = (C\Omega + D)^{-1}(A\Omega + B).$$

So the parameter space (or moduli space) for principally polarized abelian varieties of dimension g is the quotient space

$$\mathcal{A}_g = \mathcal{H}_g / \operatorname{Sp}_g(\mathbb{Z}),$$

with the same caveats as before. This has dimension $g(g+1)/2$.

Subtori and isogenies. We'll end the complex-analytic treatment of abelian varieties by a quick look at the structure of abelian varieties. Let $X = V/\Gamma$ be a compact complex torus.

A *subtorus* is a connected (closed, hence compact) complex subgroup. As we saw in Lecture 2, such a subgroup again has the form $X' = V'/\Gamma'$, and so $V' \subseteq V$ is a complex subspace, and $\Gamma' = V' \cap \Gamma$ should be a lattice in V'. We can think of a subtorus either as a discrete subgroup of Γ of some even rank $2k$, whose span is a complex subspace of dimension k; or as a complex subspace of dimension k that intersects Γ in a discrete subgroup of rank $2k$.

Example 7.2. Any holomorphic mapping $f\colon X \to Y$ from a compact complex torus to a compact complex torus that satisfies $f(0) = 0$ is a group homomorphism. So the connected component of $\ker f$ is a subtorus.

If $X' \subseteq X$ is a subtorus, then the quotient X/X' is again a compact complex torus; to see this, write the quotient as

$$X/X' \cong \frac{V/V'}{\Gamma/\Gamma'}.$$

A compact complex torus X is called *simple* if the only subtori are $\{0\}$ and X. Elliptic curves are simple (for dimension reasons); in fact, if we choose a random lattice in V, then the resulting compact complex torus will be simple.

Now we would like to prove that every abelian variety can be decomposed into simple abelian varieties. Here "decomposed" could mean "written as a product", but that doesn't quite work, so we have to settle for something a bit weaker. The relevant definition is the following.

Definition 7.3. A group homomorphism $f\colon X \to Y$ from a compact complex torus X to a compact complex torus Y is called an *isogeny* if f is surjective and $\ker f$ is a finite group. In that case, we say that X and Y are *isogenous*.

Consider an isogeny $f\colon X_1 \to X_2$. The induced map $\tilde{f}\colon V_1 \to V_2$ must be an isomorphism, and $\tilde{f}(\Gamma_1) \subseteq \Gamma_2$. Then $\ker f \cong \tilde{f}^{-1}(\Gamma_2)/\Gamma_1$, and so this must be a finite group. Equivalently, we can identify the two vector spaces using $\tilde{f}$, and after that identification, our isogeny has the form

$$V/\Gamma_1 \to V/\Gamma_2,$$

where $\Gamma_1 \subseteq \Gamma_2$ is a subgroup of finite index. This shows that being isogenous really is an equivalence relation. Indeed, if $\Gamma_1 \subseteq \Gamma_2$ is a subgroup of finite index, then Γ_2 will have finite index in $\Gamma_{1,m} = \frac{1}{m}\Gamma_1$ for some integer $m \geq 1$. Because the mapping

$$V/\Gamma_{1,m} \to V/\Gamma_1, \quad v \mapsto mv,$$

is an isomorphism, this gives us the desired isogeny

$$V/\Gamma_2 \to V/\Gamma_{1,m} \to V/\Gamma_1.$$

So working up to isogeny basically means replacing the lattice Γ by the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$.

Example 7.4. For any nonzero integer $m \in \mathbb{Z}$, the homomorphism

$$[m]\colon X \to X, \quad x \mapsto mx,$$

is an isogeny. The kernel is the set $X[m]$ of points of order m in the group X; as we saw in Lecture 2, this is a group with $m^{2\dim X}$ elements.

Example 7.5. If X is an abelian variety, and L an ample line bundle, then the homomorphism

$$\varphi_L\colon X \to \operatorname{Pic}^0(X), \quad x \mapsto t_x^* L \otimes L^{-1},$$

is an isogeny: indeed, the kernel is a finite group of order $\dim H^0(X, L)^2$.

One has the following simple structure theorem for abelian varieties. It is known as the *Poincaré complete irreducibility theorem.*

Theorem 7.6. *Every abelian variety is isogenous to a product of simple abelian varieties, and the factors are unique up to isogeny.*

Unlike other structure theorems in geometry, this one is completely elementary. We first prove the following lemma about simple tori.

Lemma 7.7. *Let X and Y be simple compact complex tori. Then any holomorphic group homomorphism $f\colon X \to Y$ is either constant or an isogeny.*

Proof. Consider a holomorphic group homomorphism $f\colon X \to Y$. The image $\operatorname{im} f$ is a subtorus of Y, and because Y is simple, we either have $\operatorname{im} f = \{0\}$ or $\operatorname{im} f = Y$. In the first case, f is constant. In the second case, the connected component of $\ker f$ is a subtorus of X, and because X is simple and f is not constant, this connected component must be trivial. But then f is surjective with finite kernel, and so it is an isogeny. $\square$

We can now prove the theorem.

Proof. This is exactly the same as the prime factorization of integers. Let's first prove existence. By induction on $\dim X$, we only need to show that if X contains a nontrivial subtorus X', then X is isogenous to $X' \times X''$ for some other subtorus X''. Write $X = V/\Gamma$ and $X' = V'/\Gamma'$, with $\Gamma' = V' \cap \Gamma$. Choose a polarization $H\colon V \times V \to \mathbb{C}$ and set $E = \operatorname{Im} H$ as usual. The orthogonal complement

$$V'' = \left\{\, v \in V \;\middle|\; H(v, v') = 0 \text{ for every } v' \in V' \,\right\}$$

satisfies $V = V' \oplus V''$ because H is positive definite. Moreover, we have

$$\Gamma'' = V'' \cap \Gamma = \left\{\, \gamma \in \Gamma \;\middle|\; E(\gamma, \gamma') = 0 \text{ for every } \gamma' \in \Gamma' \,\right\}$$

because $V' = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma'$. Since E is nondegenerate, $\Gamma' \oplus \Gamma'' \subseteq \Gamma$ is a subgroup of finite index. Therefore Γ'' is a lattice in V''; the quotient $X'' = V''/\Gamma''$ is a compact complex torus; and the induced mapping

$$X' \oplus X'' \to X$$

is an isogeny.

Now let's prove uniqueness. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be two collections of simple abelian varieties, and suppose that we have an isogeny

$$f\colon X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, consider the induced homomorphism $f_{j,i}\colon X_i \to Y_j$. By Lemma 7.7, it is either constant or an isogeny. After rearranging the order of the factors, we may assume that $X_1, \dots, X_p$ and $Y_1, \dots, Y_q$ are isogenous to each other (and therefore of the same dimension), but not isogenous to any of the other factors. If we view our isogeny

$$f\colon (X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_m) \to (Y_1 \times \cdots \times Y_q) \times (Y_{q+1} \times \cdots \times Y_n)$$

as a 2×2-matrix, it has the form

$$f = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix},$$

where $g\colon X_1 \times \cdots \times X_p \to Y_1 \times \cdots \times Y_q$ and $h\colon X_{p+1} \times \cdots \times X_m \to Y_{q+1} \times \cdots \times Y_n$ are homomorphisms. Because f is surjective with finite kernel, both g and h must be surjective with finite kernel. For dimension reasons, we get $p = q$. But $h\colon X_{p+1} \times \cdots \times X_m \to Y_{q+1} \times \cdots \times Y_n$ is still an isogeny, and so we can finish the proof by induction on the number of factors. $\square$