Lecture 6: February 20

General setup. We start working on the proof of the theorem from last time, comparing the two notions of dimension $d^B(M)$ (with respect to the Bernstein filtration) and $d^{\deg}(M)$ (with respect to the degree filtration). In order to make the result more useful, and to simplify the notation, we are going to work in the following more general setting.

Let R be a ring with 1. We assume that R is filtered; as before, this means that R comes with an exhaustive increasing filtration $F_\bullet R$, with

$$\{0\} = F_{-1}R \subseteq F_0 R \subseteq F_1 R \subseteq \cdots,$$

such that $1 \in F_0 R$ and $F_i R \cdot F_j R \subseteq F_{i+j} R$ for all $i, j \geq 0$. This makes $F_0 R$ a subring of R. We define $S = \operatorname{gr}^F R$ to be the associated graded ring, with $S_j = F_j R / F_{j-1} R$, and with the product defined by $(r + F_i R) \cdot (r' + F_j R) = (rr' + F_{i+j} R)$; note that $F_0 R = S_0$ is also a subring of S. Generalizing from what happens in the case $R = A_n$, we make the following two assumptions about S:

(A) S is a commutative noetherian ring.
(B) S is regular of dimension $\dim S = 2n$.

As in Lecture 2, the assumption (A) implies that R is left noetherian; moreover, the subring $F_0 R = S_0$ is also commutative and noetherian. The condition in (B) means concretely that for every maximal ideal $\mathfrak{m} \subseteq S$, the localization $S_\mathfrak{m}$ is a regular local ring of dimension $2n$, in the sense that

$$\dim_{S/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim S_\mathfrak{m} = 2n.$$

This implies that every finitely generated $S_\mathfrak{m}$-module has a free resolution of length at most $2n$; in fact, by a theorem of Serre, the two things are equivalent to each other. The geometric meaning of the condition in (B) is of course that the scheme $\operatorname{Spec} S$ is nonsingular of dimension $2n$.

Example 6.1. Take $R = A_n$, either with the Bernstein filtration $F_\bullet^B A_n$ or the degree filtration $F_\bullet^{\deg} A_n$. In both cases, S is the polynomial ring in $2n$ variables.

Now let M be a finitely generated left R-module. As in Lecture 3, we have the notion of a *compatible filtration* $F_\bullet M$. Recall that this means that $F_\bullet M$ is an exhaustive increasing filtration of M, such that $F_i R \cdot F_j M \subseteq F_{i+j} M$ for every $i, j \geq 0$, and such that each $F_j M$ is finitely generated over the commutative ring $F_0 R$. As before, the filtration is called *good* if the associated graded module $\operatorname{gr}^F M$ is finitely generated over $S = \operatorname{gr}^F R$. Every finitely generated R-module has a good filtration. As in the case of A_n, one shows that the ideal

$$J(M) = \sqrt{\operatorname{Ann}_S(\operatorname{gr}^F M)}$$

is independent of the choice of good filtration $F_\bullet M$. It is easy to see that a prime ideal $P \subseteq S$ contains $J(M)$ if and only if the localized module $M_P = S_P \otimes_S M$ is nonzero. The geometric interpretation is that the finitely generated S-module $\operatorname{gr}^F M$ defines a coherent sheaf on the scheme $\operatorname{Spec} S$, and the closed subscheme defined by the ideal $J(M)$ is the support of this sheaf.

Definition 6.2. Let M be a finitely generated left R-module. We set

$$d(M) = \dim S/J(M) = \dim \operatorname{Supp}(\operatorname{gr}^F M)$$
$$j(M) = \min\left\{ j \geq 0 \;\middle|\; \operatorname{Ext}^j_R(M, R) \neq 0 \right\}$$

The theorem I stated last time holds in this generality.

Theorem 6.3. *Let $(R, F_\bullet R)$ be a filtered ring satisfying the two conditions in (A) and (B). Then one has*

$$d(M) + j(M) = \dim S$$

for every finitely generated left R-module M.

This immediately implies the result I stated last time. Take $R = A_n$, and suppose that M is a finitely generated left A_n-module. The definition of the invariant $j(M)$ does not mention any filtrations, and so it is the same no matter what filtration on R we consider. If we take $F_\bullet R = F_\bullet^B A_n$, we get

$$d^B(M) + j(M) = 2n,$$

and if we take $F_\bullet R = F_\bullet^{\deg} A_n$, we get

$$d^{\deg}(M) + j(M) = 2n.$$

The two equations together give us the desired equality $d^B(M) = d^{\deg}(M)$.

The commutative case. The proof of Theorem 6.3 is going to take some time. Let us first consider what happens in the commutative case. In the general setting from above, R is of course allowed to be commutative; but to avoid any confusion, let me stick to the notation S for the commutative noetherian ring.

Proposition 6.4. *Let S be a commutative noetherian ring, regular of dimension $2n$. For any finitely generated S-module M, set $J(M) = \sqrt{\operatorname{Ann}_S M}$ and define*

$$d(M) = \dim S/J(M) \quad \text{and} \quad j(M) = \min\{\, j \geq 0 \mid \operatorname{Ext}^j_S(M, S) \neq 0 \,\}$$

Then the following is true:

- *(a) If $\operatorname{Ext}^j_S(M, S) \neq 0$, then $2n - d(M) \leq j \leq 2n$.*
- *(b) One has $d\big(\operatorname{Ext}^j_S(M, S)\big) \leq 2n - j$ for every $j \geq 0$.*
- *(c) One has $d\big(\operatorname{Ext}^{j(M)}_S(M, S)\big) = d(M)$.*
- *(d) The identity $d(M) + j(M) = 2n$ holds.*

Proof. Let me try to give at least an idea of the proof (without dotting all the i's). The first step is to reduce to the case where S is a regular local ring. We can test whether or not $\operatorname{Ext}^j_S(M, S)$ is zero by localizing at all maximal ideals of M. Let $\mathfrak{m} \subseteq S$ be any maximal ideal containing $J(M)$; in terms of the scheme $\operatorname{Spec} S$, we are choosing a closed point on the support of M. Then one has

$$S_\mathfrak{m} \otimes_S \operatorname{Ext}^j_S(M, S) \cong \operatorname{Ext}^j_{S_\mathfrak{m}}\big(S_\mathfrak{m} \otimes_S M, S_\mathfrak{m}\big).$$

After replacing S by its localization, and M by $S_\mathfrak{m} \otimes_S M$, we can therefore assume that S is a regular local ring of dimension $2n$. Geometrically, this means that we are working locally near a point of $\operatorname{Supp} M$.

We prove (a) and (b) by induction on $d = \dim S/J(M) \geq 0$. When $d = 0$, the fact that S is local implies that $J(M) = \mathfrak{m}$. Since M is finitely generated, one has $\mathfrak{m}^\ell M = 0$ for some $\ell \geq 0$. By considering the chain of submodules $M \supseteq \mathfrak{m}M \supseteq \mathfrak{m}^2 M \supseteq \cdots \supseteq \mathfrak{m}^\ell M = \{0\}$ and the long exact sequence for Ext-modules, we reduce to the case where $\mathfrak{m}M = 0$. Now M is finitely generated over the field $S/\mathfrak{m}$, and so we further reduce to the case where $M = S/\mathfrak{m}$ is the residue field of the local ring. Since S is regular, the Koszul complex (for any system of $2n$ generators for the maximal ideal) resolves $S/\mathfrak{m}$; from this resolution, one obtains

$$\operatorname{Ext}^j_S(S/\mathfrak{m}, S) = \begin{cases} S/\mathfrak{m} & \text{if } j = 2n, \\ 0 & \text{if } j \neq 2n. \end{cases}$$

This establishes (a) and (b) in the case $d = 0$. For the inductive step, it suffices (with a little bit of extra work) to consider the case where there is an element $f \in \mathfrak{m}$ that is not a zero-divisor on M. We then have a short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0,$$

and $d(M/fM) = d - 1$. The geometric picture is that $\operatorname{Supp} M$ is a closed subset of dimension d, and that the hypersurface defined by f intersects it in a subset of dimension $d - 1$; the S-module M/fM is of course representing the restriction of M to the hypersurface. Define

$$E^j = \operatorname{Ext}^j_S(M, S) \quad \text{and} \quad F^j = \operatorname{Ext}^j_S(M/fM, S).$$

By induction, we have $F^j = 0$ unless $2n - d - 1 \leq j \leq 2n$, and $d(F^j) \leq 2n - j$. The long exact cohomology sequence for Ext-modules gives

$$\cdots \to F^j \to E^j \xrightarrow{f} E^j \to F^{j+1} \to \cdots.$$

If $j \notin \{2n - d, \dots, 2n\}$, then we have $F^j = F^{j+1} = 0$, and so multiplication by f is an isomorphism from E^j to itself. Since E^j is a finitely generated S-module, and $f \in \mathfrak{m}$, this implies $E^j = 0$ by Nakayama's lemma. This proves (a). Also from the exact sequence, E^j/fE^j is isomorphic to a submodule of F^{j+1}, and therefore

$$2n - (j + 1) \geq d(F^{j+1}) \geq d(E^j/fE^j) \geq d(E^j) - 1,$$

which proves (b).

Now we turn to (c). From (a), we get $j(M) \geq 2n - d(M)$. Combined with (b), this gives

$$d(E^j) \leq 2n - j \leq 2n - j(M) \leq d(M),$$

with strict inequality for $j > j(M)$. Assume for the sake of contradiction that $d(E^{j(M)}) < d(M)$. Then $d(E^j) < d(M)$ for every $j \geq 0$. Setting

$$E = \bigoplus_{j=2n-d(M)}^{2n} E^j,$$

this gives $d(E) < d(M)$, and therefore the ideal $J(E)$ must be strictly bigger than $J(M)$. After localizing at an element $f \in J(E) \setminus J(M)$, we achieve that $M \neq 0$ but $\operatorname{Ext}^j_S(M, S) = 0$ for every $j \geq 0$. Now one can show (as an exercise) that this contradicts the fact that M is finitely generated.

It remains to deduce (d). We have already seen that $j(M) \leq 2n - d(M)$. The reverse inequality follows from (c) and (b), because

$$d(M) = d(E^{j(M)}) \leq 2n - j(M).$$

This completes the proof. □

Filtered resolutions. Now we return to the case where M is a finitely generated left R-module. Choose a good filtration $F_\bullet M$. Proposition 6.4, applied to the finitely generated S-module $\operatorname{gr}^F M$, gives

$$d(\operatorname{gr}^F M) + j(\operatorname{gr}^F M) = 2n.$$

Obviously, we have $J(M) = \sqrt{\operatorname{Ann}_S(\operatorname{gr}^F M)} = J(\operatorname{gr}^F M)$, and therefore

$$d(M) = \dim S/J(M) = d(\operatorname{gr}^F M).$$

The identity $d(M) + j(M) = 2n$ in Theorem 6.3 is therefore equivalent to

$$j(M) = j(\operatorname{gr}^F M).$$

In order to prove the theorem, we therefore need to understand the relationship between $\operatorname{Ext}^j_R(M, R)$ and $\operatorname{Ext}^j_{\operatorname{gr}^F M}(\operatorname{gr}^F M, \operatorname{gr}^F R)$. We will see next time that this involves a spectral sequence. To set it up, we need a resolution of M by free R-modules that takes into account the good filtration $F_\bullet M$.

Proposition 6.5. *Let $(M, F_\bullet M)$ be a finitely generated R-module with a good filtration. Then there exists a free resolution*

$$\cdots \to L_2 \to L_1 \to L_0 \to M \to 0$$

where each $(L_j, F_\bullet L_j)$ is a free R-module with a good filtration, and the differentials in the resolution respect the filtrations. Moreover,

(a) each $\operatorname{gr}^F L_j$ is free over S, of the same rank as L_j, and

(b) the complex of S-modules

$$\cdots \to \operatorname{gr}^F L_2 \to \operatorname{gr}^F L_1 \to \operatorname{gr}^F L_0 \to \operatorname{gr}^F M \to 0$$

is exact.

Proof. For any $e \in \mathbb{Z}$, define $R(e) = R$, but with the good filtration $F_j R(e) = F_{j+e} R$. We are going to construct a resolution in which each L_j is a direct sum of copies of $R(e)$ for various values of e.

We start by building L_0. Since $\operatorname{gr}^F M$ is a finitely generated graded S-module, we can choose homogeneous generators $[m_1], \dots, [m_r]$, of degrees $e_1, \dots, e_r$, meaning that $m_i \in F_{e_i} M$. Then

$$\operatorname{gr}_j^F M = \sum_{i=1}^r S_{j-e_i}[m_i],$$

and an easy argument shows that therefore

$$F_j M = \sum_{i=1}^r F_{j-e_i} R \cdot m_i$$

for every $j \geq 0$. This means exactly that we have a surjective morphism of left R-modules

$$L_0 = \bigoplus_{i=1}^r R(-e_i) \to M$$

compatible with the good filtrations on both terms, such that $\operatorname{gr}^F L_0 \to \operatorname{gr}^F M$ is also surjective. Let M' be the kernel of $L_0 \to M$, with the induced filtration. Then the sequence

$$0 \to \operatorname{gr}^F M' \to \operatorname{gr}^F L_0 \to \operatorname{gr}^F M \to 0$$

is short exact, and since S is noetherian, it follows that $\operatorname{gr}^F M'$ is finitely generated; in other words, M' is finitely generated, and $F_\bullet M'$ is a good filtration. Now apply the same argument to $(M', F_\bullet M')$ to construct L_1, and continue step by step to create the desired free resolution for M. ☐

Let $\cdots \to L_2 \to L_1 \to L_0$ be a filtered free resolution of M with the properties in the proposition. If we set $L_j^* = \operatorname{Hom}_R(L_j, R)$, then the complex of right R-modules

$$0 \to L_0^* \to L_1^* \to L_2^* \to \cdots$$

can be used to compute $\operatorname{Ext}_R^j(M, R)$. In fact, each term in this complex again has a natural compatible filtration (in the sense of right R-modules).

Definition 6.6. Let L be a finitely generated left R-module with a good filtration $F_\bullet L$. On the right R-module $L^* = \operatorname{Hom}_R(L, R)$, we define

$$F_j L^* = \left\{ \phi \in L^* \;\middle|\; \phi(F_i L) \subseteq F_{i+j} R \text{ for every } i \geq 0 \right\}$$

for every $j \in \mathbb{Z}$.

Lemma 6.7. *Suppose that L is a finitely generated left R-module with a good filtration $F_\bullet L$. Then L^* is a finitely generated right R-module, and the filtration $F_\bullet L^*$ is again good.*

Proof. Since L is finitely generated, L^* is clearly again finitely generated. It is easy to see that $F_j L^* \cdot F_k R \subseteq F_{j+k} L^*$. Indeed, if $\phi \in F_j L^*$ and $r \in F_k R$, then we have

$$(\phi \cdot r)(x) = \phi(x) \cdot r$$

and this belongs to $F_{i+j} R \cdot F_k R \subseteq F_{i+(j+k)} R$. We also need to prove that the filtration on L^* is exhaustive. Let $\phi \in \mathrm{Hom}_R(L, R)$ be arbitrary. Since the filtration on L is good, there exists some $j_0 \geq 0$ such that $F_{j+j_0} L = F_j R \cdot F_{j_0} L$ for every $j \geq 0$. Since ϕ is left R-linear, we get

$$\phi(F_{j+j_0} L) \subseteq F_j R \cdot \phi(F_{j_0} L).$$

Now $F_{j_0} L$ is finitely generated over $F_0 R$, and therefore $\phi(F_{j_0} L) \subseteq F_{j_1} R$ for some $j_1 \geq 0$. We now obtain

$$\phi(F_{j+j_0} L) \subseteq F_j R \cdot F_{j_1} R \subseteq F_{j+j_1} R,$$

which is enough to conclude that $\phi \in F_{j_1} L^*$. The proof that the filtration $F_\bullet L^*$ is good is left as an exercise. □

Exercises.

Exercise 6.1. Let S be a local ring, M a finitely generated S-module. Suppose that $\mathrm{Ext}_S^j(M, S) = 0$ for every $j \geq 0$. Prove that $M = 0$.

Exercise 6.2. Let $L = R(\ell)$, as a left R-module. Show that L^* is isomorphic to $R(-\ell)$ as a right R-module (with the filtration defined in class).

Exercise 6.3. Let L be a finitely generated left R-module with a good filtration $F_\bullet L$. Show that the natural morphism

$$\mathrm{gr}^F L^* \to \mathrm{Hom}_S\big(\mathrm{gr}^F L, S\big)$$

is injective, and use this to prove that $\mathrm{gr}^F L^*$ is finitely generated over S.