Lecture 5 (February 11)

Our first task is to finish the proof of Theorem 4.6 from last time. In class, I reviewed the notation and the first half of the argument; look at the notes from last time before reading on.

Step 4. Let's see what the functional equation in (4.7) tells about the Fourier coefficients of ϑ. For that, we need to rewrite the terms with $H-B$ in a more manageable way. Each $\gamma \in \Gamma$ determines a homomorphism

$$\hat{\gamma} \colon \Gamma' \to \mathbb{Z}, \quad \hat{\gamma}(\delta) = E(\gamma, \delta).$$

As we observed during Step 1 of the proof, the mapping

$$\Gamma/\Gamma' \to (\Gamma')^*, \quad \gamma + \Gamma' \mapsto \hat{\gamma},$$

is injective, and the image has index $m = \sqrt{\det E}$. Now if $\gamma \in \Gamma$ and $\delta \in \Gamma'$, then

$$H(\delta,\gamma) - B(\delta,\gamma) = \overline{H(\gamma,\delta)} - B(\gamma,\delta) = \overline{H(\gamma,\delta)} - H(\gamma,\delta) = -2iE(\gamma,\delta),$$

because $B|_{W_{\mathbb{R}} \times W_{\mathbb{R}}} = H|_{W_{\mathbb{R}} \times W_{\mathbb{R}}}$ and both B and H are $\mathbb{C}$-linear in their first argument. Consequently,

$$H(\delta,\gamma) - B(\delta,\gamma) = -2i\hat{\gamma}(\delta),$$

and because $V = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$ and everything is $\mathbb{C}$-linear, we get

$$H(v,\gamma) - B(v,\gamma) = -2i\hat{\gamma}(v) \qquad \text{for all } v \in V.$$

This allows us to rewrite (4.7) as

$$\vartheta(v+\gamma) = e^{-2\pi i \lambda(\gamma)} \alpha(\gamma) \cdot e^{-2\pi i \hat{\gamma}(v) - i\pi\hat{\gamma}(\gamma)} \cdot \vartheta(v).$$

If we now substitute the Fourier expansion for ϑ into this identity, we get

$$\sum_{\chi} c_\chi e^{2\pi i \chi(\gamma)} e^{2\pi i \chi(v)} = e^{-2\pi i \lambda(\gamma)} \alpha(\gamma) \cdot e^{-i\pi\hat{\gamma}(\gamma)} \sum_{\chi} c_\chi e^{2\pi i (\chi(v) - \hat{\gamma}(v))}.$$

Comparing coefficients on both sides, we find that

$$c_{\chi+\hat{\gamma}} = e^{2\pi i \lambda(\gamma)} \alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i \chi(\gamma)} \cdot c_\chi. \tag{5.1}$$

This shows that all the Fourier coefficients are uniquely determined once we know the values on each coset of Γ/Γ' inside $(\Gamma')^*$. Since the index of this subgroup is m, we conclude that there are at most m linearly independent solutions, and therefore

$$\dim H^0\big(X, L(H,\alpha)\big) \leq m.$$

Step 5. It remains to prove that we get exactly m linearly inpedendent theta functions. For that, we have to prove that each time we have a solution to (5.1), the corresponding Fourier series actually converges. Let's fix a homomorphism $\chi_0 \in (\Gamma')^*$, and consider its coset in $(\Gamma')^*$. We set $c_{\chi_0} = 1$, and $c_\chi = 0$ unless $\chi = \chi_0 + \hat{\gamma}$ for some $\gamma \in \Gamma$. Solving the equations in (5.1) above, we find that

$$c_{\chi_0+\hat{\gamma}} = e^{2\pi i \lambda(\gamma)} \alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i \chi_0(\gamma)}.$$

The Fourier series with these coefficients is

$$\sum_{\hat{\gamma}} e^{2\pi i \lambda(\gamma)} \alpha(\gamma)^{-1} \cdot e^{i\pi\hat{\gamma}(\gamma)} e^{2\pi i \chi_0(\gamma)} e^{2\pi i \chi_0(v) + 2\pi i \hat{\gamma}(v)}.$$

Note that each term only depends on $\hat{\gamma}$, as indicated, because all the factors where γ appears are equal to 1 when $\gamma \in \Gamma'$. Anyway, the Fourier series is clearly dominated, in absolute value, by the series

$$\sum_{\hat{\gamma}} e^{-\pi \operatorname{Im} \hat{\gamma}(\gamma)} e^{-2\pi \operatorname{Im} \chi_0(\gamma)} e^{-2\pi \operatorname{Im} \chi_0(v) - 2\pi \operatorname{Im} \hat{\gamma}(v)}.$$

We will prove in a moment that that $\operatorname{Im} \hat{\gamma}(\gamma) = H\big(q(\gamma), q(\gamma)\big)$, where $q \colon V \to W_{\mathbb{R}}$ is the projection. As long as v stays in a compact subset, the exponent in the exponential therefore looks like

$$-\pi H\big(q(\gamma), q(\gamma)\big) + O\big(\|\gamma\|\big),$$

where $\|-\|$ is any inner product on V. Because H is positive definite, and q embeds Γ/Γ' as a lattice into $W_{\mathbb{R}}$, the quadratic term is negative definite, and as in the case of the Jacobi theta function, this ensures that the series converges. Our Fourier series is therefore absolutely and uniformly convergent on compact subsets, and so each of the m linearly independent choices of Fourier coefficients gives rise to a theta function for (H, α).

Step 6. It remains to prove that

$$\operatorname{Im} \hat{\gamma}(\gamma) = H\big(q(\gamma), q(\gamma)\big).$$

Recall that $p \colon V \to W_{\mathbb{R}}$ and $q \colon V \to W_{\mathbb{R}}$ are the two projections, so $v = p(v) + Jq(v)$. We showed earlier that

$$H(v, \gamma) - B(v, \gamma) = -2i\hat{\gamma}(v).$$

Plugging in $v = \gamma$ gives

$$\operatorname{Im} \hat{\gamma}(\gamma) = \operatorname{Re} \frac{H(\gamma, \gamma) - B(\gamma, \gamma)}{2}.$$

Because H is hermitian and $J(v) = iv$, we have

$$H(\gamma, \gamma) = H\big(p(v), p(v)\big) + H\big(q(v), q(v)\big) - iH\big(p(v), q(v)\big) + iH\big(q(v), p(v)\big).$$

At the same time, B is bilinear, and equal to H on $W_{\mathbb{R}} \times W_{\mathbb{R}}$, and so

$$\begin{aligned} B(\gamma, \gamma) &= B\big(p(v), p(v)\big) - B\big(q(v), q(v)\big) + iB\big(p(v), q(v)\big) + iB\big(q(v), p(v)\big) \\ &= H\big(p(v), p(v)\big) - H\big(q(v), q(v)\big) + iH\big(p(v), q(v)\big) + iH\big(q(v), p(v)\big). \end{aligned}$$

Taking the difference, we obtain

$$\frac{H(\gamma, \gamma) - B(\gamma, \gamma)}{2} = H\big(q(v), q(v)\big) - iH\big(p(v), q(v)\big),$$

and the real part of the right-hand side is obviously $H\big(q(v), q(v)\big)$.

Riemann-Roch theorem. We can also express Theorem 4.6 in a more cohomological way, as follows. Let (H, α) be Appel-Humbert data, with H positive definite; then the line bundle $L = L(H, \alpha)$ is ample. Since the canonical bundle of X is trivial, the Kodaira vanishing theorem shows that $H^i(X, L) = 0$ for $i > 0$. Therefore the Euler characteristic of L is equal to

$$\chi(X, L) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, L) = \dim H^0(X, L).$$

Our computation for the dimension of the space of sections, together with Corollary 5.3 below, gives

$$\chi(X, L) = \sqrt{\det E} = \frac{1}{n!} c_1(L)^n.$$

Because the tangent bundle T_X is trivial, this is exactly the formula one gets from Grothendieck's Riemann-Roch theorem.

Some matrix calculations. Let $E\colon \Gamma\times\Gamma\to\mathbb{Z}$ be an alternating bilinear form, such that the induced group homomorphism

$$E\colon \Gamma\to \operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z}),\quad \gamma\mapsto E(\gamma,-),$$

is injective. For the sake of completeness, I am including proofs for the assertions about E that we used in the previous two sections. The key technical point is the following lemma.

Lemma 5.2. *There is a basis $e_1,\dots,e_{2n}\in\Gamma$ such that the $2n\times 2n$-matrix with entries $E(e_i,e_j)$ has the form*

$$\begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

for positive integers $m_1\mid m_2\mid\cdots\mid m_n$. In particular,

$$\det E=(m_1\cdots m_n)^2$$

is always the square of an integer.

Proof. Choose two vectors $e_1,e_2\in\Gamma$ such that $m_1=E(e_1,e_2)$ is the smallest possible positive integer among the values of E. For any $\gamma\in\Gamma$, we have

$$\begin{aligned} E(\gamma-ae_1-be_2,e_1)&=E(\gamma,e_1)+bm_1,\\ E(\gamma-ae_1-be_2,e_2)&=E(\gamma,e_2)-am_2. \end{aligned}$$

By minimality of m_1, both integers $E(\gamma,e_1)$ and $E(\gamma,e_2)$ must be divisible by m_1, and so we can uniquely choose $a,b\in\mathbb{Z}$ such that $\gamma-ae_1-be_2$ becomes orthogonal to e_1 and e_2. This means that $\Gamma=\mathbb{Z}e_1\oplus\mathbb{Z}e_2\oplus\Gamma'$, where Γ' is the subgroup

$$\Gamma'=\left\{\gamma\in\Gamma\;\middle|\;E(\gamma,e_1)=E(\gamma,e_2)=0\right\}.$$

Again by minimality of m_1, all values of E on Γ' must be divisible by m_1. The result we want now follows by induction on the rank of Γ. □

One consequence is that the image of the homomorphism $E\colon\Gamma\to\operatorname{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$ has index equal to $\det E$. The reason is that the image of

$$\begin{pmatrix} 0 & m\\ -m & 0\end{pmatrix}\colon \mathbb{Z}^2\to\mathbb{Z}^2$$

is the subgroup $m\mathbb{Z}^2$, which clearly has index m^2. We used this fact during the proof of Theorem 4.6. Another consequence is the following description of $\det E$ in terms of intersection numbers.

Corollary 5.3. *Set $L=L(H,\alpha)$, with H positive definite. Then*

$$\sqrt{\det E}=\frac{1}{n!}c_1(L)^n.$$

Proof. Choose a basis $e_1,\dots,e_{2n}\in\Gamma$ as in the lemma, and let $e_1^*,\cdots,e_{2n}^*\in\Gamma^*$ be the dual basis. As elements of $H^2(X,\mathbb{Z})\cong\bigwedge^2\Gamma^*$, we then have

$$c_1(L)=\sum_{j<k}E(e_j,e_k)\,e_j^*\wedge e_k^*=\sum_{i=1}^n m_i\,e_{2i-1}^*\wedge e_{2i}^*,$$

where $L=L(H,\alpha)$. Therefore

$$\frac{1}{n!}c_1(L)^n=m_1\cdots m_n\cdot e_1^*\wedge\cdots\wedge e_{2n}^*,$$

and as elements of $H^{2n}(X,\mathbb{Z}) \cong \bigwedge^{2n}\Gamma^*$, this gives

$$\frac{1}{n!}c_1(L)^n = m_1\cdots m_n = \sqrt{\det E}. \qquad \square$$

Some terminology. An *abelian variety* is by definition a compact complex torus $X = V/\Gamma$ that can be embedded into projective space. According to Theorem 4.4, this is equivalent to the existence of a positive-definite hermitian form $H\colon V\times V \to \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma\times\Gamma$. If that is the case, then any line bundle of the form $L(H,\alpha)$ is ample; for historical reasons, such a line bundle is called a *polarization*. If we choose a basis for Γ as in Lemma 5.2, such that

$$E = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix},$$

then the n-tuple of integers $(m_1, m_2, \dots, m_n)$ with $m_1 \mid m_2 \mid \cdots \mid m_n$ is called the *type* of the polarization. A polarization is called *principal* if $m_1 = \cdots = m_n = 1$; this is equivalent to saying that the homomorphism

$$E\colon \Gamma \to \Gamma^*, \quad \gamma \mapsto E(\gamma, -),$$

is an isomorphism. (In that case, E is also said to be *unimodular*.)

Exercise 5.1. If $m_1 \geq 2$, show that $L(H,\alpha)$ is the m_1-th tensor power of some other holomorphic line bundle.

Jacobians. Let C be a compact Riemann surface of genus $g \geq 1$. The most important example of a principally polarized abelian variety is the Jacobian

$$J(C) = \operatorname{Pic}^0(C) \cong H^1(C,\mathscr{O}_C)/H^1(C,\mathbb{Z}).$$

Let's verify that this is the case. The starting point is the Hodge decomposition

$$H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C) \cong H^0(C,\Omega^1_C) \oplus H^1(C,\mathscr{O}_C).$$

The mapping $H^1(C,\mathbb{R}) \to H^1(C,\mathscr{O}_C)$ is an isomorphism of $\mathbb{R}$-vector spaces: if $\alpha \in H^1(C,\mathbb{R})$, the in the Hodge decomposition $\alpha = \alpha^{1,0}+\alpha^{0,1}$, one has $\alpha^{1,0} = \overline{\alpha^{0,1}}$, and so $\alpha^{0,1} = 0$ implies $\alpha = 0$. It follows that the composition

$$H^1(C,\mathbb{Z}) \to H^1(C,\mathbb{R}) \to H^1(C,\mathscr{O}_C)$$

embeds $\Gamma = H^1(C,\mathbb{Z})$ as a lattice into $V = H^1(C,\mathscr{O}_C)$, and so the quotient is a compact complex torus of dimension g.

To show that it is an abelian variety, we need to find a positive-definite hermitian form H such that $E = \operatorname{Im} H$ is integral. Consider the alternating pairing

$$E\colon H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z}, \quad E(\gamma,\delta) = [C] \cap (\gamma \cup \delta).$$

We have $H^1(C,\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}\bigl(H_1(C,\mathbb{Z}),\mathbb{Z}\bigr)$, and by Poincaré duality, the mapping

$$H^1(C,\mathbb{Z}) \to H_1(C,\mathbb{Z}), \quad \gamma \mapsto [C]\cap\gamma,$$

is an isomorphism. Therefore E is unimodular.

Using the embedding $H^1(C,\mathbb{Z}) \hookrightarrow H^1(C,\mathbb{C})$, we can view each element γ as a de Rham cohomology class. As such, we have

$$E(\gamma,\delta) = \int_C \gamma\wedge\delta = \int_C \gamma^{0,1}\wedge\overline{\delta^{0,1}} + \int_C \overline{\gamma^{0,1}}\wedge\delta^{0,1} = 2\operatorname{Re}\int_C \gamma^{0,1}\wedge\overline{\delta^{0,1}}$$

The Hodge-Riemann bilinear relations show that the hermitian form

$$H\colon H^{0,1}(C)\times H^{0,1}(C) \to \mathbb{C}, \quad H(\gamma^{0,1},\delta^{0,1}) = -2i\int_C \gamma^{0,1}\wedge\overline{\delta^{0,1}}$$

is positive-definite. (There is again nothing deep here: locally, $\gamma^{0,1}$ looks like $f d\bar{z}$ for some function f, and therefore

$$-i\gamma^{0,1} \wedge \overline{\gamma^{0,1}} = -i|f|^2 d\bar{z} \wedge dz = 2|f|^2\, dx \wedge dy \geq 0;$$

therefore the integral is nonnegative, and vanishes iff $\gamma^{0,1} = 0$.)

The computation above tells us that

$$E(\gamma, \delta) = -\operatorname{Im} H(\gamma^{0,1}, \delta^{0,1}),$$

and so we should redefine the pairing E as

$$E\colon H^1(C, \mathbb{Z}) \times H^1(C, \mathbb{Z}) \to \mathbb{Z}, \quad E(\gamma, \delta) = -[C] \cap (\gamma \cup \delta)$$

in order for it to be the first Chern class of an ample line bundle. Since E is unimodular, the Jacobian $J(C)$ is therefore a principally polarized abelian variety.

Morphisms. Let $X_1 = V_1/\Gamma_1$ and $X_2 = V_2/\Gamma_2$ be two compact complex tori. The following simple lemma shows that, up to translation, every holomorphic mapping from X_1 to X_2 is a group homomorphism.

Lemma 5.4. *Let $f\colon X_1 \to X_2$ be a holomorphic mapping between two compact complex tori. Then f is the composition of a group homomorphism and a translation.*

Proof. If $f(0) = y$, we can compose f with the holomorphic mapping

$$X_2 \to X_2, \quad x \mapsto x - y,$$

and arrange that $f(0) = 0$. So it suffices to prove that if $f(0) = 0$, then f is a group homomorphism. Because $V_1 \to X_1$ and $V_2 \to X_2$ are the universal covering spaces, f lifts uniquely to a holomorphic mapping $\tilde{f}\colon V_1 \to V_2$ with $\tilde{f}(0) = 0$, as in the following diagram:

$$\begin{tikzcd} V_1 \arrow[r, "\tilde{f}"] \arrow[d] & V_2 \arrow[d] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

For every $\gamma \in \Gamma_1$, we must have

$$\tilde{f}(v + \gamma) - \tilde{f}(v) \in \Gamma_2,$$

and after differentiating this formula, we see that all the first derivatives of $\tilde{f}$ are holomorphic and doubly periodic, hence constant. As $\tilde{f}(0) = 0$, this implies that $\tilde{f}$ is a linear map; but then f is clearly a group homomorphism as well. $\square$