

LECTURE 4 (FEBRUARY 6)

The Appel-Humbert theorem. Last time, we described all holomorphic line bundles on a compact complex torus $X = V/\Gamma$. There were two pieces of data:

- (1) A hermitian form $H: V \times V \rightarrow \mathbb{C}$ such that $E = \text{Im } H$ takes integer values on $\Gamma \times \Gamma$. Let $\text{Herm}_{\mathbb{Z}}(V, \Gamma)$ denote the set of all such.
- (2) A mapping $\alpha: \Gamma \rightarrow U(1)$ such that

$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)e^{i\pi E(\gamma, \delta)} \quad \text{for all } \gamma, \delta \in \Gamma.$$

We call such a pair (H, α) an *Appel-Humbert datum*. Let $\text{AH}(V, \Gamma)$ be the set of Appel-Humbert data. To each $(H, \alpha) \in \text{AH}(V, \Gamma)$, we associated a holomorphic line bundle $L(H, \alpha)$ on X , defined as the quotient of $V \times \mathbb{C}$ by the Γ -action

$$\gamma \cdot (v, z) = (v + \gamma, e^{\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)} \alpha(\gamma) \cdot z).$$

We now get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\Gamma, U(1)) & \longrightarrow & \text{AH}(V, \Gamma) & \longrightarrow & \text{Herm}_{\mathbb{Z}}(V, \Gamma) \longrightarrow 0 \\ & & \downarrow & & \downarrow L & & \downarrow \\ 0 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \ker(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)) \longrightarrow 0 \end{array}$$

The first arrow in the first line sends a homomorphism α to the pair $(0, \alpha)$, and the second arrow sends an Appel-Humbert datum (H, α) to the hermitian form H . The vertical arrow in the middle sends (H, α) to the associated line bundle $L(H, \alpha)$. We could not quite state the main result last time, so here it is.

Theorem 4.1 (Appel-Humbert). *The mapping $L: \text{AH}(V, \Gamma) \rightarrow \text{Pic}(X)$ is an isomorphism of abelian groups.*

Proof. The multiplication in $\text{AH}(V, \Gamma)$ is given by the rule

$$(H_1, \alpha_1) \cdot (H_2, \alpha_2) = (H_1 + H_2, \alpha_1 \alpha_2).$$

This is compatible with the group structures on $\text{Hom}_{\mathbb{Z}}(\Gamma, U(1))$ and on $\text{Herm}_{\mathbb{Z}}(V, \Gamma)$. Now if two line bundles are represented by cocycles, in the way we introduced last time, then their tensor product is represented by the pointwise product of the two cocycles. Together with the explicit formula for $L(H, \alpha)$, this shows that

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \cong L(H_1 + H_2, \alpha_1 \alpha_2),$$

and so L is indeed a group homomorphism. We showed last time that the first and third vertical arrow in the diagram are isomorphisms; by the five lemma, the arrow in the middle is also an isomorphism. \square

Example 4.2. Let's look at the case of elliptic curves. Here $V = \mathbb{C}$, with coordinate z , and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$ is a point in the upper halfplane. The pairing $E = \text{Im } H$ is determined by the integer $m = E(\tau, 1)$, which is the first Chern class of the line bundle. As always, we extend E \mathbb{R} -linearly; then

$$m = E(\text{Re } \tau + i \text{Im } \tau, 1) = \text{Re } \tau E(1, 1) + \text{Im } \tau E(i, 1),$$

and therefore $E(i, 1) = m / \text{Im } \tau$. The hermitian pairing H is then determined by

$$H(1, 1) = E(i, 1) + iE(1, 1) = \frac{m}{\text{Im } \tau}.$$

So the quantity $\text{Im } \tau$ shows up in the Appel-Humbert description of line bundles.

Global sections. Next, we are going to compute the space of global sections of $L(H, \alpha)$, and determine under what conditions $L(H, \alpha)$ is ample. Along the way, we'll prove the following interesting fact: If L is a holomorphic line bundle on a compact complex torus X , and if $H^0(X, L) \neq 0$, then there is a surjective holomorphic group homomorphism $q: X \rightarrow Y$ to another compact complex torus Y , and an *ample* line bundle M on Y , such that $L \cong q^*M$.

Consider a line bundle of the form $L(H, \alpha)$. From the description as $V \times \mathbb{C}/\Gamma$, we see that a global section of $L(H, \alpha)$ is the same thing as a holomorphic function $\theta: V \rightarrow \mathbb{C}$ with the property that

$$(4.3) \quad \theta(v + \gamma) = e^{\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)} \alpha(\gamma) \cdot \theta(v)$$

for every $\gamma \in \Gamma$. Such functions are called *theta functions* for the pair (H, α) . We will see in a moment how these are related to the classical theta function.

It turns out that the existence or non-existence of sections depends very much on the hermitian form H . There are three cases:

Case 1. The hermitian form H is degenerate. Recall that $E = \text{Im } H$ is integral on $\Gamma \times \Gamma$. Consider the null space

$$\begin{aligned} V_0 &= \{ v \in V \mid H(v, w) = 0 \text{ for all } w \in V \} \\ &= \{ v \in V \mid E(v, \gamma) = 0 \text{ for all } \gamma \in \Gamma \}. \end{aligned}$$

The first line shows that V_0 is a complex subspace of V , and the second line shows that $\Gamma_0 = V_0 \cap \Gamma$ is again a lattice in V_0 . Define $V_1 = V/V_0$ and $\Gamma_1 = \Gamma/\Gamma_0$; then $X_1 = V_1/\Gamma_1$ is again a compact complex torus. Because V_0 is the nullspace, H descends to a nondegenerate hermitian form H_1 on V_1 .

For $\gamma \in \Gamma_0$, the transformation rule in (4.3) gives

$$\theta(v + \gamma) = \alpha(\gamma)\theta(v),$$

and since $|\alpha(\gamma)| = 1$, this shows that θ is bounded on each coset $v + V_0$. By Liouville's theorem, θ is constant, and so there is a holomorphic function

$$\theta_1: V_1 \rightarrow \mathbb{C}$$

such that $\theta(v) = \theta_1(v + V_0)$. It then follows that $\alpha(\gamma) = 1$ for $\gamma \in \Gamma_0$, and so there is a function $\alpha_1: \Gamma_1 \rightarrow U(1)$ with the property that $\alpha(\gamma) = \alpha_1(\gamma + \Gamma_0)$. If we let $q: X \rightarrow X_1$ denote the quotient mapping, this means that $L(H, \alpha) \cong q^*L(H_1, \alpha_1)$ is the pullback of a holomorphic line bundle from the smaller torus X_1 . Without loss of generality, we therefore need to consider only the case when H is nondegenerate.

Case 2. There is a nonzero vector $w \in V$ such that $H(w, w) < 0$. We are going to show that this forces $\theta = 0$. In order to use the transformation rule in (4.3), we pick a compact subset $K \subseteq V$ such that $V = K + \Gamma$. For every $t \in \mathbb{C}$, we can then write $tw = k_t + \gamma_t$, with $k_t \in K$ and $\gamma_t \in \Gamma$. Now fix a point $v \in V$ and consider the restriction of θ to the complex line $v + tw$. We have

$$|\theta(v + tw)| = |\theta(v + k_t + \gamma_t)| = |e^{\pi H(v + k_t, \gamma_t) + \frac{\pi}{2} H(\gamma_t, \gamma_t)}| \cdot |\theta(v + k_t)|$$

If we rewrite the exponent in terms of w , we get

$$\begin{aligned} \pi H(v + k_t, \gamma_t) + \frac{\pi}{2} H(\gamma_t, \gamma_t) &= \pi H(v + k_t, tw - k_t) + \frac{\pi}{2} H(tw - k_t, tw - k_t) \\ &= \pi H(w, w)|t|^2 + O(|t|), \end{aligned}$$

because $v \in V$ is fixed and $k_t \in K$ lies in a compact subset. As $H(w, w) < 0$, this expression goes to $-\infty$ when $|t| \rightarrow \infty$. Because the function $\theta(v + tw)$ is holomorphic in t , it follows that $\theta(v + tw) = 0$; but then $\theta(v) = 0$, and so $\theta = 0$. Under the assumption that H is nondegenerate, $L(H, \alpha)$ can therefore have nontrivial sections only when H is positive definite.

Case 3. The hermitian form H is positive definite. If we pick a basis $v_1, \dots, v_n \in V$, and let $z_1, \dots, z_n \in V^*$ denote the dual basis, then the first Chern class of $L(H, \alpha)$ is represented by the closed $(1, 1)$ -form

$$\frac{i}{2} \sum_{j,k=1}^n H(v_j, v_k) dz_j \wedge d\bar{z}_k.$$

This is now a *positive* form, which means that the line bundle $L(H, \alpha)$ is positive (in Kodaira's sense). According to the Kodaira embedding theorem, a sufficiently large power of $L(H, \alpha)$ will therefore embed X into projective space. (Borrowing a piece of terminology from algebraic geometry, we may say that $L(H, \alpha)$ is an *ample* line bundle.) So we have proved the following criterion for X to be projective.

Theorem 4.4. *A compact complex torus $X = V/\Gamma$ is projective iff there exists a positive definite hermitian form $H: V \times V \rightarrow \mathbb{C}$ such that $E = \text{Im } H$ takes integer values on $\Gamma \times \Gamma$.*

The discussion in Case 1 also shows that if $L = L(H, \alpha)$ is a holomorphic line bundle on X such that $H^0(X, L) \neq 0$, then there is surjective holomorphic group homomorphism $q: X \rightarrow X_1$ to a (possibly smaller) compact complex torus X_1 , and an ample line bundle $L_1 = L(H_1, \alpha_1)$, such that $L \cong q^* L_1$. Unlike in other parts of algebraic geometry, the existence of sections is therefore very closely related to ampleness.

Now let us actually determine the space of global sections of $L(H, \alpha)$, under the assumption that H is positive definite. This will also allow us to figure out exactly what power of $L(H, \alpha)$ we need to get an embedding into projective space. The proof is a bit tricky, so let's think about the classical case first.

Example 4.5. Consider $V = \mathbb{C}$ and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, with $\tau \in \mathbb{H}$. For simplicity, let's take $E(\tau, 1) = 1$, and $\alpha(1) = \alpha(\tau) = 1$; these two values determine α uniquely. We already computed that

$$H(1, 1) = \frac{1}{\text{Im } \tau},$$

and so H is positive definite. A theta function for (H, α) is an entire function $\theta: \mathbb{C} \rightarrow \mathbb{C}$ that satisfies the two functional equations

$$\begin{aligned} \theta(z+1) &= e^{\pi H(z,1) + \frac{\pi}{2} H(1,1)} \cdot \theta(z) = e^{\frac{\pi}{2}(2z+1)/\text{Im } \tau} \cdot \theta(z) \\ \theta(z+\tau) &= e^{\pi H(z,\tau) + \frac{\pi}{2} H(\tau,\tau)} \cdot \theta(z) = e^{\frac{\pi}{2}(2z\bar{\tau} + |\tau|^2)/\text{Im } \tau} \cdot \theta(z). \end{aligned}$$

Now the (very classical) idea is to make θ periodic, meaning invariant under the substitution $z \mapsto z+1$, and then to use Fourier series. We can achieve this by completing the square: consider the new entire function

$$\vartheta(z) = e^{-\frac{\pi}{2} z^2 / \text{Im } \tau} \cdot \theta(z).$$

The first functional equation then gives $\vartheta(z+1) = \vartheta(z)$, and so we can expand $\vartheta(z)$ into a Fourier series of the form

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n z},$$

with coefficients $c_n \in \mathbb{C}$. (This is actually quite elementary: using the holomorphic mapping $\mathbb{C} \rightarrow \mathbb{C}^\times$, $z \mapsto e^{2\pi i z}$, descend ϑ to a holomorphic function on \mathbb{C}^* ; the Fourier series is then just the Laurent series of this holomorphic function.)

After simplifying, the second functional equation reads

$$\vartheta(z+\tau) = e^{-i\pi\tau - 2\pi i z} \vartheta(z).$$

If we substitute the Fourier series into this equation, we get

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \tau} e^{2\pi i n z} = e^{-i\pi \tau} \sum_{n \in \mathbb{Z}} c_n e^{2\pi i (n-1)z},$$

and after comparing coefficients, we arrive at the identity

$$c_{n+1} = c_n e^{i\pi(2n+1)\tau}.$$

This shows that all the Fourier coefficients c_n are uniquely determined by c_0 . If we set $c_0 = 1$, we get $c_n = e^{i\pi n^2 \tau}$, and so

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2\pi i n z}$$

is exactly the classical *Jacobi theta function*. The series converges absolutely and uniformly on compact subsets; in fact,

$$|\vartheta(z)| \leq \sum_{n \in \mathbb{Z}} |e^{i\pi n^2 \tau + 2\pi i n z}| = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 \operatorname{Im} \tau} e^{2\pi n \operatorname{Re} z}$$

converges very rapidly on any strip of the form $|\operatorname{Re} z| \leq C$. The conclusion is that the line bundle $L(H, \alpha)$ has a unique holomorphic section, which looks like

$$\theta(z) = e^{\frac{\pi}{2} z^2 / \operatorname{Im} \tau} \cdot \vartheta(z),$$

where ϑ is Jacobi's theta function.

Now we carry out the same kind of computation in general. Let us fix a positive definite hermitian form $H: V \times V \rightarrow \mathbb{C}$ such that $E = \operatorname{Im} H$ takes integer values on $\Gamma \times \Gamma$. After choosing a basis for $\Gamma \cong \mathbb{Z}^{2n}$, we can represent E as a $2n \times 2n$ -matrix with integer entries; let's denote the determinant of this matrix by $\det E$.

Theorem 4.6. *We have $\dim H^0(X, L(H, \alpha)) = \sqrt{|\det E|}$.*

We divide the proof into six steps. The general idea is the same as in the example. We find a subgroup $\Gamma' \subseteq \Gamma$ of rank n , and complete the square in order to make θ invariant under translation by this sublattice. We then study the coefficients in the Fourier series in order to determine all possible theta functions for (H, α) .

Step 1. We find a subgroup $\Gamma' \subseteq \Gamma$ of rank n on which E is trivial. We can turn the pairing E into a group homomorphism

$$E: \Gamma \rightarrow \Gamma^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}), \quad \gamma \mapsto E(\gamma, -).$$

This is injective (because E is nondegenerate over \mathbb{R} and Γ is torsion-free), and the image has index equal to $|\det E|$. Now suppose that Γ' is any subgroup of Γ such that $E|_{\Gamma' \times \Gamma'} = 0$. We get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma' & \longrightarrow & \Gamma & \longrightarrow & \Gamma/\Gamma' \longrightarrow 0 \\ & & \downarrow & & \downarrow E & & \downarrow \\ 0 & \longrightarrow & (\Gamma/\Gamma')^* & \longrightarrow & \Gamma^* & \longrightarrow & (\Gamma')^* \end{array}$$

with exact rows; the last arrow on the bottom can fail to be surjective (because Γ/Γ' can have torsion). If $\gamma \in \Gamma$ is in the kernel of $\Gamma/\Gamma' \rightarrow (\Gamma')^*$, then $E(\gamma, \delta) = 0$ for every $\delta \in \Gamma'$, and so $\Gamma' + \mathbb{Z}\gamma$ is a bigger subgroup on which E is identically zero. So if we take Γ' to be maximal with this property, then

$$\Gamma/\Gamma' \rightarrow (\Gamma')^*$$

must be injective; consequently, Γ/Γ' is torsion-free, and $\text{rk } \Gamma = 2 \text{rk } \Gamma'$, which gives $\text{rk } \Gamma' = n$. Let m be the index of Γ/Γ' in $(\Gamma')^*$. Because the first vertical arrow in the diagram is the dual of the third one, it follows that

$$|\det E| = (\Gamma^* : \Gamma) = ((\Gamma')^* : \Gamma/\Gamma')^2 = m^2,$$

or equivalently, $m = \sqrt{|\det E|}$. So we can restate the theorem as

$$\dim H^0(X, L(H, \alpha)) = m.$$

Anyway, the subgroup Γ' will play the role that $\mathbb{Z} \subseteq \mathbb{Z} + \mathbb{Z}\tau$ played in the example.

Step 2. Now suppose that $\theta: V \rightarrow \mathbb{C}$ is a theta function for (H, α) , with

$$\theta(v + \gamma) = e^{\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)} \alpha(\gamma) \cdot \theta(v)$$

We want to make θ invariant under translation by Γ' , but in order to “complete the square”, we need to turn our hermitian form H into a quadratic form. Let

$$W_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \Gamma' \subseteq V_{\mathbb{R}}$$

be the \mathbb{R} -vector space spanned by Γ' . We have $\dim_{\mathbb{R}} W_{\mathbb{R}} = n$, and because Γ/Γ' is torsion free, we also have $W_{\mathbb{R}} \cap \Gamma = \Gamma'$. Recall that $J \in \text{End}(V_{\mathbb{R}})$ is the endomorphism $J(v) = iv$. The hermitian form H is related to E by the formula $H(v, w) = E(Jv, w) + iE(v, w)$, and so H is identically zero on $W_{\mathbb{R}} \cap J(W_{\mathbb{R}})$. Because H is positive definite, we get $W_{\mathbb{R}} \cap J(W_{\mathbb{R}}) = 0$, and therefore

$$V = W_{\mathbb{R}} \oplus J(W_{\mathbb{R}})$$

for dimension reasons. This shows that $V = \mathbb{C} \otimes_{\mathbb{R}} W_{\mathbb{R}} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$. Let $p: V \rightarrow W_{\mathbb{R}}$ and $q: V \rightarrow J(W_{\mathbb{R}})$ be the two projections; then

$$v = p(v) + Jq(v) \quad \text{for any } v \in V.$$

Now consider the restriction $H|_{W_{\mathbb{R}} \times W_{\mathbb{R}}}$. Because $E = \text{Im } H$, this is an \mathbb{R} -valued symmetric bilinear form; let $B: V \times V \rightarrow \mathbb{C}$ be the unique \mathbb{C} -valued symmetric bilinear form such that $B|_{W_{\mathbb{R}} \times W_{\mathbb{R}}} = H|_{W_{\mathbb{R}} \times W_{\mathbb{R}}}$. This will play the role that the quadratic function $z^2/\text{Im } \tau$ played in the example.

We also need to deal with the factor $\alpha(\gamma)$ that was not there in the example. For $\gamma, \delta \in \Gamma'$, we have $E(\gamma, \delta) = 0$, and therefore $\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)$. By choosing a basis for $\Gamma' \cong \mathbb{Z}^n$, we can find a homomorphism

$$\lambda: \Gamma' \rightarrow \mathbb{R}$$

with the property that $\alpha(\gamma) = e^{2\pi i \lambda(\gamma)}$ for $\gamma \in \Gamma'$. Since $V = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$, this extends uniquely to a \mathbb{C} -linear mapping $\lambda: V \rightarrow \mathbb{C}$.

Step 3. As in the example, we now consider the new holomorphic function

$$\vartheta: V \rightarrow \mathbb{C}, \quad \vartheta(v) = e^{-\frac{\pi}{2} B(v, v)} e^{-2\pi i \lambda(v)} \cdot \theta(v).$$

A brief computation shows that this satisfies the functional equation

$$\vartheta(v + \gamma) = e^{-2\pi i \lambda(\gamma)} \alpha(\gamma) \cdot e^{\pi(H(v, \gamma) - B(v, \gamma)) + \frac{\pi}{2}(H(\gamma, \gamma) - B(\gamma, \gamma))} \cdot \vartheta(v).$$

When $\gamma \in \Gamma'$, both factors are trivial, and so ϑ is invariant under translation by Γ' . We can therefore expand it into a Fourier series

$$\vartheta(v) = \sum_{\chi \in \text{Hom}_{\mathbb{Z}}(\Gamma', \mathbb{Z})} c_{\chi} e^{2\pi i \chi(v)}.$$

The Fourier coefficients $c_{\chi} \in \mathbb{C}$ are indexed by homomorphisms $\chi: \Gamma' \rightarrow \mathbb{Z}$. Note that each χ extends uniquely to a \mathbb{C} -linear mapping $\chi: V \rightarrow \mathbb{C}$, which is how we define the $\chi(v)$ in the exponent.