Lecture 28 (May 8)

The Hodge conjecture for abelian varieties. In the final lecture, I surveyed what is known about the Hodge conjecture for abelian varieties. An important role is played by abelian varieties of "Weil type", but the definition is slightly broader than the one we used during the previous lectures. Let's briefly look at this, in case you want to read some of the papers later on. Let A be an abelian variety of even dimension $2n$. Then A is said to be of *Weil type* if there is an embedding

$$\eta \colon \mathbb{Q}(\sqrt{-d}) \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of an imaginary quadratic field (with $d \geq 2$ a square-free integer) into the rational endomorphism ring of A, such that both eigenspaces for the action of $\eta(\sqrt{-d})$ on $H^{1,0}(A)$ have dimension n. Note that $\eta(\sqrt{-d})^2 = \eta(-d)$ acts on $H^1(A, \mathbb{Q})$ as multiplication by $-d$, and so the two possible eigenvalues of $\eta(\sqrt{-d})$ are exactly $\pm\sqrt{-d}$. If we set $V = H^1(A, \mathbb{Q})$, this is exactly the condition that $\dim V_s^{1,0} = \dim V_s^{0,1} = n$ for each of the two complex embeddings of $\mathbb{Q}(\sqrt{-d})$. Note that $\mathbb{Q}(\sqrt{-d})$ is a CM-field of degree 2.

A *polarization* on an abelian variety of Weil type is by definition an ample divisor class $h \in H^2(A, \mathbb{Z})$ such that $\eta(\sqrt{-d})^* h = d \cdot h$. This may look different, but it is actually the same as our condition that the Rosati involution needs to act as complex conjugation on the CM-field. Let's do the computation. The ample class h defines a polarization on $V = H^1(A, \mathbb{Q})$ by the formula

$$\psi(v, w) = [A] \cap \big(v \cup w \cup h^{2n-1}\big).$$

Here $[A]$ is the fundamental class of A; over the real or complex numbers, this is basically the integral over A. We would like to show that

$$\psi\Big(\eta(\sqrt{-d})^* v, w\Big) = \psi\Big(v, \eta(-\sqrt{-d})^* w\Big).$$

We first observe that $\eta(\sqrt{-d})$ acts on $H^{4n}(A, \mathbb{Q})$ as multiplication by d^{2n}. Indeed, $\eta(\sqrt{-d})$ must be multiplication by some positive integer N, and because $\eta(\sqrt{-d})^2 = \eta(-d)$ acts as multiplication by d^{4n}, we get $N = d^{2n}$. This gives

$$[A] \cap \Big(\eta(\sqrt{-d})^* v \cup \eta(\sqrt{-d})^* w \cup \eta(\sqrt{-d})^* h^{2n-1}\Big) = d^{2n} \cdot [A] \cap \big(v \cup w \cup h^{2n-1}\big).$$

If we now replace w by $\eta(-\sqrt{-d})^* w$, and remember that $\eta(\sqrt{-d})^* h = dh$ and $\eta(d)^* w = dw$, we obtain

$$d^{2n} \cdot [A] \cap \Big(\eta(\sqrt{-d})^* v \cup w \cup h^{2n-1}\Big) = d^{2n} \cdot [A] \cap \Big(v \cup \eta(-\sqrt{-d})^* w \cup h^{2n-1}\Big).$$

This shows that the Rosati involution for ψ is complex conjugation on $\mathbb{Q}(\sqrt{-d})$.

As in the previous lectures, the polarization ψ can be written as

$$\psi = \operatorname{Tr}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\big(\sqrt{-d}\phi\big)$$

for a unique hermitian form $\phi \colon V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(\sqrt{-d})$. The discrete invariants of the polarized abelian variety (A, h) of Weil type are therefore the integer d, as well as the discriminant $\operatorname{disc} \phi$, which is an element in $\mathbb{Q}^\times$ modulo rational numbers of the norm $a^2 + db^2$ with $a, b \in \mathbb{Q}$. (In the "split" case, which is the one we were considering earlier, the discriminant is always $(-1)^n$.)

One can show (by a dimension count) that the space of polarized abelian varieties of Weil type has dimension n^2. The 3-dimensional subspace

$$\Big\langle h^n, \bigwedge\nolimits^{2n}_{\mathbb{Q}(\sqrt{-d})} H^1(A, \mathbb{Q}) \Big\rangle \subseteq H^{2n}(A, \mathbb{Q})$$

consists of Hodge classes; these are again called Hodge classes of Weil type. For a general (A, h), one can show moreover (by computing the Mumford-Tate group) that these are all the Hodge classes in $H^{2n}(A, \mathbb{Q})$.

Remark. All Hodge classes in $H^2(A, \mathbb{Q})$ are algebraic (by the Lefschetz $(1,1)$-theorem). Since the intersection of algebraic classes is algebraic, every Hodge class in the image of $\operatorname{Sym}^2 H^2(A, \mathbb{Q}) \to H^4(A, \mathbb{Q})$ is also algebraic. Mumford constructed the first example of an abelian fourfold that has extra Hodge classes in $H^4(A, \mathbb{Q})$. Weil realized the importance of CM-fields in Mumford's construction, which is why these classes are now called Hodge classes of Weil type.

Here are some known results about the Hodge conjecture for abelian varieties. Let's write $H^{k,k}(A, \mathbb{Q}) = H^{2k}(A, \mathbb{Q}) \cap H^{k,k}(A)$ for the space of Hodge classes in $H^{2k}(A, \mathbb{Q})$. In order to know all the Hodge classes on A, it is enough to know the Mumford-Tate group $\operatorname{MT}(A) = \operatorname{MT}\big(H^1(A, \mathbb{Q})\big)$. The reason is that

$$H^{2k}(A, \mathbb{Q}) = \bigwedge^{2k} H^1(A, \mathbb{Q}),$$

and so the Hodge classes are exactly the classes in $H^{2k}(A, \mathbb{Q})$ that are invariant under the action by $\operatorname{MT}(A)$. Unfortunately, a lot of proofs in this subject work by first classifying all possible Mumford-Tate groups (and their possible representations), and then doing a case-by-case analysis.

(1) Tate proved that the Hodge conjecture is true if A is isogeneous to a product of elliptic curves.
(2) Mari Rámon proved that the Hodge conjecture is true if A is isogeneous to a product of abelian surfaces.
(3) Tankeev proved that the Hodge conjecture holds on simple abelian varieties such that $\dim A$ is a prime number.
(4) Moonen and Zarhin showed that if A is a simple abelian 4-fold such that $\operatorname{Sym}^2 H^{1,1}(A, \mathbb{Q}) \to H^{2,2}(A, \mathbb{Q})$ is not surjective, then A is of Weil type, and $H^{2,2}(A, \mathbb{Q})$ is spanned by the image of $\operatorname{Sym}^2 H^{1,1}(A, \mathbb{Q})$ together with the Hodge classes of Weil type. (Note that A can be of Weil type for several different values of d, and we are supposed to take the Hodge classes of Weil type for all such values.)
(5) Moonen and Zarhin also showed that this holds when A is isogeneous to the product of an elliptic curve with a simple abelian threefold.

Altogether, these results reduce the Hodge conjecture on abelian fourfolds to the case of abelian fourfolds of Weil type, and to proving that all Hodge classes of Weil type are algebraic. This result was recently announced by Markman, after many earlier results (especially by Schoen). Markman proves this for all imaginary quadratic fields and all values of the discriminant, by reducing the problem to abelian sixfolds of Weil type with discriminant -1. A lot of the earlier work was for specific fields and/or specific values of the discriminant. The simplest example is the following result by van der Geemen.

Example 28.1. Van der Geemen gave a nice geometric proof for the following result: On a general principally polarized abelian fourfold of Weil type, with $E = \mathbb{Q}(i)$, all Hodge classes of Weil type are algebraic. In outline, the argument goes like this. The principal polarization can be represented by a symmetric theta divisor Θ, with $h^0\big(A, \mathscr{O}_A(\Theta)\big) = 1$. The line bundle $L = \mathscr{O}_A(2\Theta)$ is then base-point free and has $h^0(A, L) = 2^4 = 16$. The endomorphism $\eta(i)$ acts on $H^0(A, L)$ as an involution, and in the eigenspace decomposition

$$H^0(A, L) = H^0(A, L)^+ \oplus H^0(A, L)^-,$$

the first summand has dimension 10, the second dimension 6. Consider now the rational mapping

$$A \to \mathbb{P}^{15} \to \mathbb{P}^5$$

given by the linear system $|2\Theta|$ followed by projection to the second summand. One can show that the closure of the image is a smooth 4-dimensional quadric. The pullback of one of the two rulings then gives a subvariety of codimension 2 in A, whose class is not a multiple of $h^2 = \Theta^2$. For general A, the space $H^{2,2}(A, \mathbb{Q})$ is generated by h^2 and Weil classes, so at least one Weil class is algebraic; one can then use the monodromy action to conclude that all Weil classes must be algebraic for general A.

Remark. Somebody asked whether the Hodge conjecture is known for Jacobians of curves. I said yes, but that was wrong: the Hodge conjecture for Jacobians is equivalent to the Hodge conjecture for symmetric products of curves, but that's only known in certain special cases.